Triple Integrals

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1. Triple Integrals

In which follows, consider a continuous function $f : D \longrightarrow \mathbb{R}$, where D is a domain of $\mathbb{R}^3$.

Triple Integral Over Rectangular Domain

If the domain D is rectangular, $D = [a, b] \times [c, d] \times [r, s]$, then

$$\begin{aligned}
\int_D f(x,y,z)dxdydz &= \int_a^b \int_c^d \int_r^s f(x,y,z)dzdydx \\
&= \int_a^b \int_r^s \int_c^d f(x,y,z)dydzdx \\
&= \int_c^d \int_a^b \int_r^s f(x,y,z)dzdxdy \\
&= \int_c^d \int_r^s \int_a^b f(x,y,z)dxdzdy \\
&= \int_r^s \int_a^b \int_c^d f(x,y,z)dydxdz \\
&= \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz.
\end{aligned}$$

This is the Fubini's Theorem.

Triple Integral Over General Bounded Domain

The general bounded domains considered in this course are of three types.
First, consider the bounded region $D \subset \mathbb{R}^3$ of the form

$$D = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in E : f_1(x, y) \leq z \leq f_2(x, y)\},$$

where E is a bounded domain in the $xy-$plane. Hence

$$\iiint_D f(x, y, z)dxdydz := \iint_E \left(\int_{f_1(x,y)}^{f_2(x,y)} f(x, y, z)dz\right) dxdy.$$

Second, consider the bounded region $D \subset \mathbb{R}^3$ of the form

$$D = \{(x, y, z) \in \mathbb{R}^3 : (x, z) \in E : f_1(x, z) \leq y \leq f_2(x, z)\},$$

where E is a bounded domain in the $xz-$plane. Hence

$$\iiint_D f(x, y, z)dxdydz := \iint_E \left(\int_{f_1(x,z)}^{f_2(x,z)} f(x, y, z)dy\right) dxdz.$$

Third, consider the bounded region $D \subset \mathbb{R}^3$ of the form

$$D = \{(x, y, z) \in \mathbb{R}^3 : (y, z) \in E : f_1(y, z) \leq x \leq f_2(y, z)\},$$

where E is a bounded domain in the $yz-$plane. Hence

$$\iiint_D f(x, y, z)dxdydz := \iint_E \left(\int_{f_1(y,z)}^{f_2(y,z)} f(x, y, z)dx \right) dydz.$$

Example

Consider the integral $I = \iiint_D f(x,y,z)dxdydz$, where D is the domain delimited by the planes of equations $x=0$, $y=0$, $z=0$ and $x+y+z=1$ and $f(x,y,z)=(x+y+z)^2$.
The projection of the domain D on the $xy-$plane is the domain E delimited by the axes and the line of equation $x+y=1$. Then if $(x,y)\in E$, we have

$$I_z = \int_0^{1-x-y}(x+y+z)^2dz = \frac{1}{3}(1-(x+y)^3).$$

Hence

$$I = \frac{1}{3}\int_0^1\int_0^{1-x}(1-(x+y)^3)dydx = \frac{1}{10}.$$

Theorem (Properties of Triple Integrals)

Let D be a closed, bounded region in space, and let D_1 and D_2 be disjoint regions such that $D = D_1 \cup D_2$. Then

$$\iiint_D dV = \iiint_{D_1} dV + \iiint_{D_2} dV.$$

Example

Consider the integral $I = \iiint_D f(x,y,z)dxdydz$, where D is the domain delimited by the planes of equations $x=0$, $y=0$, $z=0$ and $x+y+z=1$ and $f(x,y,z)=e^{x+y+z}$.
The projection of the domain D on the $xy-$plane is the domain E delimited by the axes and the line of equation $x+y=1$. Then if $(x,y)\in E$, we have

$$I_z = \int_0^{1-x-y} e^{x+y+z}dz = e - e^{x+y}.$$

Hence

$$I = \int_0^1 \int_0^{1-x} e - e^{x+y}dydx = \frac{e}{2} - 1.$$

Example

Consider the integral $I = \iiint_D f(x,y,z)dxdydz$, where
$D = \{(x,y,z) \in \mathbb{R}^3 :\ 0 \leq z \leq x^2 + y^2,\ 0 \leq y \leq x \leq 1\}$ and
$f(x,y,z) = x + y + z$.
The projection of the domain D on the $xy-$plane is the domain E delimited by the $x-$axes and the lines of equations $x = y$ and $x = 1$. Then if $(x,y) \in E$, we have

$$I_z = \int_0^{x^2+y^2} (x+y+z)dz = (x+y)(x^2+y^2) + \frac{1}{2}(x^2+y^2)^2.$$

Hence

$$I = \int_0^1 \int_0^x (x+y)(x^2+y^2) + \frac{1}{2}(x^2+y^2)^2 dydx = \frac{103}{180}.$$

Example

Consider the integral $I = \iiint_D f(x,y,z)dxdydz$, where D is the domain delimited by the planes of equations $x=0$, $y=0$, $z=0$, $x+z=1$ and $y+z=1$ and $f(x,y,z)=(x-y+z)^2$.
We separate the domain D by the plane $x=y$. On the domain D_1, $x\geq y$ and on the domain D_2, $x\leq y$.
The projection of the domain D_1 on the $xy-$plane is the domain E_1 delimited by the $x-$axes and the lines of equations $x=y$ and $x=1$. Then if $(x,y)\in E_1$, we have

$$I_{1,z} = \int_0^{1-x}(x-y+z)^2dz = \frac{1}{3}\left((1-y)^3-(x-y)^3\right).$$

Hence

$$I_1 = \frac{1}{3}\int_0^1 \int_0^x \left((1-y)^3 - (x-y)^3\right) dydx = \frac{1}{20}.$$

The projection of the domain D_2 on the $xy-$plane is the domain E_2 delimited by the $y-$axes and the lines of equations $x = y$ and $y = 1$. Then if $(x, y) \in E_y$, we have

$$I_{2,z} = \int_0^{1-y} (x-y+z)^2 dz = \frac{1}{3}\left((x+1-2y)^3 - (x-y)^3\right).$$

Hence

$$I_2 = \frac{1}{3}\int_0^1 \int_0^y \left((x+1-2y)^3 - (x-y)^3\right) dxdy = \frac{1}{60}.$$

Then $I = \iiint_D f(x, y, z) dxdydz = \frac{1}{15}$.

Example

Find the volume of the tetrahedron with corners at $(0,0,0)$, $(0,3,0)$, $(2,3,0)$, and $(2,3,5)$.
The solid is defined as follows: $0 \leq x \leq 2$, $\frac{3}{2}x \leq y \leq 3$, $0 \leq z \leq \frac{5}{2}x$. The lower y limit comes from the equation of the line $y = \frac{3}{2}x$ that forms one edge of the tetrahedron in the $x - y$ plane; the upper z limit comes from the equation of the plane $z = \frac{5}{2}x$ that forms the "upper" side of the tetrahedron. Now the volume is

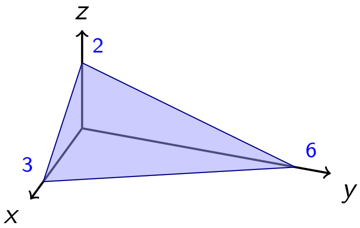
$$\begin{aligned}\int_0^2 \int_{\frac{3}{2}x}^3 \int_0^{\frac{5}{2}x} dz\,dy\,dx &= \int_0^2 \int_{\frac{3}{2}x}^3 \frac{5x}{2}\,dy\,dx \\ &= \int_0^2 \frac{15x}{2} - \frac{15x^2}{4}\,dx = 5.\end{aligned}$$

Example

Find the volume of the space region in the first octant bounded by the plane $z = 2 - \frac{y}{3} - 2\frac{x}{3}$, using the order of integration $dzdydx$.

Solution

1. Starting with the order of integration $dzdydx$, we need to first find bounds on z. The region D is bounded below by the plane $z = 0$ (because we are restricted to the first octant) and above by $z = 2 - \frac{y}{3} - 2\frac{x}{3}$, $0 \leq z \leq 2 - \frac{y}{3} - 2\frac{x}{3}$.
   To find the bounds on y and x, we project the region onto the xy plane. By setting $z = 0$, we have
   $0 = 2 - \frac{y}{3} - 2\frac{x}{3} \Rightarrow y = 6 - 2x$. Secondly, we know this is going to be a straight line between the points $(3, 0)$ and $(0, 6)$ in the xy plane.)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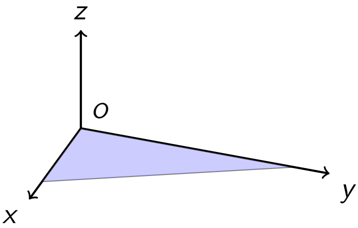
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The region R, in the integration order of $dydx$, with bounds $0 \leq y \leq 6-2x$ and $0 \leq x \leq 3$. Thus the volume V of the region D is:

$$
\begin{aligned}
V = \iiint_D dV &= \int_0^3 \int_0^{6-2x} \int_0^{2-\frac{y}{3}-\frac{2x}{3}} dzdydz \\
&= \int_0^3 \int_0^{6-2x} \left(\int_0^{2-\frac{y}{3}-\frac{2x}{3}} dz \right) dydz \\
&= \int_0^3 \int_0^{6-2x} \left(2 - \frac{y}{3} - \frac{2x}{3} \right) dydz \\
&= \int_0^3 6 - 4x + \frac{2}{3}x^2 dx = 6.
\end{aligned}
$$

2. Now consider the volume using the order of integration $dzdxdy$. The bounds on z are the same as before, $0 \leq z \leq 2 - \frac{y}{3} - 2\frac{x}{3}$. Taking the projection on the xy plane, this gives the bounds $0 \leq x \leq 3 - \frac{y}{2}$ and $0 \leq y \leq 6$. Thus the volume is given by the triple integral

$$V = \int_0^6 \int_0^{3-\frac{y}{2}} \int_0^{2-\frac{y}{3}-\frac{2x}{3}} dzdxdy.$$

3. Now consider the volume using the order of integration $dxdydz$. The bounds on x are: $0 \leq x \leq 3 - \frac{y}{2} - \frac{3z}{2}$. The projection of the region D on the yz plane, we find the equation of the line $z = 2 - \frac{y}{3}$ by setting $x = 0$ in the equation $x = 3 - \frac{y}{2} - \frac{3z}{2}$. The bound of y are $y = 0$ and $y = 6 - 3z$; the points that bound z are 0 and 2. Thus the triple integral giving volume is: $0 \leq x \leq 3 - \frac{y}{2} - \frac{3z}{2}$, $0 \leq y \leq 6 - 3z$, $0 \leq z \leq 2$. Then

$$V = \int_0^2 \int_0^{6-3z} \int_0^{3-\frac{y}{2}-\frac{3z}{2}} dx\, dy\, dz.$$

Example

Set up the triple integrals that find the volume of the space region D bounded by the surfaces $x^2 + y^2 = 1$, $z = 0$ and $z = -y$, $z \geq 0$, with the orders of integration $dzdydx$, $dydxdz$ and $dxdzdy$.

Solution

The order $dzdydx$. The region D is bounded below by the plane $z = 0$ and above by the plane $z = -y$. The cylinder $x^2 + y^2 = 1$ does not offer any bounds in the $z-$direction, as that surface is parallel to the $z-$axis. Thus $0 \leq z \leq -y$. The projection of the region on the xy plane, is a part of the circle with equation $x^2 + y^2 = 1$. As a function of x, this half circle has equation $y = -\sqrt{1 - x^2}$. Thus y is bounded below by $-\sqrt{1 - x^2}$ and above by $y = 0$: $-\sqrt{1 - x^2} \leq y \leq 0$.

The x bounds of the half circle are $-1 \leq x \leq 1$. The volume is:

$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} \int_{0}^{-y} dz\, dy\, dx = .$$

In the order $dydxdz$. The region is bounded below in the $y-$direction by the surface $x^2 + y^2 = 1$, then $y = -\sqrt{1-x^2}$ and above by the surface $y = -z$. Thus $-\sqrt{1-x^2} \leq y \leq -z$.

The projection on the xz plane gives the half circle has equation $x^2+z^2=1$. (We find this curve by solving each surface for y^2, then $y^2=1-x^2$ and $y=-z$, hence $y^2=z^2$. Thus $x^2+z^2=1$.) It is bounded below by $x=-\sqrt{1-z^2}$ and above by $x=\sqrt{1-z^2}$, where z is bounded by $0\le z\le 1$. We have: $-\sqrt{1-x^2}\le y\le -z$, $-\sqrt{1-z^2}\le x\le\sqrt{1-z^2}$, $0\le z\le 1$ and

$$V=\int_0^1\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-x^2}}^{-z}dy\,dx\,dz.$$

Example

Evaluate the integral $\displaystyle\int_0^1 \int_{x^2}^{x} \int_{x^2-y}^{2x+3y} \big(xy + 2xz\big)\, dz\, dy\, dx.$

$$
\begin{aligned}
\int_0^1 \int_{x^2}^{x} \int_{x^2-y}^{2x+3y} \big(xy + 2xz\big)\, dz\, dy\, dx &= = \int_0^1 \int_{x^2}^{x} \left(\int_{x^2-y}^{2x+3y} \big(xy + 2xz\big)\, dz \right) dy\, dx \\
&= \int_0^1 \int_{x^2}^{x} \left(\big(xyz + xz^2\big)\Big|_{x^2-y}^{2x+3y} \right) dy\, dx \\
&= \int_0^1 \int_{x^2}^{x} \Big(xy(2x+3y) + x(2x+3y)^2 - \big(xy(x^2 - y \\
&= \int_0^1 \int_{x^2}^{x} \Big(-x^5 + x^3y + 4x^3 + 14x^2y + 12xy^2 \Big)\, dy\, d \\
&= \int_0^1 \left(-\frac{7}{2}x^7 - 8x^6 - \frac{7}{2}x^5 + 15x^4 \right) dx \\
&= \frac{281}{336}.
\end{aligned}
$$

Exercises

Exercise 1 :
Evaluate $\int_0^1 \int_0^x \int_0^{x+y} 2x + y - 1 \, dz \, dy \, dx$.

Exercise 2 :
Evaluate $\int_0^2 \int_{-1}^{x^2} \int_1^{y} xyz \, dz \, dy \, dx$.

Exercise 3 :
Evaluate $\int_0^1 \int_0^x \int_0^{\ln y} e^{x+y+z} \, dz \, dy \, dx$.

Exercise 4 :
Evaluate $\int_0^1 \int_0^{y^2} \int_0^{x+y} x \, dz \, dx \, dy$.

Exercise 5 :
Evaluate $\int_1^2 \int_y^{y^2} \int_0^{\ln(y+z)} e^x \, dx \, dz \, dy$.

Exercise 6 :
Compute $\int_0^{\pi} \int_0^{\pi/2} \int_0^1 z \sin x + z \cos y \, dz \, dy \, dx$.

Exercise 7 :

Compute $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}}\int_{0}^{9-x^2-y^2} dzdxdy.$

Exercise 8 :
For each of the integrals in the previous exercises, give a description of the volume (both algebraic and geometric) that is the domain of integration.

Exercise 9 :
Compute $\int\int\int x + y + z \, dV$ over the region $x^2 + y^2 + z^2 \leq 1$ in the first octant.

Exercise 10 :
Consider the iterated integral

$$\int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{y} dzdydx.$$

The bounds for this integral describe a region in space which satisfies the 3 inequalities $-1 \leq x \leq 1$, $0 \leq y \leq 1-x^2$, and $0 \leq z \leq y$.

1. Draw the solid domain D in space described by the bounds of the iterated integral.
2. There are 5 other iterated integrals equivalent to this one. Set up the integrals that use the bounds $dydxdz$ and $dxdzdy$. We'll create the other 3 in class (though you are welcome to include them as part of your presentation).

Exercise 11 :

In each problem below, you'll be given enough information to determine a solid domain D in space. Draw the solid D and then set up an iterated integral (pick any order you want) that would give the volume of D. You don't need to evaluate the integral, rather you just need to set them up.

1. The region D under the surface $z = y^2$, above the xy-plane, and bounded by the planes $y = -1$, $y = 1$, $x = 0$, and $x = 4$.
2. The region D in the first octant that is bounded by the coordinate planes, the plane $y + z = 2$, and the surface $x = 4 - y^2$.
3. The pyramid D in the first octant that is below the planes $\dfrac{x}{3} + \dfrac{z}{2} = 1$ and $\dfrac{y}{5} + \dfrac{z}{2} = 1$. [Hint, don't let z be the inside bound.]
4. The region D that is inside both right circular cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$.

Example

Compute $\iiint_D f(x,y,z)dxdydz$, for
$D = \{0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$,
$f(x,y,z) = \dfrac{1}{(x+y+z+1)^3}$,

Example

Compute $\iiint_D f(x,y,z)dxdydz$, for

1. $D = \{x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1\}$,
   $f(x,y,z) = \dfrac{1}{(x+y+z+1)^2}$.
2. $D = \{x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1,\ f(x,y,z) = xyz,$

Example

Compute $\iiint_D f(x,y,z)dxdydz$, for $D = \left\{ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \leq 1 \right\}$,
$f(x,y,z) = x^2 + y^2$,

Example

Compute $\displaystyle\iiint_D \frac{\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z}{(1+x^2z^2)(1+y^2z^2)}$, with $D = \{(x,y,z):\ 0 \le x \le 1, 0 \le y \le 1, 0 \le z\}$.

Deduce $\displaystyle\int_0^{+\infty} \left(\frac{\tan^{-1} t}{t}\right)^2\ \mathrm{d}t$.

Example

Let $I = \displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}dx$. Compute $J = \displaystyle\iint_D \frac{x \,\mathrm{d}x \,\mathrm{d}y}{(1+x^2)(1+xy)}$, with $D = \{(x,y) :\ 0 \le x \le 1, 0 \le y \le 1\}$ in two manner and deduce the value of I.