1

Tut. Sess. 5

*** Pb 5.4 p. 69 Textbook**

Let X have a loglogistic distribution. Demonstrate that the inverse distribution also has a loglogistic distribution. Therefore, there is no need to identify a separate inverse loglogistic distⁿ.

Ans: For loglogistic γ, θ

$$F_X(x) = \frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma}$$ See p. 466

For $\tau = -1$, $F_Y(y) = 1 - F_X(y^{-1})$ theorem (inverse distⁿ)

$$\therefore F_Y(y) = 1 - \frac{(y^{-1}/\theta)^\gamma}{1+(y^{-1}/\theta)^\gamma}$$

$$= \frac{1 + \cancel{(y^{-1}/\theta)^\gamma - (y^{-1}/\theta)^\gamma}}{1+(y^{-1}/\theta)^\gamma}$$

$$F_Y(y) = \frac{1}{1+(y^{-1}/\theta)^\gamma} \quad \frac{\times (y\theta)^\gamma}{\times (y\theta)^\gamma}$$

$$\therefore F_Y(y) = \frac{(y\theta)^\gamma}{(y\theta)^\gamma + 1}$$

$$\therefore F_Y(y) = \frac{(y\theta)^\gamma}{1+(y\theta)^\gamma}$$ which is also loglogistic

distⁿ with parameters γ and $\theta \Rightarrow 1/\theta$

*** Pb 5.5 p. 69 Textbook**

Let Y have a lognormal distribution with parameters μ and σ. Let $Z = \theta Y$. Show that Z also has a lognormal distribution and, therefore, the addition of a third parameter has not created a new distribution.

Ans: $Y \sim \text{lognormal}(\mu, \sigma)$

$$\Rightarrow F_Y(y) = \Phi\left(\frac{\ln y - \mu}{\sigma}\right)$$ See p. 471

$$\therefore F_Z(z) = F_Y(z/\theta)$$ theorem 471 where $Z = \theta Y$

$$\therefore F_Z(z) = \Phi\left(\frac{\ln(z/\theta) - \mu}{\sigma}\right)$$

$$\therefore F_Z(z) = \Phi\left[\frac{\ln z - \ln\theta - \mu}{\sigma}\right]$$

$$\therefore F_Z(z) = \Phi\left[\frac{\ln z - (\mu + \ln\theta)}{\sigma}\right]$$

which is a cdf of another lognormal distⁿ with parameters $\mu \Rightarrow \ln\theta + \mu$ and $\sigma \Rightarrow \sigma$. #

2

pb 5.6 p. 69 Textbook

let X have a pareto distribution with parameters α and θ. let $Y = \ln(1 + X/\theta)$. Determine the name of the distribution of Y and its parameters.

Ans: $\because X \sim$ pareto (α, θ)

$$\therefore F_X(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}$$, See p. 465

For $Y = \ln(1 + X/\theta)$, $F_Y(y) = pr(Y \le y)$

$$\therefore F_Y(y) = pr[\ln(1 + X/\theta) \le y]$$

$$F_Y(y) = pr(1 + X/\theta \le e^y)$$

$$\therefore F_Y(y) = pr[X \le \theta(e^y - 1)]$$

$$= 1 - \left[\frac{\theta}{\theta(e^y - 1) + \theta}\right]$$

$$= 1 - \left[\frac{\cancel{\theta}}{\cancel{\theta} e^y - \cancel{\theta} + \cancel{\theta}}\right]^{\alpha}$$

$$\therefore F_Y(y) = 1 - \left(\frac{1}{e^y}\right)^{\alpha}$$

$F_Y(y) = 1 - e^{-\alpha y}$ which is the distribution function of the exponential distn with parameter $1/\alpha$.

$F_X(x) = 1 - e^{-x/\theta}$
For $X \sim \exp(\theta)$

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Venter [124] notes that if X has a transformed gamma distribution and its scale parameter θ has an inverse tranformed gamma distribution (where the parameter τ is the same in both distns), the resulting mixture has the transformed beta distribution. Demonstrate that this is true.

Ans:

Let $X|\Theta \sim$ transformed gamma (α, θ, τ)

where $\Theta \sim$ inverse transformed gamma (β, δ, τ)

$$\Rightarrow f_{X|\Theta}(x|\theta) = \frac{\tau\, x^{\tau\alpha-1}\, e^{-x^{\tau}\theta^{-\tau}}}{\theta^{\tau\alpha}\,\Gamma(\alpha)} \quad (1)$$

For tranformed gamma α, θ, τ

$$f(x) = \frac{\tau(x/\theta)^{\tau\alpha}\, e^{-(x/\theta)^{\tau}}}{x\,\Gamma(\alpha)}$$

For inv. transformed gamma α, θ, τ

$$f(x) = \frac{\tau(\theta/x)^{\tau\alpha}\, e^{-(\theta/x)^{\tau}}}{x\,\Gamma(\alpha)}$$

See p. 467

$$f_{\Theta}(\theta) = \frac{\tau(\delta/\theta)^{\tau\beta}\, e^{-(\delta/\theta)^{\tau}}}{\theta\,\Gamma(\beta)}$$

$$= \frac{\tau\,\delta^{\tau\beta}\, e^{-\delta^{\tau}\theta^{-\tau}}}{\theta^{\tau\beta+1}\,\Gamma(\beta)} \quad (2)$$

$$\because f_X(x) = \int_0^{\infty} f_{X|\Theta}(x|\theta)\, f_{\Theta}(\theta)\, d\theta \quad \text{(For Mixture distn)} \quad (3)$$

Substitute (1), (2) in (3), we get

$$f_X(x) = \frac{\tau^2 x^{\tau\alpha-1}\delta^{\tau\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{\infty} \frac{\exp[-\theta^{-\tau}(x^{\tau}+\delta^{\tau})]}{\theta^{\tau\alpha+\tau\beta+1}}\, d\theta$$

$$\therefore f_X(x) = \frac{\tau^2 x^{\tau\alpha-1}\delta^{\tau\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{\infty} \theta^{-\tau\alpha-\tau\beta-1} \cdot \exp\left[-\theta^{-\tau}(x^{\tau}+\delta^{\tau})\right] d\theta$$

Let $y = \theta^{-\tau}(x^{\tau}+\delta^{\tau}) \Rightarrow \begin{matrix}\theta: 0 \longrightarrow \infty \\ y: \infty \longrightarrow 0\end{matrix}$

$$\Rightarrow \theta = y^{-1/\tau}(x^{\tau}+\delta^{\tau})^{1/\tau}, \quad d\theta = -\tfrac{1}{\tau}\, y^{-1/\tau-1}(x^{\tau}+\delta^{\tau})^{1/\tau}\, dy$$

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$$\therefore f_X(x) = \frac{\cancel{\tau^2}\, x^{\tau\alpha-1}\, \delta^{\tau\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \left[y^{-1/\tau} (x^\tau + \delta^\tau)^{1/\tau} \right]^{-\tau\alpha-\tau\beta-1} \cdot e^{-y} \cdot \frac{1}{\tau} y^{-1/\tau-1} (x^\tau + \delta^\tau)^{1/\tau}\, dy$$

$$\therefore f_X(x) = \frac{\tau x^{\tau\alpha-1} \delta^{\tau\beta}}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{1}{(x^\tau+\delta^\tau)^{\alpha+\beta}} \cdot \int_0^\infty y^{\alpha+\beta-1} e^{-y}\, dy$$

$$(x^\tau + \delta^\tau)^{-\alpha-\beta-1/\tau} \cdot (x^\tau+\delta^\tau)^{1/\tau} = (x^\tau+\delta^\tau)^{-\alpha-\beta} = \frac{1}{(x^\tau+\delta^\tau)^{\alpha+\beta}}$$

$$\therefore f_X(x) = \frac{\Gamma(\alpha+\beta)\tau x^{\tau\alpha-1}\delta^{\tau\beta}}{\Gamma(\alpha)\Gamma(\beta)(x^\tau+\delta^\tau)^{\alpha+\beta}}$$

$$y^{\alpha+\beta+\cancel{1/\tau}} \cdot y^{\cancel{-1/\tau}-1} = y^{\alpha+\beta-1}$$

which is a transformed beta pdf

with $\gamma = \tau$, $\tau = \alpha$, $\alpha = \beta$, and $\theta = \delta$.

$\gamma \to \tau$
$\tau \to \alpha$
$\theta \to \delta$
$\alpha \to \beta$

See p. 463