Tut. Session (11)

Ch 17: Greatest Accuracy credibility

* pb 17.1 p. 432

Suppose that X and Z are independent Poisson random variables with means λ_1 and λ_2, respectively. Let $Y = X + Z$. Demonstrat that $X|Y = y$ is binomial.

Ans:

The conditional prob. mass fn of X given $Y = y$ is

$$Y = X + Z \Rightarrow y = x + z \Rightarrow z = y - x$$

$$\underset{X|Y}{pr}(x|y) = \frac{pr(X=x, Y=y)}{pr(Y=y)}$$

$$= \frac{pr(X=x, Z=y-x)}{pr(Y=y)}$$

$$= \frac{\dfrac{\lambda_1^x \cancel{e^{-\lambda_1}}}{x!} \; \dfrac{\lambda_2^{y-x} \cancel{e^{-\lambda_2}}}{(y-x)!}}{\dfrac{(\lambda_1+\lambda_2)^y \cancel{e^{-\lambda_1-\lambda_2}}}{y!}}$$

$$= \frac{y!}{x!\,(y-x)!} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{y-x}$$

for $x = 0, 1, 2, \ldots, y$

$$\therefore \underset{X|Y}{pr}(x|y) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n$$

where, $n = y$, and $p = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$.

That is, $X|Y=y$ has a binomial distribution with parameters $n = y$ and $p = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$.

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* pb 17.3 p. 432

Consider a Compound Poisson distribution with Poisson mean λ, where $X = Y_1 + Y_2 + \dots + Y_N$ with $E(Y_i) = \mu$ and $Var(Y_i) = \sigma_Y^2$. Determine the mean and Variance of X.

Ans: we have, $X = Y_1 + Y_2 + \dots + Y_N$
where, $E(Y_i) = \mu_Y$, $Var(Y_i) = \sigma_Y^2$, and

$N \sim poisson(\lambda)$
$\Rightarrow E(N) = Var(N) = \lambda$

The expectation of $E(X|N)$ is given by

$$E[E(X|N)] = E(X), \text{ See Eq. (17.3) p. 405}$$

and variance of X is given by

$$Var(X) = E[Var(X|N)] + Var[E(X|N)], \text{ See Eq. (17.6) p. 407}$$

The mean of X is

$$\begin{aligned} E(X) = E[E(X|N)] &= E[N E(Y_i)] \\ &= E[\mu_Y N] \\ &= \mu_Y E(N) = \lambda \mu_Y \end{aligned}$$

Also, the Variance of X is

$$\begin{aligned} Var(X) &= E[N Var(Y_i)] + Var[N E(Y_i)] \\ &= E[\sigma_Y^2 N] + Var[\mu_Y N] \\ &= \sigma_Y^2 E(N) + \mu_Y^2 Var(N) = \lambda \sigma_Y^2 + \lambda \mu_Y^2 \end{aligned}$$

$\therefore Var(X) = \lambda(\sigma_Y^2 + \mu_Y^2) = \lambda E(Y_i^2)$, where $\sigma_Y^2 = E(Y_i^2) - \mu_Y^2$.

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pb 17.4

Let X and Y have joint probability distribution as given in Table 17.2.

x \ y	0	1	2
0	0.20	0	0.10
1	0	0.15	0.25
2	0.05	0.15	0.10

Table 17.2 pb 17.4 p. 432

(a) Compute the marginal distributions of X and Y.
(b) Compute the Conditional distribution of X given $Y=y$ for $y=0,1,2$.
(c) Compute $E(X|y)$, $E(X^2|y)$ and $Var(X|y)$ for $y=0,1,2$.
(d) Compute $E(X)$ and $Var(X)$ using Equations (17.3), (17.6), and (C).

Ans:

(a) We Compute the marginal distributions of X and Y as follows.

x \ y	0	1	2	$f_X(x)$
0	0.20	0	0.10	0.30
1	0	0.15	0.25	0.40
2	0.05	0.15	0.10	0.30
$f_Y(y)$	0.25	0.30	0.45	

(b) $f(X|Y=y)$, $y=0,1,2$? $f(X|Y=y)=\frac{f(x,y)}{f_Y(y)}$

$f(X|Y=0)=\frac{0.20}{0.25}=\frac{4}{5}, \frac{0}{0.25}=0, \frac{0.05}{0.25}=\frac{1}{5} \Rightarrow$ sum $=1$

$f(X|Y=1)=\frac{0}{0.30}=0, \frac{0.15}{0.30}=\frac{1}{2}, \frac{0.15}{0.30}=\frac{1}{2}$ ~

and $f(X|Y=2)=\frac{0.10}{0.45}=\frac{2}{9}, \frac{0.25}{0.45}=\frac{5}{9}, \frac{0.10}{0.45}=\frac{2}{9}$ ~

$f(X|y)$

(c) $E(X|y)$, $E(X^2|y)$ and $Var(X|y)$???
for $y=0,1,2$.

$E(X|Y=0)=\cancel{0(\tfrac{4}{5})}+\cancel{1(0)}+2(\tfrac{1}{5})=\frac{2}{5}$

$E(X|Y=1)=\cancel{0(0)}+1(\tfrac{1}{2})+2(\tfrac{1}{2})=\frac{3}{2}$

$E(X|Y=2)=\cancel{0(\tfrac{2}{9})}+1(\tfrac{5}{9})+2(\tfrac{2}{9})=1$

$E(X|y)$

$$E(X^2|Y=0) = \cancel{0^2\left(\tfrac{4}{5}\right)} + \cancel{1^2(0)} + 2^2\left(\tfrac{1}{5}\right) = \frac{4}{5}$$

$$E(X^2|Y=1) = \cancel{0^2(0)} + 1^2\left(\tfrac{1}{2}\right) + 2^2\left(\tfrac{1}{2}\right) = \frac{5}{2}$$

$$E(X^2|Y=2) = \cancel{0^2\left(\tfrac{2}{9}\right)} + 1^2\left(\tfrac{5}{9}\right) + 2^2\left(\tfrac{2}{9}\right) = \frac{13}{9}$$

$E(X^2|y)$

$$Var(X|Y=0) = \frac{4}{5} - \left(\frac{2}{5}\right)^2 = \frac{16}{25}$$

$$Var(X|Y=1) = \frac{5}{2} - \left(\frac{3}{2}\right)^2 = \frac{1}{4}$$

$$Var(X|Y=2) = \frac{13}{9} - 1^2 = \frac{4}{9}$$

$Var(X|y)$

(d) $E(X)$ and $Var(X)$??

$$E(X) = E[E(X|Y)]$$

$$= \sum_y E(X|Y)\, f(y)$$

$$\therefore E(X) = \frac{2}{5}(0.25) + \frac{3}{2}(0.30) + 1(0.45) = 1$$

Also, $Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$

$$E[Var(X|Y)] = \sum_y Var(X|Y)\, f(y)$$

$$= \frac{16}{25}(0.25) + \frac{1}{4}(0.30) + \frac{4}{9}(0.45) = 0.435$$

$$Var[E(X|Y)] = E\{[E(X|Y)]^2\} - \{E[E(X|Y)]\}^2$$

$$= E\{[E(X|Y)]^2\} - [E(X)]^2$$

$$= \left(\frac{2}{5}\right)^2(0.25) + \left(\frac{3}{2}\right)^2(0.30) + 1^2(0.45) - 1^2$$

$$= 0.165$$

$$\therefore Var(X) = 0.435 + 0.165 = 0.6$$

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pb 17.7 p 433

Suppose that Θ has pdf $\pi(\theta)$, $\theta>0$ and Θ_1 has pdf $\pi_1(\theta)=\pi(\theta-\alpha)$, $\theta>\alpha>0$. If, given Θ_1, X is poisson distributed with mean Θ_1, show that X has the same distribution as $Y+Z$, where Y and Z are independent, Y is Poisson distributed with mean α, and $Z|\Theta$ is poisson distributed with mean Θ.

Ans: we have,

$\Theta \longrightarrow$ pdf $\pi(\theta)$, $\theta>0$, $\Theta_1 \longrightarrow$ pdf $\pi_1(\theta)=\pi(\theta-\alpha)$

$X|\Theta_1 \sim$ poisson (Θ_1) , $Y \sim$ poisson (α)

and $Z|\Theta \sim$ poisson (Θ)

$$\Rightarrow \quad f_X(x)=\int \frac{e^{-\theta}\theta^x}{x!}\pi(\theta-\alpha)\,d\theta, \quad \theta>\alpha>0$$

$$, \quad f_Y(y)=\frac{e^{-\alpha}\alpha^y}{y!}, \quad \alpha>0$$

$$\text{and} \quad f_Z(z)=\int \frac{e^{-\theta}\theta^z}{z!}\pi(\theta)\,d\theta, \quad \theta>0$$

let $W=Y+Z$, then

$$f_W(w)=\sum_{y=0}^{w} f_Y(y)\, f_Z(w-y)$$

$$=\sum_{y=0}^{w}\frac{e^{-\alpha}\alpha^y}{y!}\int\frac{e^{-\theta}\theta^{(w-y)}}{(w-y)!}\pi(\theta)\,d\theta$$

$$=\int \frac{e^{-(\alpha+\theta)}(\alpha+\theta)^w}{w!}\sum_{y=0}^{w}\binom{w}{y}\left(\frac{\alpha}{\alpha+\theta}\right)^y\left(\frac{\theta}{\alpha+\theta}\right)^{w-y}\pi(\theta)\,d\theta$$

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$$\therefore f_W(w)=\int\frac{e^{-(\alpha+\theta)}(\alpha+\theta)^w}{w!}\pi(\theta)\,d\theta$$

where $\displaystyle\sum_{y=0}^{w}\binom{w}{y}\left(\frac{\alpha}{\alpha+\theta}\right)^y\left(\frac{\theta}{\alpha+\theta}\right)^{w-y}=\left[\frac{\alpha}{\alpha+\theta}+\frac{\theta}{\alpha+\theta}\right]^w=1$

let $r=\alpha+\theta$

$$\therefore f_W(w)=\int\frac{e^{-r}r^w}{w!}\pi(r-\alpha)\,dr=f_X(x). \quad \#$$

Note that $\displaystyle\sum_{r=0}^{n}\binom{n}{r}a^r b^{n-r}=(a+b)^n$ Binomial formula