1

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Tut. Session (12)

pb 17.17 p. 437

Suppose that, given $\Theta = \theta$, the random variables $X_1, X_2, \dots, X_n$ are independent with poisson pf

$$f_{X_j|\theta}(x_j|\theta) = \frac{\theta^{x_j} e^{-\theta}}{x_j!}, \quad x_j = 0,1,2,\dots$$

(a) let $S = X_1 + X_2 + \dots + X_n$. Show that S has pf

$$f_S(s) = \int_0^\infty \frac{(n\theta)^s e^{-n\theta}}{s!} \pi(\theta)\,d\theta, \quad s = 0,1,2,\dots$$

where Θ has pdf $\pi(\theta)$.

(b) Show that the Bayesian premium is

$$E(X_{n+1}|X=x) = \frac{1+n\bar{x}}{n} \frac{f_S(1+n\bar{x})}{f_S(n\bar{x})}$$

(c) Evaluate the distribution of S in (a) when $\pi(\theta)$ is a gamma distribution. What type of distribution is this?

Ans:

(a) The poisson pgf (prob. generating fn) of each X_j is $P_{X_j}(z|\theta) = e^{\theta(z-1)}$

$\therefore$ The pgf of S is $P_S(z|\theta) = e^{n\theta(z-1)}$ which is a poisson with mean $n\theta$. Hence,

$$f_{S|\theta}(s|\theta) = \frac{(n\theta)^s e^{-n\theta}}{s!}$$

$$\therefore f_S(s) = \int_0^\infty \frac{(n\theta)^s}{s!} e^{-n\theta} \pi(\theta)\,d\theta, \quad s = 0,1,2,\dots \qquad (1)$$

(b) The Bayesian premium is

$$E(X_{n+1}|X) = \int \mu_{n+1}(\theta)\,\pi(\theta|X)\,d\theta = \int_0^\infty \mu(\theta)\,\pi(\theta|X)\,d\theta$$

— See, lecture (29)

where, $\mu(\theta) = E(X|\theta) = \theta$

and the posterior distn $\pi(\theta|X)$ is given by

$$\pi(\theta|X) = \left[\prod_{j=1}^n f(x_j|\theta)\right]\pi(\theta)/f(x), \quad f(x) = \int_0^\infty \left[\prod_{j=1}^n f(x_j|\theta)\right]\pi(\theta)\,d\theta$$

— See, lecture (28)

لا يكتب في هذا الهامش

$\because f(x_j|\theta) = \dfrac{\theta^{x_j} e^{-\theta}}{x_j!}$

$\therefore f(x) = \dfrac{\int_0^\infty \theta^{\sum x_j} e^{-n\theta} \pi(\theta)\, d\theta}{\prod_{j=1} x_j!}$

$\therefore \pi(\theta|X) = \dfrac{\theta^{\sum x_j} e^{-n\theta} \pi(\theta)}{\int_0^\infty \theta^{\sum x_j} e^{-n\theta} \pi(\theta)\, d\theta}$

$\therefore$ the Bayesian premium is

$$E(X|x) = \frac{\int_0^\infty \theta\, \theta^{\sum x_j} e^{-n\theta} \pi(\theta)\, d\theta}{\int_0^\infty \theta^{\sum x_j} e^{-n\theta} \pi(\theta)\, d\theta}$$

$$= \frac{\int_0^\infty \theta^{s+1} e^{-n\theta} \pi(\theta)\, d\theta}{\int_0^\infty \theta^{s} e^{-n\theta} \pi(\theta)\, d\theta}, \quad s = \sum x_j$$

$$E(X_{n+1}|x) = \frac{\frac{(s+1)!}{n^{s+1}} \int_0^\infty \frac{n^{s+1}}{(s+1)!} \theta^{s+1} e^{-n\theta} \pi(\theta)\, d\theta}{\frac{s!}{n^s} \int_0^\infty \frac{n^s}{s!} \theta^s e^{-n\theta} \pi(\theta)\, d\theta} \quad ②$$

$\therefore$ By using ① in ②, We get

$$E(X_{n+1}|x) = \frac{s+1}{n} \frac{f_S(s+1)}{f_S(s)} = \frac{1+n\bar{x}}{n} \frac{f_S(1+n\bar{x})}{f_S(n\bar{x})}$$

where, $s = \sum_j x_j = n\bar{x}$

(c) $f_S(s) = \int_0^\infty \dfrac{(n\theta)^s e^{-n\theta}}{s!} \pi(\theta)\, d\theta, \quad s = 0, 1, \ldots$

when, $\Theta \sim \text{gamma}(\alpha, \beta)$, $\pi(\theta) = \dfrac{\theta^{\alpha-1} \beta^{-\alpha} e^{-\theta/\beta}}{\Gamma(\alpha)}$

$f_S(s) = \int_0^\infty \dfrac{(n\theta)^s e^{-n\theta}}{s!} \dfrac{\beta^{-\alpha} \theta^{\alpha-1} e^{-\theta/\beta}}{\Gamma(\alpha)}\, d\theta$

$X \sim \text{gamma}(\alpha, \beta)$

$f(x) = \dfrac{(x/\beta)^\alpha e^{-x/\beta}}{x\Gamma(\alpha)}$

$x \to \theta$

$\therefore f_S(s) = \dfrac{n^s \beta^{-\alpha}}{s!\,\Gamma(\alpha)} \int_0^\infty \theta^{s+\alpha-1} e^{-(n+\beta^{-1})\theta}\, d\theta$

لا يكتب في هذا الهامش

let $(n+\beta^{-1})\theta = t \Rightarrow d\theta = \frac{dt}{n+\beta^{-1}}$

$$\therefore f_S(s) = \frac{n^s}{s!}\frac{\beta^{-\alpha}}{\Gamma(\alpha)}\int_0^\infty \left(\frac{t}{n+\beta^{-1}}\right)^{s+\alpha-1} e^{-t}\frac{dt}{(n+\beta^{-1})}$$

$$= \frac{n^s}{s!}\frac{\beta^{-\alpha}}{\Gamma(\alpha)}\frac{1}{(n+\beta^{-1})^{s+\alpha}}\int_0^\infty t^{s+\alpha-1} e^{-t}\,dt$$

$$f_S(s) = \frac{n^s}{s!}\frac{\beta^{-\alpha}}{\Gamma(\alpha)}\frac{\Gamma(s+\alpha)}{(n+\beta^{-1})^{s+\alpha}}$$

$$= \frac{n^s\beta^{-\alpha}}{s!}\frac{(s+\alpha-1)!}{(\alpha-1)!}\left(\frac{1}{n+\beta^{-1}}\right)^{\alpha}\left(\frac{1}{n+\beta^{-1}}\right)^{s}$$

$$= \binom{\alpha+s-1}{s} n^s\beta^{-\alpha}\left(\frac{1}{n+1/\beta}\right)^{\alpha}\left(\frac{1}{n+1/\beta}\right)^{s}$$

$$= \binom{\alpha+s-1}{s}\left(\frac{1}{1+n\beta}\right)^{\alpha}\left(\frac{n\beta}{1+n\beta}\right)^{s}$$

$$\therefore f_S(s) = \binom{r+s-1}{s}\left(\frac{1}{1+\beta^*}\right)^{r}\left(\frac{\beta^*}{1+\beta^*}\right)^{s}$$

where $\beta^* = n\beta$ and $r = \alpha$, which is a negative binomial dist$\underline{n}$.

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