

17

Second-Order Differential Equations



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OVERVIEW In this chapter we extend our study of differential equations to those of *second order*, equations that involve second derivatives of a function. Second-order differential equations arise in many applications in the sciences and engineering. For instance, they can be applied to the study of vibrating springs and electric circuits. You will learn how to solve such differential equations by several methods in this chapter.

17.1 Second-Order Linear Equations

An equation of the form

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = G(x), \quad (1)$$

which is linear in y and its derivatives, is called a **second-order linear differential equation**. We assume that the functions P , Q , R , and G are continuous throughout some open interval I . If $G(x)$ is identically zero on I , the equation is said to be **homogeneous**; otherwise it is called **nonhomogeneous**. Therefore, the form of a second-order linear homogeneous differential equation is

$$P(x)y'' + Q(x)y' + R(x)y = 0. \quad (2)$$

We also assume that $P(x)$ is never zero for any $x \in I$.

Two fundamental results are important to solving Equation (2). The first of these says that if we know two solutions y_1 and y_2 of the linear homogeneous equation, then any **linear combination** $y = c_1y_1 + c_2y_2$ is also a solution for any constants c_1 and c_2 .

THEOREM 1—The Superposition Principle

If $y_1(x)$ and $y_2(x)$ are two solutions to the linear homogeneous equation (2), then for any constants c_1 and c_2 , the function

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution to Equation (2).

Proof Substituting y into Equation (2), we have

$$\begin{aligned}
 P(x)y'' + Q(x)y' + R(x)y &= P(x)(c_1y_1 + c_2y_2)'' + Q(x)(c_1y_1 + c_2y_2)' + R(x)(c_1y_1 + c_2y_2) \\
 &= P(x)(c_1y_1'' + c_2y_2'') + Q(x)(c_1y_1' + c_2y_2') + R(x)(c_1y_1 + c_2y_2) \\
 &= c_1(\underbrace{P(x)y_1'' + Q(x)y_1' + R(x)y_1}_{=0, y_1 \text{ is a solution}}) + c_2(\underbrace{P(x)y_2'' + Q(x)y_2' + R(x)y_2}_{=0, y_2 \text{ is a solution}}) \\
 &= c_1(0) + c_2(0) = 0.
 \end{aligned}$$

Therefore, $y = c_1y_1 + c_2y_2$ is a solution of Equation (2). ■

Theorem 1 immediately establishes the following facts concerning solutions to the linear homogeneous equation.

1. A sum of two solutions $y_1 + y_2$ to Equation (2) is also a solution. (Choose $c_1 = c_2 = 1$.)
2. A constant multiple ky_1 of any solution y_1 to Equation (2) is also a solution. (Choose $c_1 = k$ and $c_2 = 0$.)
3. The **trivial solution** $y(x) \equiv 0$ is always a solution to the linear homogeneous equation. (Choose $c_1 = c_2 = 0$.)

The second fundamental result about solutions to the linear homogeneous equation concerns its **general solution**, or solution containing all solutions. This result says that there are two solutions y_1 and y_2 such that any solution is some linear combination of them for suitable values of the constants c_1 and c_2 . However, not just any pair of solutions will do. The solutions must be **linearly independent**, which means that neither y_1 nor y_2 is a constant multiple of the other. For example, the functions $f(x) = e^x$ and $g(x) = xe^x$ are linearly independent, whereas $f(x) = x^2$ and $g(x) = 7x^2$ are not (they are linearly dependent). These results on linear independence and the following theorem are proved in more advanced courses.

THEOREM 2 If P , Q , and R are continuous over the open interval I and $P(x)$ is never zero on I , then the linear homogeneous equation (2) has two linearly independent solutions y_1 and y_2 on I . Moreover, if y_1 and y_2 are *any* two linearly independent solutions of Equation (2), then the general solution is given by

$$y(x) = c_1y_1(x) + c_2y_2(x),$$

where c_1 and c_2 are arbitrary constants.

We now turn our attention to finding two linearly independent solutions to the special case of Equation (2) where P , Q , and R are constant functions.

Constant-Coefficient Homogeneous Equations

Suppose we wish to solve the second-order homogeneous differential equation

$$ay'' + by' + cy = 0, \quad (3)$$

where a , b , and c are constants. To solve Equation (3), we seek a function that, when multiplied by a constant and added to a constant times its first derivative plus a constant times its second derivative, sums identically to zero. One function that behaves this way is the exponential function $y = e^{rx}$, when r is a constant. Two differentiations of this exponential

function give $y' = re^{rx}$ and $y'' = r^2e^{rx}$, which are just constant multiples of the original exponential. If we substitute $y = e^{rx}$ into Equation (3), we obtain

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0.$$

Since the exponential function is never zero, we can divide this last equation through by e^{rx} . Thus, $y = e^{rx}$ is a solution to Equation (3) if and only if r is a solution to the algebraic equation

$$ar^2 + br + c = 0. \quad (4)$$

Equation (4) is called the **auxiliary equation** (or **characteristic equation**) of the differential equation $ay'' + by' + cy = 0$. The auxiliary equation is a quadratic equation with roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

There are three cases to consider, which depend on the value of the discriminant $b^2 - 4ac$.

Case 1: $b^2 - 4ac > 0$. In this case the auxiliary equation has two real and unequal roots r_1 and r_2 . Then $y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$ are two linearly independent solutions to Equation (3) because e^{r_2x} is not a constant multiple of e^{r_1x} (see Exercise 61). From Theorem 2 we conclude the following result.

THEOREM 3 If r_1 and r_2 are two real and unequal roots of the auxiliary equation $ar^2 + br + c = 0$, then

$$y = c_1e^{r_1x} + c_2e^{r_2x}$$

is the general solution to $ay'' + by' + cy = 0$.

EXAMPLE 1 Find the general solution of the differential equation

$$y'' - y' - 6y = 0.$$

Solution Substitution of $y = e^{rx}$ into the differential equation yields the auxiliary equation

$$r^2 - r - 6 = 0,$$

which factors as

$$(r - 3)(r + 2) = 0.$$

The roots are $r_1 = 3$ and $r_2 = -2$. Thus, the general solution is

$$y = c_1e^{3x} + c_2e^{-2x}. \quad \blacksquare$$

Case 2: $b^2 - 4ac = 0$. In this case $r_1 = r_2 = -b/2a$. To simplify the notation, let $r = -b/2a$. Then we have one solution $y_1 = e^{rx}$ with $2ar + b = 0$. Since multiplication of e^{rx} by a constant fails to produce a second linearly independent solution, suppose we try

multiplying by a *function* instead. The simplest such function would be $u(x) = x$, so let's see where $y_2 = xe^{rx}$ is also a solution. Substituting y_2 into the differential equation gives

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a(2re^{rx} + r^2xe^{rx}) + b(e^{rx} + rxe^{rx}) + cxe^{rx} \\ &= (2ar + b)e^{rx} + (ar^2 + br + c)xe^{rx} \\ &= 0(e^{rx}) + (0)xe^{rx} = 0. \end{aligned}$$

The first term is zero because $r = -b/2a$; the second term is zero because r solves the auxiliary equation. The functions $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are linearly independent (see Exercise 62). From Theorem 2 we conclude the following result.

THEOREM 4 If r is the only (repeated) real root of the auxiliary equation $ar^2 + br + c = 0$, then

$$y = c_1e^{rx} + c_2xe^{rx}$$

is the general solution to $ay'' + by' + cy = 0$.

EXAMPLE 2 Find the general solution to

$$y'' + 4y' + 4y = 0.$$

Solution The auxiliary equation is

$$r^2 + 4r + 4 = 0,$$

which factors into

$$(r + 2)^2 = 0.$$

Thus, $r = -2$ is a double root. Therefore, the general solution is

$$y = c_1e^{-2x} + c_2xe^{-2x}. \quad \blacksquare$$

Case 3: $b^2 - 4ac < 0$. In this case the auxiliary equation has two complex roots: $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, where α and β are real numbers and $i^2 = -1$. (These real numbers are $\alpha = -b/2a$ and $\beta = \sqrt{4ac - b^2}/2a$.) These two complex roots then give rise to two linearly independent solutions:

$$y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x}(\cos \beta x + i \sin \beta x) \quad \text{and} \quad y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x}(\cos \beta x - i \sin \beta x).$$

(The expressions involving the sine and cosine terms follow from Euler's identity, as seen in the discussion of Taylor series.) However, the solutions y_1 and y_2 are *complex valued* rather than real valued. Nevertheless, because of the superposition principle (Theorem 1), we can obtain from them the two real-valued solutions

$$y_3 = \frac{1}{2}y_1 + \frac{1}{2}y_2 = e^{\alpha x} \cos \beta x \quad \text{and} \quad y_4 = \frac{1}{2i}y_1 - \frac{1}{2i}y_2 = e^{\alpha x} \sin \beta x.$$

The functions y_3 and y_4 are linearly independent (see Exercise 63). From Theorem 2 we conclude the following result.

THEOREM 5 If $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ are two complex roots of the auxiliary equation $ar^2 + br + c = 0$, then

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

is the general solution to $ay'' + by' + cy = 0$.

EXAMPLE 3 Find the general solution to the differential equation

$$y'' - 4y' + 5y = 0.$$

Solution The auxiliary equation is

$$r^2 - 4r + 5 = 0.$$

The roots are the complex pair $r = (4 \pm \sqrt{16 - 20})/2$, or $r_1 = 2 + i$ and $r_2 = 2 - i$. Thus, $\alpha = 2$ and $\beta = 1$ give the general solution

$$y = e^{2x}(c_1 \cos x + c_2 \sin x).$$

Initial Value and Boundary Value Problems

To determine a unique solution to a first-order linear differential equation, it was sufficient to specify the value of the solution at a single point. Since the general solution to a second-order equation contains two arbitrary constants, it is necessary to specify two conditions. One way of doing this is to specify the value of the solution function and the value of its derivative at a single point: $y(x_0) = y_0$ and $y'(x_0) = y_1$. These conditions are called **initial conditions**. The following result is proved in more advanced texts and guarantees the existence of a unique solution for both homogeneous and nonhomogeneous second-order linear initial value problems.

THEOREM 6 If P , Q , R , and G are continuous throughout an open interval I , then there exists one and only one function $y(x)$ satisfying both the differential equation

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = G(x)$$

on the interval I , and the initial conditions

$$y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = y_1$$

at the specified point $x_0 \in I$.

It is important to realize that any real values can be assigned to y_0 and y_1 , and Theorem 6 applies. Here is an example of an initial value problem for a homogeneous equation.

EXAMPLE 4 Find the particular solution to the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

Solution The auxiliary equation is

$$r^2 - 2r + 1 = (r - 1)^2 = 0.$$

The repeated real root is $r = 1$, giving the general solution

$$y = c_1 e^x + c_2 x e^x.$$

Then

$$y' = c_1 e^x + c_2(x + 1)e^x.$$

From the initial conditions we have

$$1 = c_1 + c_2 \cdot 0 \quad \text{and} \quad -1 = c_1 + c_2 \cdot 1.$$

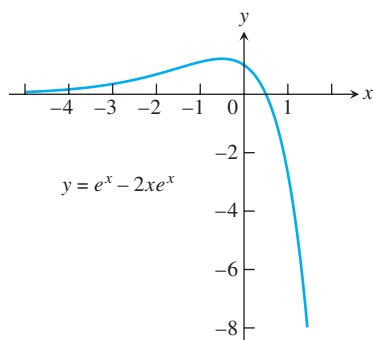


FIGURE 17.1 Particular solution curve for Example 4.

Thus, $c_1 = 1$ and $c_2 = -2$. The unique solution satisfying the initial conditions is

$$y = e^x - 2xe^x.$$

The solution curve is shown in Figure 17.1. ■

Another approach to determine the values of the two arbitrary constants in the general solution to a second-order differential equation is to specify the values of the solution function at *two different points* in the interval I . That is, we solve the differential equation subject to the **boundary values**

$$y(x_1) = y_1 \quad \text{and} \quad y(x_2) = y_2,$$

where x_1 and x_2 both belong to I . Here again the values for y_1 and y_2 can be any real numbers. The differential equation together with specified boundary values is called a **boundary value problem**. Unlike the result stated in Theorem 6, boundary value problems do not always possess a solution, or more than one solution may exist (see Exercise 65). These problems are studied in more advanced texts, but here is an example for which there is a unique solution.

EXAMPLE 5 Solve the boundary value problem

$$y'' + 4y = 0, \quad y(0) = 0, \quad y\left(\frac{\pi}{12}\right) = 1.$$

Solution The auxiliary equation is $r^2 + 4 = 0$, which has the complex roots $r = \pm 2i$. The general solution to the differential equation is

$$y = c_1 \cos 2x + c_2 \sin 2x.$$

The boundary conditions are satisfied if

$$y(0) = c_1 \cdot 1 + c_2 \cdot 0 = 0$$

$$y\left(\frac{\pi}{12}\right) = c_1 \cos\left(\frac{\pi}{6}\right) + c_2 \sin\left(\frac{\pi}{6}\right) = 1.$$

It follows that $c_1 = 0$ and $c_2 = 2$. The solution to the boundary value problem is

$$y = 2 \sin 2x. \quad \blacksquare$$

EXERCISES 17.1

In Exercises 1–30, find the general solution of the given equation.

1. $y'' - y' - 12y = 0$
2. $3y'' - y' = 0$
3. $y'' + 3y' - 4y = 0$
4. $y'' - 9y = 0$
5. $y'' - 4y = 0$
6. $y'' - 64y = 0$
7. $2y'' - y' - 3y = 0$
8. $9y'' - y = 0$
9. $8y'' - 10y' - 3y = 0$
10. $3y'' - 20y' + 12y = 0$
11. $y'' + 9y = 0$
12. $y'' + 4y' + 5y = 0$
13. $y'' + 25y = 0$
14. $y'' + y = 0$
15. $y'' - 2y' + 5y = 0$
16. $y'' + 16y = 0$
17. $y'' + 2y' + 4y = 0$
18. $y'' - 2y' + 3y = 0$
19. $y'' + 4y' + 9y = 0$
20. $4y'' - 4y' + 13y = 0$
21. $y'' = 0$
22. $y'' + 8y' + 16y = 0$

23. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$
24. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$
25. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$
26. $4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 9y = 0$
27. $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$
28. $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$
29. $9\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + y = 0$
30. $9\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$

In Exercises 31–40, find the unique solution of the second-order initial value problem.

31. $y'' + 6y' + 5y = 0, \quad y(0) = 0, \quad y'(0) = 3$
32. $y'' + 16y = 0, \quad y(0) = 2, \quad y'(0) = -2$
33. $y'' + 12y = 0, \quad y(0) = 0, \quad y'(0) = 1$

34. $12y'' + 5y' - 2y = 0$, $y(0) = 1$, $y'(0) = -1$
 35. $y'' + 8y = 0$, $y(0) = -1$, $y'(0) = 2$
 36. $y'' + 4y' + 4y = 0$, $y(0) = 0$, $y'(0) = 1$
 37. $y'' - 4y' + 4y = 0$, $y(0) = 1$, $y'(0) = 0$
 38. $4y'' - 4y' + y = 0$, $y(0) = 4$, $y'(0) = 4$
 39. $4\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 9y = 0$, $y(0) = 2$, $\frac{dy}{dx}(0) = 1$
 40. $9\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$, $y(0) = -1$, $\frac{dy}{dx}(0) = 1$

In Exercises 41–55, find the general solution.

41. $y'' - 2y' - 3y = 0$ 42. $6y'' - y' - y = 0$
 43. $4y'' + 4y' + y = 0$ 44. $9y'' + 12y' + 4y = 0$
 45. $4y'' + 20y = 0$ 46. $y'' + 2y' + 2y = 0$
 47. $25y'' + 10y' + y = 0$ 48. $6y'' + 13y' - 5y = 0$
 49. $4y'' + 4y' + 5y = 0$ 50. $y'' + 4y' + 6y = 0$
 51. $16y'' - 24y' + 9y = 0$ 52. $6y'' - 5y' - 6y = 0$
 53. $9y'' + 24y' + 16y = 0$ 54. $4y'' + 16y' + 52y = 0$
 55. $6y'' - 5y' - 4y = 0$

In Exercises 56–60, solve the initial value problem.

56. $y'' - 2y' + 2y = 0$, $y(0) = 0$, $y'(0) = 2$
 57. $y'' + 2y' + y = 0$, $y(0) = 1$, $y'(0) = 1$

58. $4y'' - 4y' + y = 0$, $y(0) = -1$, $y'(0) = 2$
 59. $3y'' + y' - 14y = 0$, $y(0) = 2$, $y'(0) = -1$
 60. $4y'' + 4y' + 5y = 0$, $y(\pi) = 1$, $y'(\pi) = 0$
 61. Prove that the two solution functions in Theorem 3 are linearly independent.
 62. Prove that the two solution functions in Theorem 4 are linearly independent.
 63. Prove that the two solution functions in Theorem 5 are linearly independent.
 64. Prove that if y_1 and y_2 are linearly independent solutions to the homogeneous equation (2), then the functions $y_3 = y_1 + y_2$ and $y_4 = y_1 - y_2$ are also linearly independent solutions.
 65. a. Show that there is no solution to the boundary value problem

$$y'' + 4y = 0, \quad y(0) = 0, y(\pi) = 1.$$

 b. Show that there are infinitely many solutions to the boundary value problem

$$y'' + 4y = 0, \quad y(0) = 0, y(\pi) = 0.$$

 66. Show that if a , b , and c are positive constants, then all solutions of the homogeneous differential equation

$$ay'' + by' + cy = 0$$

approach zero as $x \rightarrow \infty$.

17.2 Nonhomogeneous Linear Equations

In this section we study two methods for solving second-order linear nonhomogeneous differential equations with constant coefficients. These are the methods of *undetermined coefficients* and *variation of parameters*. We begin by considering the form of the general solution.

Form of the General Solution

Suppose we wish to solve the nonhomogeneous equation

$$ay'' + by' + cy = G(x), \quad (1)$$

where a , b , and c are constants and G is continuous over some open interval I . Let $y_c = c_1y_1 + c_2y_2$ be the general solution to the associated **complementary equation**

$$ay'' + by' + cy = 0. \quad (2)$$

(We learned how to find y_c in Section 17.1.) Now suppose we could somehow come up with a particular function y_p that solves the nonhomogeneous equation (1). Then the sum

$$y = y_c + y_p \quad (3)$$

also solves the nonhomogeneous equation (1) because

$$\begin{aligned} a(y_c + y_p)'' + b(y_c + y_p)' + c(y_c + y_p) &= (ay_c'' + by_c' + cy_c) + (ay_p'' + by_p' + cy_p) \\ &= 0 + G(x) \quad \text{y}_c \text{ solves Eq. (2) and } y_p \text{ solves Eq. (1)} \\ &= G(x). \end{aligned}$$

Moreover, if $y = y(x)$ is the general solution to the nonhomogeneous equation (1), it must have the form of Equation (3). The reason for this last statement follows from the observation that for any function y_p satisfying Equation (1), we have

$$\begin{aligned} a(y - y_p)'' + b(y - y_p)' + c(y - y_p) \\ &= (ay'' + by' + cy) - (ay_p'' + by_p' + cy_p) \\ &= G(x) - G(x) = 0. \end{aligned}$$

Thus, $y_c = y - y_p$ is the general solution to the homogeneous equation (2). We have established the following result.

THEOREM 7 The general solution $y = y(x)$ to the nonhomogeneous differential equation (1) has the form

$$y = y_c + y_p,$$

where the **complementary solution** y_c is the general solution to the associated homogeneous equation (2), and y_p is any **particular solution** to the nonhomogeneous equation (1).

The Method of Undetermined Coefficients

This method for finding a particular solution y_p to the nonhomogeneous equation (1) applies to special cases for which $G(x)$ is a sum of terms of various polynomials $p(x)$ multiplying an exponential with possibly sine or cosine factors. That is, $G(x)$ is a sum of terms of the following forms:

$$p_1(x)e^{rx}, \quad p_2(x)e^{\alpha x} \cos \beta x, \quad p_3(x)e^{\alpha x} \sin \beta x.$$

For instance, $1 - x$, e^{2x} , xe^x , $\cos x$, and $5e^x - \sin 2x$ represent functions in this category. (Essentially these are functions solving homogeneous linear differential equations with constant coefficients, but the equations may be of order higher than two.) We now present several examples illustrating the method.

EXAMPLE 1 Solve the nonhomogeneous equation $y'' - 2y' - 3y = 1 - x^2$.

Solution The auxiliary equation for the complementary equation $y'' - 2y' - 3y = 0$ is

$$r^2 - 2r - 3 = (r + 1)(r - 3) = 0.$$

It has the roots $r = -1$ and $r = 3$, giving the complementary solution

$$y_c = c_1 e^{-x} + c_2 e^{3x}.$$

Now $G(x) = 1 - x^2$ is a polynomial of degree 2. It would be reasonable to assume that a particular solution to the given nonhomogeneous equation is also a polynomial of degree 2 because if y is a polynomial of degree 2, then $y'' - 2y' - 3y$ is also a polynomial of degree 2. So we seek a particular solution of the form

$$y_p = Ax^2 + Bx + C.$$

We need to determine the unknown coefficients A , B , and C . When we substitute the polynomial y_p and its derivatives into the given nonhomogeneous equation, we obtain

$$2A - 2(2Ax + B) - 3(Ax^2 + Bx + C) = 1 - x^2,$$

or, collecting terms with like powers of x ,

$$-3Ax^2 + (-4A - 3B)x + (2A - 2B - 3C) = 1 - x^2.$$

This last equation holds for all values of x if its two sides are identical polynomials of degree 2. Thus, we equate corresponding powers of x to get

$$-3A = -1, \quad -4A - 3B = 0, \quad \text{and} \quad 2A - 2B - 3C = 1.$$

These equations imply in turn that $A = 1/3$, $B = -4/9$, and $C = 5/27$. Substituting these values into the quadratic expression for our particular solution gives

$$y_p = \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27}.$$

By Theorem 7, the general solution to the nonhomogeneous equation is

$$y = y_c + y_p = c_1e^{-x} + c_2e^{3x} + \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27}. \quad \blacksquare$$

EXAMPLE 2 Find a particular solution of $y'' - y' = 2 \sin x$.

Solution If we try to find a particular solution of the form

$$y_p = A \sin x$$

and substitute the derivatives of y_p in the given equation, we find that A must satisfy the equation

$$-A \sin x + A \cos x = 2 \sin x$$

for all values of x . Since this requires A to equal both -2 and 0 at the same time, we conclude that the nonhomogeneous differential equation has no solution of the form $A \sin x$.

It turns out that the required form is the sum

$$y_p = A \sin x + B \cos x.$$

The result of substituting the derivatives of this new trial solution into the differential equation is

$$-A \sin x - B \cos x - (A \cos x - B \sin x) = 2 \sin x,$$

or

$$(B - A)\sin x - (A + B)\cos x = 2 \sin x.$$

This last equation must be an identity. Equating the coefficients for like terms on each side then gives

$$B - A = 2 \quad \text{and} \quad A + B = 0.$$

Simultaneous solution of these two equations gives $A = -1$ and $B = 1$. Our particular solution is

$$y_p = \cos x - \sin x. \quad \blacksquare$$

EXAMPLE 3 Find a particular solution of $y'' - 3y' + 2y = 5e^x$.

Solution If we substitute

$$y_p = Ae^x$$

and its derivatives into the differential equation, we find that

$$Ae^x - 3Ae^x + 2Ae^x = 5e^x,$$

or

$$0 = 5e^x.$$

However, the exponential function is never zero. The trouble can be traced to the fact that $y = e^x$ is already a solution of the related homogeneous equation

$$y'' - 3y' + 2y = 0.$$

The auxiliary equation is

$$r^2 - 3r + 2 = (r - 1)(r - 2) = 0,$$

which has $r = 1$ as a root. So we would expect Ae^x to become zero when substituted into the left-hand side of the differential equation.

The appropriate way to modify the trial solution in this case is to multiply Ae^x by x . Thus, our new trial solution is

$$y_p = Axe^x.$$

The result of substituting the derivatives of this new candidate into the differential equation is

$$(Axe^x + 2Ae^x) - 3(Axe^x + Ae^x) + 2Axe^x = 5e^x,$$

or

$$-Ae^x = 5e^x.$$

Thus, $A = -5$ gives our sought-after particular solution

$$y_p = -5xe^x. \quad \blacksquare$$

EXAMPLE 4 Find a particular solution of $y'' - 6y' + 9y = e^{3x}$.

Solution The auxiliary equation for the complementary equation

$$r^2 - 6r + 9 = (r - 3)^2 = 0$$

has $r = 3$ as a repeated root. The appropriate choice for y_p in this case is neither Ae^{3x} nor Axe^{3x} because the complementary solution contains both of those terms already. Thus, we choose a term containing the next higher power of x as a factor. When we substitute

$$y_p = Ax^2e^{3x}$$

and its derivatives into the given differential equation, we get

$$(9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x}) - 6(3Ax^2e^{3x} + 2Axe^{3x}) + 9Ax^2e^{3x} = e^{3x},$$

or

$$2Ae^{3x} = e^{3x}.$$

Thus, $A = 1/2$, and the particular solution is

$$y_p = \frac{1}{2}x^2e^{3x}. \quad \blacksquare$$

When we wish to find a particular solution of Equation (1) and the function $G(x)$ is the sum of two or more terms, we choose a trial function for each term in $G(x)$ and add them.

EXAMPLE 5 Find the general solution to $y'' - y' = 5e^x - \sin 2x$.

Solution We first check the auxiliary equation

$$r^2 - r = 0.$$

Its roots are $r = 1$ and $r = 0$. Therefore, the complementary solution to the associated homogeneous equation is

$$y_c = c_1e^x + c_2.$$

We now seek a particular solution y_p . That is, we seek a function that will produce $5e^x - \sin 2x$ when substituted into the left-hand side of the given differential equation. One part of y_p is to produce $5e^x$, the other $-\sin 2x$.

Since any function of the form c_1e^x is a solution of the associated homogeneous equation, we choose our trial solution y_p to be the sum

$$y_p = Axe^x + B \cos 2x + C \sin 2x,$$

including xe^x where we might otherwise have included only e^x . When the derivatives of y_p are substituted into the differential equation, the resulting equation is

$$\begin{aligned} (Axe^x + 2Ae^x - 4B \cos 2x - 4C \sin 2x) \\ - (Axe^x + Ae^x - 2B \sin 2x + 2C \cos 2x) = 5e^x - \sin 2x, \end{aligned}$$

or

$$Ae^x - (4B + 2C)\cos 2x + (2B - 4C)\sin 2x = 5e^x - \sin 2x.$$

This equation will hold if

$$A = 5, \quad 4B + 2C = 0, \quad 2B - 4C = -1,$$

or $A = 5$, $B = -1/10$, and $C = 1/5$. Our particular solution is

$$y_p = 5xe^x - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x.$$

The general solution to the differential equation is

$$y = y_c + y_p = c_1e^x + c_2 + 5xe^x - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x. \quad \blacksquare$$

You may find the following table helpful in solving the problems at the end of this section.

TABLE 17.1 The method of undetermined coefficients for selected equations of the form

$ay'' + by' + cy = G(x).$		
If $G(x)$ has a term that is a constant multiple of...	And if...	Then include this expression in the trial function for y_p
e^{rx}	r is not a root of the auxiliary equation	Ae^{rx}
	r is a single root of the auxiliary equation	Axe^{rx}
	r is a double root of the auxiliary equation	Ax^2e^{rx}
$\sin kx, \cos kx$	ki is not a root of the auxiliary equation	$B \cos kx + C \sin kx$
$px^2 + qx + m$	0 is not a root of the auxiliary equation	$Dx^2 + Ex + F$
	0 is a single root of the auxiliary equation	$Dx^3 + Ex^2 + Fx$
	0 is a double root of the auxiliary equation	$Dx^4 + Ex^3 + Fx^2$

The Method of Variation of Parameters

This is a general method for finding a particular solution of the nonhomogeneous equation (1) once the general solution of the associated homogeneous equation is known. The method consists of replacing the constants c_1 and c_2 in the complementary solution by functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$ and requiring (in a way to be explained) that the resulting expression satisfy the nonhomogeneous equation (1). There are two functions to be determined, and requiring that Equation (1) be satisfied is only one condition. As a second condition, we also require that

$$v_1'y_1 + v_2'y_2 = 0. \quad (4)$$

Then we have

$$\begin{aligned} y &= v_1y_1 + v_2y_2, \\ y' &= v_1y_1' + v_2y_2', \\ y'' &= v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2'. \end{aligned}$$

If we substitute these expressions into the left-hand side of equation (1), we obtain

$$v_1(ay_1'' + by_1' + cy_1) + v_2(ay_2'' + by_2' + cy_2) + a(v_1'y_1' + v_2'y_2') = G(x).$$

The first two parenthetical terms are zero since y_1 and y_2 are solutions of the associated homogeneous equation (2). So the nonhomogeneous equation (1) is satisfied if, in addition to equation (4), we require that

$$a(v_1'y_1' + v_2'y_2') = G(x). \quad (5)$$

Equations (4) and (5) can be solved together as a pair

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0, \\ v_1'y_1' + v_2'y_2' &= \frac{G(x)}{a} \end{aligned}$$

for the unknown functions v_1' and v_2' . The usual procedure for solving this simple system is to use the *method of determinants* (also known as *Cramer's Rule*), which will be demonstrated in the examples to follow. Once the derivative functions v_1' and v_2' are known, the two functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$ can be found by integration. Here is a summary of the method.

Variation of Parameters Procedure

To use the method of variation of parameters to find a particular solution to the nonhomogeneous equation

$$ay'' + by' + cy = G(x),$$

we can work directly with Equations (4) and (5). It is not necessary to rederive them. The steps are as follows.

1. Solve the associated homogeneous equation

$$ay'' + by' + cy = 0$$

to find the functions y_1 and y_2 .

2. Solve the equations

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0, \\ v_1'y_1' + v_2'y_2' &= \frac{G(x)}{a} \end{aligned}$$

simultaneously for the derivative functions v_1' and v_2' .

3. Integrate v_1' and v_2' to find the functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$.
4. Write down the particular solution to the nonhomogeneous equation (1) as

$$y_p = v_1y_1 + v_2y_2.$$

EXAMPLE 6 Find the general solution to the equation

$$y'' + y = \tan x.$$

Solution The solution of the homogeneous equation

$$y'' + y = 0$$

is given by

$$y_c = c_1 \cos x + c_2 \sin x.$$

Since $y_1(x) = \cos x$ and $y_2(x) = \sin x$, the conditions to be satisfied in Equations (4) and (5) are

$$\begin{aligned} v_1' \cos x + v_2' \sin x &= 0, \\ -v_1' \sin x + v_2' \cos x &= \tan x. \quad a = 1 \end{aligned}$$

Solving this system gives

$$v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\tan x \sin x}{\cos^2 x + \sin^2 x} = \frac{-\sin^2 x}{\cos x}.$$

Likewise,

$$v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \sin x.$$

After integrating v_1' and v_2' , we have

$$\begin{aligned} v_1(x) &= \int \frac{-\sin^2 x}{\cos x} dx \\ &= -\int (\sec x - \cos x) dx \\ &= -\ln|\sec x + \tan x| + \sin x, \end{aligned}$$

and

$$v_2(x) = \int \sin x dx = -\cos x.$$

Note that we have omitted the constants of integration in determining v_1 and v_2 . They would merely be absorbed into the arbitrary constants in the complementary solution.

Substituting v_1 and v_2 into the expression for y_p in Step 4 gives

$$\begin{aligned} y_p &= [-\ln|\sec x + \tan x| + \sin x]\cos x + (-\cos x)\sin x \\ &= (-\cos x) \ln|\sec x + \tan x|. \end{aligned}$$

The general solution is

$$y = c_1 \cos x + c_2 \sin x - (\cos x) \ln|\sec x + \tan x|. \quad \blacksquare$$

EXAMPLE 7 Solve the nonhomogeneous equation

$$y'' + y' - 2y = xe^x.$$

Solution The auxiliary equation is

$$r^2 + r - 2 = (r + 2)(r - 1) = 0,$$

giving the complementary solution

$$y_c = c_1 e^{-2x} + c_2 e^x.$$

The conditions to be satisfied in Equations (4) and (5) are

$$\begin{aligned} v_1' e^{-2x} + v_2' e^x &= 0, \\ -2v_1' e^{-2x} + v_2' e^x &= x e^x. \quad a = 1 \end{aligned}$$

Solving the above system for v_1' and v_2' gives

$$v_1' = \frac{\begin{vmatrix} 0 & e^x \\ x e^x & e^x \end{vmatrix}}{\begin{vmatrix} e^{-2x} & e^x \\ -2e^{-2x} & e^x \end{vmatrix}} = \frac{-x e^{2x}}{3e^{-x}} = -\frac{1}{3} x e^{3x}.$$

Likewise,

$$v_2' = \frac{\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & x e^x \end{vmatrix}}{3e^{-x}} = \frac{x e^{-x}}{3e^{-x}} = \frac{x}{3}.$$

Integrating to obtain the parameter functions, we have

$$\begin{aligned} v_1(x) &= \int -\frac{1}{3} x e^{3x} dx \\ &= -\frac{1}{3} \left(\frac{x e^{3x}}{3} - \int \frac{e^{3x}}{3} dx \right) \\ &= \frac{1}{27} (1 - 3x) e^{3x} \end{aligned}$$

and

$$v_2(x) = \int \frac{x}{3} dx = \frac{x^2}{6}.$$

Therefore,

$$\begin{aligned} y_p &= \left[\frac{(1 - 3x) e^{3x}}{27} \right] e^{-2x} + \left(\frac{x^2}{6} \right) e^x \\ &= \frac{1}{27} e^x - \frac{1}{9} x e^x + \frac{1}{6} x^2 e^x. \end{aligned}$$

The general solution to the differential equation is

$$y = c_1 e^{-2x} + c_2 e^x - \frac{1}{9} x e^x + \frac{1}{6} x^2 e^x,$$

where the term $(1/27)e^x$ in y_p has been absorbed into the term $c_2 e^x$ in the complementary solution. ■

EXERCISES 17.2

Solve the equations in Exercises 1–16 by the method of undetermined coefficients.

1. $y'' - 3y' - 10y = -3$

2. $y'' - 3y' - 10y = 2x - 3$

3. $y'' - y' = \sin x$

4. $y'' + 2y' + y = x^2$

5. $y'' + y = \cos 3x$

6. $y'' + y = e^{2x}$

7. $7y'' - y' - 2y = 20 \cos x$

8. $y'' + y = 2x + 3e^x$

9. $y'' - y = e^x + x^2$

10. $y'' + 2y' + y = 6 \sin 2x$

11. $y'' - y' - 6y = e^{-x} - 7 \cos x$

12. $y'' + 3y' + 2y = e^{-x} + e^{-2x} - x$

$$\begin{array}{ll}
 13. \frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 15x^2 & 14. \frac{d^2y}{dx^2} - \frac{dy}{dx} = -8x + 3 \\
 15. \frac{d^2y}{dx^2} - 3\frac{dy}{dx} = e^{3x} - 12x & 16. \frac{d^2y}{dx^2} + 7\frac{dy}{dx} = 42x^2 + 5x + 1
 \end{array}$$

Solve the equations in Exercises 17–28 by variation of parameters.

$$\begin{array}{ll}
 17. y'' + y' = x & \\
 18. y'' + y = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} & \\
 19. y'' + y = \sin x & 20. y'' + 2y' + y = e^x \\
 21. y'' + 2y' + y = e^{-x} & 22. y'' - y = x \\
 23. y'' - y = e^x & 24. y'' - y = \sin x \\
 25. y'' + 4y' + 5y = 10 & 26. y'' - y' = 2^x \\
 27. \frac{d^2y}{dx^2} + y = \sec x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} & \\
 28. \frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x \cos x, \quad x > 0 &
 \end{array}$$

In each of Exercises 29–32, the given differential equation has a particular solution y_p of the form given. Determine the coefficients in y_p . Then solve the differential equation.

$$\begin{array}{ll}
 29. y'' - 5y' = xe^{5x}, \quad y_p = Ax^2e^{5x} + Bxe^{5x} & \\
 30. y'' - y' = \cos x + \sin x, \quad y_p = A \cos x + B \sin x & \\
 31. y'' + y = 2 \cos x + \sin x, \quad y_p = Ax \cos x + Bx \sin x & \\
 32. y'' + y' - 2y = xe^x, \quad y_p = Ax^2e^x + Bxe^x &
 \end{array}$$

In Exercises 33–36, solve the given differential equations (a) by variation of parameters and (b) by the method of undetermined coefficients.

$$\begin{array}{ll}
 33. \frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x + e^{-x} & 34. \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2e^{2x} \\
 35. \frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 5y = e^x + 4 & 36. \frac{d^2y}{dx^2} - 9\frac{dy}{dx} = 9e^{9x}
 \end{array}$$

Solve the differential equations in Exercises 37–46. Some of the equations can be solved by the method of undetermined coefficients, but others cannot.

$$\begin{array}{ll}
 37. y'' + y = \cot x, \quad 0 < x < \pi & \\
 38. y'' + y = \csc x, \quad 0 < x < \pi & \\
 39. y'' - 8y' = e^{8x} & 40. y'' + 4y = \sin x \\
 41. y'' - y' = x^3 & 42. y'' + 4y' + 5y = x + 2
 \end{array}$$

$$\begin{array}{ll}
 43. y'' + 2y' = x^2 - e^x & 44. y'' + 9y = 9x - \cos x \\
 45. y'' + y = \sec x \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} & \\
 46. y'' - 3y' + 2y = e^x - e^{2x} &
 \end{array}$$

The method of undetermined coefficients can sometimes be used to solve first-order ordinary differential equations. Use the method to solve the equations in Exercises 47–50.

$$\begin{array}{ll}
 47. y' - 3y = e^x & 48. y' + 4y = x \\
 49. y' - 3y = 5e^{3x} & 50. y' + y = \sin x
 \end{array}$$

Solve the differential equations in Exercises 51 and 52 subject to the given initial conditions.

$$\begin{array}{ll}
 51. \frac{d^2y}{dx^2} + y = \sec^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}; \quad y(0) = y'(0) = 1 & \\
 52. \frac{d^2y}{dx^2} + y = e^{2x}; \quad y(0) = 0, y'(0) = \frac{2}{5} &
 \end{array}$$

In Exercises 53–58, verify that the given function is a particular solution to the specified nonhomogeneous equation. Find the general solution, and evaluate its arbitrary constants to find the unique solution satisfying the equation and the given initial conditions.

$$\begin{array}{ll}
 53. y'' + y' = x, \quad y_p = \frac{x^2}{2} - x, \quad y(0) = 0, y'(0) = 0 & \\
 54. y'' + y = x, \quad y_p = 2 \sin x + x, \quad y(0) = 0, y'(0) = 0 & \\
 55. \frac{1}{2}y'' + y' + y = 4e^x(\cos x - \sin x), & \\
 \quad y_p = 2e^x \cos x, \quad y(0) = 0, y'(0) = 1 & \\
 56. y'' - y' - 2y = 1 - 2x, \quad y_p = x - 1, y(0) = 0, y'(0) = 1 & \\
 57. y'' - 2y' + y = 2e^x, \quad y_p = x^2e^x, \quad y(0) = 1, y'(0) = 0 & \\
 58. y'' - 2y' + y = x^{-1}e^x, \quad x > 0, & \\
 \quad y_p = xe^x \ln x, \quad y(1) = e, y'(1) = 0 &
 \end{array}$$

In Exercises 59 and 60, two linearly independent solutions y_1 and y_2 are given to the associated homogeneous equation of the variable-coefficient nonhomogeneous equation. Use the method of variation of parameters to find a particular solution to the nonhomogeneous equation. Assume $x > 0$ in each exercise.

$$\begin{array}{ll}
 59. x^2y'' + 2xy' - 2y = x^2, \quad y_1 = x^{-2}, y_2 = x & \\
 60. x^2y'' + xy' - y = x, \quad y_1 = x^{-1}, y_2 = x &
 \end{array}$$

17.3 Applications

In this section we apply second-order differential equations to the study of vibrating springs and electric circuits.

Vibrations

A spring has its upper end fastened to a rigid support, as shown in Figure 17.2. An object of mass m is suspended from the spring and stretches it a length s when the spring comes to rest in an equilibrium position. According to Hooke's Law (Section 6.5), the tension

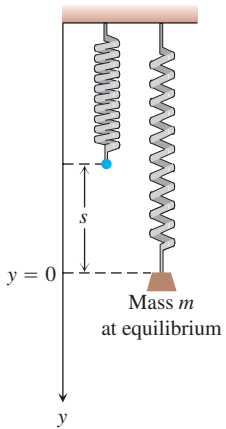


FIGURE 17.2 Mass m stretches a spring by length s to the equilibrium position at $y = 0$.

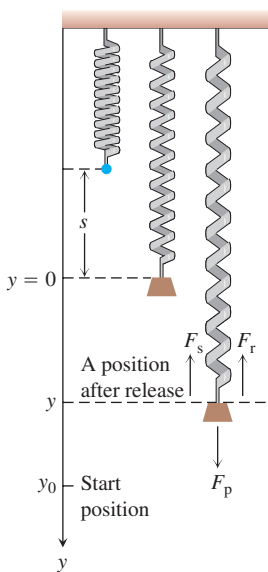


FIGURE 17.3 The propulsion force (weight) F_p pulls the mass downward, but the spring restoring force F_s and frictional force F_r pull the mass upward. The motion starts at $y = y_0$ with the mass vibrating up and down.

force in the spring is ks , where k is the spring constant. The force due to gravity pulling down on the spring is mg , and equilibrium requires that

$$ks = mg. \quad (1)$$

Suppose that the object is pulled down an additional amount y_0 beyond the equilibrium position and then released. We want to study the object's motion, that is, the vertical position of its center of mass at any future time.

Let y , with positive direction downward, denote the displacement position of the object away from the equilibrium position $y = 0$ at any time t after the motion has started. Then the forces acting on the object are (see Figure 17.3)

$$\begin{aligned} F_p &= mg, & \text{the propulsion force due to gravity,} \\ F_s &= k(s + y), & \text{the restoring force of the spring's tension,} \\ F_r &= \delta \frac{dy}{dt}, & \text{a frictional force assumed proportional to velocity.} \end{aligned}$$

The frictional force tends to slow the motion of the object. The resultant of these forces is $F = F_p - F_s - F_r$, and by Newton's second law $F = ma$, we must then have

$$m \frac{d^2 y}{dt^2} = mg - ks - ky - \delta \frac{dy}{dt}.$$

By Equation (1), $mg - ks = 0$, so this last equation becomes

$$m \frac{d^2 y}{dt^2} + \delta \frac{dy}{dt} + ky = 0, \quad (2)$$

subject to the initial conditions $y(0) = y_0$ and $y'(0) = 0$. (Here we use the prime notation to denote differentiation with respect to time t .)

You might expect that the motion predicted by Equation (2) will be oscillatory about the equilibrium position $y = 0$ and eventually damp to zero because of the frictional force. This is indeed the case, and we will show how the constants m , δ , and k determine the nature of the damping. You will also see that if there is no friction (so $\delta = 0$), then the object will simply oscillate indefinitely.

Simple Harmonic Motion

Suppose first that there is no frictional force. Then $\delta = 0$ and there is no damping. If we substitute $\omega = \sqrt{k/m}$ to simplify our calculations, then the second-order equation (2) becomes

$$y'' + \omega^2 y = 0, \quad \text{with} \quad y(0) = y_0 \quad \text{and} \quad y'(0) = 0.$$

The auxiliary equation is

$$r^2 + \omega^2 = 0,$$

which has the imaginary roots $r = \pm \omega i$. The general solution to the differential equation in (2) is

$$y = c_1 \cos \omega t + c_2 \sin \omega t. \quad (3)$$

To fit the initial conditions, we compute

$$y' = -c_1 \omega \sin \omega t + c_2 \omega \cos \omega t$$

and then substitute the conditions. This yields $c_1 = y_0$ and $c_2 = 0$. The particular solution

$$y = y_0 \cos \omega t \quad (4)$$

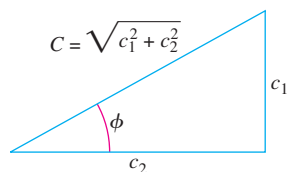


FIGURE 17.4 $c_1 = C \sin \phi$ and $c_2 = C \cos \phi$.

describes the motion of the object. Equation (4) represents **simple harmonic motion** of amplitude y_0 and period $T = 2\pi/\omega$.

The general solution given by Equation (3) can be combined into a single term by using the trigonometric identity

$$\sin(\omega t + \phi) = \cos \omega t \sin \phi + \sin \omega t \cos \phi.$$

To apply the identity, we take (see Figure 17.4)

$$c_1 = C \sin \phi \quad \text{and} \quad c_2 = C \cos \phi,$$

where

$$C = \sqrt{c_1^2 + c_2^2} \quad \text{and} \quad \phi = \tan^{-1} \frac{c_1}{c_2}.$$

Then the general solution in Equation (3) can be written in the alternative form

$$y = C \sin(\omega t + \phi). \quad (5)$$

Here C and ϕ may be taken as two new arbitrary constants, replacing the two constants c_1 and c_2 . Equation (5) represents simple harmonic motion of amplitude C and period $T = 2\pi/\omega$. The angle $\omega t + \phi$ is called the **phase angle**, and ϕ may be interpreted as its initial value. A graph of the simple harmonic motion represented by Equation (5) is given in Figure 17.5.

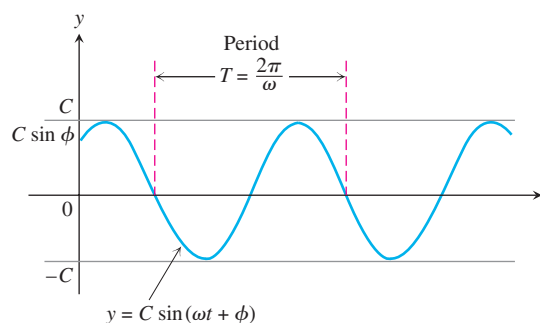


FIGURE 17.5 Simple harmonic motion of amplitude C and period T with initial phase angle ϕ (Equation 5).

Damped Motion

Assume now that there is friction in the spring system, so $\delta \neq 0$. If we substitute $\omega = \sqrt{k/m}$ and $2b = \delta/m$, then the differential equation (2) is

$$y'' + 2by' + \omega^2 y = 0. \quad (6)$$

The auxiliary equation is

$$r^2 + 2br + \omega^2 = 0,$$

with roots $r = -b \pm \sqrt{b^2 - \omega^2}$. Three cases now present themselves, depending on the relative sizes of b and ω .

Case 1: $b = \omega$. The double root of the auxiliary equation is real and equals $r = -\omega$. The general solution to Equation (6) is

$$y = (c_1 + c_2 t)e^{-\omega t}.$$

This situation of motion is called **critical damping** and is not oscillatory. Figure 17.6a shows an example of this kind of damped motion.

Case 2: $b > \omega$. The roots of the auxiliary equation are real and unequal, and they are given by $r_1 = -b + \sqrt{b^2 - \omega^2}$ and $r_2 = -b - \sqrt{b^2 - \omega^2}$. The general solution to Equation (6) is given by

$$y = c_1 e^{(-b + \sqrt{b^2 - \omega^2})t} + c_2 e^{(-b - \sqrt{b^2 - \omega^2})t}.$$

Here again the motion is not oscillatory and both r_1 and r_2 are negative. Thus y approaches zero as time goes on. This motion is referred to as **overdamping** (see Figure 17.6b).

Case 3: $b < \omega$. The roots to the auxiliary equation are complex and are given by $r = -b \pm i\sqrt{\omega^2 - b^2}$. The general solution to Equation (6) is given by

$$y = e^{-bt} (c_1 \cos \sqrt{\omega^2 - b^2} t + c_2 \sin \sqrt{\omega^2 - b^2} t).$$

This situation, called **underdamping**, represents damped oscillatory motion. It is analogous to simple harmonic motion of period $T = 2\pi/\sqrt{\omega^2 - b^2}$ except that the amplitude is not constant but damped by the factor e^{-bt} . Therefore, the motion tends to zero as t increases, so the vibrations tend to die out as time goes on. Notice that the period $T = 2\pi/\sqrt{\omega^2 - b^2}$ is larger than the period $T_0 = 2\pi/\omega$ in the friction-free system. Moreover, the larger the value of $b = \delta/(2m)$ in the exponential damping factor, the more quickly the vibrations tend to become unnoticeable. A curve illustrating underdamped motion is shown in Figure 17.6c.

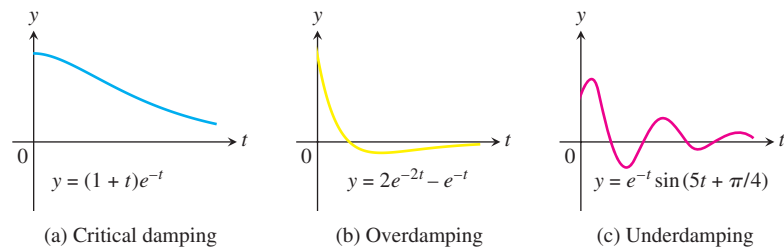


FIGURE 17.6 Three examples of damped vibratory motion for a spring system with friction, so $\delta \neq 0$.

An external force $F(t)$ can also be added to the spring system modeled by Equation (2). The forcing function may represent an external disturbance on the system. For instance, if the equation models an automobile suspension system, the forcing function might represent periodic bumps or potholes in the road affecting the performance of the suspension system. Or it might represent the effects of winds when modeling the vertical motion of a suspension bridge. Inclusion of a forcing function results in the second-order nonhomogeneous equation

$$m \frac{d^2 y}{dt^2} + \delta \frac{dy}{dt} + ky = F(t). \quad (7)$$

Such equations are studied in the theory of Differential Equations.

Electric Circuits

The basic quantity in electricity is the **charge** q (analogous to the idea of mass). In an electric field we use the flow of charge, or **current** $I = dq/dt$, as we might use velocity in a gravitational field. There are many similarities between motion in a gravitational field and the flow of electrons (the carriers of charge) in an electric field.

Consider the electric circuit shown in Figure 17.7. It consists of four components: voltage source, resistor, inductor, and capacitor. Think of electrical flow as being like a fluid flow, where the voltage source is the pump and the resistor, inductor, and capacitor tend to block the flow. A battery or generator is an example of a source, producing a

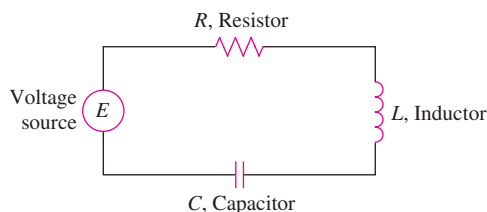


FIGURE 17.7 An electric circuit.

voltage that causes the current to flow through the circuit when the switch is closed. An electric light bulb or appliance would provide resistance. The inductance is due to a magnetic field that opposes any change in the current as it flows through a coil. The capacitance is normally created by two metal plates that alternate charges and thus reverse the current flow. The following symbols specify the quantities relevant to the circuit.

q : charge at a cross section of a conductor, measured in **coulombs** (abbreviated c)

I : current or rate of change of charge dq/dt (flow of electrons) at a cross section of a conductor, measured in **amperes** (abbreviated A)

E : electric (potential) source, measured in **volts** (abbreviated V)

V : difference in potential between two points along the conductor, measured in **volts** (V)

Ohm observed that the current I flowing through a resistor, caused by a potential difference across it, is (approximately) proportional to the potential difference (voltage drop). He named his constant of proportionality $1/R$ and called R the **resistance**. So *Ohm's law* is

$$I = \frac{1}{R}V.$$

Similarly, it is known from physics that the voltage drops across an inductor and a capacitor are, respectively,

$$L \frac{dI}{dt} \quad \text{and} \quad \frac{q}{C},$$

where L is the **inductance** and C is the **capacitance** (with q the charge on the capacitor).

The German physicist Gustav R. Kirchhoff (1824–1887) formulated the law that the sum of the voltage drops in a closed circuit is equal to the supplied voltage $E(t)$. Symbolically, this says that

$$RI + L \frac{dI}{dt} + \frac{q}{C} = E(t).$$

Since $I = dq/dt$, Kirchhoff's law becomes

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C}q = E(t). \quad (8)$$

The second-order differential equation (8), which models an electric circuit, has exactly the same form as Equation (7) modeling vibratory motion. Both models can be solved using the methods developed in Section 17.2.

Summary

The following chart summarizes our analogies between the physics of motion of an object in a spring system and the flow of charged particles in an electric circuit.

Linear Second-Order Constant-Coefficient Models

Mechanical System

$$my'' + \delta y' + ky = F(t)$$

y	displacement
y'	velocity
y''	acceleration
m	mass
δ	damping constant
k	spring constant
$F(t)$	forcing function

Electrical System

$$Lq'' + Rq' + \frac{1}{C}q = E(t)$$

q	charge
q'	current
q''	change in current
L	inductance
R	resistance
$1/C$	where C is the capacitance
$E(t)$	voltage source

EXERCISES 17.3

1. A 70-N weight is attached to the lower end of a coil spring suspended from the ceiling and having a spring constant of 15 N/m. The resistance in the spring-mass system is numerically equal to 15 times the instantaneous velocity. At $t = 0$, the weight is set in motion from a position 0.6 m below its equilibrium position by giving it a downward velocity of 0.6 m/s. Write an initial value problem that models the given situation.
2. A 36-N weight stretches a spring 1.2 m. The spring-mass system resides in a medium offering a resistance to the motion that is numerically equal to 20 times the instantaneous velocity. If the weight is released at a position 0.6 m above its equilibrium position with a downward velocity of 0.9 m/s, write an initial value problem modeling the given situation.
3. A 90-N weight is hung on a 0.4-m spring and stretches it 0.15 m. The weight is pulled down 0.1 m and 30 N are added to the weight. If the weight is now released with a downward velocity of v_0 m/s, write an initial value problem modeling the vertical displacement.
4. A 49-N weight is suspended by a spring that is stretched 0.05 m by the weight. Assume a resistance whose magnitude is $300/\sqrt{g}$ N times the instantaneous velocity v in meters per second. If the weight is pulled down 0.08 m below its equilibrium position and released, formulate an initial value problem modeling the behavior of the spring-mass system.
5. An (open) electric circuit consists of an inductor, a resistor, and a capacitor. There is an initial charge of 2 coulombs on the capacitor. At the instant the circuit is closed, a current of 3 amperes is present and a voltage of $E(t) = 20 \cos t$ is applied. In this circuit the voltage drop across the resistor is 4 times the instantaneous change in the charge, the voltage drop across the capacitor is 10 times the charge, and the voltage drop across the inductor is 2 times the instantaneous change in the current. Write an initial value problem to model the circuit.
6. An inductor of 2 henrys is connected in series with a resistor of 12 ohms, a capacitor of $1/16$ farad, and a 300-volt battery. Initially, the charge on the capacitor is zero and the current is zero. Formulate an initial value problem modeling this electric circuit.
7. A 49-N weight is attached to the lower end of a coil spring suspended from the ceiling and having a spring constant of 10 N/m. The resistance in the spring-mass system is numerically equal to 10 times the instantaneous velocity. At $t = 0$, the weight is set in motion from a position 0.6 m below its equilibrium position by giving it a downward velocity of 0.6 m/s. At the end of π s, determine whether the mass is above or below the equilibrium position and by what distance.
8. A 29.4-N weight stretches a spring 1.225 m. The spring-mass system resides in a medium offering a resistance to the motion equal to 18 times the instantaneous velocity. If the weight is released at a position 0.6 m above its equilibrium position with a downward velocity of 0.9 m/s, find its position relative to the equilibrium position 2 s later.
9. A 98-N weight is hung on a 0.6 m spring stretching it 0.2 m. The weight is pulled down 0.15 m and 49 N are added to the weight. If the weight is now released with a downward velocity of v_0 m/s, find the position of mass relative to the equilibrium in terms of v_0 and valid for any time $t \geq 0$.
10. A mass of 15 kg is attached to a spring whose constant is $375/4$ N/m. Initially the mass is released 1 m above the equilibrium position with a downward velocity of 3 m/s, and the subsequent motion takes place in a medium that offers a damping force numerically equal to 45 times the instantaneous velocity. An external force $f(t)$ is driving the system, but assume that initially $f(t) \equiv 0$. Formulate and solve an initial value problem that models the given system. Interpret your results.
11. A 50-N weight is suspended by a spring that is stretched 0.05 m by the weight. Assume a resistance whose magnitude is 100 N times the instantaneous velocity in meters per second. If the weight is pulled down 0.1 m below its equilibrium position and released, find the time required to reach the equilibrium position for the first time.

12. A weight stretches a spring 0.2 m. It is set in motion at a point 0.05 m below its equilibrium position with a downward velocity of 0.05 m/s.
 - a. When does the weight return to its equilibrium position?
 - b. When does it reach its highest point?
 - c. Show that the maximum velocity is $0.05\sqrt{10g}$ m/s.
13. A weight of 50 N stretches a spring 0.25 m. The weight is drawn down 0.05 m below its equilibrium position and given an initial velocity of 0.1 m/s. An identical spring has a different weight attached to it. This second weight is drawn down from its equilibrium position a distance equal to the amplitude of the first motion and then given an initial velocity of 0.6 m/s. If the amplitude of the second motion is twice that of the first, what weight is attached to the second spring?
14. A weight stretches one spring 0.05 m and a second weight stretches another spring 0.15 m. If both weights are simultaneously pulled down 0.02 m below their respective equilibrium positions and then released, find the first time after $t = 0$ when their velocities are equal.
15. A weight of 80 N stretches a spring 1 m. The weight is pulled down 1.5 m below the equilibrium position and then released. What initial velocity v_0 given to the weight would have the effect of doubling the amplitude of the vibration?
16. A mass weighing 40 N stretches a spring 0.1 m. The spring-mass system resides in a medium with a damping constant of 32 N-s/m. If the mass is released from its equilibrium position with a velocity of 0.1 m/s in the downward direction, find the time required for the mass to return to its equilibrium position for the first time.
17. A weight suspended from a spring executes damped vibrations with a period of 2 s. If the damping factor decreases by 90% in 10 s, find the acceleration of the weight when it is 0.1 m below its equilibrium position and is moving upward with a speed of 0.8 m/s.
18. A 50-N weight stretches a spring 0.6 m. If the weight is pulled down 0.15 m below its equilibrium position and released, find the highest point reached by the weight. Assume the spring-mass system resides in a medium offering a resistance of 30 N times the instantaneous velocity in meters per second.
19. An *LRC* circuit is set up with an inductance of $1/5$ henry, a resistance of 1 ohm, and a capacitance of $5/6$ farad. Assuming the initial charge is 2 coulombs and the initial current is 4 amperes, find the solution function describing the charge on the capacitor at any time. What is the charge on the capacitor after a long period of time?
20. An (open) electric circuit consists of an inductor, a resistor, and a capacitor. There is an initial charge of 2 coulombs on the capacitor. At the instant the circuit is closed, a current of 3 amperes is present but no external voltage is being applied. In this circuit the voltage drops at three points are numerically related as follows: across the capacitor, 10 times the charge; across the resistor, 4 times the instantaneous change in the charge; and across the inductor, 2 times the instantaneous change in the current. Find the charge on the capacitor as a function of time.
21. A 78.4-N weight stretches a spring 1.225 m. This spring-mass system is in a medium with a damping constant of 72 N-s/m, and an external force given by $f(t) = 25.6 + 6.4e^{-2t}$ (in newtons) is being applied. What is the solution function describing the position of the mass at any time if the mass is released from 0.6 m below the equilibrium position with an initial velocity of 1.2 m/s downward?
22. A 10-kg mass is attached to a spring having a spring constant of 140 N/m. The mass is started in motion from the equilibrium position with an initial velocity of 1 m/s in the upward direction and with an applied external force given by $f(t) = 5 \sin t$ (in newtons). The mass is in a viscous medium with a coefficient of resistance equal to 90 N-s/m. Formulate an initial value problem that models the given system; solve the model and interpret the results.
23. A 2-kg mass is attached to the lower end of a coil spring suspended from the ceiling. The mass comes to rest in its equilibrium position thereby stretching the spring 1.96 m. The mass is in a viscous medium that offers a resistance in newtons numerically equal to 4 times the instantaneous velocity measured in meters per second. The mass is then pulled down 2 m below its equilibrium position and released with a downward velocity of 3 m/s. At this same instant an external force given by $f(t) = 20 \cos t$ (in newtons) is applied to the system. At the end of π s determine if the mass is above or below its equilibrium position and by how much.
24. A 39.2-N weight stretches a spring 1.225 m. The spring-mass system resides in a medium offering a resistance to the motion equal to 24 times the instantaneous velocity, and an external force given by $f(t) = 28.8 + 19.2e^{-t}$ (in newtons) is being applied. If the weight is released at a position 0.6 m above its equilibrium position with downward velocity of 0.9 m/s, find its position relative to the equilibrium after 2 s have elapsed.
25. Suppose $L = 10$ henrys, $R = 10$ ohms, $C = 1/500$ farads, $E = 100$ volts, $q(0) = 10$ coulombs, and $q'(0) = i(0) = 0$. Formulate and solve an initial value problem that models the given *LRC* circuit. Interpret your results.
26. A series circuit consisting of an inductor, a resistor, and a capacitor is open. There is an initial charge of 2 coulombs on the capacitor, and 3 amperes of current is present in the circuit at the instant the circuit is closed. A voltage given by $E(t) = 20 \cos t$ is applied. In this circuit the voltage drops are numerically equal to the following: across the resistor, to 4 times the instantaneous change in the charge; across the capacitor, to 10 times the charge; and across the inductor, to 2 times the instantaneous change in the current. Find the charge on the capacitor as a function of time. Determine the charge on the capacitor and the current at time $t = 10$.

17.4 Euler Equations

In Section 17.1 we introduced the second-order linear homogeneous differential equation

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$$

and showed how to solve this equation when the coefficients P , Q , and R are constants. If the coefficients are not constant, we cannot generally solve this differential equation in terms of elementary functions we have studied in calculus. In this section you will learn how to solve the equation when the coefficients have the special forms

$$P(x) = ax^2, \quad Q(x) = bx, \quad \text{and} \quad R(x) = c,$$

where a , b , and c are constants. These special types of equations are called **Euler equations** in honor of Leonhard Euler, who studied them and showed how to solve them. Such equations arise in the study of mechanical vibrations.

The General Solution of Euler Equations

Consider the Euler equation

$$ax^2y'' + bxy' + cy = 0, \quad x > 0. \quad (1)$$

To solve Equation (1), we first make the change of variables

$$z = \ln x \quad \text{and} \quad y(x) = Y(z).$$

We next use the chain rule to find the derivatives $y'(x)$ and $y''(x)$:

$$y'(x) = \frac{d}{dx}Y(z) = \frac{d}{dz}Y(z)\frac{dz}{dx} = Y'(z)\frac{1}{x}$$

and

$$y''(x) = \frac{d}{dx}y'(x) = \frac{d}{dx}Y'(z)\frac{1}{x} = -\frac{1}{x^2}Y'(z) + \frac{1}{x}Y''(z)\frac{dz}{dx} = -\frac{1}{x^2}Y'(z) + \frac{1}{x^2}Y''(z).$$

Substituting these two derivatives into the left-hand side of Equation (1), we find

$$\begin{aligned} ax^2y'' + bxy' + cy &= ax^2\left(-\frac{1}{x^2}Y'(z) + \frac{1}{x^2}Y''(z)\right) + bx\left(\frac{1}{x}Y'(z)\right) + cY(z) \\ &= aY''(z) + (b - a)Y'(z) + cY(z). \end{aligned}$$

Therefore, the substitutions give us the second-order linear differential equation with constant coefficients

$$aY''(z) + (b - a)Y'(z) + cY(z) = 0. \quad (2)$$

We can solve Equation (2) using the method of Section 17.1. That is, we find the roots of the associated auxiliary equation

$$ar^2 + (b - a)r + c = 0 \quad (3)$$

to find the general solution for $Y(z)$. After finding $Y(z)$, we can determine $y(x)$ from the substitution $z = \ln x$.

EXAMPLE 1 Find the general solution of the equation $x^2y'' + 2xy' - 2y = 0$.

Solution This is an Euler equation with $a = 1$, $b = 2$, and $c = -2$. The auxiliary equation (3) for $Y(z)$ is

$$r^2 + (2 - 1)r - 2 = (r - 1)(r + 2) = 0,$$

with roots $r = -2$ and $r = 1$. The solution for $Y(z)$ is given by

$$Y(z) = c_1 e^{-2z} + c_2 e^z.$$

Substituting $z = \ln x$ gives the general solution for $y(x)$:

$$y(x) = c_1 e^{-2 \ln x} + c_2 e^{\ln x} = c_1 x^{-2} + c_2 x. \quad \blacksquare$$

EXAMPLE 2 Solve the Euler equation $x^2 y'' - 5xy' + 9y = 0$.

Solution Since $a = 1$, $b = -5$, and $c = 9$, the auxiliary equation (3) for $Y(z)$ is

$$r^2 + (-5 - 1)r + 9 = (r - 3)^2 = 0.$$

The auxiliary equation has the double root $r = 3$, giving

$$Y(z) = c_1 e^{3z} + c_2 z e^{3z}.$$

Substituting $z = \ln x$ into this expression gives the general solution

$$y(x) = c_1 e^{3 \ln x} + c_2 \ln x e^{3 \ln x} = c_1 x^3 + c_2 x^3 \ln x. \quad \blacksquare$$

EXAMPLE 3 Find the particular solution to $x^2 y'' - 3xy' + 68y = 0$ that satisfies the initial conditions $y(1) = 0$ and $y'(1) = 1$.

Solution Here $a = 1$, $b = -3$, and $c = 68$ substituted into the auxiliary equation (3) give

$$r^2 - 4r + 68 = 0.$$

The roots are $r = 2 + 8i$ and $r = 2 - 8i$, giving the solution

$$Y(z) = e^{2z} (c_1 \cos 8z + c_2 \sin 8z).$$

Substituting $z = \ln x$ into this expression gives

$$y(x) = e^{2 \ln x} (c_1 \cos(8 \ln x) + c_2 \sin(8 \ln x)).$$

From the initial condition $y(1) = 0$, we see that $c_1 = 0$ and

$$y(x) = c_2 x^2 \sin(8 \ln x).$$

To fit the second initial condition, we need the derivative

$$y'(x) = c_2 (8x \cos(8 \ln x) + 2x \sin(8 \ln x)).$$

Since $y'(1) = 1$, we immediately obtain $c_2 = 1/8$. Therefore, the particular solution satisfying both initial conditions is

$$y(x) = \frac{1}{8} x^2 \sin(8 \ln x).$$

Since $-1 \leq \sin(8 \ln x) \leq 1$, the solution satisfies

$$-\frac{x^2}{8} \leq y(x) \leq \frac{x^2}{8}.$$

A graph of the solution is shown in Figure 17.8. \blacksquare

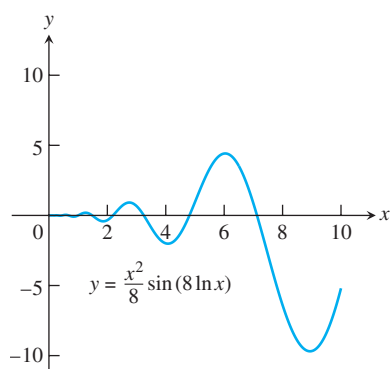


FIGURE 17.8 Graph of the solution to Example 3.

EXERCISES 17.4

In Exercises 1–24, find the general solution to the given Euler equation. Assume $x > 0$ throughout.

1. $x^2y'' + 2xy' - 2y = 0$
2. $x^2y'' + xy' - 4y = 0$
3. $x^2y'' - 6y = 0$
4. $x^2y'' + xy' - y = 0$
5. $x^2y'' - 5xy' + 8y = 0$
6. $2x^2y'' + 7xy' + 2y = 0$
7. $3x^2y'' + 4xy' = 0$
8. $x^2y'' + 6xy' + 4y = 0$
9. $x^2y'' - xy' + y = 0$
10. $x^2y'' - xy' + 2y = 0$
11. $x^2y'' - xy' + 5y = 0$
12. $x^2y'' + 7xy' + 13y = 0$
13. $x^2y'' + 3xy' + 10y = 0$
14. $x^2y'' - 5xy' + 10y = 0$
15. $4x^2y'' + 8xy' + 5y = 0$
16. $4x^2y'' - 4xy' + 5y = 0$
17. $x^2y'' + 3xy' + y = 0$
18. $x^2y'' - 3xy' + 9y = 0$
19. $x^2y'' + xy' = 0$
20. $4x^2y'' + y = 0$

21. $9x^2y'' + 15xy' + y = 0$
22. $16x^2y'' - 8xy' + 9y = 0$
23. $16x^2y'' + 56xy' + 25y = 0$
24. $4x^2y'' - 16xy' + 25y = 0$

In Exercises 25–30, solve the given initial value problem.

25. $x^2y'' + 3xy' - 3y = 0$, $y(1) = 1$, $y'(1) = -1$
26. $6x^2y'' + 7xy' - 2y = 0$, $y(1) = 0$, $y'(1) = 1$
27. $x^2y'' - xy' + y = 0$, $y(1) = 1$, $y'(1) = 1$
28. $x^2y'' + 7xy' + 9y = 0$, $y(1) = 1$, $y'(1) = 0$
29. $x^2y'' - xy' + 2y = 0$, $y(1) = -1$, $y'(1) = 1$
30. $x^2y'' + 3xy' + 5y = 0$, $y(1) = 1$, $y'(1) = 0$

17.5 Power-Series Solutions

In this section we extend our study of second-order linear homogeneous equations with variable coefficients. With the Euler equations in Section 17.4, the power of the variable x in the nonconstant coefficient had to match the order of the derivative with which it was paired: x^2 with y'' , x^1 with y' , and $x^0 (=1)$ with y . Here we drop that requirement so we can solve more general equations.

Method of Solution

The **power-series method** for solving a second-order homogeneous differential equation consists of finding the coefficients of a power series

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots \quad (1)$$

which solves the equation. To apply the method we substitute the series and its derivatives into the differential equation to determine the coefficients $c_0, c_1, c_2, \dots$. The technique for finding the coefficients is similar to that used in the method of undetermined coefficients presented in Section 17.2.

In our first example we demonstrate the method in the setting of a simple equation whose general solution we already know. This is to help you become more comfortable with solutions expressed in series form.

EXAMPLE 1 Solve the equation $y'' + y = 0$ by the power-series method.

Solution We assume the series solution takes the form of

$$y = \sum_{n=0}^{\infty} c_n x^n$$

and calculate the derivatives

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

Substitution of these forms into the second-order equation gives us

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0.$$

Next, we equate the coefficients of each power of x to zero as summarized in the following table.

Power of x	Coefficient equation		
x^0	$2(1)c_2 + c_0 = 0$	or	$c_2 = -\frac{1}{2}c_0$
x^1	$3(2)c_3 + c_1 = 0$	or	$c_3 = -\frac{1}{3 \cdot 2}c_1$
x^2	$4(3)c_4 + c_2 = 0$	or	$c_4 = -\frac{1}{4 \cdot 3}c_2$
x^3	$5(4)c_5 + c_3 = 0$	or	$c_5 = -\frac{1}{5 \cdot 4}c_3$
x^4	$6(5)c_6 + c_4 = 0$	or	$c_6 = -\frac{1}{6 \cdot 5}c_4$
$\vdots$	$\vdots$		$\vdots$
x^{n-2}	$n(n-1)c_n + c_{n-2} = 0$	or	$c_n = -\frac{1}{n(n-1)}c_{n-2}$

From the table we notice that the coefficients with even indices ($n = 2k$, $k = 1, 2, 3, \dots$) are related to each other and the coefficients with odd indices ($n = 2k + 1$) are also inter-related. We treat each group in turn.

Even indices: Here $n = 2k$, so the power is x^{2k-2} . From the last line of the table, we have

$$2k(2k-1)c_{2k} + c_{2k-2} = 0$$

or

$$c_{2k} = -\frac{1}{2k(2k-1)}c_{2k-2}.$$

From this recursive relation we find

$$\begin{aligned} c_{2k} &= \left[-\frac{1}{2k(2k-1)} \right] \left[-\frac{1}{(2k-2)(2k-3)} \right] \cdots \left[-\frac{1}{4(3)} \right] \left[-\frac{1}{2} \right] c_0 \\ &= \frac{(-1)^k}{(2k)!} c_0. \end{aligned}$$

Odd indices: Here $n = 2k + 1$, so the power is x^{2k-1} . Substituting this into the last line of the table yields

$$(2k+1)(2k)c_{2k+1} + c_{2k-1} = 0$$

or

$$c_{2k+1} = -\frac{1}{(2k+1)(2k)}c_{2k-1}.$$

Thus,

$$\begin{aligned} c_{2k+1} &= \left[-\frac{1}{(2k+1)(2k)} \right] \left[-\frac{1}{(2k-1)(2k-2)} \right] \cdots \left[-\frac{1}{5(4)} \right] \left[-\frac{1}{3(2)} \right] c_1 \\ &= \frac{(-1)^k}{(2k+1)!} c_1. \end{aligned}$$

Writing the power series by grouping its even and odd powers together and substituting for the coefficients yields

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{k=0}^{\infty} c_{2k} x^{2k} + \sum_{k=0}^{\infty} c_{2k+1} x^{2k+1} \\ &= c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}. \end{aligned}$$

From our study of Taylor series, we see that the first series on the right-hand side of the last equation represents the cosine function, and the second series represents the sine. Thus, the general solution to $y'' + y = 0$ is

$$y = c_0 \cos x + c_1 \sin x. \quad \blacksquare$$

EXAMPLE 2 Find the general solution to $y'' + xy' + y = 0$.

Solution We assume the series solution form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

and calculate the derivatives

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

Substitution of these forms into the second-order equation yields

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$$

We equate the coefficients of each power of x to zero as summarized in the following table.

Power of x	Coefficient equation
x^0	$2(1)c_2 + c_0 = 0 \quad \text{or} \quad c_2 = -\frac{1}{2}c_0$
x^1	$3(2)c_3 + c_1 + c_1 = 0 \quad \text{or} \quad c_3 = -\frac{1}{3}c_1$
x^2	$4(3)c_4 + 2c_2 + c_2 = 0 \quad \text{or} \quad c_4 = -\frac{1}{4}c_2$
x^3	$5(4)c_5 + 3c_3 + c_3 = 0 \quad \text{or} \quad c_5 = -\frac{1}{5}c_3$
x^4	$6(5)c_6 + 4c_4 + c_4 = 0 \quad \text{or} \quad c_6 = -\frac{1}{6}c_4$
$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} + (n+1)c_n = 0 \quad \text{or} \quad c_{n+2} = -\frac{1}{n+2}c_n$

From the table notice that the coefficients with even indices are interrelated and the coefficients with odd indices are also interrelated.

Even indices: Here $n = 2k - 2$, so the power is x^{2k-2} . From the last line in the table, we have

$$c_{2k} = -\frac{1}{2k}c_{2k-2}.$$

From this recurrence relation we obtain

$$\begin{aligned} c_{2k} &= \left(-\frac{1}{2k}\right)\left(-\frac{1}{2k-2}\right)\cdots\left(-\frac{1}{6}\right)\left(-\frac{1}{4}\right)\left(-\frac{1}{2}\right)c_0 \\ &= \frac{(-1)^k}{(2)(4)(6)\cdots(2k)}c_0. \end{aligned}$$

Odd indices: Here $n = 2k - 1$, so the power is x^{2k-1} . From the last line in the table, we have

$$c_{2k+1} = -\frac{1}{2k+1}c_{2k-1}.$$

From this recurrence relation we obtain

$$\begin{aligned} c_{2k+1} &= \left(-\frac{1}{2k+1}\right)\left(-\frac{1}{2k-1}\right)\cdots\left(-\frac{1}{5}\right)\left(-\frac{1}{3}\right)c_1 \\ &= \frac{(-1)^k}{(3)(5)\cdots(2k+1)}c_1. \end{aligned}$$

Writing the power series by grouping its even and odd powers and substituting for the coefficients yields

$$\begin{aligned} y &= \sum_{k=0}^{\infty} c_{2k}x^{2k} + \sum_{k=0}^{\infty} c_{2k+1}x^{2k+1} \\ &= c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2)(4)\cdots(2k)}x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(3)(5)\cdots(2k+1)}x^{2k+1}. \end{aligned}$$

EXAMPLE 3 Find the general solution to

$$(1-x^2)y'' - 6xy' - 4y = 0, \quad |x| < 1.$$

Solution Notice that the leading coefficient is zero when $x = \pm 1$. Thus, we assume the solution interval I : $-1 < x < 1$. Substitution of the series form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

and its derivatives gives us

$$\begin{aligned} (1-x^2) \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 6 \sum_{n=1}^{\infty} n c_n x^n - 4 \sum_{n=0}^{\infty} c_n x^n &= 0, \\ \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^n - 6 \sum_{n=1}^{\infty} n c_n x^n - 4 \sum_{n=0}^{\infty} c_n x^n &= 0. \end{aligned}$$

Next, we equate the coefficients of each power of x to zero as summarized in the following table.

Power of x	Coefficient equation
x^0	$2(1)c_2 - 4c_0 = 0 \quad \text{or} \quad c_2 = \frac{4}{2}c_0$
x^1	$3(2)c_3 - 6(1)c_1 - 4c_1 = 0 \quad \text{or} \quad c_3 = \frac{5}{3}c_1$
x^2	$4(3)c_4 - 2(1)c_2 - 6(2)c_2 - 4c_2 = 0 \quad \text{or} \quad c_4 = \frac{6}{4}c_2$
x^3	$5(4)c_5 - 3(2)c_3 - 6(3)c_3 - 4c_3 = 0 \quad \text{or} \quad c_5 = \frac{7}{5}c_3$
$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} - [n(n-1) + 6n + 4]c_n = 0$ $(n+2)(n+1)c_{n+2} - (n+4)(n+1)c_n = 0 \quad \text{or} \quad c_{n+2} = \frac{n+4}{n+2}c_n$

Again we notice that the coefficients with even indices are interrelated and those with odd indices are interrelated.

Even indices: Here $n = 2k - 2$, so the power is x^{2k} . From the right-hand column and the last line of the table, we get

$$\begin{aligned}
 c_{2k} &= \frac{2k+2}{2k} c_{2k-2} \\
 &= \left(\frac{2k+2}{2k}\right) \left(\frac{2k}{2k-2}\right) \left(\frac{2k-2}{2k-4}\right) \cdots \frac{6}{4} \left(\frac{4}{2}\right) c_0 \\
 &= (k+1)c_0.
 \end{aligned}$$

Odd indices: Here $n = 2k - 1$, so the power is x^{2k+1} . The right-hand column and the last line of the table give us

$$\begin{aligned}
 c_{2k+1} &= \frac{2k+3}{2k+1} c_{2k-1} \\
 &= \left(\frac{2k+3}{2k+1}\right) \left(\frac{2k+1}{2k-1}\right) \left(\frac{2k-1}{2k-3}\right) \cdots \frac{7}{5} \left(\frac{5}{3}\right) c_1 \\
 &= \frac{2k+3}{3} c_1.
 \end{aligned}$$

The general solution is

$$\begin{aligned}
 y &= \sum_{n=0}^{\infty} c_n x^n \\
 &= \sum_{k=0}^{\infty} c_{2k} x^{2k} + \sum_{k=0}^{\infty} c_{2k+1} x^{2k+1} \\
 &= c_0 \sum_{k=0}^{\infty} (k+1) x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{2k+3}{3} x^{2k+1}.
 \end{aligned}$$

EXAMPLE 4 Find the general solution to $y'' - 2xy' + y = 0$.

Solution Assuming that

$$y = \sum_{n=0}^{\infty} c_n x^n,$$

substitution into the differential equation gives us

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2 \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$$

We next determine the coefficients, listing them in the following table.

Power of x	Coefficient equation
x^0	$2(1)c_2 + c_0 = 0$ or $c_2 = -\frac{1}{2}c_0$
x^1	$3(2)c_3 - 2c_1 + c_1 = 0$ or $c_3 = \frac{1}{3 \cdot 2}c_1$
x^2	$4(3)c_4 - 4c_2 + c_2 = 0$ or $c_4 = \frac{3}{4 \cdot 3}c_2$
x^3	$5(4)c_5 - 6c_3 + c_3 = 0$ or $c_5 = \frac{5}{5 \cdot 4}c_3$
x^4	$6(5)c_6 - 8c_4 + c_4 = 0$ or $c_6 = \frac{7}{6 \cdot 5}c_4$
$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} - (2n-1)c_n = 0$ or $c_{n+2} = \frac{2n-1}{(n+2)(n+1)}c_n$

From the recursive relation

$$c_{n+2} = \frac{2n-1}{(n+2)(n+1)}c_n,$$

we write out the first few terms of each series for the general solution:

$$y = c_0 \left(1 - \frac{1}{2}x^2 - \frac{3}{4!}x^4 - \frac{21}{6!}x^6 - \dots \right) + c_1 \left(x + \frac{1}{3!}x^3 + \frac{5}{5!}x^5 + \frac{45}{7!}x^7 + \dots \right).$$

■

EXERCISES 17.5

In Exercises 1–18, use power series to find the general solution of the differential equation.

1. $y'' + 2y' = 0$

2. $y'' + 2y' + y = 0$

3. $y'' + 4y = 0$

4. $y'' - 3y' + 2y = 0$

5. $x^2 y'' - 2xy' + 2y = 0$

6. $y'' - xy' + y = 0$

7. $(1+x)y'' - y = 0$

8. $(1-x^2)y'' - 4xy' + 6y = 0$

9. $(x^2 - 1)y'' + 2xy' - 2y = 0$

10. $y'' + y' - x^2 y = 0$

11. $(x^2 - 1)y'' - 6y = 0$

12. $xy'' - (x+2)y' + 2y = 0$

13. $(x^2 - 1)y'' + 4xy' + 2y = 0$

14. $y'' - 2xy' + 4y = 0$

15. $y'' - 2xy' + 3y = 0$

16. $(1-x^2)y'' - xy' + 4y = 0$

17. $y'' - xy' + 3y = 0$

18. $x^2 y'' - 4xy' + 6y = 0$

ANSWERS TO ODD-NUMBERED EXERCISES

Chapter 17

SECTION 17.1, pp. 17-6-17-7

1. $y = c_1 e^{-3x} + c_2 e^{4x}$
3. $y = c_1 e^{-4x} + c_2 e^x$
5. $y = c_1 e^{-2x} + c_2 e^{2x}$
7. $y = c_1 e^{-x} + c_2 e^{3x/2}$
9. $y = c_1 e^{-x/4} + c_2 e^{3x/2}$
11. $y = c_1 \cos 3x + c_2 \sin 3x$
13. $y = c_1 \cos 5x + c_2 \sin 5x$
15. $y = e^x (c_1 \cos 2x + c_2 \sin 2x)$
17. $y = e^{-x} (c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$
19. $y = e^{-2x} (c_1 \cos \sqrt{5}x + c_2 \sin \sqrt{5}x)$
21. $y = c_1 + c_2 x$
23. $y = c_1 e^{-2x} + c_2 x e^{-2x}$
25. $y = c_1 e^{-3x} + c_2 x e^{-3x}$
27. $y = c_1 e^{-x/2} + c_2 x e^{-x/2}$
29. $y = c_1 e^{-x/3} + c_2 x e^{-x/3}$
31. $y = -\frac{3}{4} e^{-5x} + \frac{3}{4} e^{-x}$
33. $y = \frac{1}{2\sqrt{3}} \sin 2\sqrt{3}x$
35. $y = -\cos 2\sqrt{2}x + \frac{1}{\sqrt{2}} \sin 2\sqrt{2}x$
37. $y = (1 - 2x)e^{2x}$
39. $y = 2(1 + 2x)e^{-3x/2}$
41. $y = c_1 e^{-x} + c_2 e^{3x}$
43. $y = c_1 e^{-x/2} + c_2 x e^{-x/2}$
45. $y = c_1 \cos \sqrt{5}x + c_2 \sin \sqrt{5}x$
47. $y = c_1 e^{-x/5} + c_2 x e^{-x/5}$
49. $y = e^{-x/2} (c_1 \cos x + c_2 \sin x)$
51. $y = c_1 e^{3x/4} + c_2 x e^{3x/4}$
53. $y = c_1 e^{-4x/3} + c_2 x e^{-4x/3}$
55. $y = c_1 e^{-x/2} + c_2 e^{4x/3}$
57. $y = (1 + 2x)e^{-x}$
59. $y = \frac{15}{13} e^{-7x/3} + \frac{11}{13} e^{2x}$

SECTION 17.2, pp. 17-14-17-15

1. $y = c_1 e^{5x} + c_2 e^{-2x} + \frac{3}{10}$
3. $y = c_1 + c_2 e^x + \frac{1}{2} \cos x - \frac{1}{2} \sin x$
5. $y = c_1 \cos x + c_2 \sin x - \frac{1}{8} \cos 3x$
7. $y = c_1 e^{2x} + c_2 e^{-x} - 6 \cos x - 2 \sin x$
9. $y = c_1 e^x + c_2 e^{-x} - x^2 - 2 + \frac{1}{2} x e^x$
11. $y = c_1 e^{3x} + c_2 e^{-2x} - \frac{1}{4} e^{-x} + \frac{49}{50} \cos x + \frac{7}{50} \sin x$
13. $y = c_1 + c_2 e^{-5x} + x^3 + \frac{3}{5} x^2 - \frac{6}{25} x$
15. $y = c_1 + c_2 e^{3x} + 2x^2 + \frac{4}{3} x + \frac{1}{3} x e^{3x}$
17. $y = c_1 + c_2 e^{-x} + \frac{1}{2} x^2 - x$
19. $y = c_1 \cos x + c_2 \sin x - \frac{1}{2} x \cos x$
21. $y = (c_1 + c_2 x) e^{-x} + \frac{1}{2} x^2 e^{-x}$
23. $y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x e^x$
25. $y = e^{-2x} (c_1 \cos x + c_2 \sin x) + 2$
27. $y = A \cos x + B \sin x + x \sin x + \cos x \ln(\cos x)$
29. $y = c_1 + c_2 e^{5x} + \frac{1}{10} x^2 e^{5x} - \frac{1}{25} x e^{5x}$
31. $y = c_1 \cos x + c_2 \sin x - \frac{1}{2} x \cos x + x \sin x$

33. $y = c_1 + c_2 e^x + \frac{1}{2} e^{-x} + x e^x$
35. $y = c_1 e^{5x} + c_2 e^{-x} - \frac{1}{8} e^x - \frac{4}{5}$
37. $y = c_1 \cos x + c_2 \sin x - (\sin x)[\ln(\csc x + \cot x)]$
39. $y = c_1 + c_2 e^{8x} + \frac{1}{8} x e^{8x}$
41. $y = c_1 + c_2 e^x - x^4/4 - x^3 - 3x^2 - 6x$
43. $y = c_1 + c_2 e^{-2x} - \frac{1}{3} e^x + x^3/6 - x^2/4 + x/4$
45. $y = c_1 \cos x + c_2 \sin x + (x - \tan x) \cos x - \sin x \ln(\cos x)$
 $= c_1 \cos x + c_2' \sin x + x \cos x - (\sin x) \ln(\cos x)$
47. $y = c e^{3x} - \frac{1}{2} e^x$
49. $y = c e^{3x} + 5x e^{3x}$
51. $y = 2 \cos x + \sin x - 1 + \sin x \ln(\sec x + \tan x)$
53. $y = -e^{-x} + 1 + \frac{1}{2} x^2 - x$
55. $y = 2(e^x - e^{-x}) \cos x - 3e^{-x} \sin x$
57. $y = (1 - x + x^2) e^x$
59. $y_p = \frac{1}{4} x^2$

SECTION 17.3, pp. 17-20-17-21

1. $\frac{1}{2} y'' + y' + y = 0, \quad y(0) = 0.6, \quad y'(0) = 0.6$
3. $\frac{120}{9.8} y'' + 600y = 0, \quad y(0) = 0.05, \quad y'(0) = v_0$
5. $2q'' + 4q' + 10q = 20 \cos t, \quad q(0) = 2, \quad q'(0) = 3$
7. 0.0259 m (above equilibrium)
9. $y(t) = 0.05 \cos(5.715t) + \frac{v_0}{5.715} \sin(5.715t)$
(in meters)
11. 1.806 s
13. 45 N
15. 8.13 m/s
17. 0.6238 m/s² (acceleration upward)
19. $q(t) = -8e^{-3t} + 10e^{-2t}, \quad \lim_{t \rightarrow \infty} q(t) = 0$
21. $y(t) = 0.3 + 0.6e^{-t} - 0.1e^{-2t} - 0.2e^{-8t}$
23. $y(\pi) = -2$ m (above equilibrium)
25. $q(t) = \frac{1}{5} + \left(\frac{49\sqrt{199}}{995} \sin \frac{\sqrt{199}}{2} t + \frac{49}{5} \cos \frac{\sqrt{199}}{2} t \right) e^{-t/2}$

SECTION 17.4, p. 17-24

1. $y = \frac{c_1}{x^2} + c_2 x$
3. $y = \frac{c_1}{x^2} + c_2 x^3$
5. $y = c_1 x^2 + c_2 x^4$
7. $y = c_1 x^{-1/3} + c_2$
9. $y = x(c_1 + c_2 \ln x)$
11. $y = x[c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)]$
13. $y = \frac{1}{x} [c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)]$
15. $y = \frac{1}{\sqrt{x}} [c_1 \cos(\ln x) + c_2 \sin(\ln x)]$

$$17. y = \frac{1}{x}(c_1 + c_2 \ln x) \quad 19. y = c_1 + c_2 \ln x$$

$$21. y = \frac{1}{\sqrt[3]{x}}(c_1 + c_2 \ln x) \quad 23. y = x^{-5/4}(c_1 + c_2 \ln x)$$

$$25. y = \frac{1}{2x^3} + \frac{x}{2} \quad 27. y = x$$

$$29. y = x[-\cos(\ln x) + 2 \sin(\ln x)]$$

SECTION 17.5, p. 17-29

$$1. y = c_0 + c_1 \left(x - x^2 + \frac{2}{3}x^3 - \dots \right) = c_0 - \frac{c_1}{2}e^{-2x}$$

$$3. y = c_0(1 - 2x^2 + \dots) + c_1 \left(x - \frac{2}{3}x^3 + \dots \right) \\ = c_0 \cos 2x + c_1 \sin 2x$$

$$5. y = c_1x + c_2x^2$$

$$7. y = c_0 \left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots \right) + c_1 \left(x + \frac{1}{6}x^3 + \dots \right)$$

$$9. y = c_0 \left(1 - x^2 + \frac{5}{12}x^4 - \dots \right) + c_1x$$

$$11. y = c_0(1 - 3x^2 + \dots) + c_1(x - x^3)$$

$$13. y = c_0 \left(1 + x^2 + \frac{2}{3}x^4 + \dots \right) + c_1 \left(x + x^3 + \frac{3}{5}x^5 + \dots \right)$$

$$15. y = c_0 \left(1 - \frac{3}{2}x^2 + \dots \right) + c_1 \left(x - \frac{1}{2}x^3 + \dots \right)$$

$$17. y = c_0 \left(1 - \frac{3}{2}x^2 + \frac{1}{8}x^4 + \dots \right) + c_1 \left(x - \frac{1}{3}x^3 \right)$$

18

Complex Functions

INTRODUCTION The study of derivatives, integrals, and series can be extended from the real to the complex numbers, where it is called *complex analysis*. The study of complex functions is far from a straightforward extension of the study of real variable functions. Differentiable functions of a complex variable have many properties not found in their real relatives. For example, while we can find real functions that are once but not twice differentiable, this is not true for complex functions. If a complex function is differentiable once on an open region, then its second, third, and all higher order derivatives also exist. In Chapter 9 we saw differentiable functions that are not equal to any power series. In contrast, a complex differentiable function can always be expressed as a convergent power series. The study of complex functions gives profound new insights into the properties of the real-valued functions we have already encountered and opens the way to many new applications. The study of complex functions, particularly in higher dimensions, remains an active area of mathematical research. In this chapter we introduce the complex numbers and then study some of the many beautiful aspects of complex differentiable functions.

18.1 Complex Numbers

Complex numbers are expressed in the form $a + ib$, or $a + bi$, where a and b are real numbers and i is a symbol for a new type of number whose square equals -1 ; that is, $i^2 = -1$. We refer to a as the *real* part of the complex number and to b as the *imaginary* part. The words “real” and “imaginary” have connotations that tend to place $\sqrt{-1}$ in a less favorable position in our minds than a real number such as $\sqrt{2}$. In fact, a good deal of imagination, or inventiveness, was required to construct the real number system, which forms the basis of calculus (see Appendix A.7). We begin this chapter with a review of the successive stages of invented number systems.

The Hierarchy of Numbers

The first stage of number development was the recognition of the counting numbers $1, 2, 3, \dots$, which we now call the natural numbers or the positive integers. Certain arithmetical operations on the positive integers, such as addition and multiplication, keep us entirely within this system. That is, if m and n are any positive integers, then their sum $m + n$ and product mn are also positive integers. Some equations can be solved entirely within the system of positive integers. For example, we can solve $3 + x = 7$ using only positive integers. But other simple equations, such as $7 + x = 3$, cannot be solved if only positive integers are at our disposal. The invention of the number zero and the negative numbers allowed the solution of equations such as $7 + x = 3$. Using the integers

$$\dots, -3, -2, -1, 0, 1, 2, 3, \dots$$

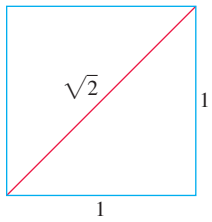


FIGURE 18.1 The diagonal of the unit square has irrational length.

we can always find the missing integer x that solves the equation $m + x = p$ when we are given the other two integers m and p in the equation. Addition and multiplication of integers always keep us within the system of integers. However, division does not, and so fractions m/n , where m and n are integers with n nonzero, were invented. The resulting system is called the rational numbers and is rich enough to perform all of the rational operations of arithmetic, including addition, subtraction, multiplication, and division (although division by zero is excluded since it is meaningless). Yet there are still simple polynomial equations that cannot be solved within the system of rational numbers. The ancient Greeks realized that there is no rational number that solves the equation $x^2 = 2$, even though the Pythagorean Theorem implies that the length x of the diagonal of the unit square satisfies this equation. (See Figure 18.1.)

To see why $x^2 = 2$ has no rational solution, consider the following argument. Suppose that there did exist some positive integers p and q with no common factor other than 1 such that the fraction $x = p/q$ satisfies $x^2 = 2$. Writing this out, we see that $p^2/q^2 = 2$, and therefore

$$p^2 = 2q^2.$$

Then p^2 is also an even integer. Since the square of an odd number is odd, we conclude that p itself must be an even number (for if p were odd, then p^2 would also be odd). So p is divisible by 2, and $p = 2k$ for some integer k . Hence $p^2 = 4k^2$. Since we already saw that $p^2 = 2q^2$, it follows that $2q^2 = p^2 = 4k^2$, and therefore $q^2 = 2k^2$.

Consequently, q^2 is an even number and this requires in turn that q itself be even. Therefore, both p and q are divisible by 2, which is contrary to our assumption that they contain no common factors other than 1. Since we have arrived at a contradiction, there cannot exist any such integers p and q , and therefore there is no rational number that solves the equation $x^2 = 2$.

The invention of real numbers addressed this issue (and others). Using real numbers, we can represent every possible physical length. Each real number can be represented as an infinite decimal $a.d_1d_2d_3d_4\dots$, where a is an integer followed by a sequence of decimal digits each between 0 and 9 (Appendix A.7). If the sequence stops or repeats in a periodic pattern, then the decimal represents a rational number. An irrational number is represented by a nonterminating and nonrepeating decimal. The rational and irrational numbers together make up the real number system. Unlike the rational numbers, the real numbers have the completeness property, meaning that there are no “holes” or “gaps” in the real line. This property is essential in calculus, implying for example that continuous functions on a closed interval have an absolute maximum value and an absolute minimum value (Extreme Value Theorem, Section 4.1), and that an increasing and bounded sequence converges (Monotonic Sequence Theorem, Section 9.1). Yet for all of its utility, there are simple equations that cannot be solved when working only within the real number system. As an example, the polynomial equation $x^2 + 1 = 0$ has no real solutions.

The Complex Numbers

We have discussed three systems of numbers that form a hierarchy, with each system containing the previous system. Each system is richer than its predecessor in that it permits additional operations to be performed without going outside the system.

1. Using the integer system, we can solve all equations of the form

$$x + a = 0, \tag{1}$$

where a is an integer.

2. Using the rational numbers, we can solve all equations of the form

$$ax + b = 0, \tag{2}$$

provided that a and b are rational numbers and $a \neq 0$.

3. Using the real numbers, we can solve all of Equations (1) and (2) and, in addition, all quadratic equations

$$ax^2 + bx + c = 0, \quad (3)$$

provided that $a \neq 0$ and $b^2 - 4ac \geq 0$.

The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (4)$$

gives the solutions to Equation (3). When $b^2 - 4ac$ is negative, there are no real number solutions to the equation $ax^2 + bx + c = 0$. In particular, the simple quadratic equation $x^2 + 1 = 0$ cannot be solved using any of the three invented systems of numbers that we have discussed. Thus we are led to the fourth system of numbers, the set of complex numbers $a + ib$. The symbol i represents a new number whose square equals -1 . We call a the *real part* and b the *imaginary part* of the complex number $a + ib$.

Real and Imaginary parts

For a complex number $z = a + ib$, we write

$$\operatorname{Re} z = a$$

and

$$\operatorname{Im} z = b.$$

Sometimes it is convenient to write $a + bi$ instead of $a + ib$; both notations describe the same complex number. We define equality and addition for complex numbers in the following way.

Equality $a + ib = c + id$ if and only if $a = c$ and $b = d$.

Two complex numbers $a + ib$ and $c + id$ are equal if and only if their real parts are equal and their imaginary parts are equal.

Addition $(a + ib) + (c + id) = (a + c) + i(b + d)$

We sum the real parts and separately sum the imaginary parts.

To multiply two complex numbers, we multiply using the distributive rule and then simplify using $i^2 = -1$.

Multiplication $(a + ib)(c + id)$

$$= ac + iad + ibc + i^2bd$$

$$= (ac - bd) + i(ad + bc) \quad i^2 = -1$$

EXAMPLE 1 Basic arithmetic operations with complex numbers

- a. The set of all complex numbers $a + i0$, where the second number b is zero, has all of the properties of the set of real numbers. For example, addition and multiplication as complex numbers give

$$(a + i0) + (c + i0) = (a + c) + i0, \quad (a + i0)(c + i0) = ac + i0,$$

which are numbers of the same type with imaginary part zero. We usually just write a instead of $a + i0$, and in this sense the complex number system contains the real number system.

- b. If we multiply a real number $a = a + i0$ by a complex number $c + id$, we get $a(c + id) = (a + i0)(c + id) = ac + iad$.

- c. The number $0 = 0 + i0$ plays the role of zero in the complex number system, and the complex number $1 = 1 + i0$ plays the role of unity, or one.
- d. The complex number $i = 0 + i1$, which has real part zero and imaginary part one, has the property that its square is $i^2 = (0 + i1)^2 = (0 + i1)(0 + i1) = (-1) + i0 = -1$. So $x = i$ is a solution to the quadratic equation $x^2 + 1 = 0$. In the complex number system there are exactly two solutions to this equation, the other solution being $x = -i = 0 + i(-1)$.
- e. We can divide any two complex numbers, as long as we do not divide by the number $0 = 0 + i0$. When $a + ib \neq 0$ (meaning that either $a \neq 0$ or $b \neq 0$ or both $a \neq 0$ and $b \neq 0$), we carry out division as follows:

$$\frac{c + id}{a + ib} = \frac{(c + id)(a - ib)}{(a + ib)(a - ib)} = \frac{(ac + bd) + i(ad - bc)}{a^2 + b^2} = \frac{ac + bd}{a^2 + b^2} + i \frac{ad - bc}{a^2 + b^2}.$$

In this way we express the quotient in the standard form $x + iy$, where $x = (ac + bd)/(a^2 + b^2)$ and $y = (ad - bc)/(a^2 + b^2)$ are real numbers. Note that $a^2 + b^2 \neq 0$ since we stipulated that a and b cannot both be zero.

- f. The number $a - ib$ that was used in part (e) as the multiplier to clear the i from the denominator is called the *complex conjugate* of $a + ib$. If we denote the original complex number by $z = a + ib$, then it is customary to write $\bar{z}$ (read “z bar”) to denote its complex conjugate:

$$z = a + ib, \quad \bar{z} = a - ib.$$

Multiplying the numerator and denominator of a fraction $(c + id)/(a + ib)$ by the complex conjugate of the denominator will always replace the denominator by a real number. ■

EXAMPLE 2 Computations with complex numbers

- a. $(2 + 3i) + (6 - 2i) = (2 + 6) + (3 - 2)i = 8 + i$
- b. $(2 + 3i) - (6 - 2i) = (2 - 6) + (3 - (-2))i = -4 + 5i$
- c. $(2 + 3i)(6 - 2i) = (2)(6) + (2)(-2i) + (3i)(6) + (3i)(-2i)$
 $= 12 - 4i + 18i - 6i^2 = 12 + 14i + 6 = 18 - 14i$
- d. $\frac{2 + 3i}{6 - 2i} = \frac{(2 + 3i)(6 + 2i)}{(6 - 2i)(6 + 2i)} = \frac{12 + 4i + 18i + 6i^2}{36 + 12i - 12i - 4i^2} = \frac{6 + 22i}{40} = \frac{3}{20} + \frac{11}{20}i$ ■

The Complex Plane

We can use the usual xy -coordinate system to represent complex numbers as points or vectors in the plane. In this context we call it the *complex plane*. There are now two geometric representations of a complex number $z = x + iy$:

1. as a point $P(x, y)$ in the xy -plane,
2. as a vector $\overrightarrow{OP}$ from the origin to P .

In each representation, the x -axis is called the *real axis* and the y -axis is the *imaginary axis*, as in Figure 18.2. Geometric representations of complex numbers are sometimes called *Argand Diagrams*.

In polar coordinates we have

$$x = r \cos \theta, \quad y = r \sin \theta,$$

and

$$z = x + iy = r(\cos \theta + i \sin \theta). \quad (5)$$

The number 0, or $0 + i0$ is a special case, with $r = 0$ but no specified θ .

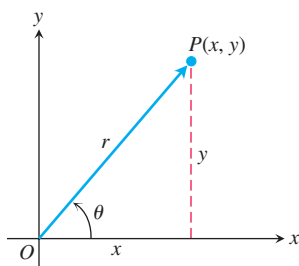


FIGURE 18.2 The complex number $z = x + iy$ can be represented both as a point $P(x, y)$ and as a vector.

We define the *absolute value* (or *magnitude*) of a complex number $x + iy$ to be the length r of a vector $\overline{OP}$ from the origin to $P(x, y)$. We denote the absolute value by vertical bars; thus,

$$|x + iy| = \sqrt{x^2 + y^2}.$$

We always choose the polar coordinates r and θ so that r is nonnegative, and then

$$r = |x + iy|.$$

The polar angle θ is called the *argument* of z and is written $\theta = \arg z$. There is more than one choice for the argument. Any integer multiple of 2π may be added to θ to produce another appropriate angle. The following equation gives a useful formula connecting a complex number z , its conjugate $\bar{z}$, and its absolute value $|z|$:

$$z \cdot \bar{z} = |z|^2.$$

Euler's Formula

The identity

$$e^{i\theta} = \cos \theta + i \sin \theta \quad (6)$$

is called Euler's Formula. The origins of this formula will be seen in Section 18.6. We show a vector representing $e^{i\theta}$ in Figure 18.3. Using Equation (6), we can write Equation (5) as

$$z = re^{i\theta}.$$

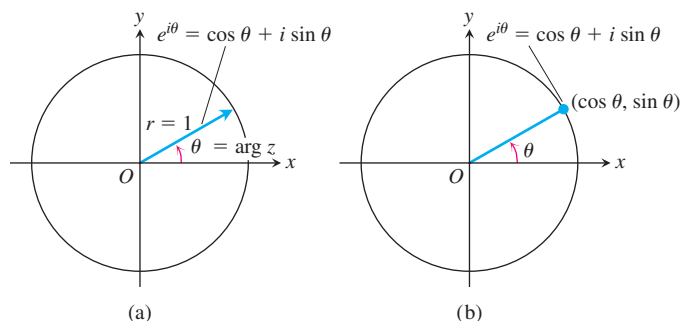


FIGURE 18.3 The complex number $z = e^{i\theta}$ as a vector and as a point.

The notation $\exp(A)$ is also used for e^A , so we could write $z = r \exp(i\theta)$.

Euler's formula leads to the following rules for calculating products, quotients, powers, and roots of complex numbers in polar form.

Products

To multiply two complex numbers in polar form, we multiply their absolute values and add their arguments. To see why, let

$$z_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2 e^{i\theta_2}, \quad (7)$$

so that

$$|z_1| = r_1, \quad \arg z_1 = \theta_1; \quad |z_2| = r_2, \quad \arg z_2 = \theta_2.$$

In the Cartesian $a + bi$ form we have

$$z_1 = r_1 \cos \theta_1 + ir_1 \sin \theta_1$$

and

$$z_2 = r_2 \cos \theta_2 + ir_2 \sin \theta_2.$$

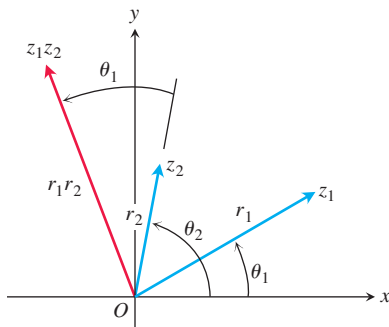


FIGURE 18.4 When z_1 and z_2 are multiplied, $|z_1 z_2| = r_1 \cdot r_2$ and $\arg(z_1 z_2) = \theta_1 + \theta_2$.

Their product is then

$$\begin{aligned} z_1 z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} \\ &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 [\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2] + i [\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2] \\ &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)) \quad \text{Angle sum formulas} \\ &= r_1 r_2 e^{i(\theta_1 + \theta_2)}. \end{aligned}$$

Therefore

$$r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}. \quad (8)$$

The absolute value of the product is $r_1 r_2$ and the argument of the product is the sum of the arguments of z_1 and z_2 . We have shown that the product of two complex numbers is represented by a vector whose length is the product of the lengths of the two factors and whose angle with the x -axis is the sum of the corresponding angles of the factors. (Figure 18.4.) In particular, Equation (8) implies that multiplying a vector by $e^{i\theta}$ causes its vector representation to be rotated counterclockwise through an angle θ . Multiplication by i rotates 90° , by -1 rotates 180° , by $-i$ rotates 270° , and so on.

EXAMPLE 3 We compute the product of $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$. We first plot these complex numbers in Figure 18.5, and we can then read off their polar representations

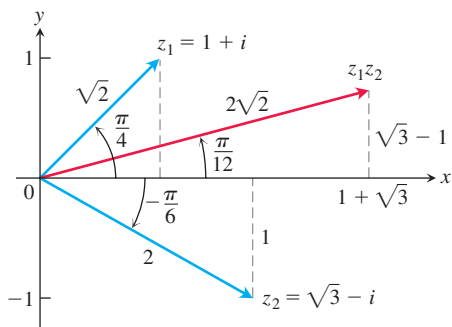


FIGURE 18.5 To multiply two complex numbers, multiply their absolute values and add their arguments.

$$\begin{aligned} z_1 \cdot z_2 &= \sqrt{2} \exp\left(\frac{i\pi}{4}\right) \cdot 2 \exp\left(-\frac{i\pi}{6}\right) \\ &= 2\sqrt{2} \exp\left(\frac{i\pi}{4} - \frac{i\pi}{6}\right) \\ &= 2\sqrt{2} \exp\left(\frac{i\pi}{12}\right) \\ &= 2\sqrt{2} \left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}\right) \\ &\approx 2.73 + 0.73i. \end{aligned}$$

EXAMPLE 4 **Computing Reciprocals** Show that if $z = re^{i\theta}$ and $r > 0$, then

$$\frac{1}{z} = \frac{1}{r} e^{-i\theta}.$$

Solution Applying the formula for products, Equation (8), we see that

$$(re^{i\theta})\left(\frac{1}{r} e^{-i\theta}\right) = 1 \cdot e^{i0} = 1.$$

Dividing both sides by $z = re^{i\theta}$ gives

$$\frac{1}{r} e^{-i\theta} = \frac{1}{re^{i\theta}} = \frac{1}{z}.$$

Quotients

We now look at how to divide $z_1 = r_1 e^{i\theta_1}$ by $z_2 = r_2 e^{i\theta_2}$ in polar form. We assume that $r_2 > 0$ so that we are not dividing by zero. To compute the quotient z_1/z_2 we multiply z_1 by $1/z_2$. This gives

$$\frac{z_1}{z_2} = z_1 \left(\frac{1}{z_2}\right) = r_1 e^{i\theta_1} \left(\frac{1}{r_2} e^{-i\theta_2}\right) = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}, \quad \text{Equation (8)}$$

so that

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2.$$

That is, we divide lengths and subtract angles to compute the quotient of complex numbers.

EXAMPLE 5 Let $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$, as in Example 3. Then

$$\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}e^{i\pi/4}}{2e^{-i\pi/6}} = \frac{\sqrt{2}}{2}e^{5\pi i/12} \approx 0.707\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right) \approx 0.183 + 0.683i. \quad \blacksquare$$

Powers

If n is a positive integer, we can apply the product formulas in Equation (8) to find the polar form of a power

$$z^n = \underbrace{z \cdot z \cdot \cdots \cdot z}_{n \text{ factors}}$$

With $z = re^{i\theta}$, we obtain

$$\begin{aligned} z^n &= (re^{i\theta})^n = r^n e^{i(\theta + \cdots + \theta)} \\ &= r^n e^{in\theta}. \quad n \text{ summands} \end{aligned} \quad (9)$$

The length $r = |z|$ is raised to the n th power, and the angle $\theta = \arg z$ is multiplied by n .

If we take $r = 1$ in Equation (9), we obtain De Moivre's Theorem.

De Moivre's Theorem

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \quad (10)$$

If we expand the left side of the equation in De Moivre's Theorem using the Binomial Theorem and reduce it to the form $a + ib$, we obtain trigonometric identities, namely formulas for $\cos n\theta$ and $\sin n\theta$ as polynomials of degree n in $\cos \theta$ and $\sin \theta$.

EXAMPLE 6 If $n = 3$ in Equation (10), we have

$$(\cos \theta + i \sin \theta)^3 = \cos(3\theta) + i \sin(3\theta).$$

The left side of this equation expands to

$$\cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta.$$

The real part of this must equal $\cos(3\theta)$ and the imaginary part must equal $\sin(3\theta)$. Therefore,

$$\begin{aligned} \cos(3\theta) &= \cos^3 \theta - 3 \cos \theta \sin^2 \theta, \\ \sin(3\theta) &= 3 \cos^2 \theta \sin \theta - \sin^3 \theta. \end{aligned} \quad \blacksquare$$

Roots

If x is a positive real number, then it has exactly one real n th root if n is an odd integer and exactly two n th roots if n is even. If x is a negative real number, then x has exactly one real n th root if n is an odd integer and no real n th root if n is even. In contrast, if $z = re^{i\theta}$ is a

complex number different from zero, then there are precisely n different complex numbers, $w_0, w_1, \dots, w_{n-1}$, that are n th roots of z . To see why, let $w = \rho e^{i\alpha}$ be an n th root of $z = re^{i\theta}$. Then

$$w^n = z$$

or

$$\rho^n e^{in\alpha} = re^{i\theta}.$$

Since both r and ρ are positive real numbers, this implies that $\rho^n = r$, and so

$$\rho = \sqrt[n]{r}$$

is the real, positive n th root of r . Although we cannot say that $n\alpha$ and θ must be equal, since the argument of a complex number is only defined up to multiples of 2π , we can say that they may differ only by an integer multiple of 2π . That is,

$$n\alpha = \theta + 2k\pi, \quad k = 0, \pm 1, \pm 2, \dots$$

Therefore,

$$\alpha = \frac{\theta}{n} + k \frac{2\pi}{n}.$$

This gives the following collection of n distinct complex numbers,

$$\sqrt[n]{re^{i\theta}} = \sqrt[n]{r} \exp\left[i\left(\frac{\theta}{n} + k \frac{2\pi}{n}\right)\right], \quad k = 0, \pm 1, \pm 2, \dots \quad (11)$$

Each of these has the property that its n th power equals z , so these n complex numbers give all of the n th roots of $z = re^{i\theta}$.

The expression $\sqrt[n]{re^{i\theta}}$ on the left of Equation (11) defines a new concept, the n th root of a complex number, while the term $\sqrt[n]{r}$ on the right refers to the real n th root. There is exactly one positive real number whose n th power equals the positive real number r , and that number gives the value of $\sqrt[n]{r}$. On the other hand, the expression $\sqrt[n]{re^{i\theta}}$ refers to a collection of n distinct numbers. In practice we often want to pick out a particular n th root, the *principal* n th root. We can do that by requiring that $z = re^{i\theta}$ have $-\pi < \theta \leq \pi$, and then taking the *principal* n th root to be

$$\sqrt[n]{r} \exp\left[i\left(\frac{\theta}{n}\right)\right].$$

There might appear to be infinitely many different answers in Equation (11), corresponding to the infinitely many possible values of k , but $k = m + n$ gives the same complex number as $k = m$. This is true because the complex number

$$\exp\left[i\left(\frac{\theta}{n} + 2\pi \frac{m+n}{n}\right)\right] = \exp\left[i\left(\frac{\theta}{n} + 2\pi \frac{m}{n} + 2\pi\right)\right]$$

and the complex number

$$\exp\left[i\left(\frac{\theta}{n} + 2\pi \frac{m}{n}\right)\right],$$

are equal, since they have arguments that differ by 2π . Therefore we need only take n consecutive values for k to obtain all the different complex numbers that are n th roots of z . For convenience, we take

$$k = 0, 1, 2, \dots, n-1.$$

All the n th roots of $re^{i\theta}$ lie on a circle centered at the origin and having radius equal to the real, positive n th root of r . One of them has argument $w_0 = \theta/n$. The others are uniformly spaced around the circle, each being separated from its neighbors by an angle equal to $2\pi/n$. (Figure 18.6 illustrates the placement of the three cube roots, w_0, w_1, w_2 , of the complex number $z = re^{i\theta}$.)

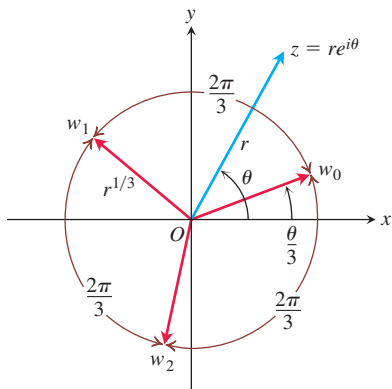


FIGURE 18.6 The three cube roots of $z = re^{i\theta}$

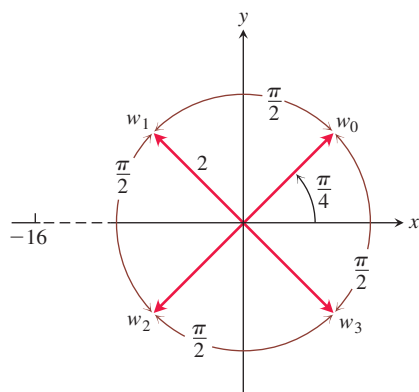


FIGURE 18.7 The four fourth roots of -16

EXAMPLE 7 Find the four fourth roots of -16 .

Solution As our first step, we plot the number -16 in the complex plane (Figure 18.7) and determine its polar representation $re^{i\theta}$. Here, $z = -16$, $r = +16$, and $\theta = \pi$. One of the fourth roots of $16e^{i\pi}$ is $2e^{i\pi/4}$. We obtain others by successive additions of $2\pi/4 = \pi/2$ to the argument of this first one. Hence,

$$\sqrt[4]{16e^{i\pi}} = 2e^{ik\pi/4}, \quad k = 1, 3, 5, 7, \quad (12)$$

and the four roots are

$$w_0 = 2 \left[\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right] = \sqrt{2}(1 + i),$$

$$w_1 = 2 \left[\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right] = \sqrt{2}(-1 + i),$$

$$w_2 = 2 \left[\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right] = \sqrt{2}(-1 - i),$$

$$w_3 = 2 \left[\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right] = \sqrt{2}(1 - i).$$

The Fundamental Theorem of Algebra

One might say that the invention of $\sqrt{2}$ is all well and good and leads to a number system that is richer than the rational number system alone, and the complex number system is richer still, but where will this process end? Will we need to develop still more systems so as to obtain $\sqrt[4]{-1}$ (a solution of $x^4 = -1$), $\sqrt[6]{-1}$ (a solution of $x^6 = -1$), and so on? It turns out this is not necessary. These numbers are already expressible in the complex number system as $a + ib$ for appropriate a and b . In fact, the Fundamental Theorem of Algebra says that with the introduction of the complex numbers, we now have enough numbers to factor every polynomial into a product of linear factors. We have enough numbers to solve every possible polynomial equation.

The Fundamental Theorem of Algebra

Every polynomial equation of the form

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0,$$

in which the coefficients $a_0, a_1, \dots, a_{n-1}, a_n$ are any complex numbers, whose degree n is greater than or equal to one, and whose leading coefficient a_n is not zero, has exactly n roots in the complex number system, provided each multiple root of multiplicity m is counted as m roots.

A proof of this theorem can be found in most texts on the theory of functions of a complex variable.

EXERCISES 18.1

Operations with Complex Numbers

1. Express the following products of complex numbers in the form $a + ib$ where a and b are real numbers.

a. $(2 + 3i)(4 - 2i)$

b. $(2 - i)(-2 - 3i)$

c. $(-1 - 2i)(2 + i)$

2. Solve the following equations for the real numbers x and y .

a. $(3 + 4i)^2 - 2(x - iy) = x + iy$

b. $\left(\frac{1+i}{1-i}\right)^2 + \frac{1}{x+iy} = 1 + i$

c. $(3 - 2i)(x + iy) = 2(x - 2iy) + 2i - 1$

Graphing and Geometry

3. How may the following complex numbers be obtained from $z = x + iy$ geometrically? Sketch.
- a. $\bar{z}$ b. $-\bar{z}$ c. $-z$ d. $1/z$
4. Show that the distance between the two points z_1 and z_2 in the complex plane is $|z_1 - z_2|$.

In Exercises 5–10, graph the points $z = x + iy$ that satisfy the given conditions.

5. a. $|z| = 2$ b. $|z| < 2$ c. $|z| > 2$
6. $|z - 1| = 2$ 7. $|z + 1| = 1$
8. $|z + 1| = |z - 1|$ 9. $|z + i| = |z - 1|$
10. $|z + 1| \geq |z - 1|$

Express the complex numbers in Exercises 11–14 in the form $re^{i\theta}$, with $r \geq 0$ and $-\pi < \theta \leq \pi$, and for each of them draw a diagram in the complex plane.

11. $(1 + \sqrt{-3})^2$ 12. $\frac{1+i}{1-i}$
13. $\frac{1+\sqrt{-3}}{1-\sqrt{-3}}$ 14. $(2+3i)(1-2i)$

Powers and Roots

Use De Moivre's Theorem to express the trigonometric functions in Exercises 15 and 16 in terms of $\cos \theta$ and $\sin \theta$.

15. $\cos 4\theta$ 16. $\sin 4\theta$
17. Find the three cube roots of 1.
18. Find the two square roots of i .
19. Find the three cube roots of $-8i$.
20. Find the six sixth roots of 64.
21. Find the four solutions of the equation $z^4 - 2z^2 + 4 = 0$.
22. Find the six solutions of the equation $z^6 + 2z^3 + 2 = 0$.

23. Find all solutions of the equation $x^4 + 4x^2 + 16 = 0$.
24. Solve the equation $x^4 + 1 = 0$.

Theory and Examples

25. **Complex numbers and vectors in the plane** Show with a diagram in the complex plane that the law for adding complex numbers is the same as the parallelogram law for adding vectors.
26. **Complex arithmetic with conjugates** Show that the conjugate of the sum (product, or quotient) of two complex numbers, z_1 and z_2 , is the same as the sum (product, or quotient) of their conjugates.
27. **Complex roots of polynomials with real coefficients come in complex-conjugate pairs**

- a. Extend the results of Exercise 26 to show that $f(\bar{z}) = \overline{f(z)}$ when

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

is a polynomial with real coefficients $a_0, \dots, a_n$.

- b. If z is a root of the equation $f(z) = 0$, where $f(z)$ is a polynomial with real coefficients as in part (a), show that the conjugate $\bar{z}$ is also a root of the equation. (Hint: Let $f(z) = u + iv = 0$; then both u and v are zero. Use the fact that $f(\bar{z}) = \overline{f(z)} = u - iv$.)

28. **Absolute value of a conjugate** Show that $|\bar{z}| = |z|$.
29. **When $z = \bar{z}$** If z and $\bar{z}$ are equal, what can you say about the location of the point z in the complex plane?
30. **Real and imaginary parts** Show that the following relations hold for any complex numbers z, z_1 , and z_2 .
- a. $z + \bar{z} = 2\operatorname{Re}(z)$
- b. $z - \bar{z} = 2i\operatorname{Im}(z)$
- c. $2\operatorname{Re}(z) \leq |z|$
- d. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1 \bar{z}_2)$
- e. $|z_1 + z_2| \leq |z_1| + |z_2|$

18.2 Functions of a Complex Variable

Complex Domains

The domains of the real variable functions discussed in Chapter 1 consisted of subsets of the real line, such as $[0, 1]$ or $(0, \infty)$. The domain of a complex function is a subset of the complex numbers, which we can visualize as a region in the complex plane.

EXAMPLE 1 Examples of complex domains

- a. The set of complex numbers z for which $|z| \leq 1$
The corresponding region in the complex plane is the set of points lying inside or on a circle of radius 1 centered at the origin O , shown in Figure 18.8(a).
- b. The set of complex numbers with $|z - z_0| \leq 1$
This corresponds to the region inside or on a circle of radius 1 centered at a complex number $z_0 = a_0 + b_0 i$, as in Figure 18.8b.
- c. A complex region can have a more general shape. Three more examples of complex domains are shown in Figure 18.8c.

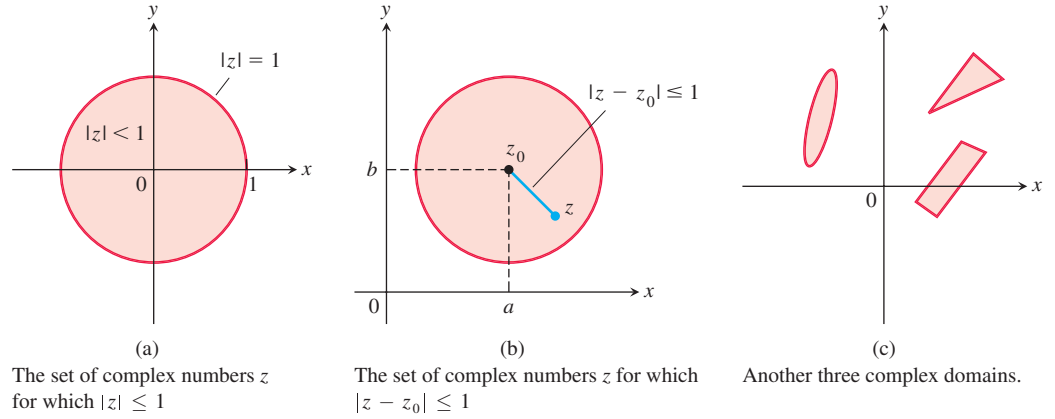


FIGURE 18.8 Some complex domains.

We usually use the letter z for a variable that describes a point in the domain of a complex function, just as we used x for a real function. We write $w = f(z)$ to describe a function that takes as input a complex number z and gives as output a complex number w .

Complex Limits

To understand complex limits, we take advantage of the fact that a complex number $z = x + iy$ can be identified with a point (x, y) in the xy -plane. This identification allows us to extend our knowledge of real limits to complex numbers.

We first consider the case where a complex number $z = z(t)$ is a function of a real variable t , which we often consider to be time. So we think of a vector whose position in the complex plane changes with time. We can write $z(t) = x(t) + iy(t)$, and then each t describes a vector $(x(t), y(t))$ in the complex plane. By describing $z(t)$ as a vector-valued function we are able to understand limits of $z(t)$ similarly to the limits of vector functions seen in Section 12.1. The meaning of an expression such as

$$\lim_{t \rightarrow 1} z(t) = z_0,$$

which states that $z(t)$ is approaching $z_0 = x_0 + iy_0$ as t approaches 1 is that

$$\lim_{t \rightarrow 1} |z(t) - z_0| = 0.$$

More specifically, we make the following definition.

DEFINITION We say that the complex-valued function $z(t) = x(t) + iy(t)$ has limit z_0 as t approaches t_0 , and write

$$\lim_{t \rightarrow t_0} z(t) = z_0$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|z(t) - z_0| < \epsilon \text{ whenever } 0 < |t - t_0| < \delta. \quad (13)$$

One way of interpreting Equation (13) is that $|z(t) - z_0|$ is small when $|t - t_0|$ is small. In other words, if we draw a circle around the complex number z_0 in the complex plane, as small as we like, then the point $z(t) = x(t) + iy(t)$ will be inside that circle if we choose t sufficiently close to t_0 . Since

$$|z(t) - z_0| = |(x(t) + iy(t)) - (x_0 + iy_0)| \leq |x(t) - x_0| + |y(t) - y_0|, \quad \text{Figure 18.9}$$

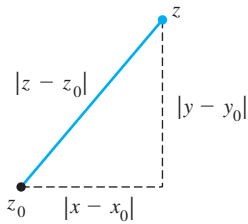


FIGURE 18.9

$$|z - z_0|^2 = |(x - x_0)|^2 + |(y - y_0)|^2$$

we see that if $x(t) - x_0 \rightarrow 0$ and $y(t) - y_0 \rightarrow 0$, then also $z(t) - z_0 \rightarrow 0$. So $z(t)$ converges to z_0 when its real part converges to the real part of z_0 and its imaginary part converges to the imaginary part of z_0 .

On the other hand, since $|z - z_0| \geq |x - x_0|$ and $|z - z_0| \geq |y - y_0|$, it follows that if $z(t) \rightarrow z_0$, then also $x(t) \rightarrow x_0$ and $y(t) \rightarrow y_0$. In summary, as $t \rightarrow t_0$, $z(t)$ approaches z_0 if and only if both $x(t)$ approaches x_0 and $y(t)$ approaches y_0 ,

$$z(t) \rightarrow z_0 \Leftrightarrow x(t) \rightarrow x_0 \text{ and } y(t) \rightarrow y_0. \quad (14)$$

This gives a practical formula that can be used to compute the complex limit of a function $z(t) = x(t) + iy(t)$:

$$\lim_{t \rightarrow t_0} z(t) = \lim_{t \rightarrow t_0} x(t) + i \lim_{t \rightarrow t_0} y(t).$$

EXAMPLE 2 For the function $z(t) = \cos t + i \sin t$, find $\lim_{t \rightarrow \pi} z(t)$.

Solution

$$\lim_{t \rightarrow \pi} z(t) = \lim_{t \rightarrow \pi} \cos t + i \lim_{t \rightarrow \pi} \sin t = -1. \quad \blacksquare$$

Functions of a Complex Variable

We say that f is a complex function on a domain S in the complex plane if to each z in S , the function f assigns a single complex number $f(z)$.

EXAMPLE 3 The function taking z to z^2

For this function, the domain S is the set of all complex numbers, or the complex plane $\mathbb{C}$, and the function sends a complex number z to the complex number z^2 .

As before, we write $f(z) = z^2$ to give an explicit name f to the function that inputs a complex number z and outputs a new complex number $f(z)$. For each point $z = x + iy$, this function f outputs the complex number $u + iv$ where

$$u + iv = (x + iy)^2 = x^2 + i(2xy) + i^2y^2 = (x^2 - y^2) + i(2xy). \quad (15)$$

The real part of the complex variable $u + iv$ is given by a real function of x and y , $u(x, y) = x^2 - y^2$, and the imaginary part by a second real function $v(x, y) = 2xy$. $\blacksquare$

We cannot draw the graph of such a function, as we did with real functions whose graphs were curves or surfaces, since both the domain and the range in this case are the complex plane. We would need four dimensions, or four perpendicular directions, to graph a complex function, but we only have three dimensions to work with. Instead, we can set $w = f(z)$ and get a feeling for what this function is doing by drawing some lines or curves in the z -plane and looking at their images in the w -plane. We can visualize the behavior of a complex function by thinking of the domain as a rubber sheet that has a grid of lines drawn on it, parallel to the x and y axes. By drawing the image of this grid under the map, we can get an understanding of how the complex function maps one domain to another.

EXAMPLE 4 Compute where $f(z) = z^2$ maps the vertical line $x = 1$. More generally, compute where it maps the vertical line $x = a$ and the horizontal line $y = b$.

Solution Looking at u and v in Equation (15), and setting the value of x to be $x = a$, we see that

$$u(a, y) = a^2 - y^2 \text{ and } v(a, y) = 2ay.$$

When $a \neq 0$ we obtain $y = v/(2a)$ in Equation (15); we have that the vertical line $x = a$ is mapped to the curve

$$u = a^2 - \frac{v^2}{4a^2}. \quad (16)$$

For each nonzero choice of constant a , this equation describes a parabola. For $a = 1$ the parabola $u = 1 - v^2/4$ in the complex w -plane is the image of the line $x = 1$ in the complex z -plane. If we chose $a = -1$ in Equation (16), the image of the line $x = -1$ is the same parabola. This is to be expected since the squares of $-z$ and z coincide under the map $z \rightarrow z^2$.

In the case $a = 0$, corresponding to the y -axis, we have $u(0, y) = -y^2$ and $v(0, y) = 0$. Thus the point $(0, y)$ on the y -axis is mapped to the point $(-y^2, 0)$. The y -axis is folded at 0 so that the positive y -axis and the negative y -axis each map onto the negative real axis.

We can get further insight into the behavior of the map $f(z) = z^2$ by examining the images of some horizontal lines. The line $y = b$ in the complex z -plane has image points satisfying

$$u(x, b) = x^2 - b^2 \text{ and } v(x, b) = 2xb.$$

We can solve for x , and when $b \neq 0$, we see that $x = v/(2b)$ and

$$u = \frac{v^2}{4b^2} - b^2. \quad (17)$$

Again this gives us a parabola for each nonzero choice of b . As before, the lines $y = \pm b$ map to the same parabola. The x -axis, corresponding to $b = 0$, is folded over at the origin. The positive real axis is mapped to itself, while the negative real axis is mapped not to itself, but also to the positive real axis. Four parabolas obtained as the images of horizontal and vertical lines are shown in Figure 18.10.

The images of two squares in the first quadrant are seen in Figure 18.11 and Figure 18.13. Green stripes are taken to green, and red to red. These figures give us a feeling for the behavior of the complex function $f(z) = z^2$. ■

Limits of a function of a complex variable $f(z) = u(x, y) + iv(x, y)$ are defined by considering the limits of the real and imaginary parts. Each of u and v is a real-valued function of two variables, and we extend the definition of limits developed for such functions in Section 13.2.

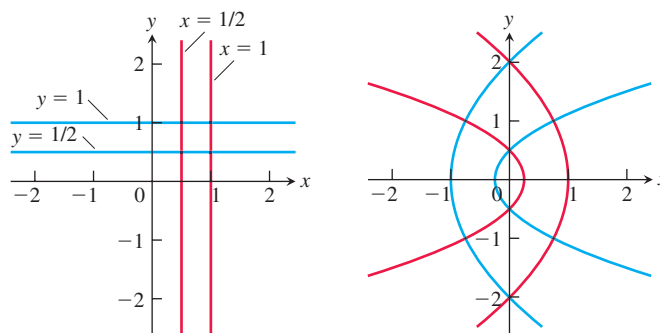


FIGURE 18.10 The images of the horizontal lines $y = 1/2$ and $y = 1$, and of the vertical lines $x = 1/2$ and $x = 1$, under the complex map $f(z) = z^2$.

DEFINITION We say that the complex-valued function $f(z) = u(x, y) + iv(x, y)$ has **limit** $w_0 = u_0 + iv_0$ as z approaches $z_0 = x_0 + iy_0$, and write

$$\lim_{z \rightarrow z_0} f(z) = w_0$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|f(z) - w_0| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta. \quad (18)$$

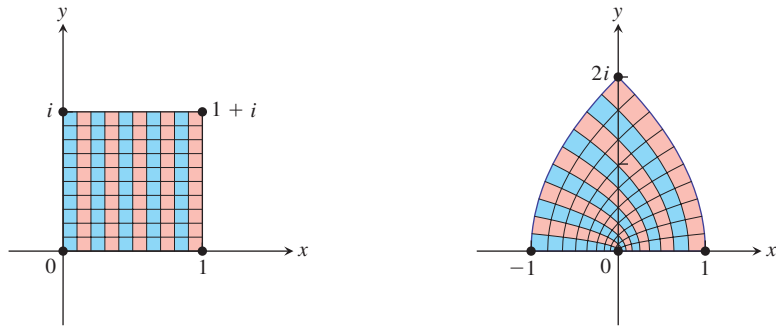


FIGURE 18.11 The image of a square with vertices at $0, 1, i, 1 + i$ under the complex map $f(z) = z^2$

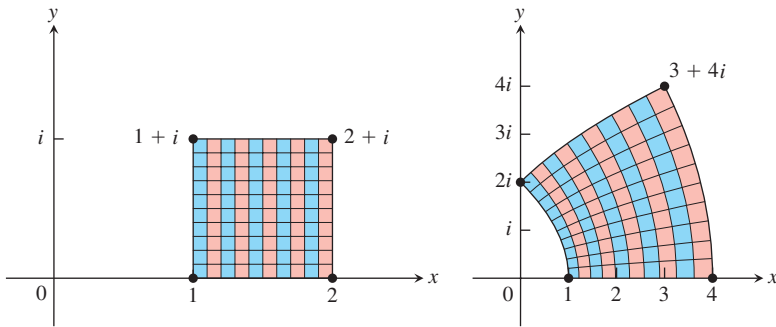


FIGURE 18.12 The image of a square with vertices at $1, 2, 1 + i, 2 + i$ under the same map

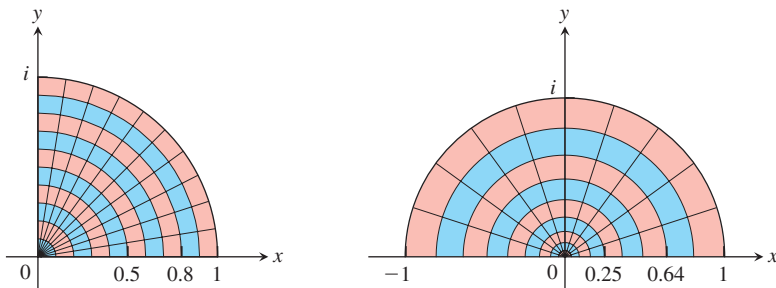


FIGURE 18.13 A quarter of the unit disk in the complex plane and its image under the complex map $f(z) = z^2$. Radial lines at angle θ are taken to new radial lines at angle 2θ , and circular arcs of radius r to circular arcs of radius r^2 .

If we express f in the form $f(x + iy) = u(x, y) + iv(x, y)$, then we have

$$|f(z) - w_0| = \sqrt{(u(x, y) - u_0)^2 + (v(x, y) - v_0)^2}.$$

For a point in the complex plane $w_0 = u_0 + iv_0$, the definition implies (Exercise 7) that $\lim_{z \rightarrow z_0} f(z) = w_0$ exactly when both

$$\lim_{(x, y) \rightarrow (x_0, y_0)} u(x, y) = u_0$$

and

$$\lim_{(x, y) \rightarrow (x_0, y_0)} v(x, y) = v_0.$$

We often use this formulation in computations. Namely the limit of a function $f(x + iy) = u(x, y) + iv(x, y)$ as $x + iy \rightarrow x_0 + iy_0$ exists if and only if the limit of $u(x, y)$ and of $v(x, y)$ each exists as $(x, y) \rightarrow (x_0, y_0)$.

EXAMPLE 5 Show that

$$\text{a. } \lim_{z \rightarrow z_0} z = z_0, \quad \text{b. } \lim_{z \rightarrow z_0} z^2 = z_0^2, \quad \text{c. } \lim_{z \rightarrow 0} \bar{z}/z \text{ does not exist.}$$

Solution

a. We express $f(z) = z$ in the form $f(z) = u(x, y) + iv(x, y)$, where $u(x, y) = x$ and $v(x, y) = y$, and compute the limit as z approaches $z_0 = x_0 + iy_0$. Then since the real-valued functions $u(x, y) = x$ and $v(x, y) = y$ are continuous,

$$\lim_{(x, y) \rightarrow (x_0, y_0)} u(x, y) = x_0$$

and

$$\lim_{(x, y) \rightarrow (x_0, y_0)} v(x, y) = y_0.$$

So $\lim_{z \rightarrow z_0} z = x_0 + iy_0 = z_0$ as claimed.

b. We express $f(z) = z^2$ as $f(z) = u(x, y) + iv(x, y)$, where $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ (Equation 15). Then since the real-valued function $u(x, y) = x^2 - y^2$ is continuous,

$$\lim_{(x, y) \rightarrow (x_0, y_0)} u(x, y) = x_0^2 - y_0^2$$

and similarly, since the real-valued function $v(x, y) = 2xy$ is continuous,

$$\lim_{(x, y) \rightarrow (x_0, y_0)} v(x, y) = 2x_0y_0.$$

So $\lim_{z \rightarrow z_0} z^2 = x_0^2 - y_0^2 + i2x_0y_0 = z_0^2$ as claimed.

c. We express $f(z) = \bar{z}/z$ in the form $f(x + iy) = u(x, y) + iv(x, y)$.

$$f(x + iy) = \frac{x - iy}{x + iy} = \frac{(x - iy)(x - iy)}{(x + iy)(x - iy)} = \frac{x^2 - y^2}{x^2 + y^2} + i \frac{-2xy}{x^2 + y^2}.$$

The function

$$u(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

is not continuous at $(0, 0)$ since it has limit equal to 1 if $(x, y) \rightarrow (0, 0)$ along the x -axis (with $y = 0$), and it has limit equal to -1 if $(x, y) \rightarrow (0, 0)$ along the y -axis (with $x = 0$). Since different limits are obtained as $(x, y) \rightarrow (0, 0)$ from different directions, no limit exists for $u(x, y)$ and therefore no limit exists for $f(z)$. ■

Continuity

A function $w = f(z)$ that is defined throughout an open set containing a point $z = z_0$ is said to be **continuous** at z_0 if

$$\lim_{z \rightarrow z_0} |f(z) - f(z_0)| = 0. \quad (19)$$

Intuitively, this means that as the point z approaches z_0 , its image $f(z)$ approaches $f(z_0)$, as shown in Figure 18.14. The notion of continuity for real-valued functions of two variables u and v was defined in Section 13.2. If we write f as a sum of its real and imaginary parts, $f(z) = u(x, y) + iv(x, y)$, then f is continuous at z_0 if and only if $u(x, y)$ and $v(x, y)$ are each continuous at (x_0, y_0) (Exercise 8).

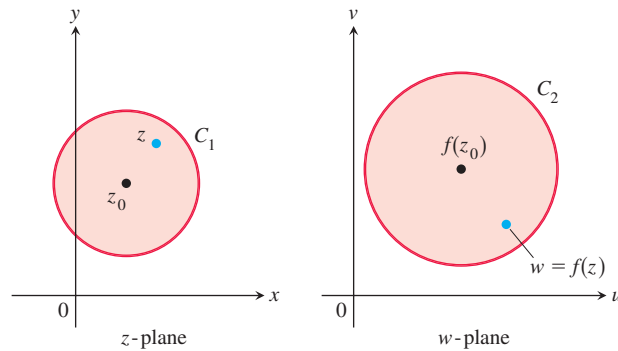


FIGURE 18.14 Choose any circle C_2 around $f(z_0)$ in the w -plane. A continuous function takes points in the z -plane that lie inside a small enough circle C_1 around z_0 to points inside C_2 .

As with functions of a real variable, polynomial functions are continuous. The proof is similar to the real case, and while we do not give it in full, we illustrate the idea with two examples.

EXAMPLE 6 Show that at any point z_0 in the complex plane, the function $f(z) = z$ is continuous.

Solution The definition of continuity given in Equation (19) requires us to show that the limit

$$\lim_{z \rightarrow z_0} |f(z) - f(z_0)| = \lim_{z \rightarrow z_0} |z - z_0| = 0.$$

Using the definition of limit in Equation (18), our challenge is to find a δ so that

$$|z - z_0| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

For this very simple function we can choose $\delta = \varepsilon$. So the requirement for continuity is satisfied, and we conclude that f is continuous at any point z_0 . ■

By similar arguments we can show that sums and products of continuous functions are continuous, as are quotients of continuous functions wherever the denominator is nonzero. These arguments combine to show that polynomial functions and rational functions are continuous on their domains. We illustrate such an argument for the function $f(z) = z^2$.

EXAMPLE 7 Show that at any point z_0 in the complex plane, the function $f(z) = z^2$ is continuous.

Solution For the function $f(z) = z^2$ we have

$$\begin{aligned} \lim_{z \rightarrow z_0} |f(z) - f(z_0)| &= \lim_{z \rightarrow z_0} |z^2 - z_0^2| \\ &= \lim_{z \rightarrow z_0} |z - z_0| |z + z_0| && \text{Factor.} \\ &= \lim_{z \rightarrow z_0} |z - z_0| \lim_{z \rightarrow z_0} |z + z_0| \\ &= 0 \cdot |2z_0| && \text{Example 1 and Exercise 9} \\ &= 0. \end{aligned}$$

The requirement for continuity given by Equation (19) is satisfied, and we conclude that f is continuous at any point z_0 . ■

Although we will not prove it here, the arguments of Examples 1 and 2 can be extended to show that all complex polynomials are continuous, as are all quotients of complex polynomials where the denominator is nonzero.

EXERCISES 18.2

- For each of the following regions in the complex plane, describe the image of the region under the map $w = z^2$. (Hint: Use polar coordinates.)
 - $|z| = 1, 0 \leq \arg z \leq \pi$
 - $|z| = 2, \pi/2 \leq \arg z \leq \pi$
 - $\arg z = \pi/4$
 - $|z| < 1, -\pi \leq \arg z \leq 0$
 - The x -axis
 - The y -axis
- Show that the two parabolas in the w -plane in Figure 18.10 intersect orthogonally (at right angles) if neither a nor b is zero.
- Show that the function $w = z^3$ maps the wedge $0 < \arg z < \pi/3$ in the z -plane onto the upper half-plane of the w -plane, the half-plane whose imaginary part satisfies $\operatorname{Im}(w) > 0$. Use polar coordinates and sketch.
- Show that $f(z) = z^3$ is continuous at $z = z_0$ for any z_0 .
- Show that $f(z) = 1/z$ is continuous at $z = z_0$ if $z_0 \neq 0$.
- The function $w = z^3$ maps the square with vertices at $0, 1, i$, and $1 + i$ to the region shown in Figure 18.15.

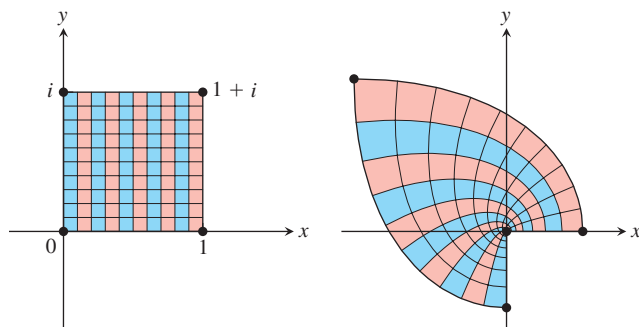


FIGURE 18.15 The image under the complex map $f(z) = z^3$ of a square with vertices at $0, 1, i$, and $1 + i$

- Determine the four complex numbers to which the vertices of the square are mapped.
 - Give parameterizations of the four curves to which the sides of the square are mapped.
- Use Formula (18) with $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$, and $w_0 = u_0 + iv_0$ to show that

$$\lim_{z \rightarrow z_0} f(z) = w_0$$

exactly when both

$$\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0$$

and

$$\lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0.$$

- Suppose that $f(z) = u(x, y) + iv(x, y)$ is defined on an open set containing the point $z_0 = x_0 + iy_0$. Show that $f(z)$ is continuous at z_0 if and only if $u(x, y)$ and $v(x, y)$ are each continuous at (x_0, y_0) .
- Show that the constant function $f(z) = z_0$ is continuous.
 - Show that if $f(z)$ and $g(z)$ are continuous, then so is $(f + g)(z)$.

18.3 Derivatives

The derivative of a complex function $w = f(z)$ at z_0 is defined in the same way as for a real-valued function of a real variable.

DEFINITION The derivative of a complex-valued function f at $z = z_0$ is

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}, \quad (20)$$

provided that this limit exists. When this limit exists we say that f is **complex differentiable** at z_0 .

As usual, we also use the equivalent notation

$$\frac{df}{dz}(z_0)$$

to denote the derivative of f at z_0 . A function that has a complex derivative at every point in its domain D is said to be *complex differentiable* or *holomorphic* on D . Due to a property of complex differentiable functions, that they are equal to their Taylor series where the Taylor series converges, they are also often called *analytic* functions.

The derivative of a real-valued function exists at a point x_0 if the derivative from the left and from the right exist and agree at x_0 . For a complex differentiable function the derivative from *any* direction must exist and agree with the derivative computed from any other direction.

There are many ways in which z can approach z_0 . It can approach z_0 on a horizontal path from the left, from the right, or along a vertical path coming from above or from below. And there are many other possibilities. It is possible for z to approach z_0 along a line coming from any direction, or even along a curved path or spiral that ends at z_0 . The existence of a single limiting value for

$$\frac{f(z) - f(z_0)}{z - z_0}$$

says that no matter how z_0 is approached, the same value is obtained. Despite this very restrictive condition, there is still a large collection of complex differentiable functions. The very special implications of differentiability and the large number of such functions make the use of complex differentiable functions an extremely powerful tool.

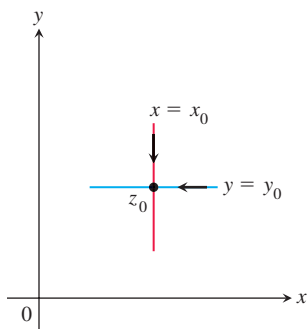


FIGURE 18.16 The limit defining a differentiable function has the same value as z approaches z_0 from any direction, including the horizontal and vertical.

EXAMPLE 1 The complex conjugation function $f(z) = \bar{z}$ takes a point $z = x + iy$ to its complex conjugate $\bar{z} = x - iy$. Show that at any point z_0 in the complex plane, this function is *not* complex differentiable.

Solution We compute the limit defining the derivative as we approach the point $z_0 = x_0 + iy_0$ from two different directions:

- along the vertical line $x = x_0$ and
- along the horizontal line $y = y_0$. See Figure 18.16.

In case (a), we fix $x = x_0$ and consider what happens as $y \rightarrow y_0$. On the vertical line $x = x_0$ we have that z approaches z_0 only from above or below. On this line $z \rightarrow z_0$ exactly when $y \rightarrow y_0$. The limit defining a derivative, Equation (20), gives

$$\begin{aligned} \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{z \rightarrow z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0} \\ &= \lim_{y \rightarrow y_0} \frac{\overline{(x_0 + iy)} - \overline{(x_0 + iy_0)}}{(x_0 + iy) - (x_0 + iy_0)} \\ &= \lim_{y \rightarrow y_0} \frac{(x_0 - iy) - (x_0 - iy_0)}{i(y - y_0)} \\ &= \lim_{y \rightarrow y_0} \frac{-i(y - y_0)}{i(y - y_0)} \\ &= \lim_{y \rightarrow y_0} \frac{-i}{i} \\ &= -1. \end{aligned}$$

In case (b), we fix $y = y_0$ and consider what happens along this horizontal line. With y fixed at y_0 , we have $z \rightarrow z_0$ exactly when $x \rightarrow x_0$, and Equation (20) becomes

$$\begin{aligned}\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{z \rightarrow z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0} \\ &= \lim_{x \rightarrow x_0} \frac{(x + iy_0) - (x_0 + iy_0)}{(x + iy_0) - (x_0 + iy_0)} \\ &= \lim_{x \rightarrow x_0} \frac{(x - iy_0) - (x_0 - iy_0)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{x - x_0}{x - x_0} \\ &= 1.\end{aligned}$$

Two different paths along which z approaches z_0 lead to two different answers. So, there is no single complex number that is equal to the limit, no matter which path is chosen. There is no number that we can call $f'(z_0)$, and we conclude that the function $w = \bar{z}$ does not have a derivative at any point z_0 in the complex plane. ■

Rules for Finding Complex Derivatives

The formulas for differentiating sums, products, quotients, and powers of the complex variable z are the same as those for a real variable x . For example, if c is any complex number, and if $f(z)$ and $g(z)$ are functions that have derivatives at $z = z_0$, then at $z = z_0$ we have:

- | | |
|---|------------------------------|
| (1) $(c)' = 0$ | Derivative of a constant c |
| (2) $(cf)'(z) = cf'(z)$ | Constant Multiple Rule |
| (3) $(f + g)'(z) = f'(z) + g'(z)$ | Sum Rule |
| (4) $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$ | Product Rule |
| (5) $(f/g)'(z) = [g(z)f'(z) - f(z)g'(z)]/g(z)^2$, if $g(z) \neq 0$ | Quotient Rule |
| (6) $(z^n)' = nz^{n-1}$, where n is any integer | Power Rule |
| (7) $(g \circ f)'(z) = g'(f(z))f'(z)$, for $f(z)$ in domain of g | Chain Rule |

These formulas are derived in the same manner as the formulas for functions of a real variable. We give an idea of the techniques used to prove them by establishing one case of the power rule.

EXAMPLE 2 Show from the definition that $f(z) = z^3$ has derivative

$$f'(z) = 3z^2.$$

Solution We compute the limit defining a derivative,

$$\begin{aligned}\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{z \rightarrow z_0} \frac{z^3 - z_0^3}{z - z_0} && \text{Factor numerator.} \\ &= \lim_{z \rightarrow z_0} \frac{(z - z_0)(z^2 + zz_0 + z_0^2)}{z - z_0} \\ &= \lim_{z \rightarrow z_0} z^2 + zz_0 + z_0^2 && \text{Each of the three terms approaches } z_0^2. \\ &= 3z_0^2.\end{aligned}$$

Note that no matter how z approaches z_0 , the terms z^2 and zz_0 each approaches z_0^2 , and the limit is $3z_0^2$. So $f'(z_0) = 3z_0^2$. ■

EXERCISES 18.3

In Exercises 1–3, find the derivative with respect to z of the given function at the point z_0 .

1. $f(z) = \frac{z+1}{z-1}$, $z_0 = 1+i$

2. $f(z) = z^3 + 3z^2 + 3z + 2$, $z_0 = -1+2i$

3. $f(z) = \sqrt{z^2 + 1}$, $z_0 = \frac{1+i}{\sqrt{2}}$

(Here we get two answers depending on our choice of the square root. Give the answer using the principal square root, as defined in Section 18.1.)

4. Use the definition of a complex derivative to find $f'(z_0)$ if $f(z) = 1/z$ and $z \neq 0$.

Theory and Examples

5. The directional derivative of a real-valued function f in the complex plane in the direction of a unit vector w is given by

$$D_w f(z_0) = \lim_{t \rightarrow 0} \frac{f(z_0 + tw) - f(z_0)}{t}. \quad (21)$$

Compare this to the derivative of a complex function $f(z)$ by computing the complex derivative of f as z approaches z_0 along the line $z = z_0 + tw$, where w now represents a complex number with magnitude 1. What are the differences between these two formulas?

18.4 The Cauchy-Riemann Equations

When we write $z = x + iy$ and the complex function $w = f(z) = u + iv$ is differentiable, the real part of f , $u(x, y)$, and the imaginary part of f , $v(x, y)$, are real-valued differentiable functions of the real variables x and y . We can compute the derivative of f at a point $z_0 = x_0 + iy_0$ by approaching z_0 from any direction we choose. By comparing the derivative $f'(z_0)$ obtained from approaching z_0 along horizontal and vertical directions, we obtain equations relating the partial derivatives of u and v .

If we approach along the horizontal line $y = y_0$, then

$$z - z_0 = (x + iy_0) - (x_0 + iy_0) = x - x_0$$

and therefore

$$\begin{aligned} f'(z_0) &= \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \\ &= \lim_{x \rightarrow x_0} \frac{f(x + iy_0) - f(x_0 + iy_0)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{u(x, y_0) + iv(x, y_0) - (u(x_0, y_0) + iv(x_0, y_0))}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i \lim_{x \rightarrow x_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0} \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0). \end{aligned}$$

Though we have approached z_0 in only one of the many possible ways, we obtained a useful formula for the derivative, equal to what we would get if we approached z_0 from any other direction. For $f(z) = u(x + iy) + iv(x + iy)$, we obtain the following explicit formula for the complex derivative of a function that has a complex derivative at z_0 .

Formula for the Complex Derivative

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) \quad (22)$$

The formula in Equation (22) does not depend on which direction is taken to approach z_0 . Instead of approaching z_0 along the x -axis, we can approach along the vertical line $x = x_0$, on which

$$z - z_0 = (x_0 + iy) - (x_0 + iy_0) = i(y - y_0).$$

In this case we have

$$\begin{aligned}
 f'(z_0) &= \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \\
 &= \lim_{y \rightarrow y_0} \frac{f(x_0 + iy) - f(x_0 + iy_0)}{i(y - y_0)} \\
 &= \lim_{y \rightarrow y_0} \frac{u(x_0, y) + iv(x_0, y) - (u(x_0, y_0) + iv(x_0, y_0))}{i(y - y_0)} \\
 &= \lim_{y \rightarrow y_0} \frac{u(x_0, y) - u(x_0, y_0)}{i(y - y_0)} + \lim_{y \rightarrow y_0} \frac{i[v(x_0, y) - v(x_0, y_0)]}{i(y - y_0)} \\
 &= \frac{\partial v}{\partial y}(x_0, y_0) - i \frac{\partial u}{\partial y}(x_0, y_0). \quad 1/i = -i
 \end{aligned}$$

Since f is complex differentiable, these two computations must give the same value for the derivative. The two real parts in the two equations must be equal, and the imaginary parts in the two equations must be the same. This gives relations between the partial derivatives of u and v that hold at any point where $f(z) = u(x, y) + iv(x, y)$ has a complex derivative.

The Cauchy-Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

EXAMPLE 1 Show that the function $f(z) = 1/z$ satisfies the Cauchy-Riemann Equations at any point in the complex plane where $z \neq 0$.

Solution Let $z = x + iy$ be a nonzero point in the complex plane, and let

$$w = f(z) = 1/z.$$

Then

$$w = u + iv = \frac{1}{x + iy} = \frac{1}{x + iy} \cdot \frac{x - iy}{x - iy} = \frac{x - iy}{x^2 + y^2},$$

and therefore

$$u = \frac{x}{x^2 + y^2}, v = \frac{-y}{x^2 + y^2}. \quad x^2 + y^2 \neq 0 \text{ since } z \neq 0.$$

After computing partial derivatives four times and simplifying, we find that

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial x}.$$

This shows that the Cauchy-Riemann Equations are satisfied at all points where $x^2 + y^2 \neq 0$. ■

The Cauchy-Riemann Equations give relationships between u and v that must be satisfied by a complex differentiable function. We cannot independently specify functions

$u(x, y)$ and $v(x, y)$ and expect the function $w = u + iv$ to be complex differentiable. However if we take functions u and v , which have continuous partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \text{ and } \frac{\partial v}{\partial y}$$

on a region R and satisfy the Cauchy-Riemann Equations on R , then a theorem that we will not prove here states that the function $w = u + iv$ is complex differentiable on R . This is a very powerful conclusion, since it means that by checking agreement of the computation of the limit

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

only as $z \rightarrow z_0$ in the horizontal and vertical directions, we know the derivative is the same as $z \rightarrow z_0$ in *every* direction.

EXERCISES 18.4

For the following functions, find the real and imaginary parts of the function $w = f(z)$, $w = u + iv$, $z = x + iy$, and show that they satisfy the Cauchy-Riemann equations.

1. z^2
2. z^3
3. z^4
4. $1/z^2, z \neq 0$
5. A real-valued function with continuous second partial derivatives $g(x, y)$ is called *harmonic* if it satisfies Laplace's Equation,

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0.$$

Show that both the real part $u = u(x, y)$ and the imaginary part $v = v(x, y)$ of a complex differentiable function $w = f(z)$ satisfy Laplace's Equation.

6. Verify that Laplace's Equation in Exercise 5 is satisfied by the real and imaginary parts of the functions
 - a. z ,
 - b. z^2 ,
 - c. z^3 .
7. Show that the following functions are complex differentiable on the complex plane by verifying that their real and imaginary parts have continuous partial derivatives and satisfy the Cauchy-Riemann Equations.
 - a. $f(z) = e^x \cos y + ie^x \sin y$
 - b. $f(z) = \sin x \cosh y + i \cos x \sinh y$

18.5 Complex Power Series

Applying the rules for finding the derivatives of sums, products, and quotients of complex functions, we can see that polynomials and rational functions (quotients of polynomials) are complex differentiable on their domains. We have extended these functions from differentiable functions on the real line to differentiable functions on the complex plane $\mathbb{C}$. We now investigate how to extend other elementary functions such as $\sin z$, $\cos z$, e^z , $\cosh z$, and so on that are defined on the real line to functions that are differentiable on $\mathbb{C}$. To do this we will need to make sense of expressions such as $\sin(2 + 3i)$ and e^{3-i} . When evaluating $\sin(2 + 3i)$ it is not helpful to try to think of $2 + 3i$ as an angle whose sine is evaluated, as we did when evaluating the sine of a real number. Rather we define the value of the sine function evaluated on a complex number by extending the expression of the sine function as a power series from real to complex numbers. In Chapter 9 we derived the Taylor series expression

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

When we use this series to calculate a value such as $\sin 2$, we don't think of $x = 2$ as a radian measure of an angle, but just consider it as a number that is input into the series. This leads us to *define* the function $f(z) = \sin z$ for a complex variable z by the formula

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}. \quad (23)$$

For real z , for which $z = x + iy$ with $y = 0$, this gives the familiar series for $\sin x$. So this new definition of $\sin z$ for complex numbers z is consistent with our previous understanding of the sine function. We now consider what it means for such a series to converge.

Sequences

As we did with real numbers, we base the concept of convergence of a complex series on the more basic concept of the convergence of complex sequences. A **complex sequence** is defined to be a function from the positive integers to the complex numbers. These sequences are often presented as an infinite list of complex numbers,

$$\{z_n\} = z_1, z_2, \dots, z_n, \dots$$

We say that a sequence $\{z_n\}$ **converges** to a complex number L if for every positive number ϵ there corresponds an integer N such that

$$|z_n - L| < \epsilon \text{ whenever } n > N.$$

If no such number exists we say that $\{z_n\}$ **diverges**.

While each number z_n in the sequence and the limiting number L is complex, the conditions for convergence involve the real numbers $|z_n - L|$, ϵ , and δ . Because of this, many of the arguments for convergence of sequences given in Section 9.1 carry over to complex sequences.

Series

A complex power series

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots \quad (24)$$

converges at a point z if the sequence of partial sums

$$s_n = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$$

has a limit $S = u + iv$ as $n \rightarrow \infty$.

If we separate s_n into its real and imaginary parts, by writing

$$s_n = u_n(x, y) + iv_n(x, y),$$

then

$$s_n \rightarrow S \text{ if and only if } u_n \rightarrow u \text{ and } v_n \rightarrow v. \quad \text{As in Equation (14)}$$

Many properties of real series proved in Chapter 9 carry over to complex series. For example, sums of convergent series converge. If

$$\sum_{n=0}^{\infty} c_n = L \quad \text{and} \quad \sum_{n=0}^{\infty} d_n = M$$

are two convergent complex series, then the series $\sum_{n=0}^{\infty} (c_n + d_n)$ also converges, and

$$\sum_{n=0}^{\infty} (c_n + d_n) = L + M.$$

Proofs of these properties are very similar to the proofs for real series.

The series $\sum_{n=0}^{\infty} a_n z^n$ is said to *converge absolutely* if the corresponding series of absolute values

$$\sum_{n=0}^{\infty} |a_n z^n| = |a_0| + |a_1 z| + |a_2 z^2| + \dots \quad (25)$$

converges. The sum in Equation (25) is a series of real numbers, so we can apply the tests developed in Chapter 9, such as the Ratio Test, Comparison Test, or Integral Test, to verify its convergence. If the series of absolute values does converge, then the following theorem tells us that the original series also converges.

THEOREM 1 — The Absolute Convergence Test for Complex Series

If a series converges absolutely, then it converges.

Proof Separate each term of the series into its real and imaginary part by writing

$$a_k z^k = c_k + i d_k, \quad k = 0, 1, 2, \dots$$

where c_k and d_k are real numbers. Suppose that the series of absolute values in Equation (25) converges. Then, since

$$|c_k| = \sqrt{c_k^2} \leq \sqrt{c_k^2 + d_k^2} = |a_k z^k| \quad \text{and} \quad |d_k| = \sqrt{d_k^2} \leq \sqrt{c_k^2 + d_k^2} = |a_k z^k|,$$

the Direct Comparison Test tells us that the series

$$\sum_{k=0}^{\infty} |c_k| \quad \text{and} \quad \sum_{k=0}^{\infty} |d_k|$$

both converge (Theorem 10 in Chapter 9). From this we can conclude that each of the series

$$\sum_{k=0}^{\infty} c_k \quad \text{and} \quad \sum_{k=0}^{\infty} d_k,$$

computed without taking absolute values, also converge. Therefore the series

$$\sum_{k=0}^{\infty} (c_k + i d_k) = \sum_{k=0}^{\infty} c_k + i \sum_{k=0}^{\infty} d_k$$

converges. ■

EXAMPLE 1 Show that the series for $\sin z$ in Equation (23) converges for all complex numbers z .

Solution We test the series of real numbers

$$\sum_{n=0}^{\infty} \left| (-1)^n \frac{z^{2n+1}}{(2n+1)!} \right| \tag{26}$$

for absolute convergence. For this series we apply the Ratio Test. Set

$$w_n = \left| (-1)^n \frac{z^{2n+1}}{(2n+1)!} \right|.$$

Then we calculate

$$\left| \frac{w_{n+1}}{w_n} \right| = \frac{|z|^2}{(2n+3)(2n+2)},$$

and

$$\lim_{n \rightarrow \infty} \left| \frac{w_{n+1}}{w_n} \right| = \lim_{n \rightarrow \infty} \frac{|z|^2}{(2n+3)(2n+2)} = 0. \quad \text{As } n \rightarrow \infty, \text{ denominator} \rightarrow \infty.$$

Since this limit is less than one, the Ratio Test implies that the series in Equation (26) converges absolutely. By Theorem 1, the series in Equation (26) itself converges, without the absolute value signs, and we have shown that the complex series for $\sin z$, the series in Equation (23), converges at every point z in the complex plane. ■

EXERCISES 18.5

In Exercises 1–5, find the regions in which each of the series converges. Sketch the region of absolute convergence.

1. $1 + z + z^2 + z^3 + \cdots$

2. $1 - 2z + 3z^2 - 4z^3 + \cdots$

3. $(z - 1) - \frac{(z - 1)^2}{2} + \frac{(z - 1)^3}{3} - \frac{(z - 1)^4}{4} + \cdots$

4. $(z + i) + \frac{(z + i)^2}{2} + \frac{(z + i)^3}{3} + \frac{(z + i)^4}{4} + \cdots$

5. $1 - \frac{(z + 1)}{2} + \frac{(z + 1)^2}{4} - \frac{(z + 1)^3}{8} + \frac{(z + 1)^4}{16} - \cdots$

6. Show that the series $\sum_{k=0}^{\infty} \frac{z^k}{k!}$ converges absolutely for all z .

7. Show that the series $\sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$ converges absolutely for all z .

8. Show that the series $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^k}{k}$ converges absolutely for $|z| < 1$.

T 9. Use the series in Equation (23) for $\sin z$ to compute $\sin(0.2 + 0.3i)$ to three decimals.

18.6 Some Complex Functions

We have seen how to define the sine function as a complex series by extending the series expansion for real numbers to complex numbers. We can similarly define other basic complex functions by extending the series formulas developed for a real variable x to complex numbers.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

$$\operatorname{Arctan} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}$$

Using the Ratio Test as in Example 1 of Section 18.5, we can show that the series for e^z and $\cos z$ converge absolutely for all z in the complex plane. To define $\operatorname{Arctan} z$ as an inverse function for the function $\tan z$, we need to restrict the domain of $\tan z$ to a region on which it is one-to-one. The formula for $\operatorname{Arctan} z$ in the above power series corresponds to one choice for such a restricted domain and is called the *principal value* of $\operatorname{arctan} z$. The Ratio Test shows that this series for $\operatorname{Arctan} z$ converges when $|z| < 1$, and diverges when $|z| > 1$. We will explore this idea in more detail when we look at the complex logarithmic function later in this section.

If there is an open region of the complex plane containing a point z in which all points except for z are in the domain of a complex differentiable function, then z is called an *isolated singularity* of f . For example, the point $z = 0$ is not in the domain of $f(z) = 1/z$ and is an isolated singularity of this complex differentiable function.

A basic theorem in the theory of complex functions says that a power series $\sum_{n=0}^{\infty} a_n z^n$ either

- (1) converges for all z , or
- (2) converges inside a circle $|z| < R$ and diverges for $|z| > R$, or
- (3) converges only at $z = 0$.

This theorem closely resembles Theorem 18 in Chapter 9, and its proof, based on the Ratio Test, is similar.

THEOREM 2—The Convergence Theorem for Complex Power Series

If a series $\sum_{n=0}^{\infty} a_n z^n$ converges at a point $z = z_0$, then it converges absolutely for all z with $|z| < |z_0|$. If the series diverges at $z = z_0$, then it diverges for all z with $|z| > |z_0|$.

EXAMPLE 1 Show that the largest radius circle centered at 0 inside which the series

$$1 - z^2 + z^4 - z^6 + \cdots$$

converges is the circle of radius $R = 1$.

Solution The absolute value of the n th term $a_n = z^{2n}$ of this series is $|z|^{2n}$. By Theorem 1, the Absolute Convergence Test, this converges for all z with $|z| < 1$. For $z = i$ the series becomes

$$1 - i^2 + i^4 - i^6 + \cdots = 1 + 1 + 1 + 1 + \cdots,$$

which diverges. By Theorem 2 the series diverges for all z with $|z| > 1$. So the largest radius circle centered at zero inside which the series converges is the circle of radius one. ■

The largest radius circle around a point z_0 inside of which a series converges is called the *circle of convergence* for z_0 . The series converges for all points inside this circle and diverges at any point outside it. Understanding the convergence for points lying on the circle of convergence is a more difficult problem that we will not discuss. The circle $|z| = 1$ is the circle of convergence centered at zero for the function in Example 1.

Many of the basic properties of trigonometric and exponential functions extend to the complex versions of these functions that are defined using series.

EXAMPLE 2 Using the series definition of the exponential function

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

show that

$$e^z e^w = e^{z+w}. \quad (27)$$

Solution We multiply the series for e^z and the series for e^w , and gather terms to get a series for the product:

$$\begin{aligned} & \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) \left(1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots\right) \\ &= 1 + (z + w) + \left(\frac{z^2}{2!} + z \cdot w + \frac{w^2}{2!}\right) + \cdots. \end{aligned}$$

The general form for the terms whose degree is n is given by

$$\frac{z^n}{n!} \cdot 1 + \frac{z^{n-1}}{(n-1)!} \cdot w + \frac{z^{n-2}}{(n-2)!} \cdot \frac{w^2}{2!} + \cdots + \frac{z^{n-k}}{(n-k)!} \cdot \frac{w^k}{k!} + \cdots + 1 \cdot \frac{w^n}{n!},$$

which in summation form equals

$$\sum_{k=0}^n \frac{z^{n-k}}{(n-k)!} \frac{w^k}{k!}. \quad (28)$$

On the other hand, the series expression for e^{z+w} is

$$e^{z+w} = 1 + z + \frac{(z+w)^2}{2!} + \frac{(z+w)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!},$$

and the term of degree n is

$$\frac{(z+w)^n}{n!}.$$

By noticing that

$$\frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \frac{(n-k)!}{(n-k)!} = \frac{n!}{k!(n-k)!}$$

we may also write this in summation form as

$$\begin{aligned}\frac{(z+w)^n}{n!} &= \frac{1}{n!} \left[z^n + nz^{n-1}w + \frac{n(n-1)}{2}z^{n-2}w^2 + \frac{n(n-1)(n-2)}{3!}z^{n-3}w^3 + \cdots + w^n \right] \\ &= \frac{1}{n!} \sum_{k=0}^n \frac{n!}{(n-k)!k!} z^{n-k} w^k \\ &= \sum_{k=0}^n \frac{z^{n-k}}{(n-k)!} \frac{w^k}{k!}.\end{aligned}$$

This last expression is identical to the one that appeared in Equation (28). We have shown that e^ze^w and e^{z+w} have power series whose terms of degree n are equal for each value of n . It follows that the power series themselves are equal, and so $e^ze^w = e^{z+w}$. ■

In a similar way, we can multiply and add power series term by term to show that

$$\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$$

and

$$\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w).$$

When we substitute iz for z in the series expansion for e^z , we obtain

$$\begin{aligned}e^{iz} &= 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \cdots \\ &= 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + \cdots && i^2 = -1, i^3 = -i, \\ &= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) && i^4 = 1, \dots \\ &= \cos z + i \sin z. && \text{Series expansions for } \cos z \text{ and } \sin z\end{aligned}$$

This establishes **Euler's Formula**, which we stated in Section 18.1,

$$e^{iz} = \cos z + i \sin z. \quad (29)$$

Euler's formula tells us how to make sense of a complex exponent. Namely if we write $z = x + iy$, then

$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y). \quad (30)$$

Thus the value of e^z is expressed in terms of sines and cosines of the real numbers x and y .

With this formula, we can begin to understand the behavior of the exponential function on the complex plane. The image of a rectangle in the complex plane under the exponential map, computed using Euler's formula, is shown in Figure 18.17. The rectangle is wrapped around the origin, which is not in the range of e^z . If the rectangle is translated upwards by or downwards by a multiple of $2\pi i$, then the formula for e^z yields the same value, as can be seen in Equation (30). The height of the rectangle in Figure 18.17 is 6, just

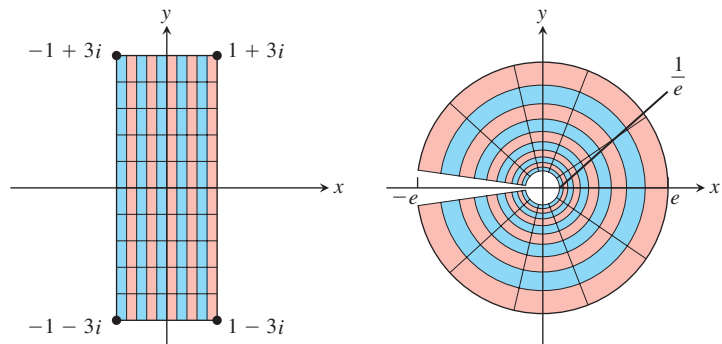


FIGURE 18.17 The image under the map $f(z) = e^z$ of a rectangle with vertices at $-1 - 3i$, $1 - 3i$, $-1 + 3i$, and $1 + 3i$

short of 2π . The top and bottom edges are wrapped around by the exponential function and almost coincide. A slightly taller rectangle with height 2π would have its top and bottom edges mapped by $w = e^z$ to coinciding radial segments.

If we replace z by $-iz$ in Euler's formula, we get the companion equation

$$e^{-iz} = \cos z - i \sin z. \quad (31)$$

By adding these two equations and dividing by two, we obtain a formula for $\cos z$,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}. \quad (32)$$

If we subtract Equation (31) from Equation (29), then we obtain a formula for $\sin z$,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}. \quad (33)$$

The complex versions of the other trigonometric functions are derived from $\sin z$ and $\cos z$ in the usual way. For example

$$\tan z = \frac{\sin z}{\cos z} = \frac{1}{i} \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}. \quad (34)$$

The usual trigonometric identities can be established by expressing the trigonometric functions as exponentials and then making use of the properties of the exponential function.

EXAMPLE 3 Show that

$$\cos^2 z + \sin^2 z = 1.$$

Solution We square both sides of Equation (32) and Equation (33) and add:

$$\begin{aligned} \cos^2 z + \sin^2 z &= \frac{e^{2iz} + 2e^0 + e^{-2iz}}{4} - \frac{e^{2iz} - 2e^0 + e^{-2iz}}{4} \\ &= \frac{4e^0}{4} = 1. \end{aligned}$$

Equations (32) and (33) also reveal the close relationship between the trigonometric and hyperbolic functions. We define the complex hyperbolic functions by extending the series formulas for the real case, so that

$$\begin{aligned} \cosh z &= \frac{e^z + e^{-z}}{2} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots, \\ \sinh z &= \frac{e^z - e^{-z}}{2} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots. \end{aligned}$$

Then Equations (32) and (33) say that

$$\cos z = \cosh iz \quad \text{and} \quad i \sin z = \sinh iz. \quad (35)$$

These relationships explain the similarities between trigonometric identities such as

$$\cos^2 z + \sin^2 z = 1,$$

and the corresponding identities for hyperbolic functions, such as

$$\cosh^2 z - \sinh^2 z = 1. \quad \blacksquare$$

The Logarithmic Function

We now define the logarithm of a complex number, extending the familiar notion of the logarithm of a real positive number. It is convenient to work with the extension of the natural logarithm, $\ln x$, and we will assume this choice of base. However we will reserve usage of the notation $\ln x$ to the case where the x is a real number.

If x is a positive real number, then the logarithm arises as the inverse of the exponential function, so that $\ln x = y$ if and only if $x = e^y$. We wish to have the complex logarithmic function be the inverse of the exponential function in a similar manner. However, when we replace x with an arbitrary complex number z , we are faced with a problem because

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z (\cos 2\pi + i \sin 2\pi) = e^z.$$

The function $w = e^z$ is not one-to-one, and therefore does not have an inverse if the domain of the exponential function is the entire complex plane. We ran into this type of problem when we defined inverse functions, such as $\arcsin x$, because $\sin x$ is not one-to-one. We handled the issue by restricting the domain of $\sin x$ to an interval on which it is one-to-one, namely $[-\pi/2, \pi/2]$. We do the same now, restricting the domain of e^z to a subset on which it is one-to-one. This allows for defining an unambiguous inverse for e^z , which we call $\text{Log } z$, written with a capital “L.” While there are many possibilities for such a subset of the complex plane, we make a choice that gives what is called the *principal branch* of the logarithm function. Other choices of notation, such as $\text{Ln } z$, are sometimes used for this function.

The idea behind the definition of $\text{Log } z$ is most easily understood if we represent a complex number using its polar coordinate form, $z = re^{i\theta}$, where $r = |z|$ and $\theta = \arg(z)$. Now recall the property $\ln ab = \ln a + \ln b$ that is satisfied by the real logarithmic function, whose domain is the positive real numbers. If we retain this property, we are led to define the complex logarithm of a nonzero complex number z to be

$$\log z = \ln r + i\theta. \quad (36)$$

There is still an ambiguity to resolve, because $\theta = \arg(z)$ is only defined up to multiples of 2π . The description of the complex number z could also have been given as $z = re^{i(\theta+2\pi)}$, or $z = re^{i(\theta+4\pi)}$, or in general

$$z = re^{i(\theta+2n\pi)}, \quad n = 0, \pm 1, \pm 2, \dots$$

The definition in Equation (36) could be applied to any of these choices that describe z , leading to infinitely many possible values for the logarithm of z , with any pair differing by a multiple of $2\pi i$. To pin down one choice, we require that the angle θ used to specify z should be chosen to satisfy $-\pi < \theta \leq \pi$. (We always take $r > 0$, which implies that 0 is not in the domain of the logarithmic function.) We call the resulting complex number $\ln r + i\theta$, $-\pi < \theta \leq \pi$, the *principal value* of the complex logarithmic function. The definition of the complex logarithmic function now becomes unambiguous,

$$\text{Log } z = \ln r + i\theta, \quad -\pi < \theta \leq \pi.$$

If we do not require that θ satisfy $-\pi < \theta \leq \pi$, then we get multiple values for $\ln r + i\theta$. We use $\log z$ to denote this entire collection of values. So $\log z$ denotes all complex numbers w satisfying $e^w = z$. A value of $\log z$ other than the principal value differs from it by a multiple of $2\pi i$. Thus all possible values w for the logarithm of z are related to the principal value by

$$w = \text{Log } z + 2n\pi i, \quad n = 0, \pm 1, \pm 2, \dots$$

EXAMPLE 4 Find all possible values for the logarithm of $1 + i$.

Solution The complex number $1 + i$ is described with polar coordinates $r = \sqrt{2}$, $\theta = \pi/4$ (Figure 18.18). Since $-\pi < \pi/4 \leq \pi$, the principal value $\text{Log}(1 + i)$ is

$$\text{Log}(1 + i) = \ln \sqrt{2} + (\pi/4)i.$$

So the possible values w for the logarithm of $1 + i$ are

$$w = \text{Log}(1 + i) + (2n\pi)i = \ln \sqrt{2} + (\pi/4)i + (2n\pi)i, \quad n = 0, \pm 1, \pm 2, \dots \quad \blacksquare$$

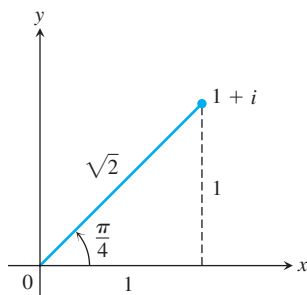


FIGURE 18.18 The polar description of $1 + i$

EXERCISES 18.6

T 1. Estimate $e^{(1+i)/\sqrt{2}}$ by using the first four terms of the series expansion for e^z .

2. Express the sines and cosines in the following identity in terms of exponentials and thereby show that

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

is a consequence of the exponential law $e^{z_1}e^{z_2} = e^{z_1+z_2}$.

3. Establish the following differentiation formulas by differentiating the appropriate series term-by-term.

a. $\frac{de^z}{dz} = e^z$

b. $\frac{d \sin z}{dz} = \cos z$

c. $\frac{d \cos z}{dz} = -\sin z$

d. $\frac{d \operatorname{Arctan} z}{dz} = \frac{1}{1+z^2}$

4. The derivative of a complex-valued function $f(x) = u(x) + iv(x)$ whose input is a real variable x is defined to be $f'(x) = u'(x) + iv'(x)$.

a. Show that $y = e^{i\omega x}$ and $y = e^{-i\omega x}$ are solutions of the differential equation

$$\frac{d^2 y}{dx^2} + \omega^2 y = 0.$$

b. Show that $y = e^{(a+ib)x}$ and $y = e^{(a-ib)x}$ are solutions of the differential equation

$$\frac{d^2 y}{dx^2} + 2a \frac{dy}{dx} + (a^2 + b^2)y = 0.$$

c. Assuming that a and b are real in part 4(b), show that both

$$e^{ax} \cos bx = \operatorname{Re}(e^{(a+ib)x})$$

and

$$e^{ax} \sin bx = \operatorname{Im}(e^{(a+ib)x})$$

are solutions of the given differential equation.

5. Find values of a such that $y = e^{ax}$ is a solution of the differential equation

a. $\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + 5y = 0,$

b. $\frac{d^4 y}{dx^4} + 4y = 0,$

c. $\frac{d^3 y}{dx^3} - 8y = 0.$

6. Show that the point $z(t)$ traverses a unit circle with angular velocity ω if $z(t) = e^{i\omega t}$ and ω is a real constant.

7. If x and y are real, show that $|e^{x+iy}| = e^x$.

8. Show that the real and imaginary parts of $w = e^z$ satisfy the Cauchy-Riemann Equations.

9. Show using the relevant series that

a. $\sin(iz) = i \sinh z,$ b. $\cos(iz) = \cosh z.$

10. Show using the results of Exercises 2 and 9 that $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$, and find a value of z such that $\sin z = 2$.

11. Show that the real and imaginary parts of $w = \sin z$ in Exercise 10 satisfy the Cauchy-Riemann equations.

12. Show that $|\sin(x + iy)| = \sqrt{\sin^2 x + \sinh^2 y}$ if x and y are real.

13. Let $z = x + iy$, $w = u + iv$, and $w = \sin z$. Show that for any nonzero real constant b , the line segment $\{-\pi < x \leq \pi, y = b\}$ in the z -plane maps to the ellipse

$$\frac{u^2}{\cosh^2 b} + \frac{v^2}{\sinh^2 b} = 1$$

in the w -plane.

14. Show that the only complex roots $z = x + iy$ (x and y real) of the equation $\sin z = 0$ are at points on the real axis ($y = 0$) at which $\sin x = 0$.

15. Show that $\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$.

16. Show that $|\cosh(x + iy)| = \sqrt{\cos^2 y + \sinh^2 x}$ if x and y are real.

17. If x and y are real and $\cosh(x + iy) = 0$, show that $x = 0$ and $\cos y = 0$.

18. What is the image in the w -plane of a line $x = \text{constant}$ in the z -plane if

a. $w = e^z,$

b. $w = \sin z?$ Sketch.

19. The integral of a complex-valued function of a real variable x , $f(x) = u(x) + iv(x)$, is defined to be $\int f(x) dx = \int u(x) dx + i \int v(x) dx$. Evaluate $\int e^{(a+ib)x} dx$ and by equating real and imaginary parts, obtain formulas for $\int e^{ax} \cos bx dx$ and $\int e^{ax} \sin bx dx$.

Logarithms

20. Find the principal value of $\log z$ for each of the following complex numbers z :

a. $2 - 2i,$

b. $\sqrt{3} + i,$

c. $-4,$

d. $4,$

e. $2i,$

f. $\frac{1+i}{1-i}.$

21. Find all values of $\log z$ for each of the following complex numbers z :

a. $2,$

b. $-2,$

c. $2i,$

d. $-2i,$

e. $i - \sqrt{3}.$

22. Express $w = \tan z$ in terms of exponentials; then solve for z in terms of w and thereby show that

$$\arctan w = \frac{1}{2i} \log \frac{1+iw}{1-iw}.$$

23. a. Show that

$$\arcsin z = -i \log(iz + \sqrt{1-z^2}).$$

b. Use part (a) to find a complex number z satisfying $\sin z = 3$. Note that since $\sin z = \sin(z + 2\pi)$ there are infinitely many choices for such a z .

24. Sketch the images in the w -plane of the following sets in the z -plane, under the mapping function $w = \ln z$:
- $|z| = \text{constant}$, $-\pi < \arg z < \pi$,
 - $\arg z = \text{constant}$, $0 < |z| < \infty$.
25. Figure 18.19 shows the image under the complex map $f(z) = \text{Log } z$ of a square with vertices at 1 , e , $1 + (e - 1)i$, and $e + (e - 1)i$.
- Find the complex number to which each of the four vertices is mapped.
 - Find parameterizations of the four curves forming the boundary of the image.

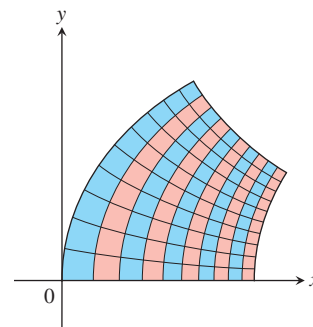


FIGURE 18.19 The map $w = \text{Log } z$ maps a square to this region.

18.7 Conformal Maps

In this section we will introduce the concept of a conformal map. Informally, these maps are functions on open regions in the plane that preserve the angles between curves. There is a close connection between this idea and properties of complex functions. It turns out that complex differentiable functions with nonzero derivative are conformal. This geometric concept plays an important role in many applications.

To measure the angle between two differentiable curves that meet at a point z_0 , we take the angle between their tangent vectors, as we did in Chapter 12. We now look at what a map $f(x, y) = (u(x, y), v(x, y))$ does to such angles. We can compare the angle between two curves $a(t)$ and $b(t)$ intersecting at z_0 with the angle between the two curves $c(t) = f(a(t))$ and $d(t) = f(b(t))$ that they are sent to by f , as in Figure 18.20. When these angles are unchanged for all choices of curves, we say that the map f is *conformal*.

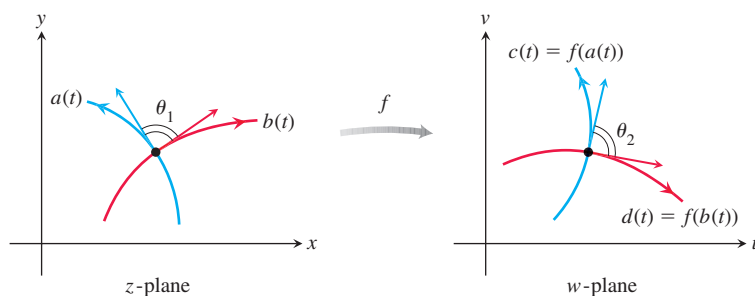


FIGURE 18.20 A conformal map preserves angles, meaning that $\theta_1 = \theta_2$ for any pair of curves in the domain of f .

Intuitively, conformal maps act at infinitesimal scales by stretching or contracting equally in all directions, and by rotating. Since angles are preserved, a conformal map f rotates the tangent vector of all curves through a point z_0 by the same angle. The length of a tangent vector can be changed by a conformal map, but any two tangent vectors are stretched or compressed by the same factor. It is a rather amazing fact, shown in Theorem 3, that all complex differentiable maps have this powerful and useful geometric property wherever their derivative is nonzero.

Figure 18.21 shows the square domain $A = \{(x, y): 0 < x < 1, 0 < y < 1\}$, which is mapped by $f(z) = z^2$ to the domain B in the complex w -plane.

Blue stripes in A are sent by f to blue stripes in B , and red stripes to red. Notice that the edges of the red and blue stripes in A meet at right angles, as do their images in B . The angle between these curves is preserved. Even though the region A itself is stretched and distorted when mapped to B , the angle between any two curves in A is the same as the angle between the images of these curves in B .

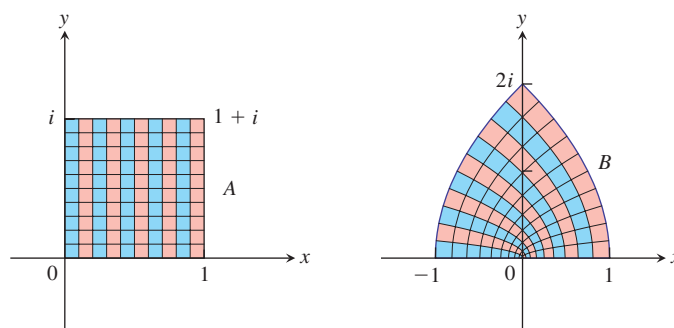


FIGURE 18.21 The map $f(z) = z^2$ takes the perpendicular horizontal and vertical lines in A to perpendicular curves in B . It preserves the angle between any pair of curves, at all points other than $z = 0$.

Cartographers making maps of the earth for navigational purposes made use of conformal maps. The most commonly used method for displaying a map of the earth, the Mercator Projection, shown in Figure 18.22, is conformal. It maps the surface of the earth, cut open from the north to the south pole along a longitude, to a rectangle in the plane. To set a course to a distant port, a navigator could draw a straight line on the Mercator map between their location and the destination. This would give a heading to follow, such as south 10 degrees west, or due east. If they then sailed their ship in that direction, they would follow the line drawn on the map till they reached their goal. This was very useful before the days of GPS. Note that while it preserves angles, the Mercator conformal map distorts lengths, and can give a distorted impression of the size of two regions. Antarctica and Greenland appear large relative to regions near the equator in this representation.



Qimono/Pixabay



NASA

FIGURE 18.22 The Mercator projection preserves angles, but distorts area. Regions near the poles are highly stretched.

Since they preserve angles, conformal maps give an accurate representation of shapes at a small scale. This makes them useful for computer graphics, shape recognition, alignment algorithms, and many other applications. Conformal maps also have many uses in solving problems in electromagnetism and heat distribution.

To compute angles between two curves, we use their tangent vectors. If $a(t) = (x(t), y(t))$ is a curve with $a(0) = (x_0, y_0)$ and $a'(0) \neq 0$, the vector $a'(0) = \langle x'(0), y'(0) \rangle$ gives the tangent vector. A second curve $b(t)$ has tangent vector $b'(0)$ at the point (x_0, y_0) . The angle between these two vectors at z_0 gives the angle between the two curves.

A complex-valued map f whose domain includes a disk around $z_0 = x_0 + iy_0$ maps $a(t)$ to a new curve, $c(t) = f(a(t))$, with tangent vector $c'(0)$ at $f(z_0)$. Similarly $d(t) = f(b(t))$ has tangent vector $d'(0)$ at $f(z_0)$. We can compare the angle between the vectors $c'(0)$ and $d'(0)$ at $f(z_0)$ with the angle between $a'(0)$ and $b'(0)$ at z_0 . In general there is no reason to expect these angles to be the same. However when f is a complex differentiable map and $f'(z_0) \neq 0$, then it turns out that these angles are indeed equal, no matter which point z_0 and which curves a and b are considered. Theorem 3 explains why this is true.

THEOREM 3 Suppose that a map $f = u + iv$ is complex differentiable and $f'(z) \neq 0$ on a domain A . Then f is conformal.

Proof We will show that f preserves the angle between the tangent vectors of any pair of differentiable curves. Suppose that $a(t)$ and $b(t)$ are two differentiable curves with $a(0) = b(0) = z_0$ and that their tangent vectors meet at an angle α . Take a_r to be a point on $a(t)$, $t > 0$, that has distance r from z_0 , and pick b_r to be a point on $b(t)$ at the same distance r from z_0 . Then we can write

$$a_r - z_0 = re^{i\theta_1} \quad \text{and} \quad b_r - z_0 = re^{i\theta_2}.$$

The quotient satisfies

$$\frac{a_r - z_0}{b_r - z_0} = e^{i(\theta_1 - \theta_2)}.$$

As $r \rightarrow 0$, the angle $\theta_1 - \theta_2$ approaches the angle α between the tangent vectors to the two curves,

$$\alpha = \lim_{r \rightarrow 0} \arg \left(\frac{a_r - z_0}{b_r - z_0} \right).$$

Set $c(t)$ and $d(t)$ to be the images of $a(t)$ and $b(t)$ under f so that this pair of curves intersects at $f(z_0)$.

The angle β at which $c(t)$ and $d(t)$ intersect at $f(z_0)$ is

$$\beta = \lim_{r \rightarrow 0} \arg \left(\frac{f(a_r) - f(z_0)}{f(b_r) - f(z_0)} \right)$$

as this limit gives the angle between tangent vectors to the two image curves. We can rewrite this limit as

$$\begin{aligned} \beta &= \lim_{r \rightarrow 0} \arg \left(\frac{f(a_r) - f(z_0)}{f(b_r) - f(z_0)} \right) \\ &= \lim_{r \rightarrow 0} \arg \left(\frac{\left(\frac{f(a_r) - f(z_0)}{a_r - z_0} \right)}{\left(\frac{f(b_r) - f(z_0)}{b_r - z_0} \right)} \right) \left(\frac{a_r - z_0}{b_r - z_0} \right). \end{aligned}$$

As $r \rightarrow 0$ we have

$$\lim_{r \rightarrow 0} \frac{f(a_r) - f(z_0)}{a_r - z_0} = f'(z_0) \quad \text{and} \quad \lim_{r \rightarrow 0} \frac{f(b_r) - f(z_0)}{b_r - z_0} = f'(z_0)$$

since the complex derivative can be computed as we approach z_0 from any direction. We assumed $f'(z_0) \neq 0$, so we can cancel these terms. Then

$$\beta = \lim_{r \rightarrow 0} \arg \left(\frac{a_r - z_0}{b_r - z_0} \right).$$

This is the same as the formula giving α , so the two angles are equal. ■

EXAMPLE 1 Conformal maps of a square

We consider the square $\{0 < x < 1, 0 < y < 1\}$, shown in Figure 18.23a, and its image under three conformal maps:

The image under the map $f(z) = \sin z$ is shown in Figure 18.23b.

The image under the map $f(z) = \cos z$ is shown in Figure 18.23c.

The image under the map $f(z) = \operatorname{Arctan} z$ is shown in Figure 18.23d.

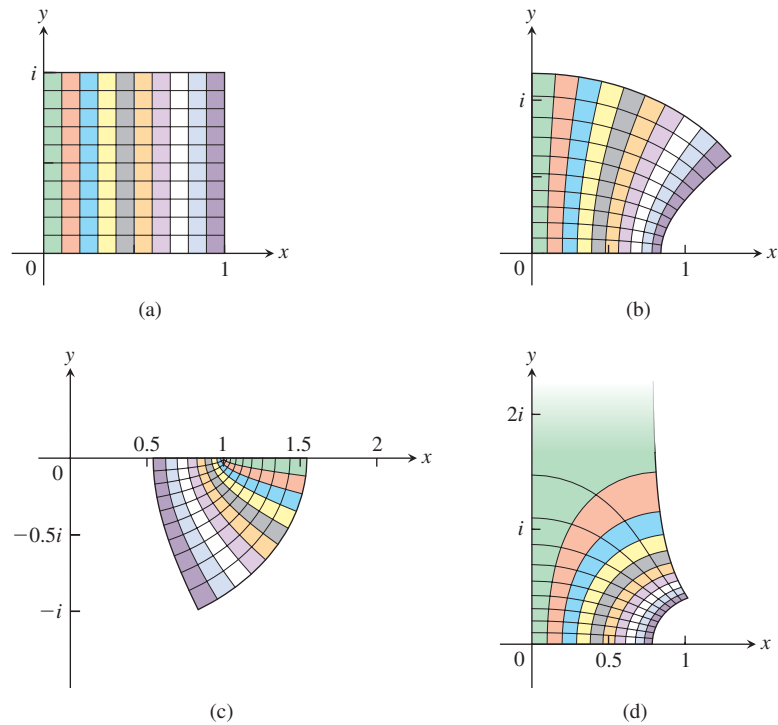


FIGURE 18.23 Conformal images of (a) a square region divided into ten vertical stripes under the conformal maps (b) $\sin z$, (c) $\cos z$, and (d) $\text{Arctan } z$.

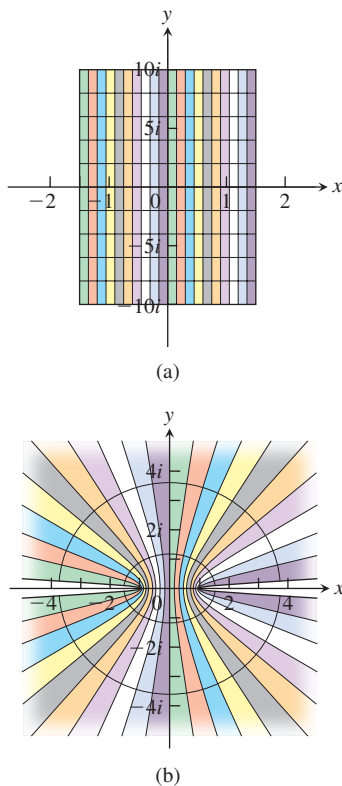


FIGURE 18.24. (a) A rectangle divided into 20 vertical stripes and (b) its image under the conformal map $f(z) = \sin z$

Only part of the image of the vertical green stripe in the square is shown in Figure 18.23d. As $z \rightarrow i$ the values of $\text{Arctan } z$ diverge, and $\text{Arctan } z$ is not defined at $z = i$. The image of the green stripe is not bounded. This illustrates that conformal maps can stretch bounded domains such as the square to unbounded domains.

Since these maps are conformal, the images of perpendicular, vertical, and horizontal lines in the square remain perpendicular in the image. In fact, all angles between pairs of curves are preserved by these conformal maps. ■

In Example 1 we see three regions in the plane that are the images of a square under a conformal map. It turns out that these types of region are not so special, and *any* simply connected open region in the plane is the image of an open square under a conformal map, with one exception. The exception is the entire complex plane. The existence of a conformal map between any other pair of simply connected open regions is given by the Riemann Mapping Theorem, a deep result from the theory of conformal maps.

EXAMPLE 2 The sine function $f(z) = \sin(z)$ has nonzero derivative when restricted to the rectangle $\{-1.5 < x < 1.5, -10 < y < 10\}$, shown in Figure 18.24a. The image of this rectangle lying within the square $\{-2.5 < x < 2.5, -2.5 < y < 2.5\}$ is shown in Figure 18.24b. If the rectangle is enlarged to the infinite rectangle $\{-\pi/2 < x < \pi/2, -\infty < y < \infty\}$, then its image under the sine function fills up the entire complex plane except for two “slits”, the intervals $(-\infty, -1]$ and $[1, \infty)$ on the x -axis.

EXERCISES 18.7

- Does the map $f(z) = z^2$ preserve the angles between the intersecting lines $y = 0$ and $y = x$? How does this fit into Theorem 3?
- For which z is the map $f(z) = e^z$ conformal?
- For which z is the map $f(z) = z^3$ conformal? What happens to angles at points where it is not conformal?
- For the map $f(x, y) = (2x, y)$, compute the angle of intersection of the images under f of the horizontal line $\{x = 1\}$ and the vertical line $\{y = 1\}$. Is this map conformal?
- Is the map $f(x, y) = (x, x + y)$ conformal? If not, find two intersecting curves whose angles are not preserved by f .
- Show that the map $f(z) = (z - i)/(z + i)$ is a conformal map from the upper half-plane $\{(x, y): y > 0\}$ to the open unit disk $\{(x, y): x^2 + y^2 < 1\}$. A deep result, the Riemann Mapping Theorem, shows that this is not unusual. Any simply connected region in the plane can be conformally mapped to the open unit disk, with the exception of the entire complex plane.
- Find the angle by which a curve through the point z_0 is rotated by the map $f(z) = z^2$ if **a.** $z_0 = 1$, **b.** $z_0 = i$, **c.** $z_0 = 1 + i$, **d.** $z_0 = -i$, **e.** $z_0 = 2i$, and **f.** $z_0 = -1$.
- The *conformal factor* of f at z_0 is the amount by which it stretches tangent vectors. Where is the conformal factor of $f(z) = z^2$ smaller than one?
- Given two complex numbers a and b , the map $f(z) = az + b$ rotates all vectors by the same angle. What is this angle?
- Given two complex numbers a and b , the map $f(z) = az + b$ stretches all vectors by the same factor. What is this factor?
- The most basic conformal maps have the form $f(z) = az + b$. Show that such a map is conformal when $a \neq 0$. When a is not real show that f has a fixed point z_0 satisfying $f(z_0) = z_0$. If $a = 1$ and $b \neq 0$, show that f does not have a fixed point.
- What is the fixed point in Exercise 11 when **a.** $f(z) = 3z + i$, **b.** $f(z) = (1 + i)z + 4 - 2i$?

CHAPTER 18 Questions to Guide Your Review

- Define the set of complex numbers.
- Define, for complex numbers, the concepts of equality, addition, multiplication, and division.
- A set of numbers is said to be *closed* under an operation $\otimes$ if $a \otimes b$ is in the set whenever a and b are. Is the set of complex numbers closed under the operations of addition, subtraction, multiplication, division, and raising to a power (including complex exponents)? (Be careful to consider all complex numbers in formulating your answer, including 0.)
- How may the complex number $a + ib$ be represented graphically in the complex plane?
- Illustrate in the complex plane how the absolute values and arguments of the product and quotient of two complex numbers z_1 and z_2 are related to the absolute values and the arguments of z_1 and z_2 .
- State De Moivre's theorem and explain how to find the n distinct complex n th roots of a nonzero complex number $a + ib$.
- Prove that if n is an even positive integer, then $\cos n\theta$ may be expressed as a polynomial with integer coefficients, in $\cos^2 \theta$. (Example: $\cos 2\theta = 2\cos^2 \theta - 1$.)
- Using vectors in the complex plane, illustrate geometrically how to find the n complex n th roots of any complex number $a + ib$.
- Using the complex plane, illustrate geometrically how the conjugate and the reciprocal of a complex number $a + ib$ are related to that number.
- If z is a complex number such that $z = \bar{z}$, what else can you conclude about z ?
- If z is a complex number such that $z = -\bar{z}$, what else can you conclude about z ?
- Describe the sets of complex numbers z satisfying the following equalities or inequalities, where $z_0 = a + ib$ is a given complex number and k is a positive real constant.
 - $|z - z_0| = k$,
 - $|z - z_0| < k$,
 - $|z - z_0| > k$.
- Define the concept of continuity for a function of a complex variable z . Define the derivative of f at $z = a + ib$.
- What are the Cauchy-Riemann Equations and when are they known to be satisfied?
- Define the convergence of a series of complex numbers.
- How do we define e^z , $\sin z$, $\cos z$, and $\tan z$ for a complex number $z = a + ib$?
- What is a conformal map and what is its relationship to a complex differentiable map?

CHAPTER 18

Additional and Advanced Exercises

1. Let $z = 2 - 2i$.
 - a. Plot the points z , $\bar{z}$, and z^2 .
 - b. Plot the three complex cube roots of z^2 (that is, $z^{2/3}$).
2. Express each of the following complex numbers in the form $re^{i\theta}$ with $r \geq 0$ and $-\pi < \theta \leq \pi$. Sketch
 - a. $(1+i)(1-i\sqrt{3})^2$,
 - b. $\sqrt[3]{2-2i}$ (three answers).
3. Express the four 4th roots of $-16i$ in the form $a + bi$.
4. a. Solve the equation $z^4 + 16 = 0$, obtaining four distinct roots.
 b. Express the five roots of the equation $z^5 + 32 = 0$ in polar form.
5. a. Find all complex numbers z such that $z^4 + 1 + i\sqrt{3} = 0$.
 b. Express the number $e^{2+\pi i/4}$ in the form $a + bi$.
6. Plot the complex number $2 - 2\sqrt{3}i$ in the complex plane and find
 - a. its two square roots,
 - b. the principal value of its logarithm.
7. Find the values of r and θ such that $3 + 4i = re^{i\theta}$.
8. If $z = 3 - 3i$, find all values of $\log z$.
9. Find a complex number $a + ib$ that satisfies the equation

$$e^{a+ib} = 1 - i\sqrt{3}.$$
10. Express the following in the form $a + bi$:
 - a. $(-1 - i)^{1/3}$ (write down all the roots).
 - b. $\log(3 + i\sqrt{3})$. (There are infinitely many possible answers. Give an answer that expresses all of these.)
 - c. $e^{2+\pi i}$.
11. Let $f(z) = \bar{z}$ (the conjugate of z).
 - a. Study the behavior of the quotient

$$\frac{f(z) - f(z_1)}{z - z_1}$$
 when $z \rightarrow z_1$ along straight lines of slope m .
 b. From the results of **part (a)** what can you conclude about the derivative of $f(z)$?
12. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f(\bar{z}) = \overline{f(z)}$, for $|z| < R$, show that the a_n are real.
13. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f(\bar{z}) = f(z)$, for $|z| < R$, show that $f(z)$ is a constant.
14. Show that $\frac{d^n}{d\theta^n}(\cos \theta + i \sin \theta) = i^n(\cos \theta + i \sin \theta)$.
15. Draw the set of points in the complex plane that satisfy
 - a. $\operatorname{Re}(z) > 0$,
 - b. $\operatorname{Im}(z) = 2, (z - i) \leq 0$,
 - c. $\left| \frac{z-i}{z+i} \right| < 1$,
 - d. $|e^z| \geq 1$,
 - e. $|\sin z| \leq 1$.
16. If $u(x, y)$ and $v(x, y)$ are the real and imaginary parts of a complex differentiable function of $z = x + iy$, show that the family of curves $u = \text{constant}$ is orthogonal (perpendicular) to the family $v = \text{constant}$ at every point of intersection where $f'(z) \neq 0$.
17. Given that $z = e^{\operatorname{Log} z}$ and $(e^{z_1})^{z_2} = e^{z_1 z_2}$, how would you compute z^w for complex numbers z and w ? Compute z^w for $z = 1 + i$, $w = 2i$.

ANSWERS TO ODD-NUMBERED EXERCISES

Chapter 18

SECTION 18.1, pp. 18-9-18-10

- (a) $14 + 8i$ (b) $-7 - 4i$ (c) $-5i$
- (a) By reflecting z across the real axis
(b) By reflecting z across the imaginary axis
(c) By reflecting z through the origin, or equivalently, rotating it by 180 degrees.
(d) By reflecting z across the real axis and then multiplying the length of the vector by $1/|z|^2$
- (a) Points on the circle $x^2 + y^2 = 4$
(b) Points inside the circle $x^2 + y^2 = 4$
(c) Points outside the circle $x^2 + y^2 = 4$
- Points on a circle of radius 1, center $(-1, 0)$
- Points on the line $y = -x$ 11. $4e^{2\pi i/3}$ 13. $1e^{2\pi i/3}$
- $\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta$
- $1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ 19. $2i, -\sqrt{3} - i, \sqrt{3} - i$
- $\frac{\sqrt{6}}{2} \pm \frac{\sqrt{2}}{2}i, -\frac{\sqrt{6}}{2} \pm \frac{\sqrt{2}}{2}i$ 23. $1 \pm \sqrt{3}i, -1 \pm \sqrt{3}i$

SECTION 18.2, p. 18-17

- (a) $|w| = 1, 0 \leq \arg w \leq 2\pi$
(b) $|w| = 4, \pi \leq \arg w \leq 2\pi$
(c) $\arg w = \pi/2$
(d) $|w| < 1, -2\pi < \arg w \leq 0$
(e) The u -axis, $u \geq 0$
(f) The u -axis, $u \leq 0$

SECTION 18.3, p. 18-20

- $\frac{-2}{(z_0 - 1)^2} = +2$
- $f'(z_0) = \frac{z_0}{\sqrt{z_0^2 + 1}}$, which gives two values;
 $\frac{1}{\sqrt[3]{2}} \exp\left(\frac{\pi i}{8}\right)$ and $\frac{1}{\sqrt[3]{2}} \exp\left(\frac{9\pi i}{8}\right)$.

SECTION 18.4, p. 18-22

- $u = x^2 - y^2, v = 2xy$
- $u = x^4 - 6x^2y^2 + y^4, v = 4x^3y - 4xy^3$

SECTION 18.5, p. 18-25

- $|z| < 1$ 3. $|z - 1| < 1$ 5. $|z + 1| < 2$
- $0.208 + 0.298i$

SECTION 18.6, pp. 18-30-18-31

- $1.59 + 1.32i$
- (a) $-1 \pm 2i$ (b) $1 \pm i, -1 \pm i$ (c) $2, -1 \pm i\sqrt{3}$
- $\int e^{az} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + C$
 $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + C$
- (a) $\ln 2 + 2n\pi i$
(b) $\ln 2 + (2n + 1)\pi i$
(c) $\ln 2 + (2n + \frac{1}{2})\pi i$
(d) $\ln 2 + (2n - \frac{1}{2})\pi i$
(e) $\ln 2 + (2n + \frac{5}{6})\pi i$,
($n = 0, \pm 1, \pm 2, \dots$)
- (b) $\left(\frac{\pi}{2} + 2n\pi\right) - i \ln(3 + 2\sqrt{2})$

SECTION 18.7, p. 18-35

- No. The angle between the lines is 90° , but the angle between their images is 180° . This is the case in Theorem 3 where $f'(z_0) = 0$.
- All $z \neq 0$ since $f'(z) \neq 0$ except at $z = 0$ and f is analytic. At $z = 0$ angles multiply by 3.
- No. The angle between $y = 0$ and $x = 0$ is $\pi/2$. These two lines are taken by f to $y = x$ and $x = 0$, which intersect at an angle of $\pi/4$.
- (a) 0 (b) $\pi/2$ (c) $\pi/4$ (d) $3\pi/2$ (e) $\pi/2$ (f) π
- $\arg a$ 11. $z_0 = b/(1 - a)$

ADDITIONAL AND ADVANCED EXERCISES, p. 18-36

- (a) $\pm(\sqrt{2 + \sqrt{2}} - i\sqrt{2 - \sqrt{2}})$, $\pm(\sqrt{2 - \sqrt{2}} + i\sqrt{2 + \sqrt{2}})$
(b) $\sin^{-1} 5 = 2n\pi + \pi/2 - i \ln(5 + \sqrt{24})$,
($n = 0, \pm 1, \pm 2, \dots$)
- (a) $2^{1/4} e^{in(2+3n)/6}$ ($n = 0, 1, 2, 3$) (b) $e^2/\sqrt{2} + i(e^2/\sqrt{2})$
- $r = 5, \theta = \tan^{-1} \frac{4}{3}$ 9. $\ln 2 - i\pi/3$
- (b) Derivative doesn't exist.

19

Fourier Series and Wavelets

BIOGRAPHY

Jean-Baptiste Joseph Fourier
(1768–1830)

Fourier was a French mathematician, who as a young boy aspired to become an army officer. He was denied that opportunity for military service and, therefore, switched his passion to mathematics.

To know more, visit the companion Website.

In Section 9.8 we saw that if a function f has derivatives $f^{(k)}$ of order $k = 1, 2, \dots, n$, then we can create a polynomial function P_n that approximates f near a chosen point. This *Taylor polynomial* gives a close fit to f near that particular point, but the error in the approximation can be large at points that are far away. In this chapter we will introduce *Fourier series*, which provide a way to approximate a function over an interval, rather than near a single point. Instead of involving polynomials, a Fourier series approximation is a sum of multiples of the functions $\sin kx$ and $\cos kx$. In a sense that we will describe, a Fourier series approximation minimizes an error that is distributed over an interval. Additionally, while Taylor polynomials cannot be used to approximate discontinuous functions, Fourier series approximations will often be applicable to such functions.

Fourier series, introduced by Joseph Fourier, are well-suited to analyzing periodic functions, such as radio signals and alternating currents. They can also be used to solve heat transfer problems; indeed, this was the problem that motivated Fourier. A large field of mathematics called *harmonic analysis* began with Fourier series. Fourier series and harmonic analysis have applications to electrical engineering, acoustics, optics, signal processing, image processing, quantum mechanics, econometrics, and many other areas in science and engineering.

19.1 Periodic Functions

For some periodic functions, such as the constant function $f(x) = 1$, there is no smallest $P > 0$ such that $f(x + P) = f(x)$ for every x .

Let f be a function whose domain is the real line. We say that f is **periodic** if there is a positive real number P such that $f(x + P) = f(x)$ for every real number x . In Section 1.3 we defined the period of f to be the smallest such number P . In this chapter we will call the smallest P the **fundamental period** of f , and say that any other positive number P that satisfies $f(x + P) = f(x)$ for every x is a **period** for f . We will focus in this chapter on functions with period $P = 2\pi$, but it is not difficult to adapt to other periods.

EXAMPLE 1

- (a) The function $f(x) = \sin x$ is periodic with period 2π because for every real number x we have

$$f(x + 2\pi) = \sin(x + 2\pi) = \sin x = f(x).$$

Likewise, $\cos x$ is periodic with period 2π , and 2π is the fundamental period for both functions.

- (b) Let k be any positive integer. Then the function $g(x) = \sin kx$ has period $2\pi/k$ because if x is any real number, then

$$g\left(x + \frac{2\pi}{k}\right) = \sin\left[k\left(x + \frac{2\pi}{k}\right)\right] = \sin(kx + 2\pi) = \sin kx = g(x).$$

The number $2\pi/k$ is the fundamental period of g , but it is not the only period. Any positive integer multiple of $2\pi/k$ is also a period. For example, 2π is a period for g because for every x we have

$$g(x + 2\pi) = \sin[k(x + 2\pi)] = \sin(kx + 2\pi k) = \sin kx = g(x).$$

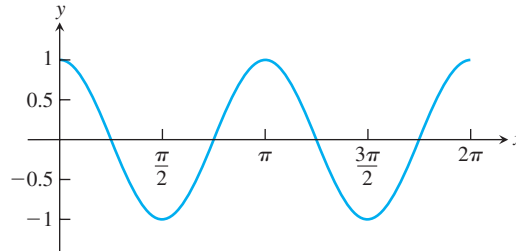
Similarly, $g(x) = \cos kx$ has fundamental period $2\pi/k$, and it is also periodic with period 2π .

- (c) We will let $\mathbf{1}$ denote the constant function that takes the value 1 at every point:

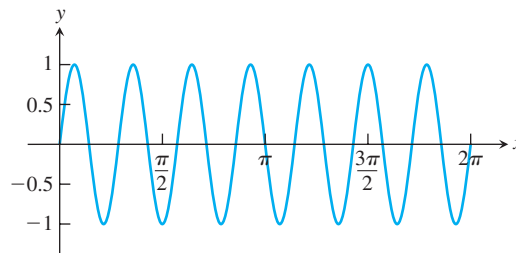
$$\mathbf{1}(x) = 1 \quad \text{for every real number } x.$$

The function $\mathbf{1}$ is periodic, and every positive real number P is a period. There is no smallest period P , so $\mathbf{1}$ does not have a fundamental period. ■

We will use the functions $\sin kx$ and $\cos kx$, along with the constant function $\mathbf{1}$, to define the Fourier series approximation of a function. Each of these functions is periodic with period 2π . The oscillatory nature of these functions is one reason that they are well-suited for forming approximations. We say that $\sin kx$ and $\cos kx$ have **frequency** k , because their graphs show k complete oscillations within the interval $[0, 2\pi]$ (Figure 19.1). Since $\cos 0x = 1 = \mathbf{1}(x)$ for every x , we say that the constant function $\mathbf{1}$ has **frequency zero**. We introduce a name for this set of functions.



(a) Graph of $\cos 2x$ over the interval $[0, 2\pi]$



(b) Graph of $\sin 7x$ over the interval $[0, 2\pi]$

FIGURE 19.1 Two functions in the trigonometric system.

DEFINITION The collection of functions

$$\mathbf{1}, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x, \sin 3x, \dots$$

is called the **trigonometric system**.

EXERCISES 19.1

Periodic Functions

In Exercises 1–6, determine whether the given function is periodic. If it is, find its fundamental period P .

1. $1 + \sin x$
2. x
3. $|x|$
4. $e^{\cos x}$
5. $\sin^2 x \cos x$
6. $\sin x^2$

Theory and Examples

7. Show that if $f(x)$ and $g(x)$ are both periodic with the same period P , then the following functions are also periodic with period P .

- a. $f(x) + g(x)$
- b. $cf(x)$, where c is a real number

c. $f(x)g(x)$

d. $e^{f(x)}$

8. **Periodic extension** Suppose that the values of $f(x)$ are specified for $0 \leq x < 2\pi$. Then for each $0 \leq x < 2\pi$ and every nonzero integer k , define $f(x + 2k\pi) = f(x)$. Show that this extends the definition of f to the entire real line, and that the resulting function is periodic with period 2π .

19.2 Summing Sines and Cosines

We discussed *Taylor series* and *Taylor polynomials* in Section 9.8. If a function f has derivatives of order k for $k = 1, 2, \dots, n$ then the **Taylor polynomial of order n** at $x = 0$ is

$$P_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \quad \text{where } a_k = \frac{f^{(k)}(0)}{k!}.$$

By choosing the coefficients a_k in this way we obtain a polynomial P_n that approximates f near the point $x = 0$.

Taylor series are based on the functions $1, x, x^2, x^3, \dots$, while Fourier series are instead based on sines and cosines. Just as a Taylor polynomial is a combination of finitely many of the functions $1, x, x^2, x^3, \dots$, we seek a combination of finitely many elements of the trigonometric system that will approximate a function. We refer to such a finite combination of sines and cosines as a **trigonometric polynomial**.

EXAMPLE 1 Define a periodic function f by letting its values on the interval $[0, 2\pi]$ be

$$f(x) = \begin{cases} x - \pi, & 0 \leq x < 2\pi, \\ -\pi, & x = 2\pi, \end{cases} \quad (1)$$

and then let those values repeat every 2π units (Figure 19.2). This gives us a function f that is periodic on the real line with period 2π . Observe that f has a jump discontinuity at $x = 2\pi k$ for every integer k .

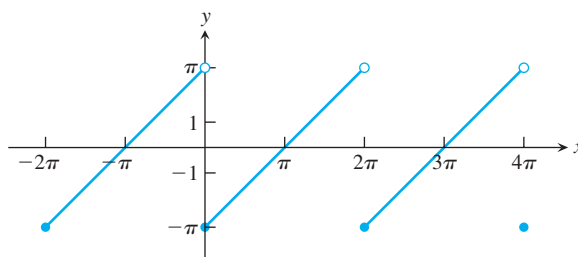


FIGURE 19.2 Graph of the periodic function f defined in Equation (1) over the interval $[-2\pi, 4\pi]$

We would like to approximate f by a trigonometric polynomial, which is a sum of finitely many multiples of the functions in the trigonometric system (the real number that we multiply $\mathbf{1}$ or $\sin kx$ or $\cos kx$ by is called the **amplitude** of that term). We cannot do a very good job using only one function from the trigonometric system (see Figure 19.3a, which compares the graphs of f and $-2 \sin x$). However, we can make a better approximation by using more elements of the trigonometric system. Figure 19.3b shows a trigonometric polynomial that is a sum of two scaled sines, while Figure 19.3c shows that we can do a better approximation using a sum of three scaled sines. As seen in Figure 19.3d, a certain combination of 9 terms gives us a trigonometric polynomial that appears to be a good approximation over most of the domain, though still not close at points near integer multiples of 2π . If we use yet more terms, we can improve the approximation. In particular, the trigonometric polynomial

$$p_n(x) = -2 \sin x - \sin 2x - \frac{2}{3} \sin 3x - \cdots - \frac{2}{n} \sin nx \quad (2)$$

is a good approximation to f using a sum of n sines. We will explain below how we knew what amplitudes to choose for the sines in this sum, and why no cosine terms are used. ■

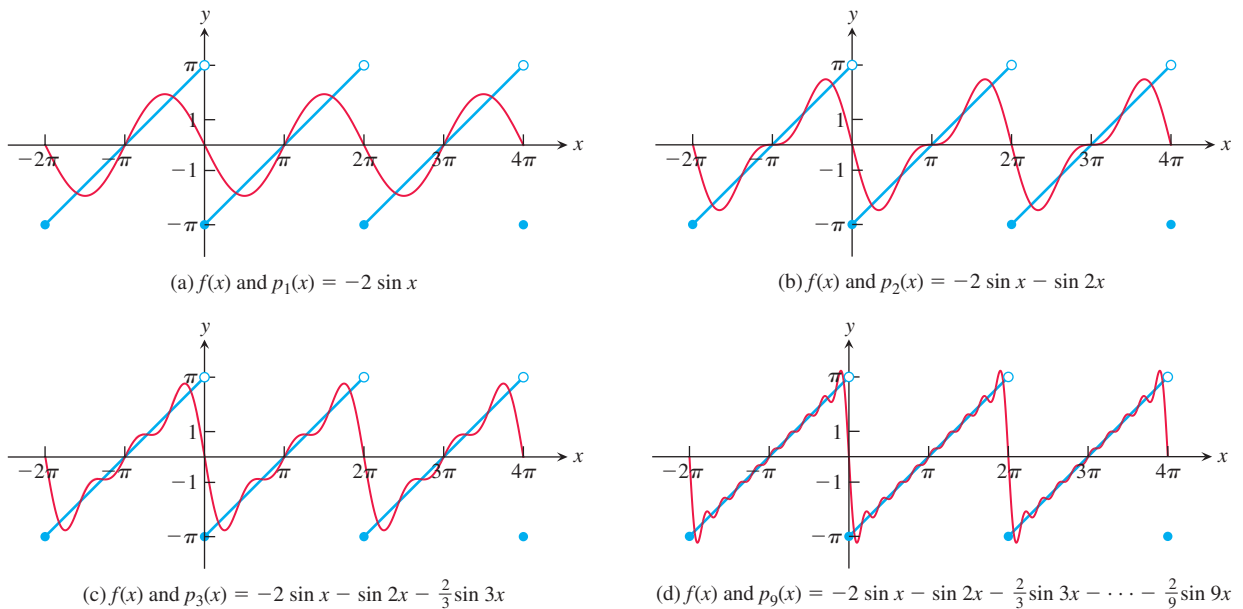


FIGURE 19.3 Trigonometric polynomial approximations to the function f defined in Equation (1) over the interval $[-2\pi, 4\pi]$, using sums of scaled sines.

We see in Example 1 that we can closely approximate a particular function over most of the interval $[0, 2\pi]$ by using sines and cosines from the trigonometric system (for that function we only needed sines). We would like to approximate other functions in a similar way. Given a function f that has period 2π , we seek a function

$$p_n(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx \quad (3)$$

that approximates f across the interval $[0, 2\pi]$. We refer to p_n as a **trigonometric polynomial** (Exercises 5–18 will explain why we use the word “polynomial” in this context). We say that a_k is the amplitude of the term $a_k \cos kx$, and similarly b_k is the amplitude of the term $b_k \sin kx$. Since $a_0 = a_0 \cos 0x$, this number is the amplitude of the zero frequency term. Recalling the constant function $\mathbf{1}$, which satisfies $\mathbf{1}(x) = 1$ for every x , we have $a_0 = a_0 \mathbf{1}(x)$. Thus we also say that a_0 is the amplitude of the constant term in the sum in Equation (3).

We can write Equation (3) more succinctly using sigma notation:

$$p_n(x) = a_0 + \sum_{k=1}^n a_k \cos kx + \sum_{k=1}^n b_k \sin kx. \quad (4)$$

We will show how to choose the amplitudes a_k and b_k so that p_n is the “best approximation” to f that is a sum of constant multiples of the constant function, the n cosines $\cos x$, $\cos 2x$, ..., $\cos nx$ and the n sines $\sin x$, $\sin 2x$, ..., $\sin nx$. To do this, we will demonstrate how to approximate vectors in three and higher dimensions in Section 19.3, and then show in Section 19.4 that ideas that work for approximating vectors by other vectors also work for functions.

EXERCISES 19.2

Theory and Examples

1. Show that a function of the form $p_n(x) = \sum_{k=1}^n b_k \sin kx$ is odd, that is, $p_n(-x) = -p_n(x)$ for every x .
2. Show that a function of the form $p_n(x) = a_0 + \sum_{k=1}^n a_k \cos kx$ is even, that is, $p_n(-x) = p_n(x)$ for every x .

Sums of Sines and Cosines

In Exercises 3 and 4, the values of a_0 , a_k , and b_k that define a trigonometric polynomial $p_n(x) = a_0 + \sum_{k=1}^n a_k \cos kx + \sum_{k=1}^n b_k \sin kx$ are given. Use a CAS to plot p_n for various values of n over the domain $[-2\pi, 4\pi]$. Does it appear that $p_n(x)$ is converging to some limit function $f(x)$ as n increases? If so, what function? Is the convergence “fast” (the limit is apparent even for small values of n) or “slow” (you must use a large value of n before the graph of p_n appears to be close to the graph of f)?

- T** 3. $a_0 = \frac{\pi}{2}$, $a_{2k-1} = \frac{4}{(2k-1)^2\pi}$, $a_{2k} = 0$, $b_k = 0$
- T** 4. $a_0 = \frac{1}{2}$, $a_{2k-1} = (-1)^k \frac{2}{(2k-1)\pi}$, $a_{2k} = 0$, $b_k = 0$

Complex Numbers and Trigonometric Polynomials

The exercises below illustrate the connection between trigonometric polynomials and complex polynomials in the variable $z = e^{ix}$. Some knowledge of complex numbers is required. In particular, we will use **Euler's Formula** $e^{ix} = \cos x + i \sin x$ and **De Moivre's Theorem**,

$$e^{inx} = (\cos x + i \sin x)^n = \cos nx + i \sin nx, \quad n \geq 0,$$

which are derived in Chapter 18, and we will use the fact that $(z^m)^n = z^{mn}$. These exercises will show that trigonometric polynomials equal “generalized” polynomials

$$a_n z^n + \cdots + a_1 z + a_0 + a_{-1} z^{-1} + \cdots + a_{-n} z^{-n}$$

that use both positive and negative powers of $z = e^{ix}$ with complex numbers a_k .

Prove the identities stated in Exercises 5–18. We assume that $z = e^{ix}$ where x is a real number.

5. $z^0 = 1$
6. $z + 1 + i = (1 + \cos x) + i(1 + \sin x)$
7. $2z^2 + 3z - 5 = (2 \cos 2x + 3 \cos x - 5) + i(2 \sin 2x + 3 \sin x)$
8. $\sum_{k=0}^n a_k z^k = a_0 + \sum_{k=0}^n a_k \cos kx + i \sum_{k=0}^n a_k \sin kx$
9. $z^{-1} = e^{-ix} = \cos x - i \sin x$
10. $z^{-n} = e^{-inx} = \cos nx - i \sin nx, \quad n \geq 0$
11. $z + z^{-1} = 2 \cos x$
12. $z - z^{-1} = 2i \sin x$
13. $z^n + z^{-n} = 2 \cos nx, \quad n \geq 0$
14. $z^n - z^{-n} = 2i \sin nx, \quad n \geq 0$
15. $2z^2 + 3z - 5 + 3z^{-1} + 2z^{-2} = 4 \cos 2x + 6 \cos x - 5$
16. $4iz^2 - 2iz + 2iz^{-1} - 4iz^{-2} = -8 \sin 2x + 4 \sin x$
17. $\cos x + 2 \sin x - 3 = (0.5 - i)z - 3 + (0.5 + i)z^{-1}$
18. $2 \cos 3x - 6 \cos 2x + 8 \sin 3x - 5 = (1 - 4i)z^3 - 3z^2 - 5 - 3z^{-2} + (1 + 4i)z^{-3}$

19.3 Vectors and Approximation in Three and More Dimensions

Vectors in two and three dimensions were introduced in Chapter 11. If $\mathbf{v}$ is a three-dimensional vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3) , then the *component form* of $\mathbf{v}$ is

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle.$$

Vectors in Three Dimensions

We review some definitions and facts about vectors in three dimensions.

DEFINITION

1. The **length** of a vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

If $|\mathbf{v}| = 1$, then we say that $\mathbf{v}$ is a **unit vector**.

2. The **distance** between two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$|\mathbf{u} - \mathbf{v}| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}.$$

Observe that $|\mathbf{u} - \mathbf{v}| = |\mathbf{v} - \mathbf{u}|$.

DEFINITION The **dot product** of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is the number

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

An important fact is that the length squared of a vector equals the dot product of the vector with itself:

$$|\mathbf{v}|^2 = v_1^2 + v_2^2 + v_3^2 = \mathbf{v} \cdot \mathbf{v}.$$

Some other properties of dot products and lengths are laid out in Exercises 21–30.

Theorem 1 in Section 11.3 shows that the dot product of two nonzero vectors satisfies

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. If $\mathbf{u}$ and $\mathbf{v}$ are perpendicular then $\theta = \pi/2$ and therefore $\mathbf{u} \cdot \mathbf{v} = 0$. The converse is also true. That is, if $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors with $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = 0$, then $\cos\theta = 0$ and therefore $\theta = \arccos 0 = \pi/2$. The following definition also allows for one or both of the vectors to be the zero vector.

DEFINITION Vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$. They are **orthonormal** if $\mathbf{u} \cdot \mathbf{v} = 0$ and additionally $|\mathbf{u}| = |\mathbf{v}| = 1$.

Often we deal with more than two vectors. We say that three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are orthogonal if the dot product of any two of them is zero: $\mathbf{u} \cdot \mathbf{v} = 0$, $\mathbf{u} \cdot \mathbf{w} = 0$, and $\mathbf{v} \cdot \mathbf{w} = 0$. If in addition each of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is a unit vector, then we say that the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are orthonormal. Sometimes we will use the longer but more precise phrasing that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is an *orthogonal set of vectors* (or *orthonormal set of vectors*). In three-dimensional space, it is not possible to have a set of more than three orthonormal vectors, but we will see below that it is possible to have larger collections in higher-dimensional space.

We will derive some formulas that relate lengths, distances, and dot products. Statement 3 of the next result is the **Pythagorean Theorem** for vectors. Exercise 25 shows that the converse of the Pythagorean Theorem holds as well.

THEOREM 1 Let $\mathbf{u}$ and $\mathbf{v}$ be three-dimensional vectors.

1. $|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2$.
2. $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2$.
3. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$.

Proof Statement 1 follows because

$$\begin{aligned}
 |\mathbf{u} - \mathbf{v}|^2 &= (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 \\
 &= u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2 + u_3^2 - 2u_3v_3 + v_3^2 \\
 &= (u_1^2 + u_2^2 + u_3^2) - 2(u_1v_1 + u_2v_2 + u_3v_3) + (v_1^2 + v_2^2 + v_3^2) \\
 &= |\mathbf{u}|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2,
 \end{aligned}$$

and the proof of Statement 2 is similar. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal then $\mathbf{u} \cdot \mathbf{v} = 0$, so Statement 3 follows from Statement 2. ■

We need one additional definition.

DEFINITION A **linear combination** of finitely many vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ is a sum where we scale each vector $\mathbf{u}_k$ by a real number a_k and add those terms together, giving the vector

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n.$$

EXAMPLE 1 The *standard unit vectors* in three-dimensional space were introduced in Section 11.2. They are

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

We can form linear combinations of one, two, or three of the standard unit vectors by multiplying the vectors by real numbers and adding them together. For example, using two of the vectors we can create linear combinations such as

$$\begin{aligned}
 2\mathbf{i} + 4\mathbf{j} &= \langle 2, 0, 0 \rangle + \langle 0, 4, 0 \rangle = \langle 2, 4, 0 \rangle, \\
 5\mathbf{i} - 7\mathbf{k} &= \langle 5, 0, 0 \rangle - \langle 0, 0, 7 \rangle = \langle 5, 0, -7 \rangle, \\
 -3\mathbf{j} - 2\mathbf{k} &= -\langle 0, 3, 0 \rangle - \langle 0, 0, 2 \rangle = \langle 0, -3, -2 \rangle.
 \end{aligned}$$

Some linear combinations of all three of the standard unit vectors are

$$\begin{aligned}
 5\mathbf{i} + 2\mathbf{j} - 7\mathbf{k} &= \langle 5, 0, 0 \rangle + \langle 0, 2, 0 \rangle - \langle 0, 0, 7 \rangle = \langle 5, 2, -7 \rangle, \\
 \pi\mathbf{i} + 6\mathbf{j} - \mathbf{k} &= \langle \pi, 0, 0 \rangle + \langle 0, 6, 0 \rangle - \langle 0, 0, 1 \rangle = \langle \pi, 6, -1 \rangle, \\
 \mathbf{i} - \sqrt{2}\mathbf{j} - 2\mathbf{k} &= \langle 1, 0, 0 \rangle - \langle 0, \sqrt{2}, 0 \rangle - \langle 0, 0, 2 \rangle = \langle 1, -\sqrt{2}, -2 \rangle.
 \end{aligned}$$

To create a linear combination of one vector, we simply multiply that vector by a real number:

$$\begin{aligned}
 3\mathbf{i} &= \langle 3, 0, 0 \rangle, \\
 \sqrt{5}\mathbf{j} &= \langle 0, \sqrt{5}, 0 \rangle, \\
 -9\mathbf{k} &= \langle 0, 0, -9 \rangle.
 \end{aligned}$$

We are not restricted to using just three vectors. For example, one linear combination of $\mathbf{u} = \langle 1, -1, 0 \rangle$, $\mathbf{v} = \langle 2, 0, 2 \rangle$, $\mathbf{w} = \langle 3, 1, 2 \rangle$, and $\mathbf{y} = \langle 0, 1, -3 \rangle$ is

$$2\mathbf{u} + 3\mathbf{v} - \mathbf{w} - 3\mathbf{y} = \langle 2, -2, 0 \rangle + \langle 6, 0, 6 \rangle - \langle 3, 1, 2 \rangle - \langle 0, 3, -9 \rangle = \langle 5, -6, 13 \rangle. \quad \blacksquare$$

Orthonormal Bases in Three Dimensions

Any vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ can be written as a linear combination of the standard unit vectors:

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\langle 1, 0, 0 \rangle + v_2\langle 0, 1, 0 \rangle + v_3\langle 0, 0, 1 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

The dot product between any two of the standard basis vectors is zero: $\mathbf{i} \cdot \mathbf{j} = 0$, $\mathbf{i} \cdot \mathbf{k} = 0$, and $\mathbf{j} \cdot \mathbf{k} = 0$. Further, each of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ is a unit vector. Therefore, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are orthonormal vectors.

Now choose any other orthonormal vectors $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$. For example, we might take the standard unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and rotate them in space around some axis that passes through the origin to obtain new vectors $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$. The lengths of the vectors and the angles between them remain unchanged under a rotation, so this process gives us orthonormal vectors. However we obtain them, if we have three orthonormal vectors $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$, then we can write any vector $\mathbf{v}$ as a linear combination of them (this is proved in texts on linear algebra). That is, if we choose a vector $\mathbf{v}$, then there must be real numbers c_1 , c_2 , and c_3 so that

$$\mathbf{v} = c_1\mathbf{r} + c_2\mathbf{s} + c_3\mathbf{t}.$$

We now determine these coefficients. Taking the dot product of $\mathbf{v}$ with $\mathbf{r}$ gives us

$$\begin{aligned} \mathbf{v} \cdot \mathbf{r} &= (c_1\mathbf{r} + c_2\mathbf{s} + c_3\mathbf{t}) \cdot \mathbf{r} & \mathbf{v} &= c_1\mathbf{r} + c_2\mathbf{s} + c_3\mathbf{t} \\ &= c_1(\mathbf{r} \cdot \mathbf{r}) + c_2(\mathbf{s} \cdot \mathbf{r}) + c_3(\mathbf{t} \cdot \mathbf{r}) & \text{distributive property of the dot product} \\ &= c_1|\mathbf{r}|^2 + 0 + 0 & \mathbf{r} \cdot \mathbf{r} = |\mathbf{r}|^2, \mathbf{s} \cdot \mathbf{r} = 0, \mathbf{t} \cdot \mathbf{r} = 0 \\ &= c_1. & \mathbf{r} \text{ has unit length.} \end{aligned}$$

A similar calculation shows that $c_2 = \mathbf{v} \cdot \mathbf{s}$ and $c_3 = \mathbf{v} \cdot \mathbf{t}$. We state these facts as a theorem.

THEOREM 2 Let $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$ be orthonormal vectors in three-dimensional space. Then every vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ can be written as

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{r})\mathbf{r} + (\mathbf{v} \cdot \mathbf{s})\mathbf{s} + (\mathbf{v} \cdot \mathbf{t})\mathbf{t}.$$

Using the standard unit vectors, we can write $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ as $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Theorem 2 tells us that if $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$ are three orthonormal vectors, then $\mathbf{v}$ is also a linear combination of $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{t}$, namely $\mathbf{v} = c_1\mathbf{r} + c_2\mathbf{s} + c_3\mathbf{t}$ where $c_1 = \mathbf{v} \cdot \mathbf{r}$, $c_2 = \mathbf{v} \cdot \mathbf{s}$ and $c_3 = \mathbf{v} \cdot \mathbf{t}$. Because every vector $\mathbf{v}$ can be written as a unique linear combination of $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{t}$, this collection of vectors is called a **basis**. Since the vectors are also orthonormal, we use the following name.

DEFINITION A set of three orthonormal vectors $\{\mathbf{r}, \mathbf{s}, \mathbf{t}\}$ in three-dimensional space is called an **orthonormal basis** for three-dimensional space.

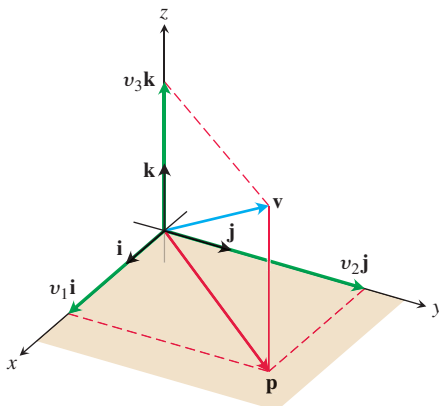


FIGURE 19.4 The vector in the xy -plane that is closest to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is $\mathbf{p} = v_1\mathbf{i} + v_2\mathbf{j}$.

Approximation Using an Orthonormal Basis in Three Dimensions

Suppose that $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is a vector. Out of all the vectors $\mathbf{p} = \langle p_1, p_2, 0 \rangle$ in the xy -plane, which one is closest to $\mathbf{v}$? It is the vector $\mathbf{p}$ that we obtain by dropping $\mathbf{v}$ straight down onto the xy -plane. That is, $\mathbf{p} = \langle v_1, v_2, 0 \rangle$ is the vector in the xy -plane that is closest to $\mathbf{v}$ (Figure 19.4). Expressing this in terms of linear combinations, if we choose any vector $\mathbf{v}$ and write it as $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then the vector in the xy -plane that is closest to $\mathbf{v}$ is $\mathbf{p} = v_1\mathbf{i} + v_2\mathbf{j}$. The distance from $\mathbf{v}$ to the xy -plane is the length of $\mathbf{v} - \mathbf{p} = v_3\mathbf{k}$, which is $|v_3|$.

We saw in Section 5 of Chapter 11 how to find the distance from a point to a generic plane in space. Now we formulate that discussion in terms of orthonormal vectors and best approximation. Suppose that $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$ are three orthonormal vectors in three-dimensional space. If we fix a vector $\mathbf{v}$, then Theorem 2 tells us that $\mathbf{v} = (\mathbf{v} \cdot \mathbf{r})\mathbf{r} + (\mathbf{v} \cdot \mathbf{s})\mathbf{s} + (\mathbf{v} \cdot \mathbf{t})\mathbf{t}$. How closely can we approximate $\mathbf{v}$ by a linear combination of just $\mathbf{r}$ and $\mathbf{s}$? The set of all possible linear combinations $a_1\mathbf{r} + a_2\mathbf{s}$ forms a plane in three-dimensional space, which

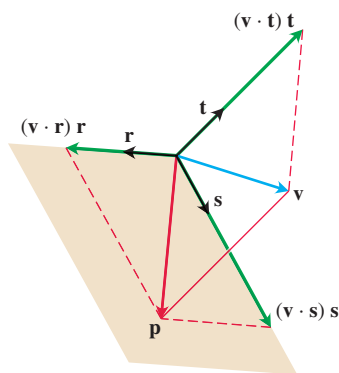


FIGURE 19.5 The vector in the plane determined by orthonormal vectors $\mathbf{r}$ and $\mathbf{s}$ that is closest to $\mathbf{v}$ is $\mathbf{p} = (\mathbf{v} \cdot \mathbf{r})\mathbf{r} + (\mathbf{v} \cdot \mathbf{s})\mathbf{s}$.

we call the plane **determined by** or **spanned by** $\mathbf{r}$ and $\mathbf{s}$. We seek to find the vector $\mathbf{p}$ in this plane that minimizes the error between $\mathbf{v}$ and $\mathbf{p}$. This error is the distance between $\mathbf{v}$ and $\mathbf{p}$, so we want to find the vector $\mathbf{p}$ in the plane that is closest to $\mathbf{v}$. We saw above how to solve this *optimization problem* for the case where our orthonormal vectors are the standard unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. A similar approach works here. As shown in Figure 19.5, we obtain the closest point by dropping $\mathbf{v}$ down perpendicularly onto the plane determined by $\mathbf{r}$ and $\mathbf{s}$, landing on the point $\mathbf{p} = (\mathbf{v} \cdot \mathbf{r})\mathbf{r} + (\mathbf{v} \cdot \mathbf{s})\mathbf{s}$. If the vector $\mathbf{v}$ happens to actually be in the plane determined by $\mathbf{r}$ and $\mathbf{s}$, then we will have $\mathbf{p} = \mathbf{v}$.

THEOREM 3 If $\{\mathbf{r}, \mathbf{s}, \mathbf{t}\}$ is an orthonormal basis for three-dimensional space and $\mathbf{v}$ is a vector, then the linear combination of $\mathbf{r}$ and $\mathbf{s}$ that is closest to $\mathbf{v}$ is

$$\mathbf{p} = (\mathbf{v} \cdot \mathbf{r})\mathbf{r} + (\mathbf{v} \cdot \mathbf{s})\mathbf{s}. \quad (5)$$

The vector $\mathbf{v} - \mathbf{p}$ is orthogonal to both $\mathbf{r}$ and $\mathbf{s}$.

Proof Let $\mathbf{v}$ be a vector and let $\mathbf{p}$ be given by Equation (5). Consider any linear combination $\mathbf{w} = a_1\mathbf{r} + a_2\mathbf{s}$ of $\mathbf{r}$ and $\mathbf{s}$. We want to show that the distance from $\mathbf{v}$ to $\mathbf{p}$ is less than or equal to the distance from $\mathbf{v}$ to $\mathbf{w}$ (less than or equal to because $\mathbf{w}$ could possibly equal $\mathbf{p}$). That is, we must prove that $|\mathbf{v} - \mathbf{p}| \leq |\mathbf{v} - \mathbf{w}|$.

Observe that $\mathbf{v} - \mathbf{w} = (\mathbf{v} - \mathbf{p}) + (\mathbf{p} - \mathbf{w})$. Since $\mathbf{p}$ and $\mathbf{w}$ are each linear combinations of $\mathbf{r}$ and $\mathbf{s}$, their difference $\mathbf{p} - \mathbf{w}$ is also a linear combination of $\mathbf{r}$ and $\mathbf{s}$. Therefore, since $\mathbf{r}$ and $\mathbf{s}$ are each orthogonal to $\mathbf{t}$, the vector $\mathbf{p} - \mathbf{w}$ is orthogonal to $\mathbf{t}$, and hence it is also orthogonal to $\mathbf{v} - \mathbf{p} = (\mathbf{v} \cdot \mathbf{t})\mathbf{t}$. Consequently,

$$\begin{aligned} |\mathbf{v} - \mathbf{w}|^2 &= |(\mathbf{v} - \mathbf{p}) + (\mathbf{p} - \mathbf{w})|^2 && \text{Add and subtract.} \\ &= |\mathbf{v} - \mathbf{p}|^2 + |\mathbf{p} - \mathbf{w}|^2 && \text{Pythagorean Theorem} \\ &\geq |\mathbf{v} - \mathbf{p}|^2. \end{aligned}$$

Taking square roots gives $|\mathbf{v} - \mathbf{p}| \leq |\mathbf{v} - \mathbf{w}|$. This shows that $\mathbf{p}$ is the linear combination of $\mathbf{r}$ and $\mathbf{s}$ that is closest to $\mathbf{v}$. Finally, $\mathbf{v} - \mathbf{p} = (\mathbf{v} \cdot \mathbf{t})\mathbf{t}$ is a multiple of $\mathbf{t}$, which is orthogonal to both $\mathbf{r}$ and $\mathbf{s}$. ■

Vectors in Higher Dimensions

Written in component form, a two-dimensional vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is an ordered pair of real numbers, while a three-dimensional vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is an ordered triple of real numbers. Although we cannot easily visualize vectors in higher dimensions, we can still define and understand them. A vector in four dimensions is an ordered quadruple $\mathbf{v} = \langle v_1, v_2, v_3, v_4 \rangle$ of real numbers. The length of such a vector is

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2}.$$

There are four axis directions, and the number v_k is the component of $\mathbf{v}$ in the direction of the k th axis.

Higher-dimensional vectors are similar. If d is a positive integer, then a d -dimensional vector is an ordered list $\mathbf{v} = \langle v_1, v_2, \dots, v_d \rangle$ of d real numbers (also called a d -tuple of numbers). Its length is

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + \dots + v_d^2} = \sqrt{\sum_{k=1}^d v_k^2}. \quad (6)$$

The distance between two d -dimensional vectors $\mathbf{u}$ and $\mathbf{v}$ is $|\mathbf{u} - \mathbf{v}|$, the length of the difference of $\mathbf{u}$ and $\mathbf{v}$. The dot product of $\mathbf{u} = \langle u_1, u_2, \dots, u_d \rangle$ with $\mathbf{v} = \langle v_1, v_2, \dots, v_d \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \dots + u_dv_d = \sum_{k=1}^d u_kv_k.$$

The length of a vector is related to the dot product of the vector with itself by the formula

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \cdots + v_d^2 = |\mathbf{v}|^2.$$

Vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$. In this case, the Pythagorean Theorem holds (Exercise 24):

$$\text{If } \mathbf{u} \cdot \mathbf{v} = 0, \text{ then } |\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2.$$

The converse of the Pythagorean Theorem also holds (Exercise 25).

As a reminder, dot products are defined in any dimension, but cross products are special to dimension 3.

Orthonormal Bases and Approximation in Higher Dimensions

The standard unit vectors in d dimensions are the d vectors

$$\begin{aligned} \mathbf{e}_1 &= \langle 1, 0, 0, \dots, 0 \rangle, \\ \mathbf{e}_2 &= \langle 0, 1, 0, \dots, 0 \rangle, \\ &\vdots \\ \mathbf{e}_d &= \langle 0, 0, 0, \dots, 1 \rangle. \end{aligned}$$

The k th vector $\mathbf{e}_k$ has a 1 as its k th component, while all other components are zero. The set of unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d\}$ is orthonormal, and additionally every vector $\mathbf{v}$ in d -dimensional space is a linear combination of these vectors. Therefore, we say that $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d\}$ forms an orthonormal basis for d -dimensional space.

Likewise, any collection of d orthonormal vectors is an orthonormal basis for d -dimensional space, and we have the following explicit formula for writing any vector as a linear combination of the basis vectors (this theorem is proved in texts on linear algebra). The coefficients in this representation are dot products of $\mathbf{v}$ with the vectors $\mathbf{u}_k$.

THEOREM 4 If $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_d$ are d orthonormal vectors in d -dimensional space (so $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_d\}$ is an orthonormal basis for d -dimensional space), then every vector $\mathbf{v} = \langle v_1, v_2, \dots, v_d \rangle$ can be written as

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{u}_1) \mathbf{u}_1 + (\mathbf{v} \cdot \mathbf{u}_2) \mathbf{u}_2 + \cdots + (\mathbf{v} \cdot \mathbf{u}_d) \mathbf{u}_d = \sum_{k=1}^d (\mathbf{v} \cdot \mathbf{u}_k) \mathbf{u}_k.$$

Now fix an integer $1 \leq n < d$. If $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_d\}$ is an orthonormal basis and $\mathbf{v}$ is a vector in d dimensions, how close can we get to $\mathbf{v}$ using a linear combination of only the first n vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$? This type of optimization problem occurs widely throughout mathematics, science, and engineering. The answer (which is often called the *least squares solution*) is given in the next theorem. The proof is similar to the proof of Theorem 3 (Exercise 31).

THEOREM 5 Assume that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_d\}$ is an orthonormal basis for d -dimensional space. If $\mathbf{v}$ is a vector and $1 \leq n < d$, then the linear combination of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ that is closest to $\mathbf{v}$ is

$$\mathbf{p} = (\mathbf{v} \cdot \mathbf{u}_1) \mathbf{u}_1 + (\mathbf{v} \cdot \mathbf{u}_2) \mathbf{u}_2 + \cdots + (\mathbf{v} \cdot \mathbf{u}_n) \mathbf{u}_n = \sum_{k=1}^n (\mathbf{v} \cdot \mathbf{u}_k) \mathbf{u}_k. \quad (7)$$

The vector $\mathbf{v} - \mathbf{p}$ is orthogonal to $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.

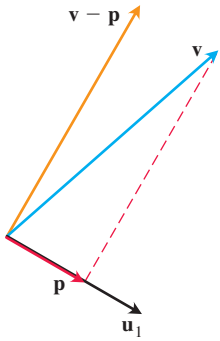


FIGURE 19.6 The vector on the line through $\mathbf{u}_1$ that is closest to $\mathbf{v}$ is $\mathbf{p} = (\mathbf{v} \cdot \mathbf{u}_1) \mathbf{u}_1$. The vector $\mathbf{v} - \mathbf{p}$ is orthogonal to $\mathbf{u}_1$.

The set of all possible linear combinations $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$ of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ is called the **subspace** spanned by $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. The vector $\mathbf{p}$ given in Equation (7) is the vector in this subspace that is closest to $\mathbf{v}$. It is the vector in that subspace that *minimizes the distance* between $\mathbf{v}$ and the vectors in the subspace, and it is called the **orthogonal projection of $\mathbf{v}$ onto the span of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$** . Geometrically, Theorem 5 tells us that to find the closest vector $\mathbf{p}$, we should simply drop down perpendicularly from $\mathbf{v}$ onto the subspace spanned by $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.

EXAMPLE 2 Assume that $\mathbf{u}_1$ is a unit vector in d -dimensional space. A linear combination of the single vector $\mathbf{u}_1$ is simply a multiple $a\mathbf{u}_1$, where a is a real number. Therefore, the subspace spanned by $\mathbf{u}_1$ is the set of all multiples $a\mathbf{u}_1$ of $\mathbf{u}_1$. We also call this the **line through $\mathbf{u}_1$** . If $\mathbf{v}$ is a vector in d -dimensional space, what is the vector $\mathbf{p}$ on the line through $\mathbf{u}_1$ that is closest to $\mathbf{v}$?

Solution We will apply Theorem 5 with $n = 1$. By results from linear algebra, starting with a unit vector $\mathbf{u}_1$ we can find vectors $\mathbf{u}_2, \dots, \mathbf{u}_d$ so that $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_d$ forms an orthonormal basis for d -dimensional space. Then, by Equation (7) with $n = 1$, the vector on the line through $\mathbf{u}_1$ that is closest to $\mathbf{v}$ is

$$\mathbf{p} = (\mathbf{v} \cdot \mathbf{u}_1) \mathbf{u}_1.$$

This agrees with the formula that we found in Section 11.3 for vectors in two- or three-dimensional space, where we denoted the vector $\mathbf{p}$ by $\text{proj}_{\mathbf{u}_1} \mathbf{v}$ (compare Figure 19.6 to Figure 11.27).

EXERCISES 19.3

Orthogonal Vectors in Three Dimensions

In Exercises 1–6, find

- the dot products $\mathbf{r} \cdot \mathbf{s}$, $\mathbf{r} \cdot \mathbf{t}$, and $\mathbf{s} \cdot \mathbf{t}$,
- the lengths $|\mathbf{r}|$, $|\mathbf{s}|$, $|\mathbf{t}|$.

Also determine whether $\{\mathbf{r}, \mathbf{s}, \mathbf{t}\}$ is

- an orthogonal set of vectors,
- an orthonormal set of vectors.

- $\mathbf{r} = \langle 1, 1, 0 \rangle$, $\mathbf{s} = \langle 1, -1, 0 \rangle$, $\mathbf{t} = \langle 0, 0, 1 \rangle$.
- $\mathbf{r} = \langle 1/\sqrt{2}, 1/\sqrt{2}, 0 \rangle$, $\mathbf{s} = \langle 1/\sqrt{2}, -1/\sqrt{2}, 0 \rangle$, $\mathbf{t} = \langle 0, 0, 1 \rangle$.
- $\mathbf{r} = \langle 1, 1, 1 \rangle$, $\mathbf{s} = \langle 1, 1, -2 \rangle$, $\mathbf{t} = \langle 0, -1, 1 \rangle$.
- $\mathbf{r} = \langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3} \rangle$, $\mathbf{s} = \langle 1/\sqrt{6}, 1/\sqrt{6}, -\sqrt{2}/\sqrt{3} \rangle$, $\mathbf{t} = \langle 0, -1/\sqrt{2}, 1/\sqrt{2} \rangle$.
- $\mathbf{r} = \langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3} \rangle$, $\mathbf{s} = \langle 1/\sqrt{6}, 1/\sqrt{6}, -\sqrt{2}/\sqrt{3} \rangle$, $\mathbf{t} = \langle 1/\sqrt{2}, -1/\sqrt{2}, 0 \rangle$.
- $\mathbf{r} = \langle 2/3, -2/3, 1/3 \rangle$, $\mathbf{s} = \langle 1/\sqrt{2}, 1/\sqrt{2}, 0 \rangle$, $\mathbf{t} = \langle -1/(3\sqrt{2}), 1/(3\sqrt{2}), 2\sqrt{2}/3 \rangle$.

Closest Point in a Plane

In Exercises 7–10,

- verify that $\{\mathbf{r}, \mathbf{s}, \mathbf{t}\}$ is an orthonormal basis for three-dimensional space,
- find the linear combination $\mathbf{p}$ of $\mathbf{r}$ and $\mathbf{s}$ that is closest to $\mathbf{v}$,
- find the distance from $\mathbf{v}$ to $\mathbf{p}$.

- $\mathbf{r} = \langle 1, 0, 0 \rangle$, $\mathbf{s} = \langle 0, -1, 0 \rangle$, $\mathbf{t} = \langle 0, 0, -1 \rangle$, $\mathbf{v} = \langle 1, 2, 3 \rangle$.
- $\mathbf{r} = \langle 2/3, -2/3, -1/3 \rangle$, $\mathbf{s} = \langle 1/3, 2/3, -2/3 \rangle$, $\mathbf{t} = \langle 2/3, 1/3, 2/3 \rangle$, $\mathbf{v} = \langle 3, 6, 3 \rangle$.
- $\mathbf{r} = \langle 2/3, -2/3, 1/3 \rangle$, $\mathbf{s} = \langle 1/\sqrt{2}, 1/\sqrt{2}, 0 \rangle$, $\mathbf{t} = \langle -1/(3\sqrt{2}), 1/(3\sqrt{2}), 2\sqrt{2}/3 \rangle$, and $\mathbf{v} = \langle 3, 0, 3 \rangle$.
- $\mathbf{r} = \langle 3/5, 3/5, \sqrt{7}/5 \rangle$, $\mathbf{s} = \langle -\sqrt{7}/4, 0, 3/4 \rangle$, $\mathbf{t} = \langle 9/20, -4/5, 3\sqrt{7}/20 \rangle$, $\mathbf{v} = \langle 0, 1, 0 \rangle$.

Vectors in Five Dimensions

In Exercises 11–14, let $\mathbf{u} = \langle 2, 0, -1, 3, 2 \rangle$, $\mathbf{v} = \langle -1, 1, 1, -1, 0 \rangle$, and $\mathbf{w} = \langle 3, -1, 2, -1, 1 \rangle$.

- Find the linear combinations $2\mathbf{u} + \mathbf{v} - 3\mathbf{w}$ and $-\mathbf{u} + 5\mathbf{v} - \mathbf{w}$.
- Find the lengths $|\mathbf{u}|$, $|\mathbf{v}|$, and $|\mathbf{w}|$.
- Find the dot products $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \cdot \mathbf{w}$, and $\mathbf{v} \cdot \mathbf{w}$.
- Find the linear combination $(2\mathbf{w} \cdot \mathbf{u})\mathbf{u} - (\mathbf{w} \cdot 3\mathbf{v})\mathbf{v} + (\mathbf{u} \cdot \mathbf{v})4\mathbf{w}$.

In Exercises 15–20, let $\mathbf{u}_1 = \langle 1/\sqrt{2}, 1/\sqrt{2}, 0, 0, 0 \rangle$, $\mathbf{u}_2 = \langle 1/\sqrt{2}, -1/\sqrt{2}, 0, 0, 0 \rangle$, $\mathbf{u}_3 = \langle 0, 0, 1/\sqrt{2}, 1/\sqrt{2}, 0 \rangle$, $\mathbf{u}_4 = \langle 0, 0, 1/\sqrt{2}, -1/\sqrt{2}, 0 \rangle$, $\mathbf{u}_5 = \langle 0, 0, 0, 0, 1 \rangle$, and $\mathbf{v} = \langle 2, -4, 4, 2, 1 \rangle$.

- Verify that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}$ is an orthonormal basis for five-dimensional space.
- Find the vector $\mathbf{p}$ on the line through $\mathbf{u}_1$ that is closest to $\mathbf{v}$, and find the distance from $\mathbf{v}$ to $\mathbf{p}$. Verify that $\mathbf{v} - \mathbf{p}$ is orthogonal to $\mathbf{u}_1$.

17. Find the linear combination $\mathbf{p}$ of $\mathbf{u}_1$ and $\mathbf{u}_2$ that is closest to $\mathbf{v}$, and find the distance from $\mathbf{v}$ to $\mathbf{p}$. Verify that $\mathbf{v} - \mathbf{p}$ is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$.
18. Find the linear combination $\mathbf{p}$ of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ that is closest to $\mathbf{v}$, and find the distance from $\mathbf{v}$ to $\mathbf{p}$. Verify that $\mathbf{v} - \mathbf{p}$ is orthogonal to $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.
19. Find the linear combination $\mathbf{p}$ of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$ that is closest to $\mathbf{v}$, and find the distance from $\mathbf{v}$ to $\mathbf{p}$. Verify that $\mathbf{v} - \mathbf{p}$ is orthogonal to $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$.
20. Verify that
- $$(\mathbf{v} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v} \cdot \mathbf{u}_2)\mathbf{u}_2 + (\mathbf{v} \cdot \mathbf{u}_3)\mathbf{u}_3 + (\mathbf{v} \cdot \mathbf{u}_4)\mathbf{u}_4 + (\mathbf{v} \cdot \mathbf{u}_5)\mathbf{u}_5$$
- equals $\mathbf{v}$.

Theory and Examples

In Exercises 21–30, $\mathbf{u} = \langle u_1, u_2, \dots, u_d \rangle$, $\mathbf{v} = \langle v_1, v_2, \dots, v_d \rangle$, and $\mathbf{w} = \langle w_1, w_2, \dots, w_d \rangle$ are d -dimensional vectors.

21. **Symmetry** Prove that $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
22. **Distributive properties** Prove that if a and b are real numbers, then
- $$(\mathbf{a}\mathbf{u} + b\mathbf{v}) \cdot \mathbf{w} = a(\mathbf{u} \cdot \mathbf{w}) + b(\mathbf{v} \cdot \mathbf{w}) \quad \text{and} \\ \mathbf{u} \cdot (\mathbf{a}\mathbf{v} + b\mathbf{w}) = a(\mathbf{u} \cdot \mathbf{v}) + b(\mathbf{u} \cdot \mathbf{w}).$$
23. Prove that $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2$.
24. **Pythagorean Theorem** Prove that if $\mathbf{u} \cdot \mathbf{v} = 0$, then $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$.

25. **Converse of the Pythagorean Theorem** Prove that if $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$, then $\mathbf{u} \cdot \mathbf{v} = 0$.
26. **Parallelogram Law** Prove that $|\mathbf{u} + \mathbf{v}|^2 + |\mathbf{u} - \mathbf{v}|^2 = 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2$.
27. **Cauchy–Schwarz Inequality** Let $a = |\mathbf{u}|^2$, $b = |\mathbf{v}|^2$, and $c = \mathbf{u} \cdot \mathbf{v}$. Set $\mathbf{w} = b\mathbf{u} - c\mathbf{v}$, so $b\mathbf{u} = \mathbf{w} + c\mathbf{v}$.
- Prove that $\mathbf{w}$ and $c\mathbf{v}$ are orthogonal.
 - Prove that $b^2a = |\mathbf{w} + c\mathbf{v}|^2$, and use the Pythagorean Theorem to show that $|\mathbf{w} + c\mathbf{v}|^2 = |\mathbf{w}|^2 + c^2|\mathbf{v}|^2 \geq c^2b$.
 - Prove that $|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|$.
28. **Triangle Inequality** Use Exercises 23 and 27 to prove that $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$.
29. **Reverse Triangle Inequality** Prove that $|\mathbf{u}| = |\mathbf{u} - \mathbf{v} + \mathbf{v}| \leq |\mathbf{u} - \mathbf{v}| + |\mathbf{v}|$ and $|\mathbf{v}| = |\mathbf{v} - \mathbf{u} + \mathbf{u}| \leq |\mathbf{v} - \mathbf{u}| + |\mathbf{u}|$. Use these facts to prove that
- $$||\mathbf{u}| - |\mathbf{v}|| \leq |\mathbf{u} - \mathbf{v}|.$$
30. **Parseval Equality** Assume that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_d\}$ is an orthonormal basis for d -dimensional space. Use Theorem 4 to prove that
- $$|\mathbf{v}|^2 = (\mathbf{v} \cdot \mathbf{u}_1)^2 + (\mathbf{v} \cdot \mathbf{u}_2)^2 + \dots + (\mathbf{v} \cdot \mathbf{u}_d)^2 \\ = \sum_{k=1}^d (\mathbf{v} \cdot \mathbf{u}_k)^2.$$
31. Prove Theorem 5.

19.4 Approximation of Functions

The Norm of a Function

The main goal of Fourier series is to find the best approximation of a function by a trigonometric polynomial. We have seen that for vectors, it is easy to find best approximations when we have an orthonormal collection $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_d$ of vectors. If we had an *orthonormal collection of functions*, then we could use similar ideas to find the best approximation. We will show below that the trigonometric system is, after being appropriately rescaled, an analogue of such an orthonormal basis. Before doing this, we must explain what we mean by the *norm of a function* (because we need unit functions) and the *inner product of two functions* (because we need orthogonal functions). We will assume that the functions we are working with are periodic with period 2π . Any such function simply repeats the values that it takes in the interval $[0, 2\pi]$ in any other interval $[2n\pi, 2(n+1)\pi]$ with n an integer.

Vectors and functions are more similar than might be apparent at first glance. A vector $\mathbf{v} = \langle v_1, v_2, \dots, v_d \rangle$ in d dimensions is an ordered list of numbers $v_1, v_2, \dots, v_d$. For each integer $k = 1, 2, \dots, d$ we have an associated number v_k . If f is a function on $[0, 2\pi]$, then for each real number x in $[0, 2\pi]$ we have an associated number $f(x)$.

Many of the things that we have done with vectors can also be done with functions if we use the analogy that the function values $f(x)$ are like the components v_k of a vector. There are infinitely many function values instead of finitely many components, so where we used *sums* before, we now use “continuous sums,” or *integrals*.

There is one technical issue here that does not arise with vectors. While we can always sum up finitely many numbers, there are functions whose integrals do not exist. We cannot define the norm of such functions. Instead, we restrict our attention to *integrable* functions, those whose integral does exist. Fortunately, most of the functions that we encounter in real world problems are integrable. All of the functions that we will see in later examples or exercises will be integrable. In particular, any function that is continuous on a finite interval $[a, b]$ is integrable. More generally, any function that is bounded and piecewise continuous on $[a, b]$ (continuous at all but finitely many points) is integrable.

The norm of a function is analogous to the length of a vector. The length of a vector $\mathbf{v} = \langle v_1, v_2, \dots, v_d \rangle$ is

$$\|\mathbf{v}\| = \sqrt{\sum_{k=1}^d v_k^2}.$$

We define the norm of a function using a similar formula, replacing components v_k with function values $f(x)$, and a finite sum with an integral.

DEFINITION

1. The **norm** of an integrable function f on the domain $[0, 2\pi]$ is

$$\|f\| = \sqrt{\int_0^{2\pi} f(x)^2 dx}.$$

We say that f is a **unit function** if $\|f\| = 1$.

2. The **distance** between two integrable functions f and g on the domain $[0, 2\pi]$ is

$$\|f - g\| = \sqrt{\int_0^{2\pi} (f(x) - g(x))^2 dx}.$$

Our definition of the norm of a function is suited to the applications that we consider in this chapter, but it is only one of many ways to extend the idea of length from vectors to functions. The notion of a norm is explored in the mathematical subject called Real Analysis.

We compute the norms of some functions, including the functions that belong to the trigonometric system.

EXAMPLE 1

- (a) Let f be the function defined in Equation (1). Observe that $f(x) = x - \pi$ for all but one point in the interval $[0, 2\pi]$. Since changing the value of a function at a single point does not change the value of its integral, we can ignore that exceptional point and compute that

$$\|f\|^2 = \int_0^{2\pi} |f(x)|^2 dx = \int_0^{2\pi} (x - \pi)^2 dx = \frac{2\pi^3}{3}.$$

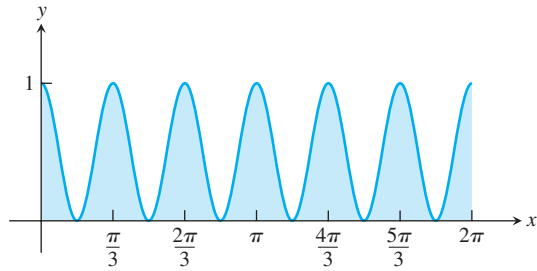
Taking square roots, $\|f\| = \sqrt{2/3} \pi^{3/2} \approx 4.55$.

- (b) Let k be any positive integer. The norm of the function $\sin kx$ is

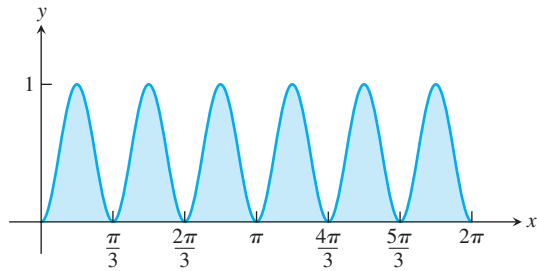
$$\|\sin kx\| = \sqrt{\int_0^{2\pi} \sin^2 kx dx}.$$

We can use trigonometric identities to evaluate this integral, but an easier way is to note that the graph of $\cos kx$ equals the graph of $\sin kx$ translated $\pi/(2k)$ units to the left, and therefore the area under the graph of $\cos^2 kx$ over $[0, 2\pi]$ is the same as the area under the graph of $\sin^2 kx$ over $[0, 2\pi]$; see Figure 19.7. Therefore, using the identity $\sin^2 x + \cos^2 x = 1$, we see that

$$\begin{aligned} 2\|\sin kx\|^2 &= \int_0^{2\pi} \sin^2 kx dx + \int_0^{2\pi} \sin^2 kx dx \\ &= \int_0^{2\pi} \sin^2 kx dx + \int_0^{2\pi} \cos^2 kx dx \\ &= \int_0^{2\pi} (\sin^2 kx + \cos^2 kx) dx \\ &= \int_0^{2\pi} 1 dx = 2\pi. \end{aligned}$$



(a) Graph of $\cos^2 3x$ over the interval $[0, 2\pi]$



(b) Graph of $\sin^2 3x$ over the interval $[0, 2\pi]$

FIGURE 19.7 The area under the graph of $\cos^2 3x$ between 0 and 2π is the same as the area under the graph of $\sin^2 3x$ between 0 and 2π .

The norm of the constant function $\mathbf{1}$ is not 1 because the length of the interval $[0, 2\pi]$ is 2π , not 1.

Therefore, $\|\sin kx\|^2 = \pi$ for each positive integer k . A similar computation applies to $\cos kx$, so by taking square roots we conclude that $\|\sin kx\| = \sqrt{\pi} = \|\cos kx\|$.

(c) The norm of the constant function $\mathbf{1}$ on the domain $[0, 2\pi]$ is

$$\|\mathbf{1}\| = \sqrt{\int_0^{2\pi} \mathbf{1}(x)^2 dx} = \sqrt{\int_0^{2\pi} 1^2 dx} = \sqrt{2\pi}.$$

The Inner Product of Functions

The inner product of functions is analogous to the dot product of vectors. The dot product of a vector $\mathbf{u} = \langle u_1, u_2, \dots, u_d \rangle$ with $\mathbf{v} = \langle v_1, v_2, \dots, v_d \rangle$ is the number

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \dots + u_d v_d = \sum_{k=1}^d u_k v_k.$$

The inner product of two functions is a number that is defined by a similar formula. However, because the notation $f \cdot g$ resembles the definition of the product function $f(x)g(x)$ rather than a number, we denote the inner product of f and g by $\langle f, g \rangle$.

The notation $\mathbf{u} = \langle u_1, u_2 \rangle$ denotes the components of a two-dimensional vector, while $\langle f, g \rangle$ denotes the inner product of functions f and g . Despite the similar notation, we can tell from context which meaning is intended.

DEFINITION The **inner product** of two integrable functions f and g on $[0, 2\pi]$ is

$$\langle f, g \rangle = \int_0^{2\pi} f(x) g(x) dx.$$

The properties of the inner product of functions are similar to the properties of the dot product of vectors. For example, just as the dot product of a vector with itself is the square of its length, the inner product of a function with itself is the square of its norm:

$$\langle f, f \rangle = \int_0^{2\pi} f(x)^2 dx = \|f\|^2. \quad (8)$$

Hence f is a unit function if and only if $\langle f, f \rangle = 1$.

The definition of orthogonality for functions is similar to the definition for vectors.

DEFINITION Let f and g be integrable functions on $[0, 2\pi]$. Then f and g are **orthogonal** if $\langle f, g \rangle = 0$. They are **orthonormal** if they are orthogonal and, additionally, $\|f\| = \|g\| = 1$.

Often we deal with more than two functions. We say that a collection of functions is orthogonal if any two distinct functions in the collection are orthogonal. If in addition each function in the collection is a unit function, then we say that the collection is orthonormal.

The next theorem is the analogue for functions of Theorem 1 in Section 19.3. Statement 3 of this theorem is the **Pythagorean Theorem** for functions.

THEOREM 6 Let f and g be integrable functions on $[0, 2\pi]$.

1. $\|f - g\|^2 = \|f\|^2 - 2\langle f, g \rangle + \|g\|^2$
2. $\|f + g\|^2 = \|f\|^2 + 2\langle f, g \rangle + \|g\|^2$
3. If f and g are orthogonal, then $\|f + g\|^2 = \|f\|^2 + \|g\|^2$.

Proof Statement 1 follows because

$$\begin{aligned} \|f - g\|^2 &= \int_0^{2\pi} (f(x) - g(x))^2 dx \\ &= \int_0^{2\pi} f(x)^2 dx - 2 \int_0^{2\pi} f(x) g(x) dx + \int_0^{2\pi} g(x)^2 dx \\ &= \|f\|^2 - 2\langle f, g \rangle + \|g\|^2, \end{aligned}$$

and the proof of Statement 2 is similar. If f and g are orthogonal, then $\langle f, g \rangle = 0$, so Statement 3 follows from Statement 2. ■

Orthogonality of the Trigonometric System

We will show that the trigonometric system is an orthogonal collection of functions. It is not an orthonormal collection because the functions $\mathbf{1}$, $\cos kx$ and $\sin kx$ do not have norm 1.

EXAMPLE 2

(a) If k is any positive integer, then the inner product of the constant function $\mathbf{1}$ with $\sin kx$ is

$$\langle \mathbf{1}, \sin kx \rangle = \int_0^{2\pi} 1 \cdot \sin kx dx = \int_0^{2\pi} \sin kx dx.$$

We see by looking at the graph of $\sin kx$ on the interval $[0, 2\pi]$ that this integral is zero since it has as much area below as above the x -axis, and therefore the signed areas cancel. We demonstrate this explicitly by computing that

$$\langle \mathbf{1}, \sin kx \rangle = \int_0^{2\pi} \sin kx dx = -\frac{\cos kx}{k} \Big|_0^{2\pi} = \frac{-\cos 2k\pi}{k} + \frac{\cos 0}{k} = \frac{-1}{k} + \frac{1}{k} = 0.$$

Consequently, the constant function is orthogonal to $\sin kx$ for every integer $k > 0$. A similar calculation shows that **1** is orthogonal to $\cos kx$ for every positive k .

- (b) We show next that $\sin x$ and $\cos x$ are orthogonal. Using the u -substitution $u = \sin x$, their inner product is

$$\langle \sin x, \cos x \rangle = \int_0^{2\pi} \sin x \cos x \, dx = \frac{\sin^2 x}{2} \Big|_0^{2\pi} = \frac{\sin^2 2\pi - \sin^2 0}{2} = 0.$$

Looking at their graphs, there is no obvious reason why we should expect that $\sin x$ and $\cos x$ should be orthogonal. We can understand visually what it means for two vectors $\mathbf{u}$ and $\mathbf{v}$ in three-dimensional space to be perpendicular (there is a 90-degree angle between them), but there is no easy way to see from their graphs why two functions are orthogonal.

- (c) We compute another special case. The inner product of $\sin x$ and $\sin 2x$ is zero because

$$\langle \sin x, \sin 2x \rangle = \int_0^{2\pi} \sin x (2 \sin x \cos x) \, dx = 2 \frac{\sin^3 x}{3} \Big|_0^{2\pi} = 2 \frac{\sin^3 2\pi - \sin^3 0}{3} = 0.$$

Therefore, $\sin x$ and $\sin 2x$ are orthogonal.

- (d) We have shown that the constant function **1** is orthogonal to $\sin kx$ and $\cos kx$ for every positive integer k . To complete the proof that the trigonometric system is a collection of orthogonal functions, it remains to prove that for all positive integers m and n we have:

$$\langle \sin mx, \sin nx \rangle = \int_0^{2\pi} \sin mx \sin nx \, dx = 0 \quad \text{whenever } m \neq n,$$

$$\langle \cos mx, \cos nx \rangle = \int_0^{2\pi} \cos mx \cos nx \, dx = 0 \quad \text{whenever } m \neq n,$$

$$\langle \sin mx, \cos nx \rangle = \int_0^{2\pi} \sin mx \cos nx \, dx = 0 \quad \text{for all } m \text{ and } n.$$

This can be done through the use of trigonometric identities (Exercises 31–33). ■

The functions in the trigonometric system do not have unit norm. However, if we divide each function by its norm, then we obtain functions whose norm is 1. We set

$$u_0(x) = \frac{1}{\sqrt{2\pi}},$$

and for each integer $k > 0$ let

$$u_k(x) = \frac{\cos kx}{\sqrt{\pi}} \quad \text{and} \quad v_k(x) = \frac{\sin kx}{\sqrt{\pi}}.$$

Each function u_0 , u_k , and v_k is a unit function, and the set of these functions is orthonormal.

DEFINITION The **normalized trigonometric system** is the collection of orthonormal functions u_0 , u_1 , v_1 , u_2 , v_2 , ... Explicitly, these are

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\cos 3x}{\sqrt{\pi}}, \frac{\sin 3x}{\sqrt{\pi}}, \dots$$

Approximation of Functions

Now we find the best approximation to a function using the normalized constant function u_0 , the first n normalized cosine functions $u_1, u_2, \dots, u_n$, and the first n normalized sine functions $v_1, v_2, \dots, v_n$. That is, out of all of the linear combinations

$$w = a_0 u_0 + \sum_{k=1}^n a_k u_k + \sum_{k=1}^n b_k v_k, \tag{9}$$

we identify the one whose distance from f is the smallest. We let p_n denote this closest function.

THEOREM 7 If f is an integrable function on $[0, 2\pi]$ and n is a positive integer, then the linear combination of u_0 and $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ that is closest to f is

$$p_n = \langle f, u_0 \rangle u_0 + \sum_{k=1}^n \langle f, u_k \rangle u_k + \sum_{k=1}^n \langle f, v_k \rangle v_k. \quad (10)$$

Proof Let p_n be the function in Equation (10), and let w be any linear combination of the form given in Equation (9). We must prove that $\|f - p_n\| \leq \|f - w\|$.

To simplify the notation, let

$$q = f - p_n \quad \text{and} \quad z = p_n - w.$$

Also let $c_0 = \langle f, u_0 \rangle$, and for each positive integer k set $c_k = \langle f, u_k \rangle$ and $d_k = \langle f, v_k \rangle$, so

$$p_n = c_0 u_0 + \sum_{k=1}^n c_k u_k + \sum_{k=1}^n d_k v_k.$$

The inner product satisfies a distributive law, just as the dot product of vectors does (Exercise 24). Using this and the fact that u_0 is orthogonal to all of the other functions u_k and v_k with $k > 0$, the inner product of p_n with u_0 is

$$\begin{aligned} \langle p_n, u_0 \rangle &= \langle c_0 u_0 + \sum_{k=1}^n c_k u_k + \sum_{k=1}^n d_k v_k, u_0 \rangle && \text{Substitute} \\ &= c_0 \langle u_0, u_0 \rangle + \sum_{k=1}^n c_k \langle u_k, u_0 \rangle + \sum_{k=1}^n d_k \langle v_k, u_0 \rangle && \text{Distributive laws} \\ &= c_0 \|u_0\|^2 + \sum_{k=1}^n c_k \cdot 0 + \sum_{k=1}^n d_k \cdot 0 && \text{Orthogonality} \\ &= c_0 \cdot 1 + 0 + 0 = c_0. && u_0 \text{ is a unit function} \end{aligned}$$

It follows from this that q is orthogonal to u_0 because $q = f - p_n$, and therefore

$$\langle q, u_0 \rangle = \langle f - p_n, u_0 \rangle = \langle f, u_0 \rangle - \langle p_n, u_0 \rangle = c_0 - c_0 = 0.$$

A similar calculation holds for inner products of q with each of the functions $u_1, u_2, \dots, u_n$ and each of $v_1, v_2, \dots, v_n$. That is, q is orthogonal to every one of these functions. Using the distributive law again, it follows that q is orthogonal to every linear combination of u_0 and $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$. Now, p_n is a linear combination of those functions, and so is w , so q is orthogonal to both p_n and w . Hence

$$\langle q, z \rangle = \langle q, p_n - w \rangle = \langle q, p_n \rangle - \langle q, w \rangle = 0 - 0 = 0,$$

and so we see that q and z are orthogonal. Therefore, we can apply the Pythagorean Theorem for functions:

$$\begin{aligned} \|f - w\|^2 &= \|q + z\|^2 && q = f - p_n, z = p_n - w \\ &= \|q\|^2 + \|z\|^2 && \text{Pythagorean Theorem} \\ &\geq \|q\|^2 + 0 && \|z\|^2 \geq 0 \\ &= \|f - p_n\|^2. && q = f - p_n \end{aligned}$$

Taking square roots, it follows that $\|f - p_n\| \leq \|f - w\|$. ■

We now show how to write the function p_n that is closest to f directly in terms of sines and cosines. We call the function p_n given in Equation (11) a **trigonometric polynomial of degree n** . The numbers a_0 , a_k , and b_k defined in this theorem are called the **Fourier coefficients** of f .

THEOREM 8 Let f be an integrable function on $[0, 2\pi]$ and let n be a positive integer. Define real numbers

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \langle f, \mathbf{1} \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx, \\ a_k &= \frac{1}{\pi} \langle f, \cos kx \rangle = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx dx, \\ b_k &= \frac{1}{\pi} \langle f, \sin kx \rangle = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx dx, \end{aligned}$$

where k is a positive integer. Then the best approximation to f by a linear combination of $\mathbf{1}$, $\cos x$, ..., $\cos nx$, $\sin x$, ..., $\sin nx$ is

$$p_n(x) = a_0 + \sum_{k=1}^n a_k \cos kx + \sum_{k=1}^n b_k \sin kx. \quad (11)$$

Proof By Theorem 7, the best approximation is the function p_n given in Equation (10). Expanding and simplifying shows that this function satisfies Equation (11):

$$\begin{aligned} p_n(x) &= \langle f, u_0 \rangle u_0(x) + \sum_{k=1}^n \langle f, u_k \rangle u_k(x) + \sum_{k=1}^n \langle f, v_k \rangle v_k(x) \\ &= \left\langle f, \frac{\mathbf{1}}{\sqrt{2\pi}} \right\rangle \frac{\mathbf{1}}{\sqrt{2\pi}} + \sum_{k=1}^n \left\langle f, \frac{\cos kx}{\sqrt{\pi}} \right\rangle \frac{\cos kx}{\sqrt{\pi}} + \sum_{k=1}^n \left\langle f, \frac{\sin kx}{\sqrt{\pi}} \right\rangle \frac{\sin kx}{\sqrt{\pi}} \\ &= \frac{1}{2\pi} \langle f, \mathbf{1} \rangle \mathbf{1} + \sum_{k=1}^n \frac{1}{\pi} \langle f, \cos kx \rangle \cos kx + \sum_{k=1}^n \frac{1}{\pi} \langle f, \sin kx \rangle \sin kx \\ &= a_0 + \sum_{k=1}^n a_k \cos kx + \sum_{k=1}^n b_k \sin kx. \quad \blacksquare \end{aligned}$$

EXAMPLE 3 Let f be the function defined in Equation (1). We will verify that the function p_n given in Equation (2) is the best approximation to f by a trigonometric polynomial of degree n . We must compute the numbers a_0 , a_k , and b_k given in Theorem 8. If k is a positive integer, then we use Equation 86 from the Table of Integrals to compute that

$$\begin{aligned} b_k &= \frac{1}{\pi} \int_0^{2\pi} (x - \pi) \sin kx dx \\ &= \frac{1}{\pi} \int_0^{2\pi} x \sin kx dx - \int_0^{2\pi} \sin kx dx \\ &= \frac{1}{\pi} \left[\frac{\sin kx}{k^2} - \frac{x \cos kx}{k} \right]_0^{2\pi} - 0 = \frac{1}{\pi} \left(\frac{-2\pi}{k} \right) = -\frac{2}{k}. \end{aligned}$$

A similar calculation shows that $a_0 = 0$ and $a_k = 0$ for every $k > 0$. Therefore, the best approximation uses amplitude zero for the constant term and the cosine terms, and amplitude $-2/k$ for the sine term of frequency k , so

$$p_n(x) = -2 \left(\sin x + \frac{\sin 2x}{2} + \cdots + \frac{\sin nx}{n} \right) = -2 \sum_{k=1}^n \frac{\sin kx}{k}.$$

This is the trigonometric polynomial given in Equation (2). ■

EXAMPLE 4 Let f be the step function

$$f(x) = \begin{cases} 1, & 0 \leq x < \pi \text{ or } x = 2\pi, \\ 2, & \pi \leq x < 2\pi. \end{cases}$$

Find the best approximation to f by a trigonometric polynomial of degree n .

Solution We compute the Fourier coefficients for f . If k is a positive integer, then

$$\begin{aligned} b_k &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} \sin kx \, dx + \frac{1}{\pi} \int_{\pi}^{2\pi} 2 \sin kx \, dx \\ &= \frac{1}{\pi} \left(\left[\frac{-\cos kx}{k} \right]_0^{\pi} + \left[\frac{-2 \cos kx}{k} \right]_{\pi}^{2\pi} \right) \\ &= \frac{\cos k\pi - 1}{k\pi} = \frac{(-1)^k - 1}{k\pi} = \begin{cases} -\frac{2}{k\pi}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases} \end{aligned}$$

A similar calculation shows that $a_0 = 3/2$, but $a_k = 0$ for every $k > 0$. Therefore, the best approximation to f uses only a constant term and sine terms with odd frequencies. Specifically, if n is odd then the linear combination of $\mathbf{1}$, $\sin x$, $\dots$, $\sin nx$ that is closest to f is

$$p_n(x) = \frac{3}{2} - \frac{2}{\pi} \left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots + \frac{\sin nx}{n} \right).$$

At the point $x = \pi$, where $f(x)$ jumps from 1 to 2, all of the sine terms vanish, so $p_n(\pi) = 3/2$ for every n . This is not the value of $f(\pi)$, nor does it approach closer to $f(\pi)$ as n increases. This does not contradict the fact that p_n is the best approximation to f because best does not mean that $p_n(x)$ is close to $f(x)$ at every point. Instead, p_n is the best approximation because the *error* is being measured here by an integral. The error in the approximation is the distance between f and p_n , which is

$$\|f - p_n\| = \sqrt{\int_0^{2\pi} (f(x) - p_n(x))^2 \, dx}.$$

Out of all linear combinations using frequencies of at most n , the trigonometric polynomial p_n minimizes this error, which is defined by an integral that takes all of the values of $f(x)$ and $p_n(x)$ across the entire domain $[0, 2\pi]$ into account (see Figure 19.8). ■

Convergence of Fourier Series

If f is an integrable function on $[0, 2\pi]$, then Theorem 8 gives us the best approximation p_n to f by a linear combination of $\mathbf{1}$, $\cos x$, $\dots$, $\cos nx$, $\sin x$, $\dots$, $\sin nx$. The only information that we needed to prove this theorem is that the functions in the trigonometric system are orthogonal. Looking at Figures 19.3 and 19.7, it appears that the approximation p_n becomes closer to f in some sense as n increases. The next theorem makes this precise. Specifically, it is the distance $\|f - p_n\|$ between f and p_n that converges to zero as $n \rightarrow \infty$.

THEOREM 9 Let f be an integrable function on $[0, 2\pi]$ and for each positive integer n let p_n be the function defined in Equation (11). Then the distance between f and p_n converges to zero as $n \rightarrow \infty$:

$$\lim_{n \rightarrow \infty} \|f - p_n\| = \lim_{n \rightarrow \infty} \sqrt{\int_0^{2\pi} (f(x) - p_n(x))^2 \, dx} = 0.$$

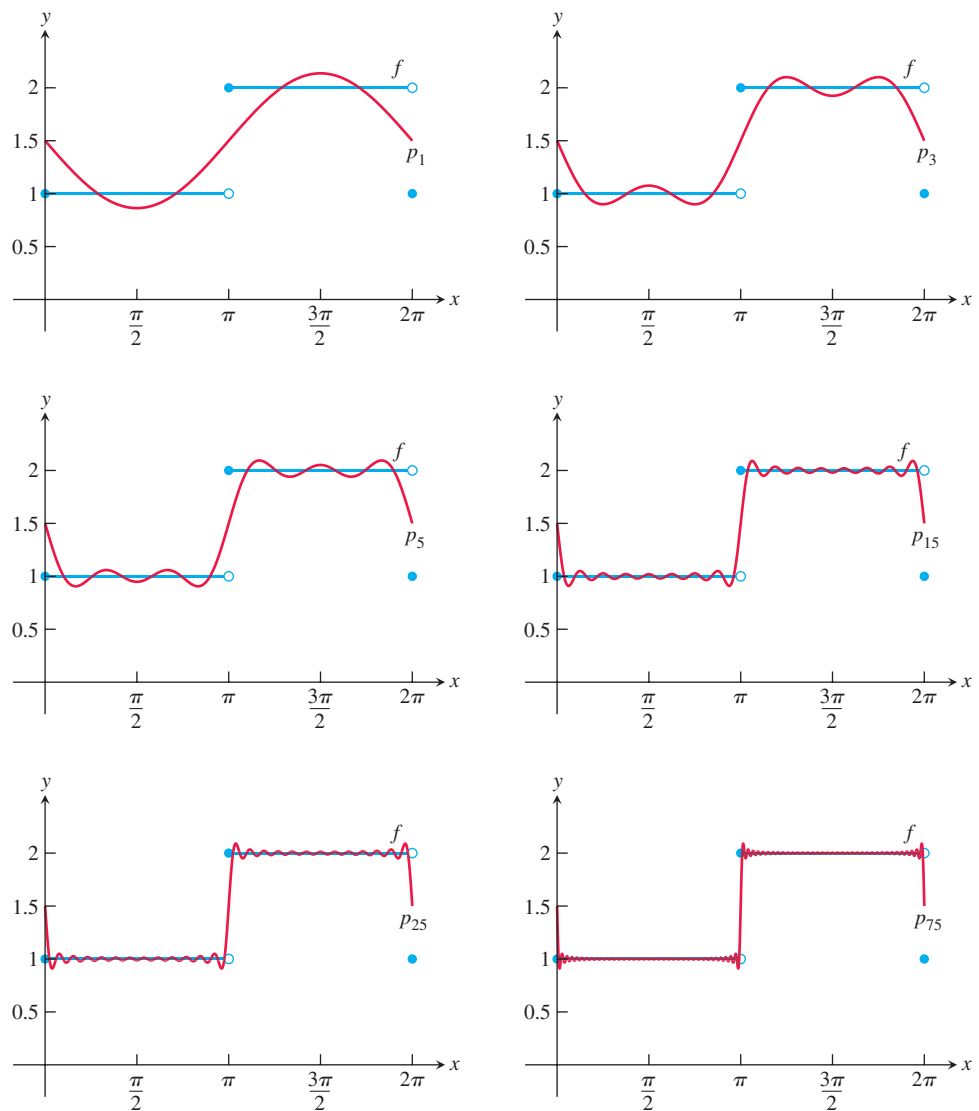


FIGURE 19.8 The trigonometric polynomials p_1 , p_3 , p_5 , p_{15} , p_{25} , and p_{75} approximating the function f from Example 4

The proof of Theorem 9 is harder than the proof of Theorem 8. It can be found in advanced texts on real analysis. We can get some insight into why the proof is difficult by recalling our discussion of orthonormal bases in three dimensions. Whenever we have three orthonormal vectors $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$ in three-dimensional space, we know that these three vectors form an orthonormal basis, and therefore every vector $\mathbf{v}$ can be written as a linear combination of those three vectors, as shown in Theorem 2. A similar fact holds in higher-dimensional space. That is, a set of d orthonormal vectors in d -dimensional space will be an orthonormal basis for that space (Theorem 4). However, the normalized trigonometric system is a set of *infinitely many* orthonormal functions. Just having infinitely many orthonormal functions is not enough to ensure that they form the correct analogue of an orthonormal basis. For example, the cosine functions $\cos x$, $\cos 2x$, $\cos 3x$, $\dots$ are infinitely many orthogonal functions, but they are only “half” of the trigonometric system. The difficulty in the proof of Theorem 9 is in showing that the trigonometric system contains “enough” functions.

Let a_0 , a_k , and b_k be the Fourier coefficients of f as given in Theorem 8, and let

$$p_n(x) = a_0 + \sum_{k=1}^n a_k \cos kx + \sum_{k=1}^n b_k \sin kx$$

be the trigonometric polynomial defined in Equation (11). Theorem 9 says that if f is an integrable function, then p_n converges to f in the sense that the distance $\|f - p_n\| = \sqrt{\int_0^{2\pi} (f(x) - p_n(x))^2 dx}$ between p_n and f converges to zero as $n \rightarrow \infty$. This does not imply that $p_n(x)$ must converge to $f(x)$ at each point x . We often call

$$a_0 + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx \quad (12)$$

the **Fourier series** for f , but it is important to note that this series need not converge. Under the right hypothesis, it will be true that $p_n(x)$ converges to $f(x)$, and hence $f(x)$ will equal its Fourier series at x . Loosely speaking, the smoother that f is, the “better” the convergence of $p_n(x)$ to $f(x)$ will be. The following result, whose proof we will omit, states that if f is smooth enough as a periodic function on the real line, then convergence will hold at every point, and the maximum difference between $f(x)$ and $p_n(x)$ shrinks as n increases.

THEOREM 10 Let f be a function on $[0, 2\pi]$ such that $f(0) = f(2\pi)$, and extend f to be a periodic function on the real line by repeating values every 2π units. If f is differentiable at every point and f' is continuous, then we have for every x that

$$f(x) = \lim_{n \rightarrow \infty} p_n(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx.$$

Further, if for each n we let M_n be the maximum value of $|f(x) - p_n(x)|$ over all x , then

$$\lim_{n \rightarrow \infty} M_n = 0.$$

The function f pictured in Figure 19.8 does not satisfy the hypotheses of Theorem 10, and it can be shown that for that function the maximum M_n defined in Theorem 10 does not decrease to zero as n increases (this is known as the **Gibbs phenomenon**).

The Parseval Equality for Fourier Series

By Exercise 30 of Section 19.3, if $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_d\}$ is an orthonormal basis for d -dimensional space, then the *Parseval Equality* holds for every vector $\mathbf{v}$ in d -dimensional space:

$$\|\mathbf{v}\|^2 = \sum_{k=1}^d (\mathbf{v} \cdot \mathbf{u}_k)^2.$$

We will prove an analogous result for Fourier series.

THEOREM 11 – The Parseval Equality

If f is an integrable function on $[0, 2\pi]$, then

$$\|f\|^2 = 2\pi a_0^2 + \pi \sum_{k=1}^{\infty} a_k^2 + \pi \sum_{k=1}^{\infty} b_k^2.$$

Proof Let p_n be the trigonometric polynomial of degree n given in Equation (11). According to Exercise 34, we have $\langle p_n, \mathbf{1} \rangle = 2\pi a_0$, while $\langle p_n, \cos kx \rangle = \pi a_k$ and $\langle p_n, \sin kx \rangle = \pi b_k$ for $k = 1, \dots, n$. Therefore,

$$\|p_n\|^2 = \langle p_n, p_n \rangle \quad \text{Equation (8)}$$

$$= \left\langle p_n, a_0 + \sum_{k=1}^n a_k \cos kx + \sum_{k=1}^n b_k \sin kx \right\rangle \quad \text{Equation (11)}$$

$$= \langle p_n, a_0 \mathbf{1} \rangle + \sum_{k=1}^n a_k \langle p_n, \cos kx \rangle + \sum_{k=1}^n b_k \langle p_n, \sin kx \rangle \quad \text{Distributive property}$$

$$= a_0 \langle p_n, \mathbf{1} \rangle + \sum_{k=1}^n a_k \pi a_k + \sum_{k=1}^n b_k \pi b_k. \quad \text{Exercise 34}$$

$$= 2\pi a_0^2 + \pi \sum_{k=1}^n a_k^2 + \pi \sum_{k=1}^n b_k^2. \quad \text{Exercise 34 and simplify}$$

Since Theorem 9 tells us that $\|f - p_n\| \rightarrow 0$, it follows that

$$\|f\|^2 = \lim_{n \rightarrow \infty} \|p_n\|^2 \quad \text{Exercise 28}$$

$$= \lim_{n \rightarrow \infty} \left(2\pi a_0^2 + \pi \sum_{k=1}^n a_k^2 + \pi \sum_{k=1}^n b_k^2 \right) \quad \text{Calculated above}$$

$$= 2\pi a_0^2 + \pi \sum_{k=1}^{\infty} a_k^2 + \pi \sum_{k=1}^{\infty} b_k^2. \quad \text{Monotonic Sequence Theorem (Theorem 6 in Section 9.1)} \quad \blacksquare$$

EXAMPLE 5 Let f be the function from Example 4. The square of the norm of f is

$$\|f\|^2 = \int_0^\pi 1^2 dx + \int_\pi^{2\pi} 2^2 dx = 5\pi.$$

We saw in Example 4 that $a_0 = 3/2$, $a_k = 0$ for all $k > 0$, $b_k = -2/(k\pi)$ if k is odd, and $b_k = 0$ if k is even. Theorem 10 therefore implies that

$$\begin{aligned} 5\pi = \|f\|^2 &= 2\pi a_0^2 + \pi \sum_{k=1}^{\infty} a_k^2 + \pi \sum_{k=1}^{\infty} b_k^2 \\ &= 2\pi \frac{9}{4} + 0 + \pi \sum_{k \text{ odd}} \left(\frac{-2}{k\pi} \right)^2 \\ &= \frac{18\pi}{4} + 0 + \frac{4}{\pi} \sum_{k \text{ odd}} \frac{1}{k^2}. \end{aligned}$$

Simplifying, we obtain

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \sum_{k \text{ odd}} \frac{1}{k^2} = \frac{\pi^2}{8}.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{8} + \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

This gives us

$$\frac{3}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{8},$$

and hence

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \quad (13) \quad \blacksquare$$

EXERCISES 19.4

Norms of Functions

In Exercises 1–6, a periodic function with period 2π is defined by specifying its values on the interval $[0, 2\pi]$. Find the norm of each function.

$$1. f(x) = \begin{cases} x, & 0 \leq x \leq \pi \\ 2\pi - x, & \pi < x \leq 2\pi \end{cases}$$

$$2. g(x) = \begin{cases} 1, & \pi/2 < x < 3\pi/2 \\ 0, & 0 \leq x \leq \pi/2 \text{ or } 3\pi/2 \leq x \leq 2\pi \end{cases}$$

$$3. h(x) = (x - \pi)^2$$

$$4. j(x) = \begin{cases} x^2, & 0 \leq x \leq \pi \\ 0, & \pi < x \leq 2\pi \end{cases}$$

$$5. k(x) = \begin{cases} e^x, & 0 \leq x < 2\pi \\ 1, & x = 2\pi \end{cases}$$

$$6. \ell(x) = \begin{cases} \cos x, & 0 \leq x \leq \pi \\ 1, & \pi < x \leq 2\pi \end{cases}$$

Inner Product of Functions

In Exercises 7–12, use the functions defined in Exercises 1–6 to compute the given inner product. Are the two functions orthogonal?

$$7. \langle g, j \rangle$$

$$8. \langle k, \cos x \rangle$$

$$9. \langle h, 1 \rangle$$

$$10. \langle \ell, 1 \rangle$$

$$11. \langle f, \sin 3x \rangle$$

$$12. \langle \ell, \cos x \rangle$$

Finding Fourier Series

In Exercises 13–18, find the Fourier coefficients a_0 , a_k , and b_k and the trigonometric polynomial p_n of degree n for the given function. If available, use a CAS to plot p_n for various values of n over the domain $[-2\pi, 4\pi]$.

13. The function f from Exercise 1.

14. The function g from Exercise 2.

15. The function h from Exercise 3.

16. The function j from Exercise 4.

17. The function k from Exercise 5.

18. The function ℓ from Exercise 6.

The Parseval Equality

19. Apply the Parseval Equality to the function f defined in Equation (11) to give another proof that $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$.

20. Apply the Parseval Equality to the function f from Exercise 1 to prove that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{96}.$$

Use that equality to prove that

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

21. Apply the Parseval Equality to the function h from Exercise 3 to give another proof of the equalities specified in Exercise 20.

22. Define a function f on $[0, 2\pi]$ by

$$f(x) = \begin{cases} x(\pi - x), & 0 \leq x \leq \pi \\ (x - \pi)(x - 2\pi), & \pi < x \leq 2\pi \end{cases}$$

Apply the Parseval Equality to this function f to prove that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^6} = \frac{\pi^6}{960}.$$

Use that equality to prove that

$$\sum_{k=1}^{\infty} \frac{1}{k^6} = \frac{\pi^6}{945}.$$

Theory and Examples

In Exercises 23–28, f , g , and h denote integrable functions on $[0, 2\pi]$.

23. **Symmetry** Prove that $\langle f, g \rangle = \langle g, f \rangle$.

24. **Distributive properties** Prove that if a and b are real numbers, then

$$\begin{aligned} \langle af + bg, h \rangle &= a \langle f, h \rangle + b \langle g, h \rangle \quad \text{and} \\ \langle f, ag + bh \rangle &= a \langle f, g \rangle + b \langle f, h \rangle. \end{aligned}$$

25. **Cauchy–Schwarz Inequality** Prove that $|\langle f, g \rangle| \leq \|f\| \|g\|$. (Hint: Follow the approach given in Exercise 27 of Section 19.3.)

26. **Triangle Inequality** Prove that $\|f + g\| \leq \|f\| + \|g\|$. (Hint: Compare Exercise 28 in Section 19.3.)

27. **Reverse Triangle Inequality** Prove that $|\|f\| - \|g\|| \leq \|f - g\|$. (Hint: Compare Exercise 29 in Section 19.3.)

28. **Convergence of norms** Use Theorem 9 and the Reverse Triangle Inequality (Exercise 27) to prove that $\lim_{n \rightarrow \infty} \|p_n\| = \|f\|$.

29. Let $f(x) = 1 + \cos x + \sin x$. Find the Fourier coefficients a_0 , a_k , and b_k and the trigonometric polynomial p_n of degree n corresponding to f .

30. Suppose that $f(x) = c_0 + \sum_{j=1}^m c_j \cos jx + \sum_{j=1}^m d_j \sin jx$. What are the Fourier coefficients a_0 , a_k , and b_k for f ? Prove that p_n is equal to f for all $n \geq m$.

Establish the identities stated in Exercises 31–33, where m and n are positive integers.

31. If $m \neq n$, then $\langle \sin mx, \sin nx \rangle = \int_0^{2\pi} \sin mx \sin nx \, dx = 0$. (Hint: $\sin A \sin B = (\cos(A - B) - \cos(A + B))/2$.)

32. If $m \neq n$, then $\langle \cos mx, \cos nx \rangle = \int_0^{2\pi} \cos mx \cos nx \, dx = 0$. (Hint: $\cos A \cos B = (\cos(A + B) + \cos(A - B))/2$.)

33. $\langle \sin mx, \cos nx \rangle = \int_0^{2\pi} \sin mx \cos nx \, dx = 0$. (Hint: $\sin A \cos B = (\sin(A + B) + \sin(A - B))/2$.)

34. Let f be an integrable function on $[0, 2\pi]$, and let p_n be the trigonometric polynomial defined in Equation (11). Prove that:

$$\text{a. } \langle p_n, 1 \rangle = 2\pi a_0.$$

$$\text{b. } \langle p_n, \cos jx \rangle = \pi a_k \text{ for } j = 1, \dots, n.$$

$$\text{c. } \langle p_n, \sin jx \rangle = \pi b_k \text{ for } j = 1, \dots, n.$$

19.5 Advanced Topic: The Haar System and Wavelets

Fourier series approximations are based on the trigonometric system, which is an orthogonal collection of functions. Today many sets of orthonormal functions are used in mathematics, science, and engineering. We will discuss a family of systems collectively known as *wavelets*, which are especially useful in signal and image processing, data analysis, and related areas. The JPEG 2000 image compression standard uses wavelets.

In contrast to Fourier series approximations, which are best suited to applications involving periodic functions, wavelets can be used to approximate functions that are not periodic. The domain of the functions in the wavelet systems that we will discuss will be the entire real line, but each individual wavelet function that we will consider is identically zero outside of some finite interval, and is integrable on that interval.

We say that a function that is integrable on an interval $[a, b]$ and identically zero outside of that interval is **supported in** $[a, b]$. The function may be zero at points within $[a, b]$. If f is supported in $[a, b]$, then its **norm** is

$$\|f\| = \sqrt{\int_a^b |f(x)|^2 dx}.$$

Since f is zero outside of $[a, b]$, we often write this integral as

$$\|f\| = \sqrt{\int_{-\infty}^{\infty} |f(x)|^2 dx}.$$

We say that f is a **unit function** if $\|f\| = 1$.

If g is another integrable function that is supported in an interval $[c, d]$ (which need not equal $[a, b]$), then the **inner product** of f and g is

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x) dx. \quad (14)$$

The product $f(x)g(x)$ is identically zero outside of the intersection of the intervals $[a, b]$ and $[c, d]$.

We say that two functions f and g are **orthogonal** if $\langle f, g \rangle = 0$. Functions f and g are **orthonormal** if they are orthogonal and both are unit functions ($\|f\| = \|g\| = 1$). A collection of functions is orthogonal if any two different functions from the collection are orthogonal, and the collection is orthonormal if it is orthogonal and every function in the collection is a unit function.

The Haar System

The first wavelet system was introduced by Alfréd Haar (1885–1933). The construction is based on two simple functions that are each supported in the interval $[0, 1]$. The first is the **box function** or **Haar scaling function**,

$$\varphi(x) = \begin{cases} 1, & 0 \leq x < 1, \\ 0, & x < 0 \text{ or } x \geq 1, \end{cases}$$

and the second is the **square wave** or **Haar wavelet**,

$$\psi(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq x < 1, \\ 0, & x < 0 \text{ or } x \geq 1. \end{cases}$$

These functions are shown in Figure 19.9.

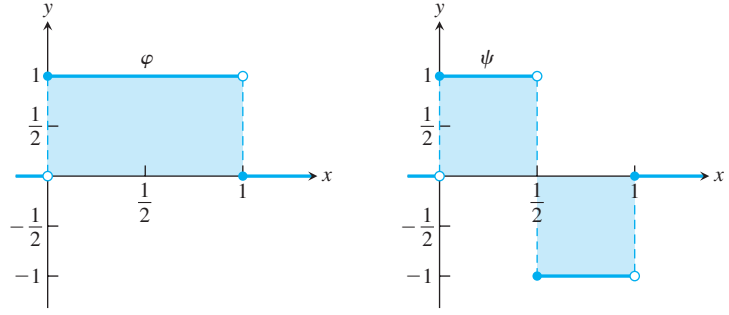


FIGURE 19.9 The Haar scaling function φ and the Haar wavelet ψ . The region between the graphs and the x -axis is shaded.

We have $\|\varphi\| = 1$ and $\|\psi\| = 1$, so φ and ψ are unit functions. Further, they are orthogonal, because their inner product is zero:

$$\begin{aligned}\langle \varphi, \psi \rangle &= \int_{-\infty}^{\infty} \varphi(x) \psi(x) dx \\ &= \int_0^1 \varphi(x) \psi(x) dx \\ &= \int_0^{1/2} 1 \cdot 1 dx + \int_{1/2}^1 1 \cdot (-1) dx = \frac{1}{2} - \frac{1}{2} = 0.\end{aligned}$$

Now we create more functions based on φ and ψ . For each integer k , we horizontally translate φ by k units to obtain the function

$$\varphi_k(x) = \varphi(x - k) = \begin{cases} 1, & k \leq x < k + 1, \\ 0, & x < k \text{ or } x \geq k + 1. \end{cases}$$

We say that φ_k is the **box function on** $[k, k + 1)$. If $j \neq k$, then the product $\varphi_j(x) \varphi_k(x)$ is identically zero. Therefore, $\langle \varphi_j, \varphi_k \rangle = 0$ whenever $j \neq k$.

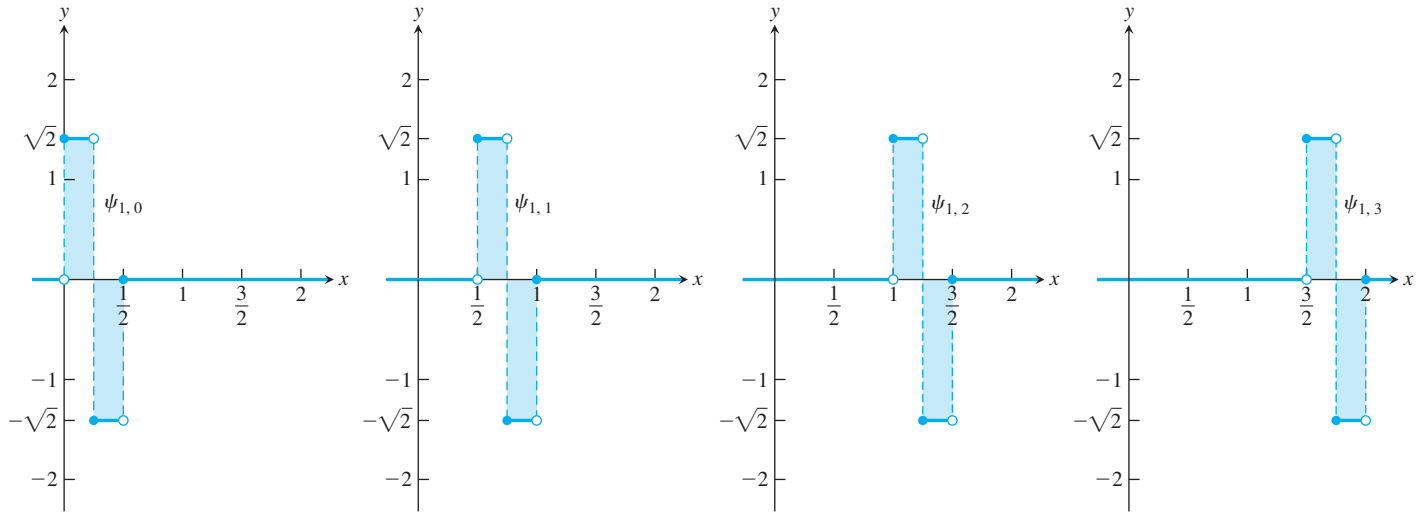
Next, we translate and also *dilate* (stretch or compress) the Haar wavelet to create new functions $\psi_{n,k}$, where n and k are integers. For $n = 0$ there is no dilation, only translation:

$$\psi_{0,k}(x) = \psi(x - k) = \begin{cases} 1, & k \leq x < k + \frac{1}{2}, \\ -1, & k + \frac{1}{2} \leq x < k + 1, \\ 0, & x < k \text{ or } x \geq k + 1. \end{cases}$$

For $n = 1$ we horizontally shrink ψ by a factor of 2 and horizontally translate by $k/2$ units, also vertically rescaling so that the resulting function has norm 1:

$$\psi_{1,k}(x) = 2^{1/2} \psi(2x - k) = 2^{1/2} \psi(2(x - k/2)) = \begin{cases} \sqrt{2}, & \frac{k}{2} \leq x < \frac{k}{2} + \frac{1}{4}, \\ -\sqrt{2}, & \frac{k}{2} + \frac{1}{4} \leq x < \frac{k}{2} + \frac{1}{2}, \\ 0, & x < \frac{k}{2} \text{ or } x \geq \frac{k}{2} + \frac{1}{2}. \end{cases}$$

The functions $\psi_{1,k}$ for $k = 0, 1, 2, 3$ are shown in Figure 19.10.

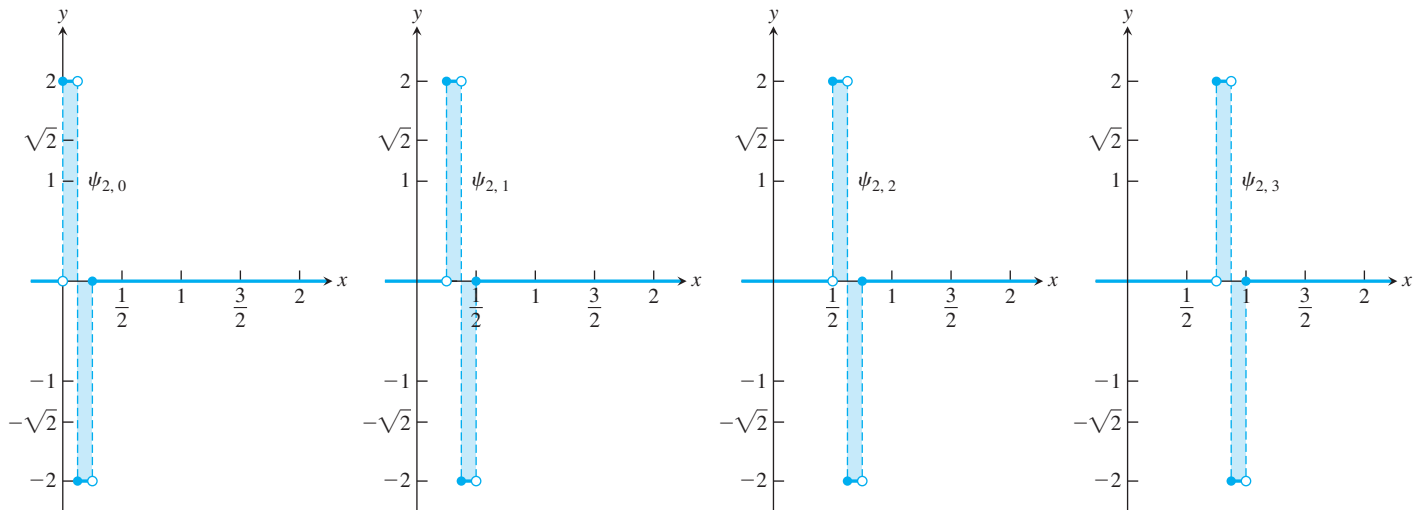

 FIGURE 19.10 The functions $\psi_{1,0}$, $\psi_{1,1}$, $\psi_{1,2}$, and $\psi_{1,3}$.

For arbitrary integers n and k we define

$$\psi_{n,k}(x) = 2^{n/2} \psi(2^n x - k) = \begin{cases} 2^{n/2}, & 2^{-n}k \leq x < 2^{-n}\left(k + \frac{1}{2}\right), \\ -2^{n/2}, & 2^{-n}\left(k + \frac{1}{2}\right) \leq x < 2^{-n}(k + 1), \\ 0, & x < 2^{-n}k \text{ or } x \geq 2^{-n}(k + 1). \end{cases}$$

We refer to $\psi_{n,k}$ as a **wavelet**. It is supported in the interval $[2^{-n}k, 2^{-n}(k + 1)]$, which has length 2^{-n} . We have $\psi_{n,k}$ take the values $\pm 2^{n/2}$ instead of ± 1 so that it is a unit function (Exercise 19).

For $n \geq 0$, the wavelet $\psi_{n,k}$ is obtained by compressing the Haar wavelet ψ and translating and rescaling it, while $n < 0$ corresponds to a stretching of ψ along with a translation and rescaling. It is useful in some applications to consider $\psi_{n,k}$ with negative n , but for our purposes we will define the Haar system using only $n \geq 0$. The functions $\psi_{2,k}$ for $k = 0, 1, 2, 3$ are shown in Figure 19.11.


 FIGURE 19.11 The functions $\psi_{2,0}$, $\psi_{2,1}$, $\psi_{2,2}$, and $\psi_{2,3}$.

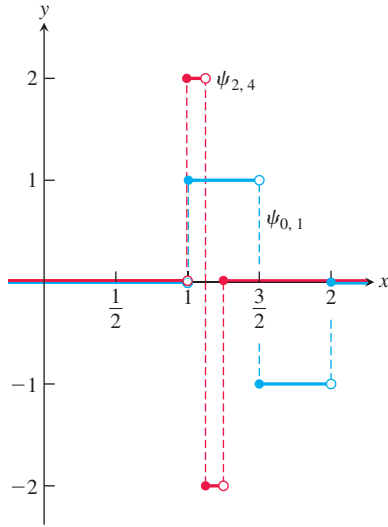


FIGURE 19.12 The functions $\psi_{0,1}$ and $\psi_{2,4}$. The second function completes its oscillation on an interval where the first function is constant.

DEFINITION The **Haar system** is the collection of functions φ_k for integers k , together with $\psi_{n,k}$ for integers k and integers $n \geq 0$.

Each function in the Haar system is a unit function. Further, any two different elements are orthogonal. To illustrate why this is, consider the specific two wavelets $\psi_{0,1}$ and $\psi_{2,4}$ shown in Figure 19.12. The wavelet $\psi_{0,1}$ is supported in the interval $[1, 2]$, while $\psi_{2,4}$ is supported in $[1, 5/4]$. Therefore, their product is $\psi_{0,1}(x) \psi_{2,4}(x) = 0$ for every x outside $[1, 5/4]$. For all x inside $[1, 5/4]$ we have $\psi_{0,1}(x) = 1$, so for these x the product is $\psi_{0,1}(x) \psi_{2,4}(x) = \psi_{2,4}(x)$. Consequently, their inner product is

$$\langle \psi_{0,1}, \psi_{2,4} \rangle = \int_{-\infty}^{\infty} \psi_{0,1}(x) \psi_{2,4}(x) dx = \int_1^{5/4} \psi_{2,4}(x) dx = 0.$$

This shows that $\psi_{0,1}$ and $\psi_{2,4}$ are orthogonal. A similar argument shows that any two different functions $\psi_{n,k}$ and $\psi_{m,j}$ are orthogonal, and that $\psi_{n,k}$ is orthogonal to every function φ_j (Exercise 20). Therefore, the Haar system is an orthonormal collection of functions.

DEFINITION Let f be an integrable function that is supported in a finite interval. The **Haar coefficients** of f are the numbers

$$\begin{aligned} a_k &= \langle f, \varphi_k \rangle \quad \text{for integer } k, \\ b_{n,k} &= \langle f, \psi_{n,k} \rangle \quad \text{for integer } k \text{ and nonnegative integer } n. \end{aligned}$$

Since φ_k is identically 1 on $[k, k+1)$ and 0 elsewhere, the Haar coefficient a_k is given by

$$a_k = \langle f, \varphi_k \rangle = \int_{-\infty}^{\infty} f(x) \varphi_k(x) dx = \int_k^{k+1} f(x) dx. \quad (15)$$

This is the average value of f on the interval $[k, k+1)$.

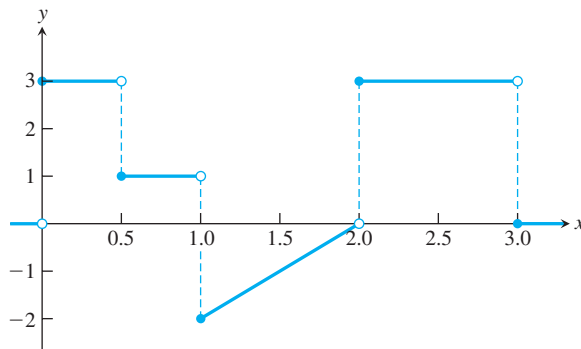
The function $\psi_{n,k}(x)$ takes the value $2^{n/2}$ for $2^{-n}k \leq x < 2^{-n}(k + \frac{1}{2})$, the value $-2^{n/2}$ for $2^{-n}(k + \frac{1}{2}) \leq x < 2^{-n}(k+1)$, and 0 elsewhere. Therefore, the Haar coefficient $b_{n,k}$ is given by

$$\begin{aligned} b_{n,k} &= \langle f, \psi_{n,k} \rangle = \int_{-\infty}^{\infty} f(x) \psi_{n,k}(x) dx \\ &= 2^{n/2} \int_{2^{-n}k}^{2^{-n}(k+1/2)} f(x) dx - 2^{n/2} \int_{2^{-n}(k+1/2)}^{2^{-n}(k+1)} f(x) dx. \end{aligned} \quad (16)$$

EXAMPLE 1 Find the Haar coefficients for the function

$$f(x) = \begin{cases} 3, & 0 \leq x < 1/2 \\ 1, & 1/2 \leq x < 1 \\ 2x - 4, & 1 \leq x < 2 \\ 3, & 2 \leq x < 3 \\ 0, & \text{otherwise.} \end{cases}$$

This function is pictured in Figure 19.13.

FIGURE 19.13 The function f from Example 1.

Solution By Equation (15), the Haar coefficient $a_k = \langle f, \varphi_k \rangle$ is the average value of f on $[k, k + 1]$. Since f is supported in the interval $[0, 3]$, the coefficient $\langle f, \varphi_k \rangle$ can only be nonzero for $k = 0, 1$, and 2 . For these k we compute that

$$a_0 = \langle f, \varphi_0 \rangle = \int_0^1 f(x) dx = \int_0^{1/2} 3 dx + \int_{1/2}^1 1 dx = 2$$

$$a_1 = \langle f, \varphi_1 \rangle = \int_1^2 f(x) dx = \int_1^2 (2x - 4) dx = -1$$

$$a_2 = \langle f, \varphi_2 \rangle = \int_2^3 f(x) dx = \int_2^3 3 dx = 3.$$

Now consider the coefficients $b_{n,k} = \langle f, \psi_{n,k} \rangle$ with $n = 0$. The function $\psi_{0,k}$ is supported in $[k, k + 1]$ while f is supported in $[0, 3]$, so $\langle f, \psi_{0,k} \rangle$ can only be nonzero for $k = 0, 1$, and 2 . For these k we use Equation (16) to compute that

$$b_{0,0} = \langle f, \psi_{0,0} \rangle = \int_0^{1/2} f(x) dx - \int_{1/2}^1 f(x) dx = \int_0^{1/2} 3 dx - \int_{1/2}^1 1 dx = 1$$

$$b_{0,1} = \langle f, \psi_{0,1} \rangle = \int_1^{3/2} f(x) dx - \int_{3/2}^2 f(x) dx = \int_1^{3/2} (2x - 4) dx - \int_{3/2}^2 (2x - 4) dx = -\frac{1}{2}$$

$$b_{0,2} = \langle f, \psi_{0,2} \rangle = \int_2^{5/2} f(x) dx - \int_{5/2}^3 f(x) dx = \int_2^{5/2} 3 dx - \int_{5/2}^3 3 dx = 0.$$

Notice that $b_{0,2}$ is zero because f is constant on the interval $[2, 3)$ while $\psi_{0,2}$ is a square wave on that interval.

Next we compute $b_{n,k} = \langle f, \psi_{n,k} \rangle$ for $n = 1$. Since f is supported in $[0, 3]$ while $\psi_{1,k}$ is supported in $[k/2, (k + 1)/2]$, this coefficient will be zero if these intervals do not overlap. Hence $b_{1,k} = 0$ for all $k < 0$ and $k > 5$. However, even for k between 0 and 5 the Haar coefficient could be zero. For example, f is constant on $[0, 1/2)$ while $\psi_{1,0}$ is a square wave on $[0, 1/2)$, so

$$\begin{aligned} b_{1,0} &= \langle f, \psi_{1,0} \rangle = 2^{1/2} \int_0^{1/4} f(x) dx - 2^{1/2} \int_{1/4}^{1/2} f(x) dx \\ &= 2^{1/2} \int_0^{1/4} 3 dx - 2^{1/2} \int_{1/4}^{1/2} 3 dx = 0. \end{aligned}$$

For the same reason, $b_{1,1}$, $b_{1,4}$, and $b_{1,5}$ are all zero. For $k = 2$ we have

$$\begin{aligned} b_{1,2} &= \langle f, \psi_{1,2} \rangle = 2^{1/2} \int_1^{5/4} f(x) dx - 2^{1/2} \int_{5/4}^{3/2} f(x) dx \\ &= 2^{1/2} \int_1^{5/4} (2x - 4) dx - 2^{1/2} \int_{5/4}^{3/2} (2x - 4) dx = -\frac{1}{4\sqrt{2}}, \end{aligned}$$

and for $k = 3$,

$$\begin{aligned} b_{1,3} &= \langle f, \psi_{1,3} \rangle = 2^{1/2} \int_{3/2}^{7/4} f(x) dx - 2^{1/2} \int_{7/4}^2 f(x) dx \\ &= 2^{1/2} \int_{3/2}^{7/4} (2x - 4) dx - 2^{1/2} \int_{7/4}^2 (2x - 4) dx = -\frac{1}{4\sqrt{2}}. \end{aligned}$$

For larger n , most of the Haar coefficients $b_{n,k} = \langle f, \psi_{n,k} \rangle$ are zero; in particular, this includes all of those for which the support of $\psi_{n,k}$ intersects $[0, 1]$ or $[2, 3]$. The coefficient $b_{n,k}$ is nonzero only for those k for which the support of $\psi_{n,k}$ intersects $[1, 2]$. ■

Approximation Using the Haar System

Now we describe how to use Haar system to create approximations of a function f . We do this in a series of steps where we first create a “low-resolution” approximation to f and then iteratively add finer and finer “details” to improve the approximation.

In the first step we find the linear combination of box functions φ_k that is closest to f . Since φ_k is 1 on $[k, k + 1]$ and zero elsewhere, this approximation, which we call p_0 , is a step function with jumps at integer points k . Because the φ_k are orthonormal functions, we know how to compute p_0 .

THEOREM 12 If f is an integrable function that is supported in a finite interval, then the linear combination of the box functions φ_k that is closest to f is

$$p_0(x) = \sum_{k=-\infty}^{\infty} a_k \varphi_k(x), \quad (17)$$

$$\text{where } a_k = \langle f, \varphi_k \rangle = \int_k^{k+1} f(x) dx.$$

The proof of Theorem 12 is similar to the proof of Theorem 7. Because f is identically zero outside of a finite interval, only finitely many Haar coefficients a_k will be nonzero, so the series in Equation (17) is actually a finite sum. In fact, if $M < N$ are integers and f is identically zero outside of the interval $[M, N]$, then $a_k = 0$ for all $k < M$ and $k \geq N$, so in this case the series in Equation (17) becomes

$$p_0(x) = \sum_{k=M}^{N-1} a_k \varphi_k(x).$$

Consider the function f from Example 1. Since f is identically zero outside of $[0, 3]$, all Haar coefficients a_k with $k < 0$ and $k \geq 3$ are zero. Using the values for a_0 , a_1 , and a_2 that we computed in that example, the best approximation p_0 to f is

$$p_0(x) = \sum_{k=0}^2 a_k \varphi_k(x) = 2\varphi_0(x) - \varphi_1(x) + 3\varphi_2(x).$$

This function is shown in Figure 19.14. As we can see, p_0 takes the average value of f on each interval $[k, k + 1]$. It is the best approximation to f by a step function that has jumps at integers. Since f is constant on the interval $[2, 3)$, the approximation p_0 equals f on that interval.

The function p_0 gives only a coarse approximation to f , analogous to representing an image using very large pixels. We refer to p_0 as the *approximation at resolution level zero*. The approximation p_0 to a function g that varies considerably across the interval $[0, 3]$ is shown in Figure 19.15.

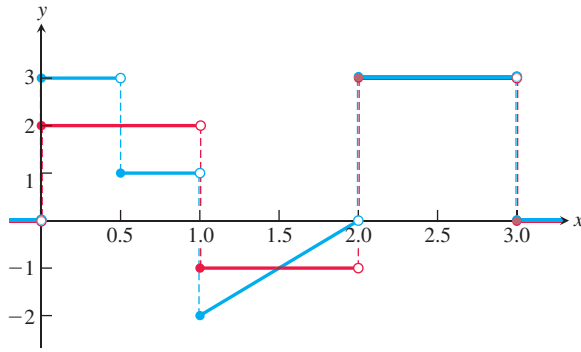


FIGURE 19.14 The function f from Example 1 (blue) and its approximation $p_0 = 2\varphi_0 - \varphi_1 + 3\varphi_2$ (red) at resolution level zero. The two functions are equal on the interval $[2, 3)$.

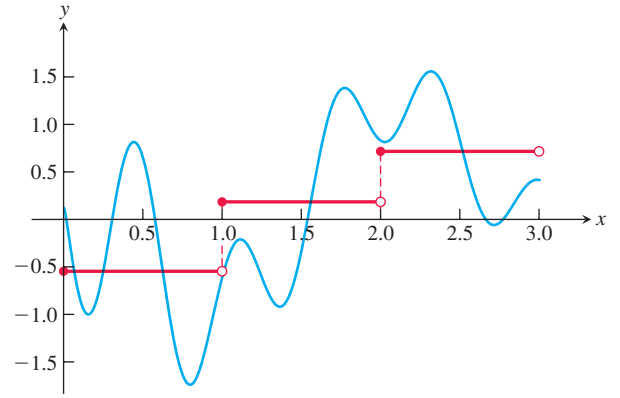


FIGURE 19.15 A function g (blue) and its approximation p_0 (red) at resolution level zero. p_0 takes the average value of g on each interval $[k, k+1)$.

Adding Details at Resolution Level Zero

Now we will see how to create an approximation p_1 at *resolution level 1* that is better than the approximation p_0 at resolution level zero. To do this, we first create a function q_0 that represents the “details in f ” at resolution level zero. This function is

$$q_0(x) = \sum_{k=-\infty}^{\infty} b_{0,k} \psi_{0,k}(x), \quad (18)$$

where $b_{0,k} = \langle f, \psi_{0,k} \rangle$. Since f is supported in a finite interval, only finitely many of the Haar coefficients $b_{0,k}$ can be nonzero, so the series in Equation (18) is actually a finite sum. Hence the detail function q_0 is a linear combination of finitely many of the wavelets $\psi_{0,k}$.

In Example 1 we computed the Haar coefficients $b_{0,k}$ for the function f shown in Figure 19.13. Using those coefficients, we see that the detail function for this f is

$$q_0(x) = b_{0,0} \psi_{0,0}(x) + b_{0,1} \psi_{0,1}(x) + b_{0,2} \psi_{0,2}(x) = \psi_{0,0}(x) - \frac{1}{2} \psi_{0,1}(x).$$

This function is shown in Figure 19.16.

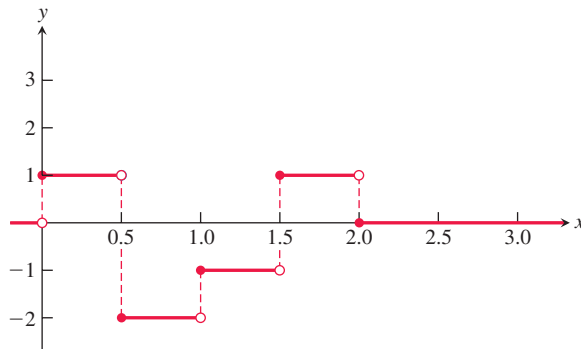


FIGURE 19.16 The detail function $q_0 = \psi_{0,0} - \frac{1}{2}\psi_{0,1}$ for the function f from Example 1.

The detail function q_0 is not itself a good approximation to f ; instead, it gives us the information needed to move from the coarse approximation p_0 to a better approximation p_1 . Specifically, p_1 is obtained by adding the detail function q_0 to the approximation p_0 :

$$p_1(x) = p_0(x) + q_0(x) = \sum_{k=-\infty}^{\infty} a_k \varphi_k(x) + \sum_{k=-\infty}^{\infty} b_{0,k} \psi_{0,k}(x).$$

As before, only finitely many terms in each series can be nonzero, so these are actually finite sums. This function p_1 is the best approximation to f using *both* the box functions φ_k and the Haar wavelets $\psi_{0,k}$. We say that p_1 is the approximation to f at *resolution level 1*.

The improved approximation p_1 for the function f from Example 1 is shown in Figure 19.17. Like p_0 , it is a step function. However, while p_0 is a step function with jumps at integers k , the new approximation p_1 is a step function with jumps at points $k/2$ where k is an integer. Because f is constant on each of the intervals $[0, 1/2)$, $[1/2, 1)$, $[2, 5/2)$, and $[5/2, 3)$, the approximation p_1 equals f on those intervals. On the intervals $[1, 3/2)$ and $[3/2, 2)$ it takes the average value of f on those intervals.

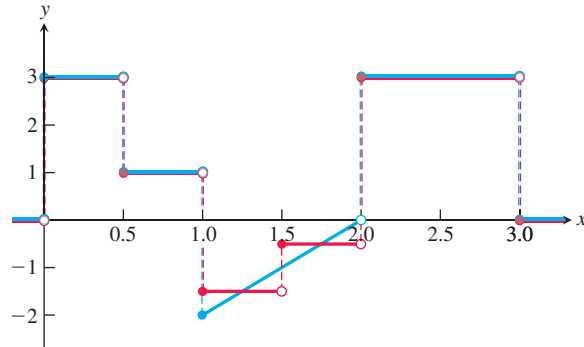


FIGURE 19.17 The approximation p_1 (red) to the function f (blue) from Example 1.

Figure 19.18 shows a superposition of p_0 and p_1 . Comparing this to Figure 19.16, we see how p_1 is obtained by adding scaled Haar wavelets $b_{0,k} \psi_{0,k}$ to p_0 .

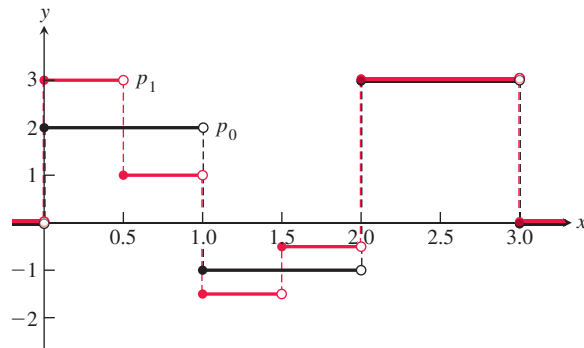


FIGURE 19.18 The approximations p_0 (black) and p_1 (red) to the function f from Example 1. We obtain p_1 by adding $q_0 = \psi_{0,0} - \frac{1}{2}\psi_{0,1}$ to p_0 .

The Multiresolution Recursion

We iterate the process described above to further improve the level of resolution in the approximation to f . The function p_1 gave us the approximation at resolution level 1. The detail function at this level is

$$q_1(x) = \sum_{k=-\infty}^{\infty} b_{1,k} \psi_{1,k}(x),$$

where $b_{1,k} = \langle f, \psi_{1,k} \rangle$. Adding q_1 to p_1 gives us the best approximation to f at resolution level 2:

$$\begin{aligned} p_2(x) &= p_1(x) + q_1(x) \\ &= p_0(x) + q_0(x) + q_1(x) \\ &= \sum_{k=-\infty}^{\infty} a_k \varphi_k(x) + \sum_{k=-\infty}^{\infty} b_{0,k} \psi_{0,k}(x) + \sum_{k=-\infty}^{\infty} b_{1,k} \psi_{1,k}(x). \end{aligned}$$

This is a step function with jumps at points $k/4$ where k is an integer. The approximation p_0 at resolution level zero for a function g was shown in Figure 19.15. Figure 19.19 shows the approximation p_1 at resolution level 1 and the approximation p_2 at resolution level 2 for this same function g .

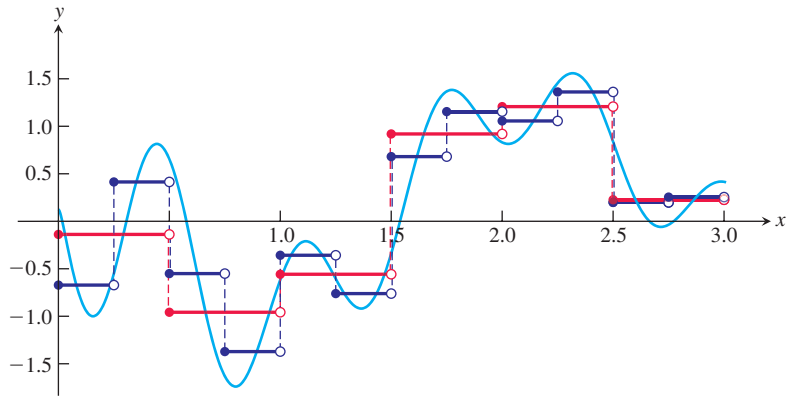


FIGURE 19.19 The approximations p_1 (red) and p_2 (dark blue) for the function g from Figure 19.15.

In general, once we have the approximation p_n at resolution level n , the detail function at that resolution level is

$$q_n(x) = \sum_{k=-\infty}^{\infty} b_{n,k} \psi_{n,k}(x) = \sum_{k=-\infty}^{\infty} \langle f, \psi_{n,k} \rangle \psi_{n,k}(x),$$

and the best approximation at the next resolution level $n + 1$ is

$$p_{n+1}(x) = p_n(x) + q_n(x).$$

Expanding,

$$p_{n+1}(x) = p_0(x) + q_0(x) + q_1(x) + \cdots + q_n(x).$$

We see that the approximation $p_{n+1}(x)$ is obtained by progressively adding finer and finer details to $p_0(x)$. The next theorem states that the distance between f and the approximation p_n at resolution level n converges to zero as $n \rightarrow \infty$. The proof of Theorem 13 can be found in more advanced texts on wavelets or real analysis.

THEOREM 13 Let f be an integrable function on the real line that is supported in a finite interval, and for each positive integer n let p_n be the function defined by the recursive process given above. That is, p_n is the linear combination of the functions $\varphi_k, \psi_{0,k}, \psi_{1,k}, \dots, \psi_{n-1,k}$ (k integer) that is closest to f . Then the distance between f and p_n converges to zero as $n \rightarrow \infty$:

$$\lim_{n \rightarrow \infty} \|f - p_n\| = \lim_{n \rightarrow \infty} \sqrt{\int_{-\infty}^{\infty} (f(x) - p_n(x))^2 dx} = 0.$$

Wavelets

The Haar system and the trigonometric system are not well-suited to every application. Each system is better suited for certain types of applications than the other. Fourier series often provide good approximations when dealing with periodic functions. The Haar system is typically a better choice when an application has a multiresolution aspect. However, there are other wavelet systems that can be even more advantageous in such applications.

One drawback of the Haar system is that the box functions φ_k and the wavelets $\psi_{n,k}$ are discontinuous. To explain another drawback, consider the function

$$f(x) = \begin{cases} 1, & 0 \leq x < m, \\ 0, & x < 0 \text{ or } x \geq m, \end{cases}$$

where m is an integer. Using the Haar system, the approximation to f at resolution level zero is f itself. Specifically,

$$p_0(x) = \sum_{k=0}^{m-1} \varphi_k(x) = f(x).$$

However, only functions that, like this f , are “locally constant” can be well represented at this low resolution level. In contrast, the approximation p_0 to

$$g(x) = \begin{cases} x, & 0 \leq x < m, \\ 0, & x < 0 \text{ or } x \geq m, \end{cases}$$

is not g itself, but rather a step function with jumps at integer points.

Today we have constructions of orthonormal wavelet systems that consist of continuous, differentiable, or smoother functions, and these systems provide good representations of functions that are locally linear, locally quadratic, or locally polynomial of higher order. In particular, for each even integer $2N$ (where $N = 1, 2, 3, \dots$), Ingrid Daubechies constructed functions D_{2N} and W_{2N} that can be used instead of the box function φ and the Haar wavelet ψ to generate orthonormal systems. Indeed, φ and ψ are precisely the functions D_2 and W_2 that correspond to $N = 1$. The function D_{2N} is called a **Daubechies scaling function** and W_{2N} is a **Daubechies wavelet**. The functions D_4 and W_4 , corresponding to $N = 2$, are shown in Figures 19.20 and 19.21. They cannot be given by closed-form formulas, but Exercises 11–15 in the Additional and Advanced Exercises discuss the recursive process by which they are defined.

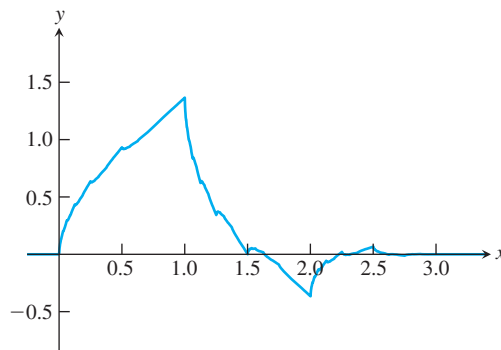


FIGURE 19.20 The Daubechies scaling function D_4 . This function is identically zero outside of the interval $[0, 3]$.

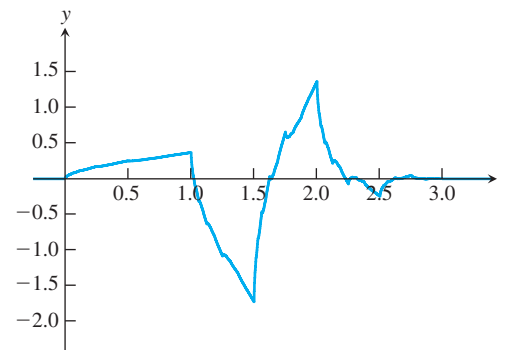


FIGURE 19.21 The Daubechies wavelet W_4 . This function is identically zero outside of the interval $[0, 3]$.

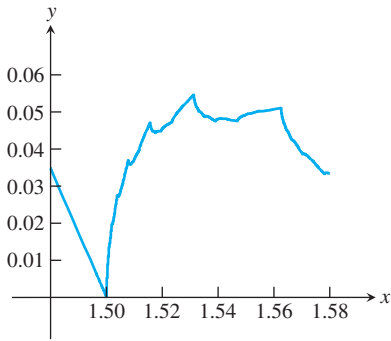


FIGURE 19.22 An enlargement of the graph of D_4 .

It is not apparent from their graphs, but D_4 and W_4 satisfy the same orthogonality relations as the box function φ and the Haar wavelet ψ . The translated box functions $\varphi_k(x) = \varphi(x - k)$ are orthogonal, and likewise the translated functions $D_4(x - k)$ are orthogonal. It is easy to see that the functions φ_k are orthogonal because their supports do not overlap. In contrast, the supports of the functions $D_4(x - k)$ do overlap, but even so the inner product $\langle D_4(x - j), D_4(x - k) \rangle$ is zero whenever $j \neq k$. Further, the collection of translates and dilates $2^{n/2}W_4(2^n x - k)$ over all integer k and nonnegative n is orthonormal, and we can substitute D_4 and W_4 for φ and ψ in the multiresolution approximation process that we described earlier.

The Daubechies scaling function D_4 and wavelet W_4 are each supported in the interval $[0, 3]$, and they are continuous, but not differentiable. Indeed, their graphs have a fractal appearance in the sense that no matter how much they are magnified, they are still jagged. Figure 19.22 shows an enlargement of the graph of D_4 centered on the interval $[1.48, 1.58]$. Linear combinations of the translates $D_4(x - k)$ can represent functions that are locally constant as well as functions that are locally linear. As N increases, D_{2N} and W_{2N} become smoother but more spread out (supported in $[0, 2N - 1]$), and can locally represent polynomials of higher order. The proof of these facts about Daubechies' constructions are much more involved than for the Haar wavelets, and can be found in texts on wavelet theory.

EXERCISES 19.5

Norms of Functions

In Exercises 1–6, the values of a function on the interval $[0, 2]$ are given, and we assume that the function is identically zero outside of that interval. Find the norm of each function.

$$1. f(x) = \begin{cases} 5, & 0 \leq x < 1 \\ 1, & 1 \leq x \leq 2 \end{cases}$$

$$2. g(x) = \begin{cases} 5, & 0 \leq x < 1/2 \\ 1, & 1/2 \leq x < 1 \\ -1, & 1 \leq x \leq 2 \end{cases}$$

$$3. h(x) = x, \quad 0 \leq x \leq 2$$

$$4. j(x) = x^2, \quad 0 \leq x \leq 2$$

$$5. \ell(x) = \begin{cases} 1, & 0 \leq x < 1 \\ \sin 2\pi x, & 1 \leq x \leq 2 \end{cases}$$

$$6. m(x) = \begin{cases} 2, & 0 \leq x < 1/2 \\ -2, & 1/2 \leq x < 1 \\ \cos 2\pi x, & 1 \leq x \leq 2 \end{cases}$$

Inner Product of Functions

In Exercises 7–12, use the functions defined in Exercises 1–6 to compute the given inner product. Are the two functions orthogonal?

$$7. \langle g, j \rangle$$

$$8. \langle \ell, \psi \rangle$$

$$9. \langle g, m \rangle$$

$$10. \langle h, \psi_{2,1} \rangle$$

$$11. \langle f, \varphi_1 \rangle$$

$$12. \langle m, \psi_{2,3} \rangle$$

Finding Haar Coefficients

In Exercises 13–18, find the Haar coefficients $a_k = \langle f, \varphi_k \rangle$ and $b_{n,k} = \langle f, \psi_{n,k} \rangle$ for the given function. If available, use a CAS to plot the approximation p_n at resolution levels $n = 0, 1$, and 2 over the domain $[0, 2]$.

13. The function f from Exercise 1.

14. The function g from Exercise 2.

15. The function h from Exercise 3.

16. The function j from Exercise 4.

17. The function ℓ from Exercise 5.

18. The function m from Exercise 6.

Theory and Examples

19. Explain why the Haar wavelet $\psi_{n,k}$ is supported in the interval $[2^{-n}k, 2^{-n}(k+1)]$ and why it is a unit function.

20. Prove that for integers j and k and nonnegative integers m and n we have

$$\text{a. } \langle \varphi_j, \varphi_k \rangle = 0 \quad \text{for all } j \neq k,$$

$$\text{b. } \langle \varphi_j, \psi_{n,k} \rangle = 0 \quad \text{for all } n, j, \text{ and } k,$$

$$\text{c. } \langle \psi_{m,j}, \psi_{n,k} \rangle = 0 \quad \text{for all } n \text{ and all } j \neq k,$$

$$\text{d. } \langle \psi_{m,j}, \psi_{n,k} \rangle = 0 \quad \text{for all } j \text{ and } k \text{ and all } m \neq n.$$

Explain why this implies that the Haar system is an orthonormal collection of functions.

CHAPTER 19 Questions to Guide Your Review

Vectors

Questions 1–5 are about vectors in d -dimensional space.

1. What is the length of a vector in d -dimensional space?
2. How do you determine the dot product of vectors in d -dimensional space? Which algebraic properties are satisfied by the dot product?
3. What does it mean for a collection of vectors in d -dimensional space to be orthogonal? orthonormal?
4. What is an orthonormal basis of vectors for d -dimensional space?
5. Assume that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_d\}$ is an orthonormal basis for d -dimensional space and $1 \leq n < d$. If $\mathbf{v}$ is a vector in d -dimensional space, what is the linear combination of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ that is closest to $\mathbf{v}$?

Periodic Functions

Questions 6–11 are about periodic functions.

6. What is a periodic function?
7. What is the norm of an integrable function on the domain $[0, 2\pi]$? What is a unit function?
8. What is the inner product of two integrable functions on the domain $[0, 2\pi]$? When are two such functions orthogonal? orthonormal?

9. What does it mean for a collection of integrable functions on the domain $[0, 2\pi]$ to be orthogonal? orthonormal?
10. What are the Fourier coefficients of an integrable function f that is periodic with period 2π ?
11. If f is an integrable function on $[0, 2\pi]$, how do you find the trigonometric polynomial p_n that is the best approximation to f by a linear combination of $1, \cos x, \dots, \cos nx, \sin x, \dots, \sin nx$?

Functions on the Real Line

Questions 12–16 are about functions whose domain is the real line.

12. What does it mean to say that an integrable function f is supported in an interval $[a, b]$? What is the norm of f ? What is a unit function?
13. What is the inner product of two integrable functions that are each supported in a finite interval? When are two such functions orthogonal? orthonormal?
14. What interval is the function φ_k supported in? $\psi_{n,k}$?
15. What are the Haar coefficients of an integrable function f that is supported in a finite interval?
16. If f is an integrable function that is supported in a finite interval, how do you find the Haar approximation p_n to f at resolution level n ? What is the detail function q_n at resolution level n ?

CHAPTER 19 Additional and Advanced Exercises

Finding More Orthonormal Vectors

1. Let $\mathbf{u}_1 = \langle 1/\sqrt{5}, 2/\sqrt{5} \rangle$ and $\mathbf{v} = \langle 3, -1 \rangle$.
 - a. Show that $\mathbf{u}_1$ is a unit vector.
 - b. Find the vector $\mathbf{p}$ on the line through $\mathbf{u}_1$ that is closest to $\mathbf{v}$.
 - c. Verify that $\mathbf{q} = \mathbf{v} - \mathbf{p}$ is orthogonal to $\mathbf{u}_1$.
 - d. Set $\mathbf{u}_2 = \mathbf{q}/\|\mathbf{q}\|$, and explain why $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis for two-dimensional space.
2. Let $\mathbf{u}_1 = \langle 1/\sqrt{6}, -1/\sqrt{6}, \sqrt{2}/\sqrt{3} \rangle$, $\mathbf{u}_2 = \langle 1/\sqrt{2}, 1/\sqrt{2}, 0 \rangle$, and $\mathbf{v} = \langle 3, -1, 1 \rangle$.
 - a. Show that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthonormal.
 - b. Find the linear combination $\mathbf{p}$ of $\mathbf{u}_1$ and $\mathbf{u}_2$ that is closest to $\mathbf{v}$.
 - c. Verify that $\mathbf{q} = \mathbf{v} - \mathbf{p}$ is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$.
 - d. Set $\mathbf{u}_3 = \mathbf{q}/\|\mathbf{q}\|$, and explain why $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for three-dimensional space.

Orthogonal Polynomials

The functions in Exercises 3–9 are all supported in the interval $[-1, 1]$. The norm of such a function f is

$$\|f\| = \sqrt{\int_{-1}^1 f(x)^2 dx},$$

and the inner product of f with g is

$$\langle f, g \rangle = \int_{-1}^1 f(x) g(x) dx.$$

For each integer $n \geq 0$ we let v_n be the function defined by $v_n(x) = x^n$ for $-1 \leq x \leq 1$ and $v_n(x) = 0$ for $|x| > 1$.

3. Find the norm of v_n for each $n \geq 0$.
4. Find the inner product of v_m with v_n for $m, n \geq 0$. For which m and n is v_m orthogonal to v_n ? Is $\{v_n\}_{n \geq 0}$ an orthogonal collection of functions?

In Exercises 5–9, for $-1 \leq x \leq 1$ define

$$q_0(x) = 1, \quad q_1(x) = x, \quad q_2(x) = 3x^2 - 1,$$

and set $q_0(x) = q_1(x) = q_2(x) = 0$ for $|x| > 1$. Also define

$$u_0(x) = \frac{q_0(x)}{\|q_0\|}, \quad u_1(x) = \frac{q_1(x)}{\|q_1\|}, \quad u_2(x) = \frac{q_2(x)}{\|q_2\|}.$$

5. Find $\|q_n\|$ for $n = 0, 1$, and 2 .
6. Prove that $\{q_0, q_1, q_2\}$ is an orthogonal set of functions.
7. Find explicit formulas for u_0, u_1 , and u_2 , and prove that $\{u_0, u_1, u_2\}$ is an orthonormal set of functions.
8. Find the function

$$w_3(x) = \langle v_3, u_0 \rangle u_0(x) + \langle v_3, u_1 \rangle u_1(x) + \langle v_3, u_2 \rangle u_2(x).$$

This is the linear combination of u_0, u_1 , and u_2 that is closest to the function v_3 .

9. Using the function w_3 from Exercise 8, let

$$q_3(x) = v_3(x) - w_3(x).$$

Find $\|q_3\|$, and set

$$u_3(x) = \frac{q_3(x)}{\|q_3\|}.$$

Prove that $\{u_0, u_1, u_2, u_3\}$ is an orthonormal set of functions.

The Haar System

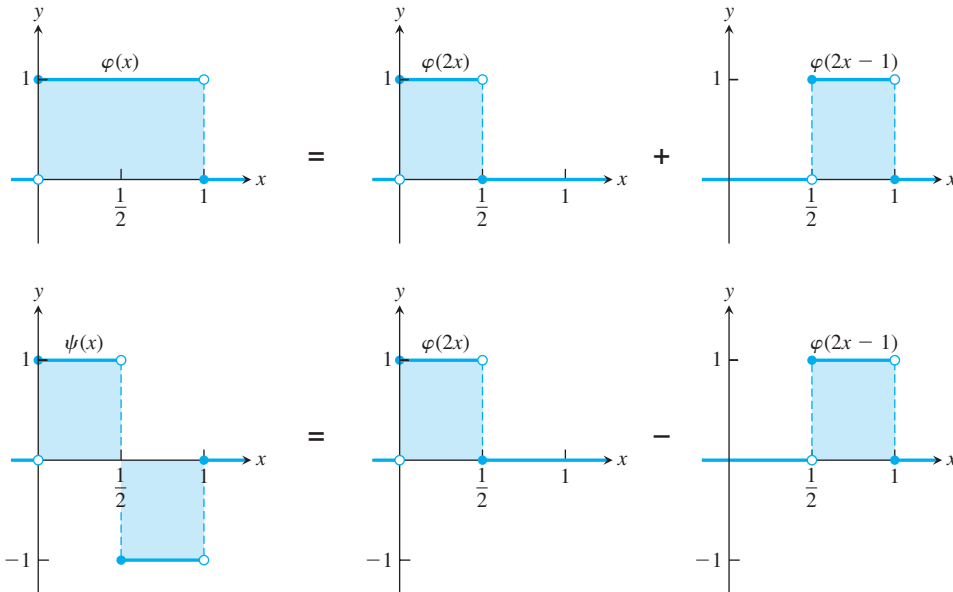
10. Prove that the box function φ satisfies the **refinement equation**

$$\varphi(x) = \varphi(2x) + \varphi(2x - 1),$$

and the Haar wavelet ψ can be obtained from the box function φ by

$$\psi(x) = \varphi(2x) - \varphi(2x - 1).$$

See the accompanying figure.

The Daubechies Scaling Function D_4

It is proved on texts in wavelet theory that the Daubechies scaling function D_4 is a continuous function that is supported in a finite interval and satisfies the refinement equation

$$D_4(x) = c_0 D_4(2x) + c_1 D_4(2x - 1) + c_2 D_4(2x - 2) + c_3 D_4(2x - 3),$$

where

$$c_0 = \frac{1 + \sqrt{3}}{4}, \quad c_1 = \frac{3 + \sqrt{3}}{4}, \quad c_2 = \frac{3 - \sqrt{3}}{4}, \quad c_3 = \frac{1 - \sqrt{3}}{4}.$$

We will use this refinement equation in Exercises 11–15.

11. The Daubechies scaling function D_4 is supported in some finite interval $[a, b]$. Use the refinement equation given above to prove that D_4 is actually supported in the interval $[0, 3]$. (Hint: Let $[a, b]$ be the smallest interval such that $D_4(x) = 0$ for every x outside of $[a, b]$. Use the refinement equation to prove that if $x \notin [a/2, (b+3)/2]$, then $2x < a$ and $2x - 3 > b$, and therefore $D_4(x) = 0$.)

12. Since D_4 is continuous and supported in the interval $[0, 3]$, we have $D_4(0) = D_4(3) = 0$. Let $a = D_4(1)$ and $b = D_4(2)$, and use the refinement equation to show that a and b satisfy the equations

$$\begin{aligned} a &= c_0 b + c_1 a, \\ b &= c_3 b + c_2 a. \end{aligned}$$

Prove that

$$a = \frac{1 + \sqrt{3}}{2}, \quad b = \frac{1 - \sqrt{3}}{2}$$

solves this system of equations and also satisfies $a + b = 1$.

13. Exercise 12 computed the values $D_4(1)$ and $D_4(2)$. Since D_4 is continuous and identically zero outside of $[0, 3]$, we know that $D_4(k) = 0$ for every integer k other than $k = 1$ and $k = 2$. Use the refinement equation to compute $D_4(k/2)$ for integer k .

T 14. This exercise will explain how Figure 19.20 was generated. Exercises 12 and 13 computed the values $D_4(k)$ and $D_4(k/2)$ for integer k . Use a CAS to compute the values $D_4(k/4)$ for integer k and plot the result. Then compute $D_4(k/8)$ for integer k and plot the result. Iterate this process until a sufficiently detailed plot of the graph of D_4 is obtained.

T 15. The Daubechies wavelet W_4 is derived from the Daubechies scaling function D_4 by the equation

$$W_4(x) = -c_3 D_4(2x) + c_2 D_4(2x - 1) - c_1 D_4(2x - 2) + c_0 D_4(2x - 3).$$

Use the values for $D_4(x)$ obtained in Exercise 14 to make a plot of the graph of W_4 .

T 16. Using the values for $D_4(x)$ obtained in Exercise 14, plot the function

$$F(x) = \sum_{k=-2}^4 D_4(x - k)$$

on the interval $[-2, 7]$ (remember that $D_4(x) = 0$ if $x \leq 0$ or $x \geq 3$). Numerically, it appears that $F(x)$ is constant on the interval $[0, 5]$.

T 17. Using the values for $D_4(x)$ obtained in Exercise 14, plot the function

$$G(x) = \sum_{k=-2}^4 k D_4(x - k)$$

on the interval $[-2, 7]$ (remember that $D_4(x) = 0$ if $x \leq 0$ or $x \geq 3$). Numerically, it appears that $G(x)$ is linear on the interval $[0, 5]$; that is, there are constants c and d such that $G(x) = cx + d$ for $0 \leq x \leq 5$.

ANSWERS TO ODD-NUMBERED EXERCISES

Chapter 19

SECTION 19.1, p. 19-3

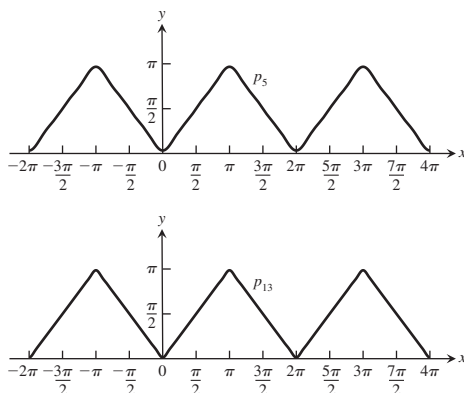
1. 2π 3. Not periodic 5. 2π
 7. (a) Since $f(x + P) = f(x)$ and $g(x + P) = g(x)$ for every x , it follows that $f(x + P) + g(x + P) = f(x) + g(x)$ for every x .

SECTION 19.2, p. 19-5

1. This follows from the fact that $\sin(-kx) = -\sin kx$.
 3. The graphs of p_5 and p_{13} are shown in the accompanying figure. It appears that p_n is converging to the function f whose values on the domain $[0, 2\pi]$ are

$$f(x) = \begin{cases} x, & 0 \leq x \leq \pi, \\ 2\pi - x, & \pi < x \leq 2\pi, \end{cases}$$

with these values then repeated every 2π units. The convergence is “fast” since the graph of p_n is already close to the graph of f when n is small.



5. $z^0 = \cos 0x + i \sin 0x = 1$
 7. $2z^2 + 3z - 5 = 2e^{i2x} + 3e^{ix} - 5 = 2(\cos 2x + i \sin 2x) + 3(\cos x + i \sin x) - 5$
 9. $z^{-1} = (e^{ix})^{-1} = e^{-ix} = \cos(-x) + i \sin(-x) = \cos x - i \sin x$
 11. $z + z^{-1} = (\cos x + i \sin x) + (\cos x - i \sin x) = 2 \cos x$
 13. $z^n + z^{-n} = (\cos nx + i \sin nx) + (\cos nx - i \sin nx) = 2 \cos nx$
 15. $2z^2 + 3z - 5 + 3z^{-1} + 2z^{-2} = 2(z^2 + z^{-2}) + 3(z + z^{-1}) - 5 = 2 \cdot 2 \cos 2x + 3 \cdot 2 \cos x - 5$
 17. $\cos x + 2 \sin x - 3 = \frac{1}{2}(z + z^{-1}) - i(z - z^{-1}) - 3 = (0.5 - i)z - 3 + (0.5 + i)z^{-1}$

SECTION 19.3, pp. 19-11-19-12

1. (a) $\mathbf{r} \cdot \mathbf{s} = 0$, $\mathbf{r} \cdot \mathbf{t} = 0$, $\mathbf{s} \cdot \mathbf{t} = 0$
 (b) $|\mathbf{r}| = \sqrt{2}$, $|\mathbf{s}| = \sqrt{2}$, $|\mathbf{t}| = 1$
 (c) Yes (d) No
 3. (a) $\mathbf{r} \cdot \mathbf{s} = 0$, $\mathbf{r} \cdot \mathbf{t} = 0$, $\mathbf{s} \cdot \mathbf{t} = -3$
 (b) $|\mathbf{r}| = \sqrt{3}$, $|\mathbf{s}| = \sqrt{6}$, $|\mathbf{t}| = \sqrt{2}$
 (c) No (d) No

5. (a) $\mathbf{r} \cdot \mathbf{s} = 0$, $\mathbf{r} \cdot \mathbf{t} = 0$, $\mathbf{s} \cdot \mathbf{t} = 0$

(b) $|\mathbf{r}| = 1$, $|\mathbf{s}| = 1$, $|\mathbf{t}| = 1$

(c) Yes (d) Yes

7. (b) $\mathbf{p} = \langle 1, 2, 0 \rangle$ (c) 3

9. (b) $\mathbf{p} = \langle 7/2, -1/2, 1 \rangle$ (c) $3/\sqrt{2}$

11. $2\mathbf{u} + \mathbf{v} - 3\mathbf{w} = \langle -6, 4, -7, 8, 1 \rangle$,
 $-\mathbf{u} + 5\mathbf{v} - \mathbf{w} = \langle -10, 6, 4, -7, -3 \rangle$

13. $\mathbf{u} \cdot \mathbf{v} = -6$, $\mathbf{u} \cdot \mathbf{w} = 3$, $\mathbf{v} \cdot \mathbf{w} = -1$

15. $\mathbf{u}_j \cdot \mathbf{u}_k = 0$ when $j \neq k$, and $|\mathbf{u}_k| = 1$ for $k = 1, 2, 3, 4, 5$

17. $\mathbf{p} = (\mathbf{v} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v} \cdot \mathbf{u}_2)\mathbf{u}_2 = \langle 2, -4, 0, 0, 0 \rangle$,

$|\mathbf{v} - \mathbf{p}| = \sqrt{21} \approx 4.58$, $(\mathbf{v} - \mathbf{p}) \cdot \mathbf{u}_1 = (\mathbf{v} - \mathbf{p}) \cdot \mathbf{u}_2 = 0$

19. $\mathbf{p} = (\mathbf{v} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v} \cdot \mathbf{u}_2)\mathbf{u}_2 + (\mathbf{v} \cdot \mathbf{u}_3)\mathbf{u}_3 + (\mathbf{v} \cdot \mathbf{u}_4)\mathbf{u}_4 = \langle 2, -4, 4, 2, 0 \rangle$, $|\mathbf{v} - \mathbf{p}| = 1$, $(\mathbf{v} - \mathbf{p}) \cdot \mathbf{u}_1 = (\mathbf{v} - \mathbf{p}) \cdot \mathbf{u}_2 = (\mathbf{v} - \mathbf{p}) \cdot \mathbf{u}_3 = (\mathbf{v} - \mathbf{p}) \cdot \mathbf{u}_4 = 0$

21. $\mathbf{u} \cdot \mathbf{v} = \sum_{k=1}^d u_k v_k = \sum_{k=1}^d v_k u_k = \mathbf{v} \cdot \mathbf{u}$

23. By the symmetry and distributive properties (Exercises 21 and 22),
 $|\mathbf{u} + \mathbf{v}|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = |\mathbf{u}|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2$.

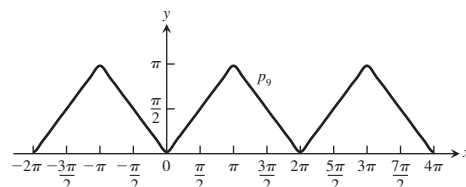
SECTION 19.4, p. 19-23

1. $\sqrt{2/3} \pi^{3/2}$ 3. $\sqrt{2/5} \pi^{5/2}$ 5. $\sqrt{(e^{4\pi} - 1)/2}$

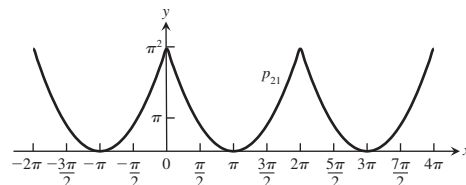
7. $7\pi^3/24$, no 9. $2\pi^3/3$, no 11. 0, yes

13. $a_0 = \frac{\pi}{2}$, $a_{2k-1} = -\frac{4}{(2k-1)^2 \pi}$, $a_{2k} = 0$, $b_k = 0$,

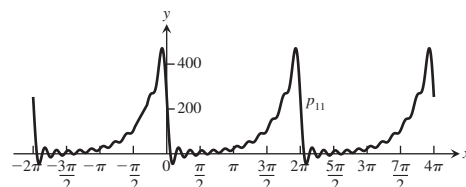
$p_{2n}(x) = p_{2n-1}(x) = \frac{\pi}{2} - \sum_{k=1}^n \frac{4}{(2k-1)^2 \pi} \cos(2k-1)x$



15. $a_0 = \frac{\pi^2}{3}$, $a_k = \frac{4}{k^2}$, $b_k = 0$, $p_n(x) = \frac{\pi^2}{3} + \sum_{k=1}^n \frac{4}{k^2} \cos kx$



17. $a_0 = \frac{e^{2\pi} - 1}{2\pi}$, $a_k = \frac{e^{2\pi} - 1}{(k^2 + 1)\pi}$, $b_k = -\frac{k(e^{2\pi} - 1)}{(k^2 + 1)\pi}$,
 $p_n(x) = \frac{e^{2\pi} - 1}{2\pi} + \sum_{k=1}^n \frac{e^{2\pi} - 1}{(k^2 + 1)\pi} \cos kx - \sum_{k=1}^n \frac{k(e^{2\pi} - 1)}{(k^2 + 1)\pi} \sin kx$



23. $\langle f, g \rangle = \int_0^{2\pi} f(x)g(x) dx = \int_0^{2\pi} g(x)f(x) dx = \langle g, f \rangle$
29. $a_0 = a_1 = b_1 = 1$, and $a_k = b_k = 0$ for all $k \geq 2$ so $p_n = 1 + \cos(x) + \sin(x) = f(x)$ for all n .
31. Assume that $m \neq n$ are positive integers. The integral of $\cos kx$ over $[0, 2\pi]$ is zero when k is a nonzero integer. Therefore, since $m - n \neq 0$ and $m + n \neq 0$, we have

$$\int_0^{2\pi} \cos(m-n)x dx = 0 \quad \text{and} \quad \int_0^{2\pi} \cos(m+n)x dx = 0.$$

Applying the trigonometric identity given in the hint, it follows that

$$\begin{aligned} \int_0^{2\pi} \sin mx \sin nx dx &= \frac{1}{2} \int_0^{2\pi} \cos(m-n)x dx \\ &\quad - \frac{1}{2} \int_0^{2\pi} \cos(m+n)x dx = 0 - 0 = 0. \end{aligned}$$

33. Assume that m and n are positive integers. The integral of $\sin kx$ over $[0, 2\pi]$ is zero for every integer k . Applying the trigonometric identity given in the hint, it follows that

$$\begin{aligned} \int_0^{2\pi} \sin mx \cos nx dx &= \frac{1}{2} \int_0^{2\pi} \sin(m+n)x dx \\ &\quad + \frac{1}{2} \int_0^{2\pi} \sin(m-n)x dx = 0 + 0 = 0. \end{aligned}$$

SECTION 19.5, p. 19-34

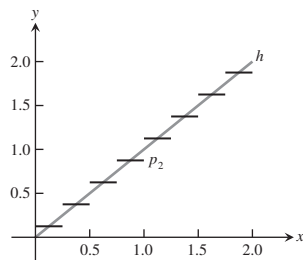
1. $\sqrt{26}$ 3. $2\sqrt{2/3}$ 5. $\sqrt{3/2}$
7. $-11/6$, no 9. 4, no 11. 1, no
13. $a_0 = 5$, $a_1 = 1$, and all other Haar coefficients are zero.
15. If $k < 0$ or $k > 1$, then $a_k = 0$. For $k = 0$ or $k = 1$,

$$a_k = \int_k^{k+1} x dx = k + \frac{1}{2}.$$

Given $n \geq 0$, if $k < 0$ or $k \geq 2^{n+1}$, then the support of $\psi_{n,k}$ does not intersect $[0, 2]$, and therefore $b_{n,k} = 0$ for those k . For all other k ,

$$b_{n,k} = 2^{n/2} \int_{2^{-n}k}^{2^{-n}(k+1/2)} x dx - 2^{n/2} \int_{2^{-n}(k+1/2)}^{2^{-n}(k+1)} x dx = -\frac{1}{4 \cdot 2^{3n/2}}.$$

This graph shows h and p_2 :

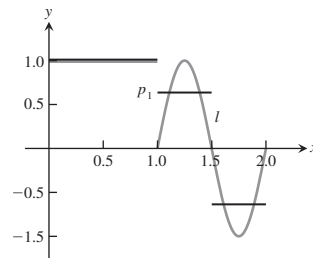


17. $a_0 = 1$, $a_1 = 0$. For $n \geq 0$, if $2^n \leq k < 2^{n+1}$, then the support of $\psi_{n,k}$ intersects the interval $[1, 2]$. For these k ,

$$\begin{aligned} b_{n,k} &= 2^{n/2} \int_{2^{-n}k}^{2^{-n}(k+1/2)} \sin 2\pi x dx - 2^{n/2} \int_{2^{-n}(k+1/2)}^{2^{-n}(k+1)} \sin 2\pi x dx \\ &= -\frac{2^{n/2}}{2\pi} \left(2 \cos 2\pi 2^{-n}(k+1/2) - \cos 2\pi 2^{-n}k \right. \\ &\quad \left. - \cos 2\pi 2^{-n}(k+1) \right) \end{aligned}$$

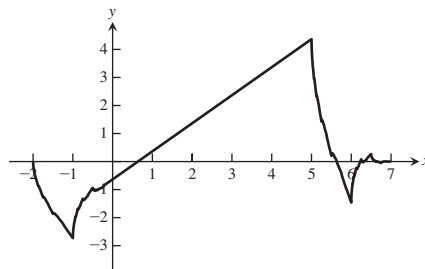
All other Haar coefficients are zero.

This graph shows ℓ and p_1 ; they are equal on $[0, 1]$:



ADDITIONAL AND ADVANCED EXERCISES, pp. 19-35–19-36

1. (a) $|\mathbf{u}_1|^2 = 1$ (b) $\mathbf{p} = \langle 1/5, 2/5 \rangle$
 (c) $\mathbf{q} = \langle 14/5, -7/5 \rangle$, $\mathbf{q} \cdot \mathbf{u}_1 = 0$
 (d) $|\mathbf{q}| = 7/\sqrt{5}$, $\mathbf{u}_2 = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$
3. $\|v_n\| = \sqrt{\frac{2}{2n+1}}$
5. $\|q_0\| = \sqrt{2}$, $\|q_1\| = \sqrt{2/3}$, $\|q_2\| = 2\sqrt{2/5}$.
7. $u_0(x) = \frac{1}{\sqrt{2}}$, $u_1(x) = \frac{x}{\sqrt{2/3}}$, $u_2(x) = \frac{3x^2 - 1}{2\sqrt{2/5}}$.
9. $q_3(x) = \frac{1}{5}(5x^3 - 3x)$, $\|q_3\| = \frac{2}{5}\sqrt{\frac{2}{7}}$,
 $u_3(x) = \frac{1}{2}\sqrt{\frac{7}{2}}(5x^3 - 3x)$.
13. $D_4(1/2) = \frac{2+\sqrt{3}}{4}$, $D_4(3/2) = 0$, $D_4(5/2) = \frac{2-\sqrt{3}}{4}$
17. The graph of G :



Appendix A

A.1 Real Numbers and the Real Line

This section reviews real numbers, inequalities, intervals, and absolute values.

Real Numbers

Much of calculus is based on properties of the real number system. **Real numbers** are numbers that can be expressed as decimals, such as

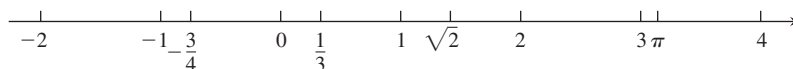
$$-\frac{3}{4} = -0.75000 \dots$$

$$\frac{1}{3} = 0.33333 \dots$$

$$\sqrt{2} = 1.4142 \dots$$

The dots ... in each case indicate that the sequence of decimal digits goes on forever. Every conceivable decimal expansion represents a real number; although, some numbers have two representations. For instance, the infinite decimals .999 ... and 1.000 ... represent the same real number 1. A similar statement holds for any number with an infinite tail of 9's.

The real numbers can be represented geometrically as points on a number line called the **real line**.



RULES FOR INEQUALITIES

If a , b , and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \frac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \frac{1}{b} < \frac{1}{a}$.

The symbol $\mathbb{R}$ denotes either the real number system or, equivalently, the real line.

The properties of the real number system fall into three categories: algebraic properties, order properties, and completeness. The **algebraic properties** say that the real numbers can be added, subtracted, multiplied, and divided (except by 0) to produce more real numbers under the usual rules of arithmetic. *You can never divide by 0.*

The **order properties** of real numbers are given in Appendix A.7. The useful rules at the left can be derived from them, where the symbol $\Rightarrow$ means “implies.”

Notice the rules for multiplying an inequality by a number. Multiplying by a positive number preserves the inequality; multiplying by a negative number reverses the inequality. Also, reciprocation reverses the inequality for numbers of the same sign. For example, $2 < 5$ but $-2 > -5$ and $1/2 > 1/5$.

The **completeness property** of the real number system is deeper and harder to define precisely. However, the property is essential to the idea of a limit (Chapter 2). Roughly speaking, it says that there are enough real numbers to “complete” the real number line, in

the sense that there are no “holes” or “gaps” in it. Many theorems of calculus would fail if the real number system were not complete. Appendix A.7 introduces the ideas involved and discusses how the real numbers are constructed.

We distinguish three special subsets of real numbers.

1. The **natural numbers**, namely 1, 2, 3, 4, ...
2. The **integers**, namely 0, ± 1 , ± 2 , ± 3 , ...
3. The **rational numbers**, namely the numbers that can be expressed in the form of a fraction m/n , where m and n are integers and $n \neq 0$. Examples are

$$\frac{1}{3}, \quad -\frac{4}{9} = \frac{-4}{9} = \frac{4}{-9}, \quad \frac{200}{13}, \quad \text{and} \quad 57 = \frac{57}{1}.$$

The rational numbers are precisely the real numbers with decimal expansions that are either

- a. terminating (ending in an infinite string of zeros), for example,

$$\frac{3}{4} = 0.75000 \dots = 0.75 \quad \text{or}$$

- b. eventually repeating (ending with a block of digits that repeats over and over), for example,

$$\frac{23}{11} = 2.090909 \dots = 2.\overline{09} \quad \text{The bar indicates the block of repeating digits.}$$

A terminating decimal expansion is a special type of repeating decimal, since the ending zeros repeat.

The set of rational numbers has all the algebraic and order properties of the real numbers but lacks the completeness property. For example, there is no rational number whose square is 2; there is a “hole” in the rational line where $\sqrt{2}$ should be.

Real numbers that are not rational are called **irrational numbers**. They are characterized by having nonterminating and nonrepeating decimal expansions. Examples are π , $\sqrt{2}$, $\sqrt[3]{5}$, and $\log_{10} 3$. Since every decimal expansion represents a real number, there are infinitely many irrational numbers. Both rational and irrational numbers are found arbitrarily close to any given point on the real line.

Set notation is very useful for specifying sets of real numbers. A **set** is a collection of objects, and these objects are the **elements** of the set. If S is a set, the notation $a \in S$ means that a is an element of S , and $a \notin S$ means that a is not an element of S . If S and T are sets, then $S \cup T$ is their **union** and consists of all elements belonging to either S or T (or to both S and T). The **intersection** $S \cap T$ consists of all elements belonging to both S and T . The **empty set** $\emptyset$ is the set that contains no elements. For example, the intersection of the rational numbers and the irrational numbers is the empty set.

Some sets can be described by *listing* their elements in braces. For instance, the set A consisting of the natural numbers (or positive integers) less than 6 can be expressed as

$$A = \{1, 2, 3, 4, 5\}.$$

The entire set of integers is written as

$$\{0, \pm 1, \pm 2, \pm 3, \dots\}.$$

Another way to describe a set is to enclose in braces a rule that generates all the elements of the set. For instance, the set

$$A = \{x \mid x \text{ is an integer and } 0 < x < 6\}$$

is the set of positive integers less than 6.

The process of finding the interval or intervals of numbers that satisfy an inequality in x is called **solving** the inequality.

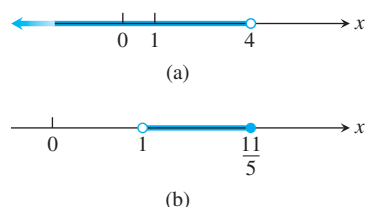


FIGURE A.1 Solution sets for the inequalities in Example 1. Hollow circles indicate endpoints that are not included in the interval, and solid dots indicate included endpoints.

Solution

$$\begin{aligned}
 \text{(a)} \quad 2x - 1 &< x + 3 \\
 2x &< x + 4 && \text{Add 1 to both sides.} \\
 x &< 4 && \text{Subtract } x \text{ from both sides.}
 \end{aligned}$$

The solution set is the open interval $(-\infty, 4)$ (Figure A.1a).

(b) The inequality $6/(x - 1) \geq 5$ can hold only if $x > 1$ because otherwise $6/(x - 1)$ is undefined or negative. Therefore, $(x - 1)$ is positive, and the inequality will be preserved if we multiply both sides by $(x - 1)$:

$$\begin{aligned}
 \frac{6}{x - 1} &\geq 5 \\
 6 &\geq 5x - 5 && \text{Multiply both sides by } (x - 1). \\
 11 &\geq 5x && \text{Add 5 to both sides.} \\
 \frac{11}{5} &\geq x. && \text{Or } x \leq \frac{11}{5}.
 \end{aligned}$$

The solution set is the half-open interval $(1, 11/5]$ (Figure A.1b). ■

Absolute Value

The **absolute value** of a number x , denoted by $|x|$, is defined by the formula

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

EXAMPLE 2 $|3| = 3$, $|0| = 0$, $|-5| = -(-5) = 5$, $|-|a|| = |a|$ ■

Geometrically, the absolute value of x is the distance from x to 0 on the real number line. Since distances are always positive or 0, we see that $|x| \geq 0$ for every real number x , and $|x| = 0$ if and only if $x = 0$. Also,

$$|x - y| = \text{the distance between } x \text{ and } y$$

on the real line (Figure A.2).

Since the symbol $\sqrt{a}$ always denotes the *nonnegative* square root of a , an alternate definition of $|x|$ is

$$|x| = \sqrt{x^2}.$$

It is important to remember that $\sqrt{a^2} = |a|$. Do not write $\sqrt{a^2} = a$ unless you already know that $a \geq 0$.

The absolute value function has the following properties. (You are asked to prove these properties in the exercises.)

Absolute Value Properties

1. $|-a| = |a|$ A number and its negative have the same absolute value.
2. $|ab| = |a||b|$ The absolute value of a product is the product of the absolute values.
3. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ The absolute value of a quotient is the quotient of the absolute values.
4. $|a + b| \leq |a| + |b|$ The **triangle inequality**. The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

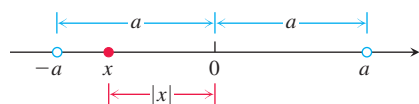


FIGURE A.3 $|x| < a$ means x lies between $-a$ and a .

Note that $|-a| \neq -|a|$. For example, $|-3| = 3$, whereas $-|3| = -3$. If a and b differ in sign, then $|a + b|$ is less than $|a| + |b|$. In all other cases, $|a + b|$ equals $|a| + |b|$. Absolute value bars in expressions like $|-3 + 5|$ work like parentheses: We do the arithmetic inside *before* taking the absolute value.

EXAMPLE 3

$$\begin{aligned} |-3 + 5| &= |2| = 2 < |-3| + |5| = 8 \\ |3 + 5| &= |8| = |3| + |5| \\ |-3 - 5| &= |-8| = 8 = |-3| + |-5| \end{aligned}$$

FURTHER PROPERTIES: ABSOLUTE VALUES AND INTERVALS

If a is any positive number, then:

5. $|x| = a \Leftrightarrow x = \pm a$
6. $|x| < a \Leftrightarrow -a < x < a$
7. $|x| > a \Leftrightarrow x > a \text{ or } x < -a$
8. $|x| \leq a \Leftrightarrow -a \leq x \leq a$
9. $|x| \geq a \Leftrightarrow x \geq a \text{ or } x \leq -a$

The inequality $|x| < a$ says that the distance from x to 0 is less than the positive number a . This means that x must lie between $-a$ and a , as we can see from Figure A.3.

Statements 5–9 in the table at left are all consequences of the definition of absolute value and are often helpful when solving equations or inequalities involving absolute values. The symbol $\Leftrightarrow$ that appears in the table is often used by mathematicians to denote the “if and only if” logical relationship. It also means “implies and is implied by.”

EXAMPLE 4 Solve the equation $|2x - 3| = 7$.

Solution By Property 5, $2x - 3 = \pm 7$, so there are two possibilities:

$$\begin{array}{lll} 2x - 3 = 7 & 2x - 3 = -7 & \text{Equivalent equations without absolute values} \\ 2x = 10 & 2x = -4 & \text{Solve as usual.} \\ x = 5 & x = -2 & \end{array}$$

The solutions of $|2x - 3| = 7$ are $x = 5$ and $x = -2$.

EXAMPLE 5 Solve the inequality $\left|5 - \frac{2}{x}\right| < 1$.

Solution We have

$$\begin{aligned} \left|5 - \frac{2}{x}\right| < 1 &\Leftrightarrow -1 < 5 - \frac{2}{x} < 1 && \text{Property 6} \\ &\Leftrightarrow -6 < -\frac{2}{x} < -4 && \text{Subtract 5.} \\ &\Leftrightarrow 3 > \frac{1}{x} > 2 && \text{Multiply by } -\frac{1}{2}. \\ &\Leftrightarrow \frac{1}{3} < x < \frac{1}{2}. && \text{Take reciprocals.} \end{aligned}$$

Notice how the various rules for inequalities were used here. Multiplying by a negative number reverses the inequality. So does taking reciprocals in an inequality in which both sides are positive. The original inequality holds if and only if $(1/3) < x < (1/2)$. The solution set is the open interval $(1/3, 1/2)$.

EXERCISES A.1

- Express $1/9$ as a repeating decimal, using a bar to indicate the repeating digits. What are the decimal representations of $2/9$? $3/9$? $8/9$? $9/9$?
- If $2 < x < 6$, which of the following statements about x are necessarily true, and which are not necessarily true?
 - $0 < x < 4$
 - $0 < x - 2 < 4$
 - $1 < \frac{x}{2} < 3$
 - $\frac{1}{6} < \frac{1}{x} < \frac{1}{2}$
 - $1 < \frac{6}{x} < 3$
 - $|x - 4| < 2$
 - $-6 < -x < 2$
 - $-6 < -x < -2$

In Exercises 3–6, solve the inequalities and show the solution sets on the real line.

- $-2x > 4$
- $5x - 3 \leq 7 - 3x$
- $2x - \frac{1}{2} \geq 7x + \frac{7}{6}$
- $\frac{4}{5}(x - 2) < \frac{1}{3}(x - 6)$

Solve the equations in Exercises 7–9.

- $|y| = 3$
- $|2t + 5| = 4$
- $|8 - 3s| = \frac{9}{2}$

Solve the inequalities in Exercises 10–17, expressing the solution sets as intervals or unions of intervals. Also, show each solution set on the real line.

- $|x| < 2$
- $|t - 1| \leq 3$
- $|3y - 7| < 4$
- $\left|\frac{z}{5} - 1\right| \leq 1$
- $\left|3 - \frac{1}{x}\right| < \frac{1}{2}$
- $|2s| \geq 4$
- $|1 - x| > 1$
- $\left|\frac{r+1}{2}\right| \geq 1$

Solve the inequalities in Exercises 18–21. Express the solution sets as intervals or unions of intervals and show them on the real line. Use the result $\sqrt{a^2} = |a|$ as appropriate.

- $x^2 < 2$
- $4 < x^2 < 9$
- $(x - 1)^2 < 4$
- $x^2 - x < 0$
- Do not fall into the trap of thinking $|-a| = a$. For what real numbers a is this equation true? For what real numbers is it false?
- Solve the equation $|x - 1| = 1 - x$.
- A proof of the triangle inequality** Give the reason justifying each of the numbered steps in the following proof of the triangle inequality.

$$|a + b|^2 = (a + b)^2 \quad (1)$$

$$= a^2 + 2ab + b^2$$

$$\leq a^2 + 2|a||b| + b^2 \quad (2)$$

$$= |a|^2 + 2|a||b| + |b|^2 \quad (3)$$

$$= (|a| + |b|)^2$$

$$|a + b| \leq |a| + |b| \quad (4)$$

- Prove that $|ab| = |a||b|$ for any numbers a and b .
- If $|x| \leq 3$ and $x > -1/2$, what can you say about x ?
- Graph the inequality $|x| + |y| \leq 1$.
- For any number a , prove that $|-a| = |a|$.
- Let a be any positive number. Prove that $|x| > a$ if and only if $x > a$ or $x < -a$.
- a.** If b is any nonzero real number, prove that $|1/b| = 1/|b|$.
b. Prove that $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ for any numbers a and $b \neq 0$.

A.2 Graphing with Software

Many computers, calculators, and smartphones have graphing applications that enable us to graph very complicated functions with high precision. Many of these functions could not otherwise be easily graphed. However, some care must be taken when using such graphing software, and in this section we address some of the issues that can arise. In Chapter 4 we will see how calculus helps us determine that we are accurately viewing the important features of a function's graph.

Graphing Windows

When software is used for graphing, a portion of the graph is visible in a **display** or **viewing window**. Depending on the software, the default window may give an incomplete or misleading picture of the graph. We use the term *square window* when the units or scales used on both axes are the same. This term does not mean that the display window itself is square (usually it is rectangular), but instead it means that the x -unit is the same length as the y -unit.

When a graph is displayed in the default mode, the x -unit may differ from the y -unit of scaling in order to capture essential features of the graph. This difference in scaling can cause visual distortions that may lead to erroneous interpretations of the function's behavior. Some graphing software enables us to set the viewing window by specifying one or both of the intervals, $a \leq x \leq b$ and $c \leq y \leq d$, and it may allow for equalizing the scales used for the axes as well. The software selects equally spaced x -values in $[a, b]$ and then plots the points $(x, f(x))$. A point is plotted if and only if x lies in the domain of the function and $f(x)$ lies within the interval $[c, d]$. A short line segment is then drawn between each plotted point and its next neighboring point. We now give illustrative examples of some common problems that may occur with this procedure.

EXAMPLE 1 Graph the function $f(x) = x^3 - 7x^2 + 28$ in each of the following display or viewing windows:

- (a) $[-10, 10]$ by $[-10, 10]$ (b) $[-4, 4]$ by $[-50, 10]$ (c) $[-4, 10]$ by $[-60, 60]$

Solution

- (a) We select $a = -10$, $b = 10$, $c = -10$, and $d = 10$ to specify the interval of x -values and the range of y -values for the window. The resulting graph is shown in Figure A.4a. It appears that the window is cutting off the bottom and top parts of the graph and that the interval of x -values is too large. Let's try the next window.
- (b) We see some new features of the graph (Figure A.4b), but the top is still missing and we need to view more to the right of $x = 4$ as well. The next window should help.

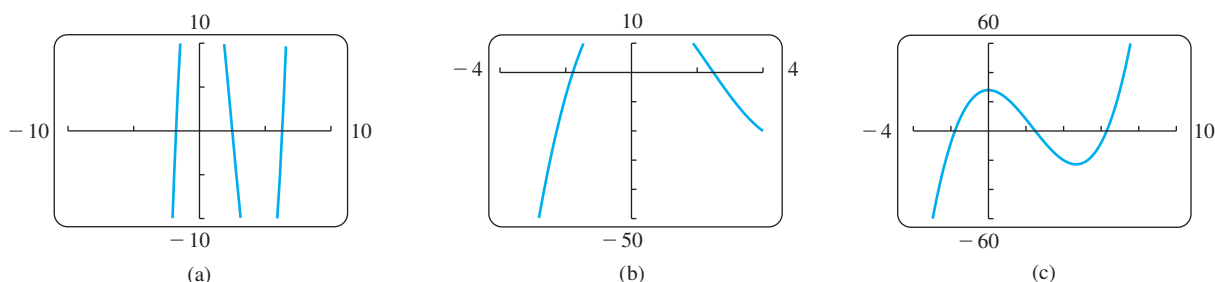


FIGURE A.4 The graph of $f(x) = x^3 - 7x^2 + 28$ in different viewing windows. Selecting a window that gives a clear picture of a graph is often a trial-and-error process (Example 1). The default window used by the software may automatically display the graph in (c).

- (c) Figure A.4c shows the graph in this new viewing window. Observe that we get a more complete picture of the graph in this window, and it is a reasonable graph of a third-degree polynomial. ■

EXAMPLE 2 When a graph is displayed, the x -unit may differ from the y -unit, as in the graphs shown in Figure A.4. The result is distortion in the picture, which may be misleading. The display window can be made square by compressing or stretching the units on one axis to match the scale on the other, giving the true graph. Many software systems have built-in options to make the window “square.” If yours does not, you may have to bring to your viewing some foreknowledge of the true picture.

Figure A.5a shows the graphs of the perpendicular lines $y = x$ and $y = -x + 3\sqrt{2}$, together with the semicircle $y = \sqrt{9 - x^2}$, in a nonsquare $[-4, 4]$ by $[-6, 8]$ display window. Notice the distortion. The lines do not appear to be perpendicular, and the semicircle appears to be elliptical in shape.

Figure A.5b shows the graphs of the same functions in a square window in which the x -units are scaled to be the same as the y -units. Notice that the scaling on the x -axis for Figure A.5a has been compressed in Figure A.5b to make the window square. Figure A.5c gives an enlarged view of Figure A.5b with a square $[-3, 3]$ by $[0, 4]$ window. ■

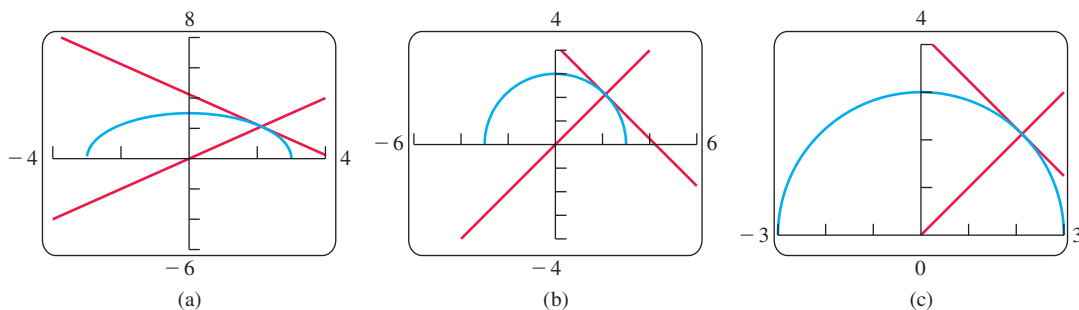


FIGURE A.5 Graphs of the perpendicular lines $y = x$ and $y = -x + 3\sqrt{2}$ and of the semicircle $y = \sqrt{9 - x^2}$ appear distorted (a) in a nonsquare window, but clear (b) and (c) in square windows (Example 2). Some software may not provide options for the views in (b) or (c).

If the denominator of a rational function is zero at some x -value within the viewing window, graphing software may produce a steep, near-vertical line segment from the top to the bottom of the window. Example 3 illustrates steep line segments.

Sometimes the graph of a trigonometric function oscillates very rapidly. When graphing software plots the points of the graph and connects them, many of the maximum and minimum points are actually missed. The resulting graph is then very misleading.

EXAMPLE 3 Graph the function $f(x) = \sin 100x$.

Solution Figure A.6a shows the graph of f in the viewing window $[-12, 12]$ by $[-1, 1]$. We see that the graph looks very strange because the sine curve should oscillate periodically between -1 and 1 . This behavior is not exhibited in Figure A.6a. We might experiment with a smaller viewing window, say $[-6, 6]$ by $[-1, 1]$, but the graph is not better (Figure A.6b). The difficulty is that the period of the trigonometric function $y = \sin 100x$ is very small ($2\pi/100 \approx 0.063$). If we choose the much smaller viewing window $[-0.1, 0.1]$ by $[-1, 1]$ we get the graph shown in Figure A.6c. This graph reveals the expected oscillations of a sine curve. ■

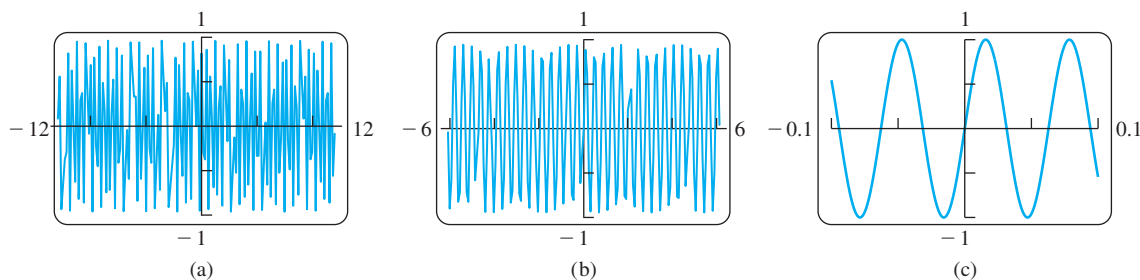


FIGURE A.6 Graphs of the function $y = \sin 100x$ in three viewing windows. Because the period is $2\pi/100 \approx 0.063$, the smaller window in (c) best displays the true aspects of this rapidly oscillating function (Example 3).

EXAMPLE 4 Graph the function $y = \cos x + \frac{1}{200}\sin 200x$.

Solution In the viewing window $[-6, 6]$ by $[-1, 1]$ the graph appears much like the cosine function with some very small sharp wiggles on it (Figure A.7a). We get a better

look when we significantly reduce the window to $[-0.2, 0.2]$ by $[0.97, 1.01]$, obtaining the graph in Figure A.7b. We now see the small but rapid oscillations of the second term, $(1/200) \sin 200x$, added to the comparatively larger values of the cosine curve. ■

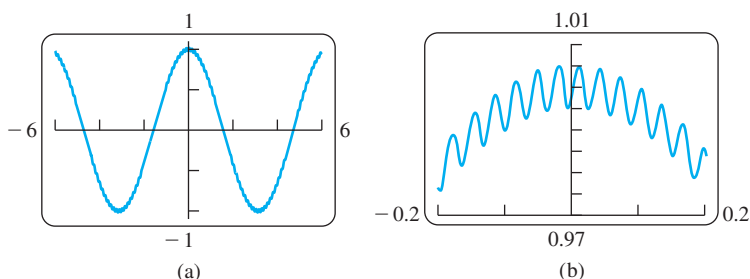


FIGURE A.7 (a) The function $y = \cos x + \frac{1}{200} \sin 200x$. (b) A close-up view, blown up near the y -axis. The term $\cos x$ clearly dominates the second term, $\frac{1}{200} \sin 200x$, which produces the rapid oscillations along the cosine curve. Both views are needed for a clear idea of the graph (Example 4).

Obtaining a Complete Graph

Some graphing software will not display the portion of a graph for $f(x)$ when $x < 0$. Usually that happens because of the algorithm the software is using to calculate the function values. Sometimes we can obtain the complete graph by defining the formula for the function in a different way, as illustrated in the next example.

EXAMPLE 5 Graph the function $y = x^{1/3}$.

Solution Some graphing software displays the graph shown in Figure A.8a. When we compare it with the graph of $y = x^{1/3} = \sqrt[3]{x}$ in Figure 1.17, we see that the left branch for $x < 0$ is missing. The reason the graphs differ is that the software algorithm calculates $x^{1/3}$ as $e^{(1/3)\ln x}$. Since the logarithmic function is not defined for negative values of x , the software can produce only the right branch, where $x > 0$. (Logarithmic and exponential functions are introduced in the next two sections.)

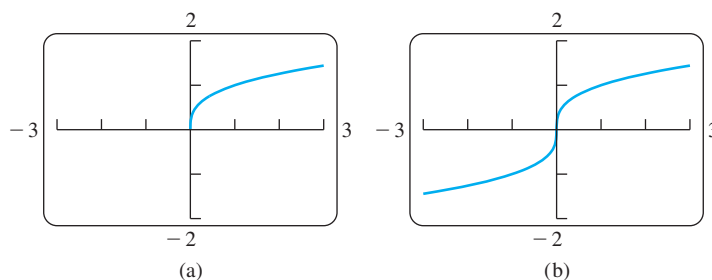


FIGURE A.8 The graph of $y = x^{1/3}$ is missing the left branch in (a). In (b) we graph the function $f(x) = \frac{x}{|x|} \cdot |x|^{1/3}$, obtaining both branches. (See Example 5.)

To obtain the full picture showing both branches, we can graph the function

$$f(x) = \frac{x}{|x|} \cdot |x|^{1/3}.$$

This function equals $x^{1/3}$ except at $x = 0$ (where f is undefined, although $0^{1/3} = 0$). A graph of f is displayed in Figure A.8b. ■

EXERCISES **A.2**

Choosing a Viewing Window

T In Exercises 1–4, use graphing software to determine which of the given viewing windows displays the most appropriate graph of the specified function.

1. $f(x) = x^4 - 7x^2 + 6x$

a. $[-1, 1]$ by $[-1, 1]$ b. $[-2, 2]$ by $[-5, 5]$

c. $[-10, 10]$ by $[-10, 10]$ d. $[-5, 5]$ by $[-25, 15]$

2. $f(x) = x^3 - 4x^2 - 4x + 16$

a. $[-1, 1]$ by $[-5, 5]$ b. $[-3, 3]$ by $[-10, 10]$

c. $[-5, 5]$ by $[-10, 20]$ d. $[-20, 20]$ by $[-100, 100]$

3. $f(x) = 5 + 12x - x^3$

a. $[-1, 1]$ by $[-1, 1]$ b. $[-5, 5]$ by $[-10, 10]$

c. $[-4, 4]$ by $[-20, 20]$ d. $[-4, 5]$ by $[-15, 25]$

4. $f(x) = \sqrt{5 + 4x - x^2}$

a. $[-2, 2]$ by $[-2, 2]$ b. $[-2, 6]$ by $[-1, 4]$

c. $[-3, 7]$ by $[0, 10]$ d. $[-10, 10]$ by $[-10, 10]$

Finding a Viewing Window

T In Exercises 5–30, find an appropriate graphing software viewing window for the given function and use it to display that function's graph. The window should give a picture of the overall behavior of the function. There is more than one choice, but incorrect choices can miss important aspects of the function.

5. $f(x) = x^4 - 4x^3 + 15$ 6. $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 1$

7. $f(x) = x^5 - 5x^4 + 10$ 8. $f(x) = 4x^3 - x^4$

9. $f(x) = x\sqrt{9 - x^2}$ 10. $f(x) = x^2(6 - x^3)$

11. $y = 2x - 3x^{2/3}$

13. $y = 5x^{2/5} - 2x$

15. $y = |x^2 - 1|$

17. $y = \frac{x+3}{x+2}$

19. $f(x) = \frac{x^2+2}{x^2+1}$

21. $f(x) = \frac{x-1}{x^2-x-6}$

23. $f(x) = \frac{6x^2-15x+6}{4x^2-10x}$

25. $y = \sin 250x$

27. $y = \cos\left(\frac{x}{50}\right)$

29. $y = x + \frac{1}{10}\sin 30x$

12. $y = x^{1/3}(x^2 - 8)$

14. $y = x^{2/3}(5 - x)$

16. $y = |x^2 - x|$

18. $y = 1 - \frac{1}{x+3}$

20. $f(x) = \frac{x^2-1}{x^2+1}$

22. $f(x) = \frac{8}{x^2-9}$

24. $f(x) = \frac{x^2-3}{x-2}$

26. $y = 3\cos 60x$

28. $y = \frac{1}{10}\sin\left(\frac{x}{10}\right)$

30. $y = x^2 + \frac{1}{50}\cos 100x$

Use graphing software to graph the functions specified in Exercises 31–36. Select a viewing window that reveals the key features of the function.

31. Graph the lower half of the circle defined by the equation $x^2 + 2x = 4 + 4y - y^2$.

32. Graph the upper branch of the hyperbola $y^2 - 16x^2 = 1$.

33. Graph four periods of the function $f(x) = -\tan 2x$.

34. Graph two periods of the function $f(x) = 3\cot\frac{x}{2} + 1$.

35. Graph the function $f(x) = \sin 2x + \cos 3x$.

36. Graph the function $f(x) = \sin^3 x$.

A.3 Mathematical Induction

Many formulas, like

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

can be shown to hold for every positive integer n by applying an axiom called the *mathematical induction principle*. A proof that uses this axiom is called a *proof by mathematical induction* or a *proof by induction*.

The steps in proving a formula by induction are the following:

1. Check that the formula holds for $n = 1$.
2. Prove that *if* the formula holds for any positive integer $n = k$, *then* it also holds for the next integer, $n = k + 1$.

The induction axiom says that once these steps are completed, the formula holds for all positive integers n . By Step 1 it holds for $n = 1$. By Step 2 it holds for $n = 2$, and therefore by Step 2 also for $n = 3$, and by Step 2 again for $n = 4$, and so on. If the first domino

falls, and if the k th domino always knocks over the $(k + 1)$ st when it falls, then all the dominoes fall.

EXAMPLE 1 Use mathematical induction to prove that for every positive integer n ,

$$1 + 2 + \cdots + n = \frac{n(n + 1)}{2}.$$

Solution We accomplish the proof by carrying out the two steps above.

1. The formula holds for $n = 1$ because

$$1 = \frac{1(1 + 1)}{2}.$$

2. If the formula holds for $n = k$, does it also hold for $n = k + 1$? The answer is yes, as we now show. If it is the case that

$$1 + 2 + \cdots + k = \frac{k(k + 1)}{2},$$

then it follows that

$$\begin{aligned} 1 + 2 + \cdots + k + (k + 1) &= \frac{k(k + 1)}{2} + (k + 1) = \frac{k^2 + k + 2k + 2}{2} \\ &= \frac{(k + 1)(k + 2)}{2} = \frac{(k + 1)((k + 1) + 1)}{2}. \end{aligned}$$

The last expression in this string of equalities is the expression $n(n + 1)/2$ for $n = (k + 1)$.

The mathematical induction principle now guarantees the original formula for all positive integers n . ■

In Example 4 of Section 5.2 we gave another proof for the formula giving the sum of the first n positive integers. However, proofs by mathematical induction can also be used to find the sums of the squares and cubes of the first n positive integers (Exercises 9 and 10). Here is another example of a proof by induction.

EXAMPLE 2 Show by mathematical induction that for all positive integers n ,

$$\frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}.$$

Solution We accomplish the proof by carrying out the two steps of mathematical induction.

1. The formula holds for $n = 1$ because

$$\frac{1}{2^1} = 1 - \frac{1}{2^1}.$$

2. If it is the case that

$$\frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} = 1 - \frac{1}{2^k},$$

then it follows that

$$\begin{aligned}\frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} &= 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1 \cdot 2}{2^k \cdot 2} + \frac{1}{2^{k+1}} \\ &= 1 - \frac{2}{2^{k+1}} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}.\end{aligned}$$

Thus, the original formula holds for $n = (k + 1)$ whenever it holds for $n = k$.

With these steps verified, the mathematical induction principle now guarantees the formula for every positive integer n . ■

Other Starting Integers

Instead of starting at $n = 1$, some induction arguments start at another integer. The steps for such an argument are as follows.

1. Check that the formula holds for $n = n_1$ (the first appropriate integer).
2. Prove that if the formula holds for any integer $n = k \geq n_1$, then it also holds for $n = (k + 1)$.

Once these steps are completed, the mathematical induction principle guarantees the formula for all $n \geq n_1$.

EXAMPLE 3 Show that $n! > 3^n$ if n is large enough.

Solution How large is large enough? We experiment:

n	1	2	3	4	5	6	7
$n!$	1	2	6	24	120	720	5040
3^n	3	9	27	81	243	729	2187

It looks as if $n! > 3^n$ for $n \geq 7$. To be sure, we apply mathematical induction. We take $n_1 = 7$ in Step 1 and complete Step 2.

Suppose $k! > 3^k$ for some $k \geq 7$. Then

$$(k + 1)! = (k + 1)(k!) > (k + 1)3^k > 7 \cdot 3^k > 3^{k+1}.$$

Thus, for $k \geq 7$,

$$k! > 3^k \quad \text{implies} \quad (k + 1)! > 3^{k+1}.$$

The mathematical induction principle now guarantees $n! \geq 3^n$ for all $n \geq 7$. ■

Proof of the Derivative Sum Rule for Sums of Finitely Many Functions

We prove the statement

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}$$

by mathematical induction. The statement is true for $n = 2$, as was proved in Section 3.3. This is Step 1 of the induction proof.

Step 2 is to show that if the statement is true for any positive integer $n = k$, where $k \geq n_0 = 2$, then it is also true for $n = k + 1$. So suppose that

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_k) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx}. \quad (1)$$

Then

$$\begin{aligned}
 & \frac{d}{dx}(\underbrace{u_1 + u_2 + \cdots + u_k}_{\text{Call the function defined by this sum } u.} + \underbrace{u_{k+1}}_{\text{Call this function } v.}) \\
 &= \frac{d}{dx}(u_1 + u_2 + \cdots + u_k) + \frac{du_{k+1}}{dx} \quad \text{Sum Rule for } \frac{d}{dx}(u + v) \\
 &= \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx} + \frac{du_{k+1}}{dx}. \quad \text{Eq. (1)}
 \end{aligned}$$

With these steps verified, the mathematical induction principle now guarantees the Sum Rule for every integer $n \geq 2$.

EXERCISES A.3

1. Assuming that the triangle inequality $|a + b| \leq |a| + |b|$ holds for any two numbers a and b , show that

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

for any n numbers.

2. Show that if $r \neq 1$, then

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

for every positive integer n .

3. Use the Product Rule, $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$, and the fact that $\frac{d}{dx}(x) = 1$ to show that $\frac{d}{dx}(x^n) = nx^{n-1}$ for every positive integer n .

4. Suppose that a function $f(x)$ has the property that $f(x_1x_2) = f(x_1) + f(x_2)$ for any two positive numbers x_1 and x_2 . Show that

$$f(x_1x_2 \cdots x_n) = f(x_1) + f(x_2) + \cdots + f(x_n)$$

for the product of any n positive numbers $x_1, x_2, \dots, x_n$.

5. Show that

$$\frac{2}{3^1} + \frac{2}{3^2} + \cdots + \frac{2}{3^n} = 1 - \frac{1}{3^n}$$

for all positive integers n .

6. Show that $n! > n^3$ if n is large enough.

7. Show that $2^n > n^2$ if n is large enough.

8. Show that $2^n \geq 1/8$ for $n \geq -3$.

9. **Sums of squares** Show that the sum of the squares of the first n positive integers is

$$\frac{n\left(n + \frac{1}{2}\right)(n + 1)}{3}.$$

10. **Sums of cubes** Show that the sum of the cubes of the first n positive integers is $(n(n + 1)/2)^2$.

11. **Rules for finite sums** Show that the following finite sum rules hold for every positive integer n . (See Section 5.2.)

a. $\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$

b. $\sum_{k=1}^n (a_k - b_k) = \sum_{k=1}^n a_k - \sum_{k=1}^n b_k$

c. $\sum_{k=1}^n ca_k = c \cdot \sum_{k=1}^n a_k$ (any number c)

d. $\sum_{k=1}^n a_k = n \cdot c$ (if a_k has the constant value c)

12. Show that $|x^n| = |x|^n$ for every positive integer n and every real number x .

A.4 Lines, Circles, and Parabolas

This section reviews coordinates, lines, distance, circles, and parabolas in the plane. The notion of increment is also discussed.

Cartesian Coordinates in the Plane

In Appendix A.1 we identified the points on the line with real numbers by assigning them coordinates. Points in the plane can be identified with ordered pairs of real numbers. To begin, we draw two perpendicular coordinate lines that intersect at the 0-point of each line.

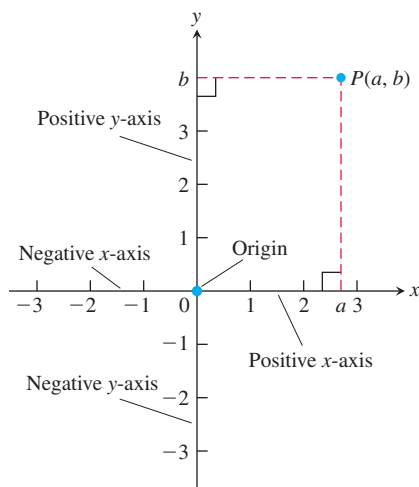


FIGURE A.9 Cartesian coordinates in the plane are based on two perpendicular axes intersecting at the origin.

HISTORICAL BIOGRAPHY

René Descartes

(1596–1650)

The Cartesian coordinate system is named in honor of Descartes. Legend tells us that he thought of this coordinate system while watching a fly crawl around on the ceiling. He noticed that the path of the fly could be described by the fly's distances from each of the walls. His discovery of analytic geometry was hailed as one of the most remarkable feats in mathematical history.

To know more, visit the companion Website.

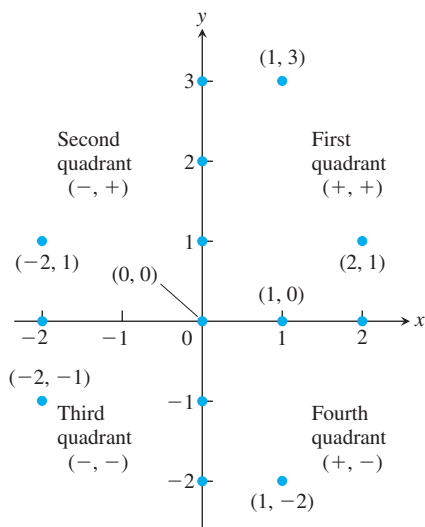


FIGURE A.10 Points labeled in the xy -coordinate or Cartesian plane. The points on the axes all have coordinate pairs but are usually labeled with single real numbers, (so $(1, 0)$ on the x -axis is labeled as 1). Notice the coordinate sign patterns of the quadrants.

These lines are called **coordinate axes**. On the horizontal x -axis, numbers are denoted by x and increase to the right. On the vertical y -axis, numbers are denoted by y and increase upward (Figure A.9). Thus “upward” and “to the right” are positive directions, whereas “downward” and “to the left” are considered negative. The **origin** O , also labeled 0, of the coordinate system is the point in the plane where x and y are both zero.

If P is any point in the plane, it can be located by exactly one ordered pair of real numbers in the following way. Draw lines through P perpendicular to the two coordinate axes. These lines intersect the axes at points with coordinates a and b (Figure A.9). The ordered pair (a, b) is assigned to the point P and is called its **coordinate pair**. The first number a is the **x -coordinate** (or **abscissa**) of P ; the second number b is the **y -coordinate** (or **ordinate**) of P . The x -coordinate of every point on the y -axis is 0. The y -coordinate of every point on the x -axis is 0. The origin is the point $(0, 0)$.

Starting with an ordered pair (a, b) , we can reverse the process and arrive at a corresponding point P in the plane. Often we identify P with the ordered pair and write $P(a, b)$. We sometimes also refer to “the point (a, b) ” and it will be clear from the context when (a, b) refers to a point in the plane and not to an open interval on the real line. Several points labeled by their coordinates are shown in Figure A.10.

This coordinate system is called the **rectangular coordinate system** or **Cartesian coordinate system** (after the sixteenth-century French mathematician René Descartes). The coordinate axes of this coordinate or Cartesian plane divide the plane into four regions called **quadrants**, numbered counterclockwise as shown in Figure A.10.

The **graph** of an equation or inequality in the variables x and y is the set of all points $P(x, y)$ in the plane whose coordinates satisfy the equation or inequality. When we plot data in the coordinate plane or graph formulas whose variables have different units of measure, we do not need to use the same scale on the two axes. If we plot time vs. thrust for a rocket motor, for example, there is no reason to place the mark that shows 1 s on the time axis the same distance from the origin as the mark that shows 1 N on the thrust axis.

Usually when we graph functions whose variables do not represent physical measurements and when we draw figures in the coordinate plane to study their geometry and trigonometry, we make the scales on the axes identical. A vertical unit of distance then looks the same as a horizontal unit. As on a surveyor's map or a scale drawing, line segments that are supposed to have the same length will look as if they do, and angles that are supposed to be congruent will look congruent.

Computer displays and calculator displays are another matter. The vertical and horizontal scales on machine-generated graphs usually differ, and there are corresponding distortions in distances, slopes, and angles. Circles may look like ellipses, rectangles may look like squares, right angles may appear to be acute or obtuse, and so on. We discuss these displays and distortions in greater detail in Appendix A.2.

Increments and Straight Lines

When a particle moves from one point in the plane to another, the net changes in its coordinates are called **increments**. They are calculated by subtracting the coordinates of the starting point from the coordinates of the ending point. If x changes from x_1 to x_2 , the **increment in x** is

$$\Delta x = x_2 - x_1.$$

EXAMPLE 1 As shown in Figure A.11, in going from the point $A(4, -3)$ to the point $B(2, 5)$, the increments in the x - and y -coordinates are

$$\Delta x = 2 - 4 = -2, \quad \Delta y = 5 - (-3) = 8.$$

From $C(5, 6)$ to $D(5, 1)$ the coordinate increments are

$$\Delta x = 5 - 5 = 0, \quad \Delta y = 1 - 6 = -5.$$

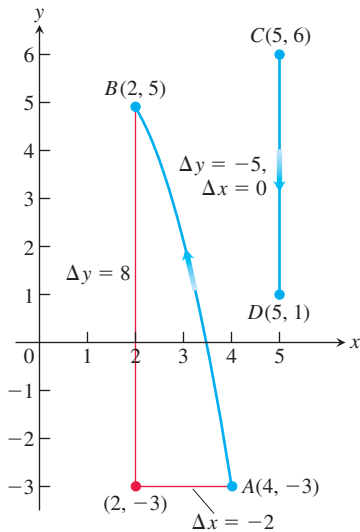


FIGURE A.11 Coordinate increments may be positive, negative, or zero (Example 1).

Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the plane, we call the increments $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ the **run** and the **rise**, respectively, between P_1 and P_2 . Two such points always determine a unique straight line (usually called simply a line) passing through them both. We call the line P_1P_2 .

Any nonvertical line in the plane has the property that the ratio

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

has the same value for every choice of the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the line (Figure A.12). This is because the ratios of corresponding sides for similar triangles are equal.

DEFINITION The constant ratio

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

is the **slope** of the nonvertical line P_1P_2 .

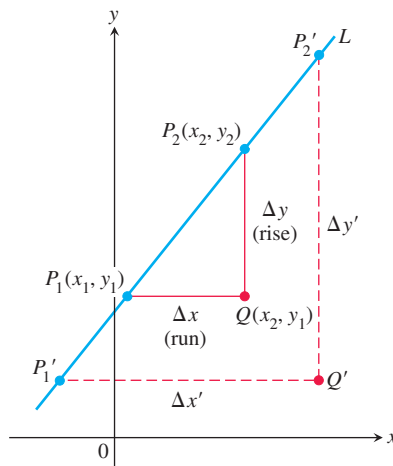


FIGURE A.12 Triangles P_1QP_2 and $P_1'Q'P_2'$ are similar, so the ratio of their sides has the same value for any two points on the line. This common value is the line's slope.

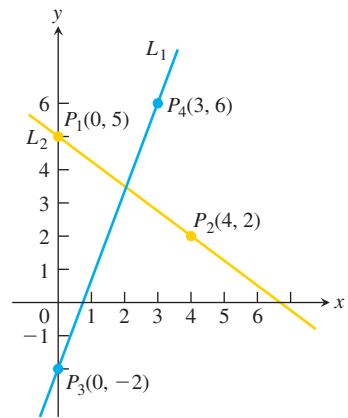


FIGURE A.13 The slope of L_1 is

$$m = \frac{\Delta y}{\Delta x} = \frac{6 - (-2)}{3 - 0} = \frac{8}{3}.$$

That is, y increases 8 units every time x increases 3 units. The slope of L_2 is

$$m = \frac{\Delta y}{\Delta x} = \frac{2 - 5}{4 - 0} = \frac{-3}{4}.$$

That is, y decreases 3 units every time x increases 4 units.

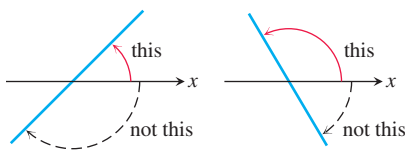


FIGURE A.14 Angles of inclination are measured counterclockwise from the x -axis.

The slope tells us the direction (uphill, downhill) and steepness of a line. A line with positive slope rises uphill to the right; one with negative slope falls downhill to the right (Figure A.13). The greater the absolute value of the slope, the more rapid the rise or fall. The slope of a vertical line is *undefined*. Since the run Δx is zero for a vertical line, we cannot form the slope ratio m .

The direction and steepness of a line can also be measured with an angle. The **angle of inclination** of a line that crosses the x -axis is the smallest counterclockwise angle from the x -axis to the line (Figure A.14). The inclination of a horizontal line is 0° . The inclination of a vertical line is 90° . If ϕ (the Greek letter phi) is the inclination of a line, then $0 \leq \phi < 180^\circ$.

The relationship between the slope m of a nonvertical line and the line's angle of inclination ϕ is shown in Figure A.15:

$$m = \tan \phi.$$

Straight lines have relatively simple equations. All points on the *vertical line* through the point a on the x -axis have x -coordinates equal to a . Thus, $x = a$ is an equation for the vertical line. Similarly, $y = b$ is an equation for the *horizontal line* meeting the y -axis at b . (See Figure A.16.)

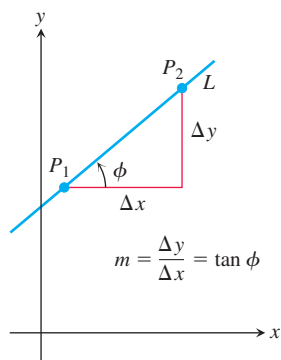


FIGURE A.15 The slope of a nonvertical line is the tangent of its angle of inclination.

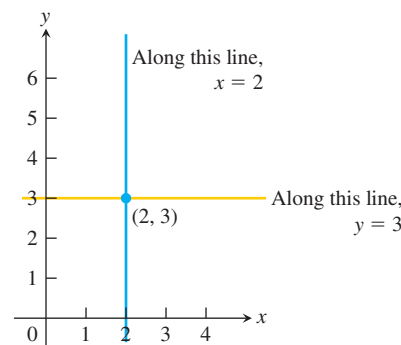


FIGURE A.16 The standard equations for the vertical and horizontal lines through $(2, 3)$ are $x = 2$ and $y = 3$.

We can write an equation for a nonvertical straight line L if we know its slope m and the coordinates of one point $P_1(x_1, y_1)$ on it. If $P(x, y)$ is *any* other point on L , then we can use the two points P_1 and P to compute the slope,

$$m = \frac{y - y_1}{x - x_1}$$

so that

$$y - y_1 = m(x - x_1), \quad \text{or} \quad y = y_1 + m(x - x_1).$$

The equation

$$y = y_1 + m(x - x_1)$$

is the **point-slope equation** of the line that passes through the point (x_1, y_1) and has slope m .

EXAMPLE 2 Write an equation for the line through the point $(2, 3)$ with slope $-3/2$.

Solution We substitute $x_1 = 2$, $y_1 = 3$, and $m = -3/2$ into the point-slope equation and obtain

$$y = 3 - \frac{3}{2}(x - 2), \quad \text{or} \quad y = -\frac{3}{2}x + 6. \quad \blacksquare$$

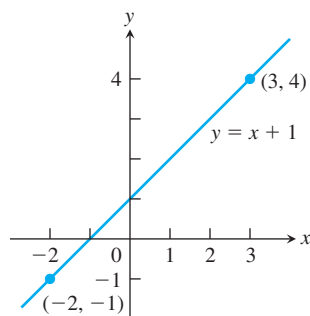
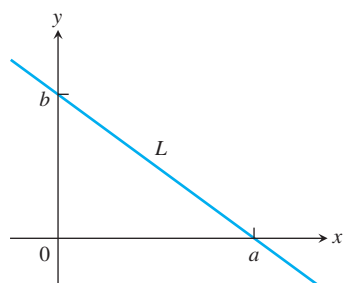


FIGURE A.17 The line in Example 3.

FIGURE A.18 Line L has x -intercept a and y -intercept b .

EXAMPLE 3 Write an equation for the line through $(-2, -1)$ and $(3, 4)$.

Solution The line's slope is

$$m = \frac{-1 - 4}{-2 - 3} = \frac{-5}{-5} = 1.$$

We can use this slope with either of the two given points in the point-slope equation:

With $(x_1, y_1) = (-2, -1)$

$$y = -1 + 1 \cdot (x - (-2))$$

$$y = -1 + x + 2$$

$$y = x + 1$$

With $(x_1, y_1) = (3, 4)$

$$y = 4 + 1 \cdot (x - 3)$$

$$y = 4 + x - 3$$

$$y = x + 1$$

Same result

Either way, we see that $y = x + 1$ is an equation for the line (Figure A.17). ■

The y -coordinate of the point where a nonvertical line intersects the y -axis is called the **y -intercept** of the line. Similarly, the **x -intercept** of a nonhorizontal line is the x -coordinate of the point where it crosses the x -axis (Figure A.18). A line with slope m and y -intercept b passes through the point $(0, b)$, so it has equation

$$y = b + m(x - 0), \quad \text{or, more simply,} \quad y = mx + b.$$

The equation

$$y = mx + b$$

is called the **slope-intercept equation** of the line with slope m and y -intercept b .

Lines with equations of the form $y = mx$ have y -intercept 0 and so pass through the origin. Equations of lines are called **linear** equations.

The equation

$$Ax + By = C \quad (A \text{ and } B \text{ not both } 0)$$

is called the **general linear equation** in x and y because its graph always represents a line and every line has an equation in this form (including lines with undefined slope).

Parallel and Perpendicular Lines

Lines that are parallel have equal angles of inclination, so they have the same slope (if they are not vertical). Conversely, lines with equal slopes have equal angles of inclination and so are parallel.

If two nonvertical lines L_1 and L_2 are perpendicular, their slopes m_1 and m_2 satisfy $m_1 m_2 = -1$, so each slope is the *negative reciprocal* of the other:

$$m_1 = -\frac{1}{m_2}, \quad m_2 = -\frac{1}{m_1}.$$

To see this, notice by inspecting similar triangles in Figure A.19 that $m_1 = a/h$, and $m_2 = -h/a$. Hence, $m_1 m_2 = (a/h)(-h/a) = -1$.

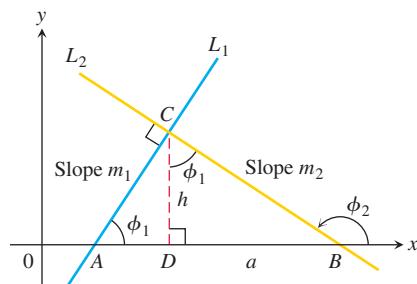


FIGURE A.19 $\triangle ADC$ is similar to $\triangle CDB$. Hence ϕ_1 is also the upper angle in $\triangle CDB$. From the sides of $\triangle CDB$, we read $\tan \phi_1 = a/h$.

Distance and Circles in the Plane

The distance between points in the plane is calculated with a formula that comes from the Pythagorean theorem (Figure A.20).

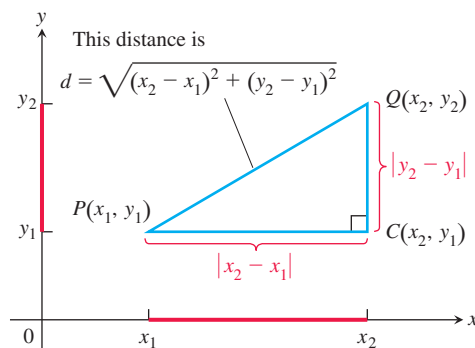


FIGURE A.20 To calculate the distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$, apply the Pythagorean theorem to triangle PCQ .

Distance Formula for Points in the Plane

The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

By definition, a **circle** of radius a is the set of all points $P(x, y)$ whose distance from some center $C(h, k)$ equals a (Figure A.21). From the distance formula, P lies on the circle if and only if

$$\sqrt{(x - h)^2 + (y - k)^2} = a,$$

so

$$(x - h)^2 + (y - k)^2 = a^2. \quad (1)$$

Equation (1) is the **standard equation** of a circle with center (h, k) and radius a . The circle of radius $a = 1$ and centered at the origin is the **unit circle** with equation

$$x^2 + y^2 = 1.$$

EXAMPLE 4

(a) The standard equation for the circle of radius 2 centered at $(3, 4)$ is

$$(x - 3)^2 + (y - 4)^2 = 2^2 = 4.$$

(b) The circle

$$(x - 1)^2 + (y + 5)^2 = 3$$

has $h = 1$, $k = -5$, and $a = \sqrt{3}$. The center is the point $(h, k) = (1, -5)$ and the radius is $a = \sqrt{3}$. ■

If an equation for a circle is not in standard form, we can find the circle's center and radius by first converting the equation to standard form. The algebraic technique for doing so is *completing the square*.

EXAMPLE 5 Find the center and radius of the circle

$$x^2 + y^2 + 4x - 6y - 3 = 0.$$

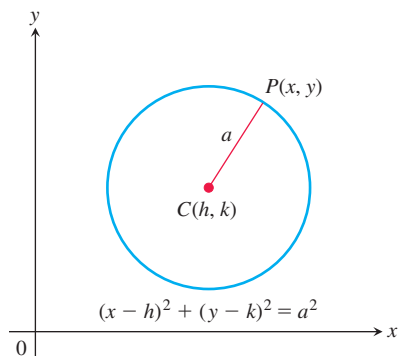


FIGURE A.21 A circle of radius a in the xy -plane, with center at (h, k) .

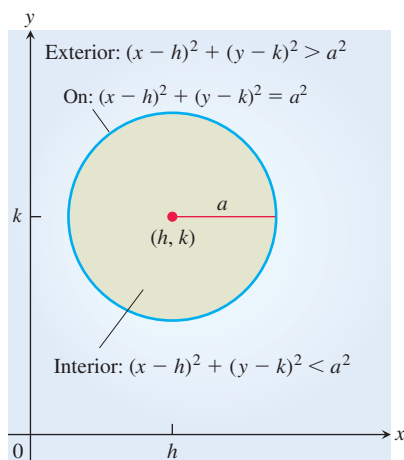


FIGURE A.22 The interior and exterior of the circle $(x - h)^2 + (y - k)^2 = a^2$.

Solution We convert the equation to standard form by completing the squares in x and y :

$$x^2 + y^2 + 4x - 6y - 3 = 0$$

$$(x^2 + 4x) + (y^2 - 6y) = 3$$

$$\left(x^2 + 4x + \left(\frac{4}{2}\right)^2\right) + \left(y^2 - 6y + \left(\frac{-6}{2}\right)^2\right)$$

$$= 3 + \left(\frac{4}{2}\right)^2 + \left(\frac{-6}{2}\right)^2$$

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) = 3 + 4 + 9$$

$$(x + 2)^2 + (y - 3)^2 = 16$$

The center is $(-2, 3)$ and the radius is $a = 4$.

Start with the given equation.
Gather terms. Move the constant to the right-hand side.

Add the square of half the coefficient of x to each side of the equation. Do the same for y . The parenthetical expressions on the left-hand side are now perfect squares.

Write each quadratic as a squared linear expression.

The points (x, y) satisfying the inequality

$$(x - h)^2 + (y - k)^2 < a^2$$

make up the **interior** region of the circle with center (h, k) and radius a (Figure A.22). The circle's **exterior** consists of the points (x, y) satisfying

$$(x - h)^2 + (y - k)^2 > a^2.$$

Parabolas

The geometric definition and properties of general parabolas are reviewed in Chapter 10. Here we look at parabolas arising as the graphs of equations of the form $y = ax^2 + bx + c$.

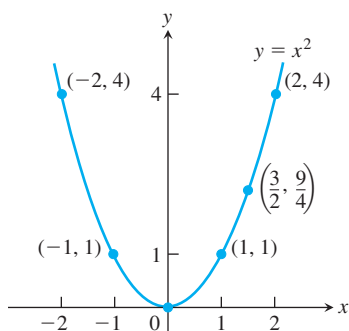


FIGURE A.23 The parabola $y = x^2$ (Example 6).

EXAMPLE 6 Consider the equation $y = x^2$. Some points whose coordinates satisfy this equation are $(0, 0)$, $(1, 1)$, $\left(\frac{3}{2}, \frac{9}{4}\right)$, $(-1, 1)$, $(2, 4)$, and $(-2, 4)$. These points (and all others satisfying the equation) make up a smooth curve called a parabola (Figure A.23). ■

The graph of an equation of the form

$$y = ax^2$$

is a **parabola** whose **axis** (axis of symmetry) is the y -axis. The parabola's **vertex** (point where the parabola and axis cross) lies at the origin. The parabola opens upward if $a > 0$ and downward if $a < 0$. The larger the value of $|a|$, the narrower the parabola (Figure A.24).

Generally, the graph of $y = ax^2 + bx + c$ is a shifted and scaled version of the parabola $y = x^2$. We discuss shifting and scaling of graphs in more detail in Section 1.2.

The Graph of $y = ax^2 + bx + c$, $a \neq 0$

The graph of the equation $y = ax^2 + bx + c$, $a \neq 0$, is a parabola. The parabola opens upward if $a > 0$ and downward if $a < 0$. The **axis** is the line

$$x = -\frac{b}{2a}. \quad (2)$$

The **vertex** of the parabola is the point where the axis and parabola intersect. Its x -coordinate is $x = -b/2a$; its y -coordinate is found by substituting $x = -b/2a$ in the parabola's equation.

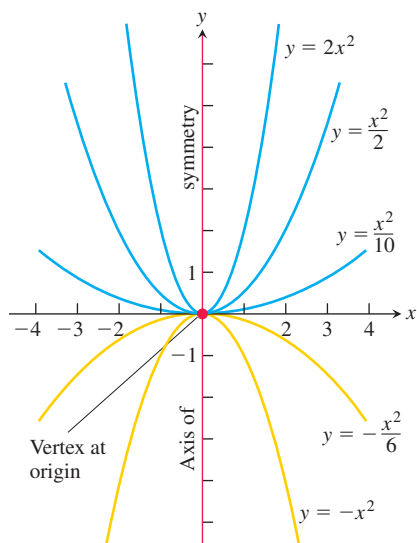


FIGURE A.24 Besides determining the direction in which the parabola $y = ax^2$ opens, the number a is a scaling factor. The parabola widens as a approaches zero and narrows as $|a|$ becomes large.

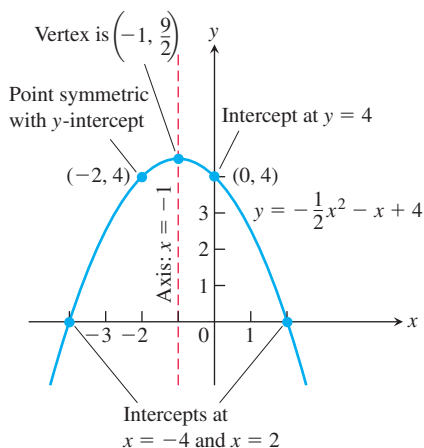


FIGURE A.25 The parabola in Example 7.

Notice that if $a = 0$, then we have $y = bx + c$, which is an equation for a line. The axis, given by Equation (2), can be found by completing the square.

EXAMPLE 7 Graph the equation $y = -\frac{1}{2}x^2 - x + 4$.

Solution Comparing the equation with $y = ax^2 + bx + c$ we see that

$$a = -\frac{1}{2}, \quad b = -1, \quad c = 4.$$

Since $a < 0$, the parabola opens downward. From Equation (2) the axis is the vertical line

$$x = -\frac{b}{2a} = -\frac{(-1)}{2(-1/2)} = -1.$$

When $x = -1$, we have

$$y = -\frac{1}{2}(-1)^2 - (-1) + 4 = \frac{9}{2}.$$

The vertex is $(-1, 9/2)$.

The x -intercepts are where $y = 0$:

$$\begin{aligned} -\frac{1}{2}x^2 - x + 4 &= 0 \\ x^2 + 2x - 8 &= 0 \\ (x - 2)(x + 4) &= 0 \\ x &= 2, \quad x = -4 \end{aligned}$$

We plot some points, sketch the axis, and use the direction of opening to complete the graph in Figure A.25. ■

Ellipses

The geometric definition and properties of general ellipses are reviewed in Chapter 10. Here we relate them to circles. Although they are not the graphs of functions, circles can be stretched horizontally or vertically in the same way as the graphs of functions. The standard equation for a circle of radius r centered at the origin is

$$x^2 + y^2 = r^2.$$

Substituting cx for x in the standard equation for a circle (Figure A.26) gives

$$c^2x^2 + y^2 = r^2. \quad (3)$$

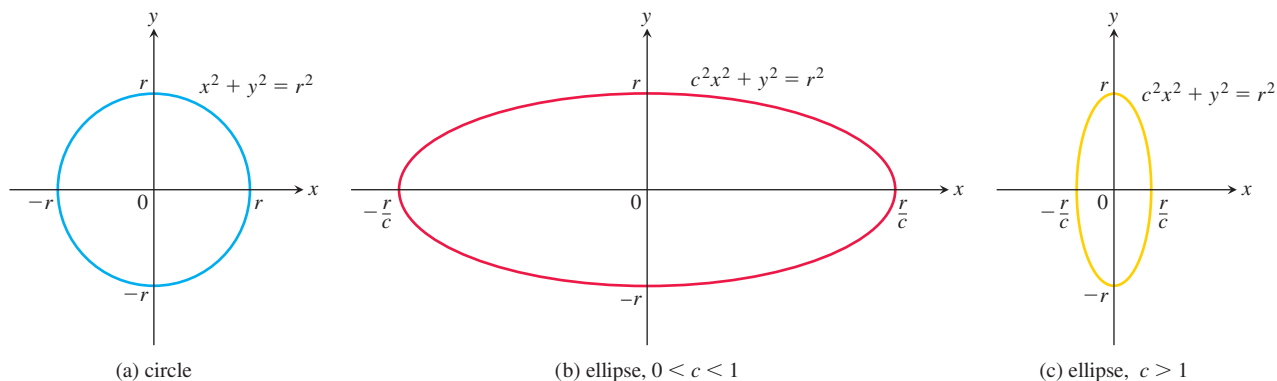


FIGURE A.26 Horizontal stretching or compression of a circle produces graphs of ellipses.

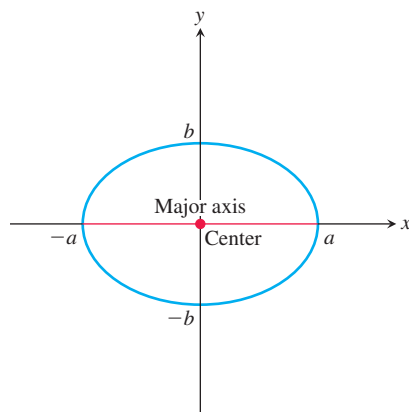


FIGURE A.27 Graph of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, where the major axis is horizontal.

If $0 < c < 1$, the graph of Equation (3) horizontally stretches the circle; if $c > 1$ the circle is compressed horizontally. In either case, the graph of Equation (3) is an ellipse (Figure A.26). Notice in Figure A.26 that the y -intercepts of all three graphs are always $-r$ and r . In Figure A.26b, the line segment joining the points $(\pm r/c, 0)$ is called the **major axis** of the ellipse; the **minor axis** is the line segment joining $(0, \pm r)$. The axes of the ellipse are reversed in Figure A.26c: The major axis is the line segment joining the points $(0, \pm r)$, and the minor axis is the line segment joining the points $(\pm r/c, 0)$. In both cases, the major axis is the longer line segment.

If we divide both sides of Equation (3) by r^2 , we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (4)$$

where $a = r/c$ and $b = r$. If $a > b$, the major axis is horizontal; if $a < b$, the major axis is vertical. The **center** of the ellipse given by Equation (4) is the origin (Figure A.27).

Substituting $x - h$ for x , and $y - k$ for y , in Equation (4) results in

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1. \quad (5)$$

Equation (5) is the **standard equation of an ellipse** with center at (h, k) .

EXERCISES A.4

Distance, Slopes, and Lines

In Exercises 1 and 2, a particle moves from A to B in the coordinate plane. Find the increments Δx and Δy in the particle's coordinates. Also find the distance from A to B .

1. $A(-3, 2)$, $B(-1, -2)$ 2. $A(-3.2, -2)$, $B(-8.1, -2)$

Describe the graphs of the equations in Exercises 3 and 4.

3. $x^2 + y^2 = 1$ 4. $x^2 + y^2 \leq 3$

Plot the points in Exercises 5 and 6 and find the slope (if any) of the line they determine. Also find the common slope (if any) of the lines perpendicular to line AB .

5. $A(-1, 2)$, $B(-2, -1)$ 6. $A(2, 3)$, $B(-1, 3)$

In Exercises 7 and 8, find an equation for (a) the vertical line and (b) the horizontal line through the given point.

7. $(-1, 4/3)$ 8. $(0, -\sqrt{2})$

In Exercises 9–15, write an equation for each line described.

9. Passes through $(-1, 1)$ with slope -1
 10. Passes through $(3, 4)$ and $(-2, 5)$
 11. Has slope $-5/4$ and y -intercept 6
 12. Passes through $(-12, -9)$ and has slope 0
 13. Has y -intercept 4 and x -intercept -1
 14. Passes through $(5, -1)$ and is parallel to the line $2x + 5y = 15$
 15. Passes through $(4, 10)$ and is perpendicular to the line $6x - 3y = 5$

In Exercises 16 and 17, find the line's x - and y -intercepts and use this information to graph the line.

16. $3x + 4y = 12$ 17. $\sqrt{2}x - \sqrt{3}y = \sqrt{6}$

18. Is there anything special about the relationship between the lines $Ax + By = C_1$ and $Bx - Ay = C_2$ ($A \neq 0$, $B \neq 0$)? Give reasons for your answer.

19. A particle starts at $A(-2, 3)$ and its coordinates change by increments $\Delta x = 5$, $\Delta y = -6$. Find its new position.

20. The coordinates of a particle change by $\Delta x = 5$ and $\Delta y = 6$ as it moves from $A(x, y)$ to $B(3, -3)$. Find x and y .

Circles

In Exercises 21–23, find an equation for the circle with the given center $C(h, k)$ and radius a . Then sketch the circle in the xy -plane. Include the circle's center in your sketch. Also, label the circle's x - and y -intercepts, if any, with their coordinate pairs.

21. $C(0, 2)$, $a = 2$ 22. $C(-1, 5)$, $a = \sqrt{10}$
 23. $C(-\sqrt{3}, -2)$, $a = 2$

Graph the circles whose equations are given in Exercises 24–26. Label each circle's center and intercepts (if any) with their coordinate pairs.

24. $x^2 + y^2 + 4x - 4y + 4 = 0$
 25. $x^2 + y^2 - 3y - 4 = 0$ 26. $x^2 + y^2 - 4x + 4y = 0$

Parabolas

Graph the parabolas in Exercises 27–30. Label the vertex, axis, and intercepts in each case.

27. $y = x^2 - 2x - 3$ 28. $y = -x^2 + 4x$
 29. $y = -x^2 - 6x - 5$ 30. $y = \frac{1}{2}x^2 + x + 4$

Inequalities

Describe the regions defined by the inequalities and pairs of inequalities in Exercises 31–34.

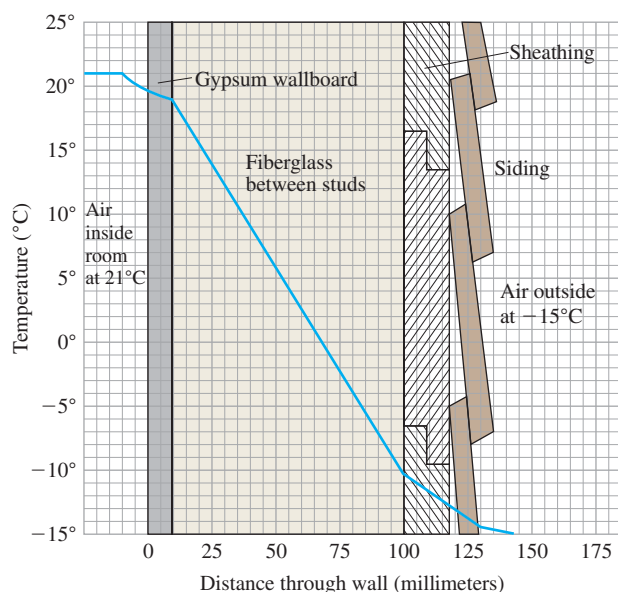
31. $x^2 + y^2 > 7$ 32. $(x - 1)^2 + y^2 \leq 4$
 33. $x^2 + y^2 > 1$, $x^2 + y^2 < 4$
 34. $x^2 + y^2 + 6y < 0$, $y > -3$
 35. Write an inequality that describes the points that lie inside the circle with center $(-2, 1)$ and radius $\sqrt{6}$.
 36. Write a pair of inequalities that describe the points that lie inside or on the circle with center $(0, 0)$ and radius $\sqrt{2}$, and on or to the right of the vertical line through $(1, 0)$.

Theory and Examples

In Exercises 37–40, graph the two equations and find the points at which the graphs intersect.

37. $y = 2x$, $x^2 + y^2 = 1$ 38. $y - x = 1$, $y = x^2$
 39. $y = -x^2$, $y = 2x^2 - 1$
 40. $x^2 + y^2 = 1$, $(x - 1)^2 + y^2 = 1$

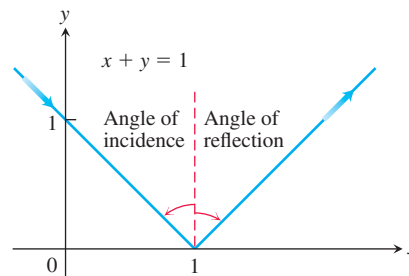
41. **Insulation** By measuring slopes in the figure, estimate the temperature change in degrees per centimeter for (a) the gypsum wallboard; (b) the fiberglass insulation; (c) the wood sheathing.



The temperature changes in the wall in Exercises 41 and 42.

42. **Insulation** According to the figure in Exercise 41, which of the materials is the best insulator? The poorest? Explain.
 43. **Pressure under water** The pressure p experienced by a diver under water is related to the diver's depth d by an equation of the form $p = kd + 1$ (k a constant). At the surface, the pressure is 1 atmosphere. The pressure at 100 meters is about 10.94 atmospheres. Find the pressure at 50 meters.
 44. **Reflected light** A ray of light comes in along the line $x + y = 1$ from the second quadrant and reflects off the x -axis (see the accompanying figure). The angle of incidence is equal to the

angle of reflection. Write an equation for the line along which the departing light travels.



The path of the light ray in Exercise 44.

Angles of incidence and reflection are measured from the perpendicular.

45. **Fahrenheit vs. Celsius** In the FC -plane, sketch the graph of the equation

$$C = \frac{5}{9}(F - 32)$$

linking Fahrenheit and Celsius temperatures. On the same graph sketch the line $C = F$. Is there a temperature at which a Celsius thermometer gives the same numerical reading as a Fahrenheit thermometer? If so, find it.

46. **The Mt. Washington Cog Railway** Civil engineers calculate the slope of roadbed as the ratio of the distance it rises or falls to the distance it runs horizontally. They call this ratio the **grade** of the roadbed, usually written as a percentage. Along the coast, commercial railroad grades are usually less than 2%. In the mountains, they may go as high as 4%. Highway grades are usually less than 5%.

The steepest part of the Mt. Washington Cog Railway in New Hampshire has an exceptional 37.1% grade. Along this part of the track, the seats in the front of the car are 4 m above those in the rear. About how far apart are the front and rear rows of seats?

47. By calculating the lengths of its sides, show that the triangle with vertices at the points $A(1, 2)$, $B(5, 5)$, and $C(4, -2)$ is isosceles but not equilateral.
 48. Show that the triangle with vertices $A(0, 0)$, $B(1, \sqrt{3})$, and $C(2, 0)$ is equilateral.
 49. Show that the points $A(2, -1)$, $B(1, 3)$, and $C(-3, 2)$ are vertices of a square, and find the fourth vertex.
 50. Three different parallelograms have vertices at $(-1, 1)$, $(2, 0)$, and $(2, 3)$. Sketch them and find the coordinates of the fourth vertex of each.
 51. For what value of k is the line $2x + ky = 3$ perpendicular to the line $4x + y = 1$? For what value of k are the lines parallel?
 52. **Midpoint of a line segment** Show that the point with coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

is the midpoint of the line segment joining $P(x_1, y_1)$ to $Q(x_2, y_2)$.

A.5

Proofs of Limit Theorems

This appendix proves Theorem 1, Parts 2–5, and Theorem 4 from Section 2.2.

THEOREM 1—Limit Laws

If L , M , c , and k are real numbers and

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = M, \quad \text{then}$$

- | | |
|-----------------------------------|---|
| 1. <i>Sum Rule:</i> | $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$ |
| 2. <i>Difference Rule:</i> | $\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$ |
| 3. <i>Constant Multiple Rule:</i> | $\lim_{x \rightarrow c} (kf(x)) = kL$ |
| 4. <i>Product Rule:</i> | $\lim_{x \rightarrow c} (f(x)g(x)) = LM$ |
| 5. <i>Quotient Rule:</i> | $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$ |
| 6. <i>Power Rule:</i> | $\lim_{x \rightarrow c} [f(x)]^n = L^n, n \text{ a positive integer}$ |
| 7. <i>Root Rule:</i> | $\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, n \text{ a positive integer}$ |
- (If n is even, we assume that $\lim_{x \rightarrow c} f(x) = L > 0$.)

We proved the Sum Rule in Section 2.3, and the Power and Root Rules are proved in more advanced texts. We obtain the Difference Rule by replacing $g(x)$ by $-g(x)$ and M by $-M$ in the Sum Rule. The Constant Multiple Rule is the special case $g(x) = k$ of the Product Rule. This leaves only the Product and Quotient Rules.

Proof of the Limit Product Rule We show that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x in the intersection D of the domains of f and g ,

$$|f(x)g(x) - LM| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Suppose then that ε is a positive number, and write $f(x)$ and $g(x)$ as

$$f(x) = L + (f(x) - L), \quad g(x) = M + (g(x) - M).$$

Multiply these expressions together and subtract LM :

$$\begin{aligned} f(x)g(x) - LM &= (L + (f(x) - L))(M + (g(x) - M)) - LM \\ &= LM + L(g(x) - M) + M(f(x) - L) \\ &\quad + (f(x) - L)(g(x) - M) - LM \\ &= L(g(x) - M) + M(f(x) - L) + (f(x) - L)(g(x) - M). \quad (1) \end{aligned}$$

Since f and g have limits L and M as $x \rightarrow c$, there exist positive numbers δ_1 , δ_2 , δ_3 , and δ_4 such that

$$\begin{aligned} |f(x) - L| &< \sqrt{\varepsilon/3} && \text{whenever} && 0 < |x - c| < \delta_1 \\ |g(x) - M| &< \sqrt{\varepsilon/3} && \text{whenever} && 0 < |x - c| < \delta_2 \\ |f(x) - L| &< \varepsilon/(3(1 + |M|)) && \text{whenever} && 0 < |x - c| < \delta_3 \\ |g(x) - M| &< \varepsilon/(3(1 + |L|)) && \text{whenever} && 0 < |x - c| < \delta_4. \end{aligned} \quad (2)$$

If we take δ to be the smallest of the numbers δ_1 through δ_4 , the inequalities on the right-hand side of the Implications (2) will hold simultaneously for $0 < |x - c| < \delta$. Therefore, for all x in D , if $0 < |x - c| < \delta$ then

$$\begin{aligned}
 |f(x)g(x) - LM| & \quad \text{Triangle inequality applied to Eq. (1)} \\
 & \leq |L||g(x) - M| + |M||f(x) - L| + |f(x) - L||g(x) - M| \\
 & \leq (1 + |L|)|g(x) - M| + (1 + |M|)|f(x) - L| + |f(x) - L||g(x) - M| \\
 & < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sqrt{\frac{\varepsilon}{3}}\sqrt{\frac{\varepsilon}{3}} = \varepsilon. \quad \text{Values from (2)}
 \end{aligned}$$

This completes the proof of the Limit Product Rule. ■

Proof of the Limit Quotient Rule We show that $\lim_{x \rightarrow c} (1/g(x)) = 1/M$. We can then conclude that

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \left(f(x) \cdot \frac{1}{g(x)} \right) = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} \frac{1}{g(x)} = L \cdot \frac{1}{M} = \frac{L}{M}$$

by the Limit Product Rule.

Let $\varepsilon > 0$ be given. To show that $\lim_{x \rightarrow c} (1/g(x)) = 1/M$, we need to show that there exists a $\delta > 0$ such that

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Since g has the limit M as $x \rightarrow c$ and since $|M| > 0$, there exists a positive number δ_1 such that

$$|g(x) - M| < \frac{M}{2} \quad \text{whenever} \quad 0 < |x - c| < \delta_1. \quad (3)$$

For any numbers A and B , the triangle inequality implies that $|A| - |B| \leq |A - B|$ and $|B| - |A| \leq |A - B|$, from which it follows that $||A| - |B|| \leq |A - B|$. With $A = g(x)$ and $B = M$, this becomes

$$||g(x)| - |M|| \leq |g(x) - M|,$$

which can be combined with the inequality on the right in Implication (3) to get, in turn,

$$\begin{aligned}
 ||g(x)| - |M|| & < \frac{|M|}{2} \\
 -\frac{|M|}{2} & < |g(x)| - |M| < \frac{|M|}{2} \\
 \frac{|M|}{2} & < |g(x)| < \frac{3|M|}{2} \\
 |M| & < 2|g(x)| < 3|M| \\
 \frac{1}{|g(x)|} & < \frac{2}{|M|} < \frac{3}{|g(x)|}. \quad (4)
 \end{aligned}$$

Therefore, $0 < |x - c| < \delta_1$ implies that

$$\begin{aligned}
 \left| \frac{1}{g(x)} - \frac{1}{M} \right| & = \left| \frac{M - g(x)}{Mg(x)} \right| \leq \frac{1}{|M|} \cdot \frac{1}{|g(x)|} \cdot |M - g(x)| \\
 & < \frac{1}{|M|} \cdot \frac{2}{|M|} \cdot |M - g(x)|. \quad \text{Inequality (4)} \quad (5)
 \end{aligned}$$

Since $(1/2)|M|^2\varepsilon > 0$, there exists a number $\delta_2 > 0$ such that

$$|M - g(x)| < \frac{\varepsilon}{2}|M|^2 \quad \text{whenever} \quad 0 < |x - c| < \delta_2. \quad (6)$$

If we take δ to be the smaller of δ_1 and δ_2 , the conclusions in (5) and (6) both hold whenever $0 < |x - c| < \delta$. Combining these conclusions gives

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

This concludes the proof of the Limit Quotient Rule. ■

THEOREM 4—The Sandwich Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval I containing c , except possibly at $x = c$ itself. Suppose also that $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$. Then $\lim_{x \rightarrow c} f(x) = L$.

Proof for Right-Hand Limits Suppose $\lim_{x \rightarrow c^+} g(x) = \lim_{x \rightarrow c^+} h(x) = L$. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that the interval $(c, c + \delta)$ is contained in I , and

$$L - \varepsilon < g(x) < L + \varepsilon \quad \text{and} \quad L - \varepsilon < h(x) < L + \varepsilon$$

whenever $c < x < c + \delta$. Since we always have $g(x) \leq f(x) \leq h(x)$ it follows that if $c < x < c + \delta$, then

$$\begin{aligned} L - \varepsilon &< g(x) \leq f(x) \leq h(x) < L + \varepsilon, \\ L - \varepsilon &< f(x) < L + \varepsilon, \\ -\varepsilon &< f(x) - L < \varepsilon. \end{aligned}$$

Therefore, $|f(x) - L| < \varepsilon$ whenever $c < x < c + \delta$.

Proof for Left-Hand Limits Suppose $\lim_{x \rightarrow c^-} g(x) = \lim_{x \rightarrow c^-} h(x) = L$. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that the interval $(c - \delta, c)$ is contained in I , and

$$L - \varepsilon < g(x) < L + \varepsilon \quad \text{and} \quad L - \varepsilon < h(x) < L + \varepsilon$$

whenever $c - \delta < x < c$. We conclude as before that $|f(x) - L| < \varepsilon$ whenever $c - \delta < x < c$.

Proof for Two-Sided Limits If $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$, then $g(x)$ and $h(x)$ both approach L as $x \rightarrow c^+$ and as $x \rightarrow c^-$; so $\lim_{x \rightarrow c^+} f(x) = L$ and $\lim_{x \rightarrow c^-} f(x) = L$. Hence $\lim_{x \rightarrow c} f(x)$ exists and equals L . ■

EXERCISES A.5

1. Suppose that functions $f_1(x)$, $f_2(x)$, and $f_3(x)$ have limits L_1 , L_2 , and L_3 , respectively, as $x \rightarrow c$. Show that their sum has limit $L_1 + L_2 + L_3$. Use mathematical induction (Appendix A.3) to generalize this result to the sum of any finite number of functions.

2. Use mathematical induction and the Limit Product Rule in Theorem 1 to show that if functions $f_1(x)$, $f_2(x)$, $\dots$, $f_n(x)$ have limits L_1 , L_2 , $\dots$, L_n as $x \rightarrow c$, then

$$\lim_{x \rightarrow c} f_1(x) \cdot f_2(x) \cdot \dots \cdot f_n(x) = L_1 \cdot L_2 \cdot \dots \cdot L_n.$$

3. Use the fact that $\lim_{x \rightarrow c} x = c$ and the result of Exercise 2 to show that $\lim_{x \rightarrow c} x^n = c^n$ for any integer $n > 1$.

4. **Limits of polynomials** Use the fact that $\lim_{x \rightarrow c} (k) = k$ for any number k , together with the results of Exercises 1 and 3, to show that $\lim_{x \rightarrow c} f(x) = f(c)$ for any polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

5. **Limits of rational functions** Use Theorem 1 and the result of Exercise 4 to show that if $f(x)$ and $g(x)$ are polynomial functions and $g(c) \neq 0$, then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}.$$

6. **Composites of continuous functions** Figure A.28 gives the diagram for a proof that the composite of two continuous functions is continuous. Reconstruct the proof from the diagram. The statement to be proved is this: If f is continuous at $x = c$ and g is continuous at $f(c)$, then $g \circ f$ is continuous at c .

Assume that c is an interior point of the domain of f and that $f(c)$ is an interior point of the domain of g . This will make the limits involved two-sided. (The arguments for the cases that involve one-sided limits are similar.)

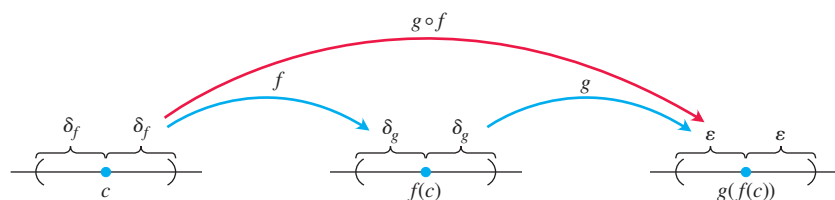


FIGURE A.28 The diagram for a proof that the composite of two continuous functions is continuous.

A.6 Commonly Occurring Limits

This appendix verifies limits (4)–(6) in Theorem 5 of Section 9.1.

Limit 4: If $|x| < 1$, $\lim_{n \rightarrow \infty} x^n = 0$ We need to show that to each $\varepsilon > 0$ there corresponds an integer N so large that $|x^n| < \varepsilon$ for all n greater than N . Since $\varepsilon^{1/n} \rightarrow 1$, while $|x| < 1$, there exists an integer N for which $\varepsilon^{1/N} > |x|$. In other words,

$$|x^N| = |x|^N < \varepsilon. \quad (1)$$

This is the integer we seek because, if $|x| < 1$, then

$$|x^n| < |x^N| \quad \text{for all } n > N. \quad (2)$$

Combining (1) and (2) produces $|x^n| < \varepsilon$ for all $n > N$, concluding the proof. ■

Limit 5: For any number x , $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ Let

$$a_n = \left(1 + \frac{x}{n}\right)^n.$$

Then

$$\ln a_n = \ln \left(1 + \frac{x}{n}\right)^n = n \ln \left(1 + \frac{x}{n}\right) \rightarrow x,$$

as we can see by the following application of L'Hôpital's Rule, in which we differentiate with respect to n :

$$\begin{aligned} \lim_{n \rightarrow \infty} n \ln \left(1 + \frac{x}{n}\right) &= \lim_{n \rightarrow \infty} \frac{\ln(1 + x/n)}{1/n} \\ &= \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{1 + x/n}\right) \cdot \left(-\frac{x}{n^2}\right)}{-1/n^2} = \lim_{n \rightarrow \infty} \frac{x}{1 + x/n} = x. \end{aligned}$$

Apply Theorem 3, Section 9.1, with $f(x) = e^x$ to conclude that

$$\left(1 + \frac{x}{n}\right)^n = a_n = e^{\ln a_n} \rightarrow e^x. \quad \blacksquare$$

Limit 6: For any number x , $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$ Since

$$-\frac{|x|^n}{n!} \leq \frac{x^n}{n!} \leq \frac{|x|^n}{n!},$$

all we need to show is that $|x|^n/n! \rightarrow 0$. We can then apply the Sandwich Theorem for Sequences (Section 9.1, Theorem 2) to conclude that $x^n/n! \rightarrow 0$.

The first step in showing that $|x|^n/n! \rightarrow 0$ is to choose an integer $M > |x|$, so that $(|x|/M) < 1$. By Limit 4, just proved, we then have $(|x|/M)^n \rightarrow 0$. We then restrict our attention to values of $n > M$. For these values of n , we can write

$$\begin{aligned} \frac{|x|^n}{n!} &= \frac{|x|^n}{1 \cdot 2 \cdot \cdots \cdot M \cdot \underbrace{(M+1) \cdot (M+2) \cdot \cdots \cdot n}_{(n-M) \text{ factors}}} \\ &\leq \frac{|x|^n}{M! M^{n-M}} = \frac{|x|^n M^M}{M! M^n} = \frac{M^M}{M!} \left(\frac{|x|}{M}\right)^n. \end{aligned}$$

Thus,

$$0 \leq \frac{|x|^n}{n!} \leq \frac{M^M}{M!} \left(\frac{|x|}{M}\right)^n.$$

Now, the constant $M^M/M!$ does not change as n increases. Thus the Sandwich Theorem tells us that $|x|^n/n! \rightarrow 0$ because $(|x|/M)^n \rightarrow 0$. $\blacksquare$

A.7 Theory of the Real Numbers

A rigorous development of calculus is based on properties of the real numbers. Many results about functions, derivatives, and integrals would be false if stated for functions defined only on the rational numbers. In this appendix we briefly examine some basic concepts of the theory of the reals that hint at what might be learned in a deeper, more theoretical study of calculus.

Three types of properties make the real numbers what they are. These are the **algebraic**, **order**, and **completeness** properties. The algebraic properties involve addition and multiplication, subtraction and division. They apply to rational or complex numbers (discussed in Chapter 18) as well as to the reals.

The structure of numbers is built around a set with addition and multiplication operations. The following properties are required of addition and multiplication.

- A1** $a + (b + c) = (a + b) + c$ for all a, b, c .
- A2** $a + b = b + a$ for all a, b .
- A3** There is a number called “0” such that $a + 0 = a$ for all a .
- A4** For each number a , there is a number b such that $a + b = 0$.
- M1** $a(bc) = (ab)c$ for all a, b, c .
- M2** $ab = ba$ for all a, b .
- M3** There is a number called “1” such that $a \cdot 1 = a$ for all a .
- M4** For each nonzero number a , there is a number b such that $ab = 1$.
- D** $a(b + c) = ab + ac$ for all a, b, c .

A1 and M1 are *associative laws*, A2 and M2 are *commutativity laws*, A3 and M3 are *identity laws*, and D is the *distributive law*. Sets that have these algebraic properties are examples of **fields**; they are studied in depth in the area of theoretical mathematics called abstract algebra.

The **order** properties enable us to compare the sizes of any two numbers. The order properties are

- O1** For any a and b , either $a \leq b$ or $b \leq a$ or both.
- O2** If $a \leq b$ and $b \leq a$ then $a = b$.
- O3** If $a \leq b$ and $b \leq c$ then $a \leq c$.
- O4** If $a \leq b$ then $a + c \leq b + c$.
- O5** If $a \leq b$ and $0 \leq c$ then $ac \leq bc$.

O3 is the *transitivity law*, and O4 and O5 relate ordering to addition and multiplication.

We can order the reals, the integers, and the rational numbers, but we cannot order the complex numbers (there is no reasonable way to decide whether a number like $i = \sqrt{-1}$ is bigger or smaller than zero). A field in which the size of any two elements can be compared as above is called an **ordered field**. Both the rational numbers and the real numbers are ordered fields, and there are many others.

We can think of real numbers geometrically, lining them up as points on a line. The **completeness property** says that the real numbers correspond to all points on the line, with no “holes” or “gaps.” The rationals, in contrast, omit points such as $\sqrt{2}$ and π , and the integers even leave out fractions like $1/2$. The reals, having the completeness property, omit no points.

What exactly do we mean by this vague idea of missing holes? To answer this we must give a more precise description of completeness. A number M is an **upper bound** for a set of numbers if all numbers in the set are smaller than or equal to M . M is a **least upper bound** if it is the smallest upper bound. For example, $M = 2$ is an upper bound for the negative numbers. So is $M = 1$, showing that 2 is not a least upper bound. The least upper bound for the set of negative numbers is $M = 0$. We define a **complete** ordered field to be one in which every nonempty set bounded above has a least upper bound.

If we work with just the rational numbers, the set of numbers less than $\sqrt{2}$ is bounded, but it does not have a rational least upper bound, since any rational upper bound M must be larger than $\sqrt{2}$ and can be replaced by a slightly smaller rational number that is still larger than $\sqrt{2}$. So the rationals are not complete. In the real numbers, a set that is bounded above always has a least upper bound. The reals are a complete ordered field.

The completeness property is at the heart of many results in calculus. One example occurs when we search for a maximum value for a function on a closed interval $[a, b]$, as in Section 4.1. The function $y = x - x^3$ has a maximum value on $[0, 1]$ at the point x satisfying $1 - 3x^2 = 0$, or $x = \sqrt{1/3}$. If we limited our consideration to functions defined only on rational numbers, we would have to conclude that the function has no maximum, since $\sqrt{1/3}$ is irrational (Figure A.29). The Extreme Value Theorem (Section 4.1), which implies that continuous functions on closed intervals $[a, b]$ have a maximum value, is not true for functions defined only on the rationals.

The Intermediate Value Theorem implies that a continuous function f on an interval $[a, b]$ with $f(a) < 0$ and $f(b) > 0$ must be zero somewhere in $[a, b]$. The function values cannot jump from negative to positive without there being some point x in $[a, b]$ where $f(x) = 0$. The Intermediate Value Theorem also relies on the completeness of the real numbers and is false for continuous functions defined only on the rationals. The function $f(x) = 3x^2 - 1$ has $f(0) = -1$ and $f(1) = 2$, but if we consider f only on the rational numbers, it never equals zero. The only value of x for which $f(x) = 0$ is $x = \sqrt{1/3}$, an irrational number.

We have captured the desired properties of the reals by saying that the real numbers are a complete ordered field. But we’re not quite finished. Greek mathematicians in the

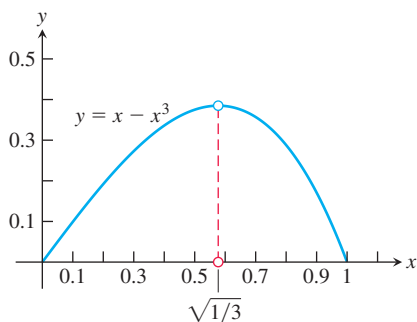


FIGURE A.29 The maximum value of $y = x - x^3$ on $[0, 1]$ occurs at the irrational number $x = \sqrt{1/3}$.

school of Pythagoras tried to impose another property on the numbers of the real line, the condition that all numbers are ratios of integers. They learned that their effort was doomed when they discovered irrational numbers such as $\sqrt{2}$. How do we know that our efforts to specify the real numbers are not also flawed, for some unseen reason? The artist Escher drew optical illusions of spiral staircases that went up and up until they rejoined themselves at the bottom. An engineer trying to build such a staircase would find that no structure realized the plans the architect had drawn. Could it be that our design for the reals contains some subtle contradiction, and that no construction of such a number system can be made?

We resolve this issue by giving a specific description of the real numbers and verifying that the algebraic, order, and completeness properties are satisfied in this model. This is called a **construction** of the reals, and just as stairs can be built with wood, stone, or steel, there are several approaches to constructing the reals. One construction treats the reals as all the infinite decimals,

$$a.d_1d_2d_3d_4\cdots.$$

In this approach a real number is an integer a followed by a sequence of decimal digits $d_1, d_2, d_3, \dots$, each between 0 and 9. This sequence may stop, or repeat in a periodic pattern, or keep going forever with no pattern. In this form, 2.00, 0.333333 $\dots$ and 3.1415926535898 $\dots$ represent three familiar real numbers. The real meaning of the dots “ $\dots$ ” following these digits requires development of the theory of sequences and series, as in Chapter 9. Each real number is constructed as the limit of a sequence of rational numbers given by its finite decimal approximations. An infinite decimal is then the same as a series

$$a + \frac{d_1}{10} + \frac{d_2}{100} + \cdots.$$

This decimal construction of the real numbers is not entirely straightforward. It’s easy enough to check that it gives numbers that satisfy the completeness and order properties, but verifying the algebraic properties is rather involved. Even adding or multiplying two numbers requires an infinite number of operations. Making sense of division requires a careful argument involving limits of rational approximations to infinite decimals.

A different approach was taken by Richard Dedekind (1831–1916), a German mathematician, who gave the first rigorous construction of the real numbers in 1872. Given any real number x , we can divide the rational numbers into two sets: those less than or equal to x and those greater. Dedekind cleverly reversed this reasoning and defined a real number to be a division of the rational numbers into two such sets. This seems like a strange approach, but such indirect methods of constructing new structures from old are powerful tools in theoretical mathematics.

These and other approaches can be used to construct a system of numbers having the desired algebraic, order, and completeness properties. A final issue that arises is whether all the constructions give the same thing. Is it possible that different constructions result in different number systems satisfying all the required properties? If yes, which of these comprises the real numbers? Fortunately, the answer turns out to be no. The reals are the only number system satisfying the algebraic, order, and completeness properties.

Confusion about the nature of the numbers and about limits caused considerable controversy in the early development of calculus. Calculus pioneers such as Newton, Leibniz, and their successors, when looking at what happens to the difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

as each of Δy and Δx approach zero, talked about the resulting derivative being a quotient of two infinitely small quantities. These “infinitesimals,” written dx and dy , were thought to be some new kind of number, smaller than any fixed number but not zero. Similarly, a definite integral was thought of as a sum of an infinite number of infinitesimals

$$f(x) \cdot dx$$

as x varied over a closed interval. The approximating difference quotients $\Delta y/\Delta x$ were understood much as today, but it was the quotient of infinitesimal quantities, rather than a limit, that was thought to encapsulate the meaning of the derivative. This way of thinking led to logical difficulties, as attempted definitions and manipulations of infinitesimals ran into contradictions and inconsistencies. The more concrete and computable difference quotients did not cause such trouble, but they were thought of merely as useful calculation tools. Difference quotients were used to work out the numerical value of the derivative and to derive general formulas for calculation, but they were not considered to be at the heart of what the derivative actually was. Today we realize that the logical problems associated with infinitesimals can be avoided by *defining* the derivative to be the limit of its approximating difference quotients. The ambiguities of the old approach are no longer present, and in the standard theory of calculus, infinitesimals are neither needed nor used.

A.8 Probability

The outcome of some events, such as a heavy rock falling from a great height, can be modeled so that we can predict with high accuracy what will happen. On the other hand, many events have more than one possible outcome and which one of them will occur is uncertain. If we toss a coin, a head or a tail will result with each outcome being equally likely, but we do not know in advance which one it will be. If we randomly select and then weigh a person from a large population, there are many possible weights the person might have, and it is not certain whether the weight will be between 80 and 85 kg. We are told it is highly likely, but not known for sure, that an earthquake of magnitude 6.0 or greater on the Richter scale will occur near a major population area in California within the next one hundred years. Events having more than one possible outcome are *probabilistic* in nature, and when modeling them we assign a *probability* to the likelihood that a particular outcome may occur. In this section we show how calculus plays a central role in making predictions with probabilistic models.

Random Variables

We begin our discussion with some familiar examples of uncertain events for which the collection of all possible outcomes is finite.

EXAMPLE 1

- (a) If we toss a coin once, there are two possible outcomes $\{H, T\}$, where H represents the coin landing head face up and T a tail landing face up. If we toss a coin three times, there are eight possible outcomes, taking into account the order in which a head or tail occurs. The set of outcomes is $\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.
- (b) If we roll a six-sided die once, the set of possible outcomes is $\{1, 2, 3, 4, 5, 6\}$ representing the six faces of the die.
- (c) If we select at random two cards from a 52-card deck, there are 52 possible outcomes for the first card drawn and then 51 possibilities for the second card. Since the order of the cards does not matter, there are $(52 \cdot 51)/2 = 1,326$ possible outcomes altogether. ■

It is customary to refer to the set of all possible outcomes as the *sample space* for an event. With an uncertain event we are usually interested in which outcomes, if any, are more likely to occur than others, and to how large an extent. In tossing a coin three times, is it more likely that two heads or that one head will result? To answer such questions, we need a way to quantify the outcomes.

DEFINITION A **random variable** is a function X that assigns a numerical value to each outcome in a sample space.

Random variables that have only finitely many values are called **discrete** random variables. A **continuous random variable** can take on values in an entire interval, and it is associated with a *distribution function*, which we explain later.

EXAMPLE 2

- (a) Suppose we toss a coin three times giving the possible outcomes $\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$. Define the random variable X to be the number of heads that appear. So $X(HHT) = 2$, $X(THT) = 1$, and so forth. Since X can only assume the values 0, 1, 2, or 3, it is a discrete random variable.
- (b) We spin an arrow anchored by a pin located at the origin. The arrow can wind up pointing in any possible direction, and we define the random variable X as the radian angle the arrow makes with the positive x -axis, measured counterclockwise. In this case, X is a continuous random variable that can take on any value in the interval $[0, 2\pi)$.
- (c) The weight of a randomly selected person in a given population is a continuous random variable W . The cholesterol level of a randomly chosen person, and the waiting time for service of a person in a queue at a bank, are also continuous random variables.
- (d) The scores on the national ACT Examination for college admissions in a particular year are described by a discrete random variable S taking on integer values between 1 and 36. If the number of outcomes is large, or for reasons involving statistical analysis, discrete random variables such as test scores are often modeled as continuous random variables (Example 13).
- (e) We roll a pair of dice and define the random variable X to be the sum of the numbers on the top faces. This sum can only assume the integer values from 2 through 12, so X is a discrete random variable.
- (f) A tire company produces tires for mid-sized sedans. The tires are guaranteed to last for 50,000 kilometers, but some will fail sooner, and some will last many more kilometers beyond 50,000 kilometers. The lifetime in kilometers of a tire is described by a continuous random variable L . ■

Probability Distributions

A *probability distribution* describes the probabilistic behavior of a random variable. Our chief interest is in probability distributions associated with continuous random variables, but to gain some perspective we first consider a distribution for a discrete random variable.

Suppose we toss a coin three times, with each side H or T equally likely to occur on a given toss. We define the discrete random variable X that assigns the number of heads appearing in each outcome, giving the following.

	{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}							
X	↓	↓	↓	↓	↓	↓	↓	↓
	3	2	2	1	2	1	1	0

Next we count the *frequency* or number of times a specific value of X occurs. Because each of the eight outcomes is equally likely to occur, we can calculate the probability of the random variable X by dividing the frequency of each value by the total number of outcomes. We summarize our results as follows:

Value of X	0	1	2	3
Frequency	1	3	3	1
$P(X)$	1/8	3/8	3/8	1/8

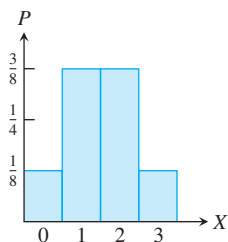


FIGURE A.30 Probability bar graph for the random variable X when tossing a fair coin three times.

We display this information in a probability bar graph of the discrete random variable X , as shown in Figure A.30. The values of X are portrayed by intervals of length 1 on the x -axis so the area of each bar in the graph is the probability of the corresponding outcome. For instance, the probability that exactly two heads occurs in the three tosses of the coin is the area of the bar associated with the value $X = 2$, which is $3/8$. Similarly, the probability that two or more heads occurs is the sum of areas of the bars associated with the values $X = 2$ and $X = 3$, or $4/8$. The probability that either zero or three heads occurs is $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$, and so forth. Note that the total area of all the bars in the graph is 1, which is the sum of all the probabilities for X .

With a continuous random variable, even when the outcomes are equally likely, we cannot simply count the number of outcomes in the sample space or the frequencies of outcomes that lead to a specific value of X . In fact, the probability that X takes on any particular one of its values is zero. What *is* meaningful to ask is how probable it is that the random variable takes on a value within some specified *interval* of values.

We capture the information we need about the probabilities of X in a function whose graph behaves much like the bar graph in Figure A.30. That is, we take a nonnegative function f defined over the range of the random variable with the property that the total area beneath the graph of f is 1. The probability that a value of the random variable X lies within some specified interval $[c, d]$ is then the area under the graph of f over that interval. The following definition assumes the range of the continuous random variable X is any real value, but the definition is general enough to account for random variables having a range of finite length.

DEFINITIONS A **probability density function** for a continuous random variable is a function f defined over $(-\infty, \infty)$ and having the following properties:

1. f is continuous, except possibly at a finite number of points.
2. f is nonnegative, so $f \geq 0$.
3. $\int_{-\infty}^{\infty} f(x) dx = 1$.

If X is a continuous random variable with probability density function f , the **probability** that X assumes a value in the interval between $X = c$ and $X = d$ is given by the integral

$$P(c \leq X \leq d) = \int_c^d f(x) dx.$$

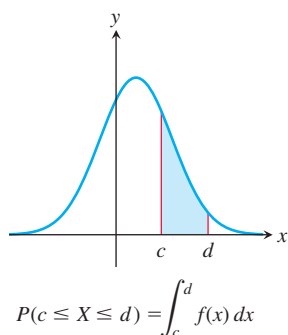
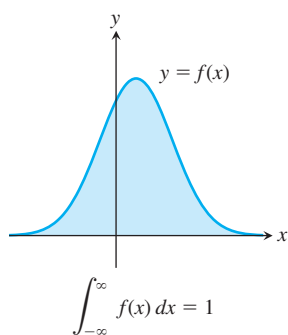


FIGURE A.31 A probability density function for the continuous random variable X .

The probability that a continuous random variable X assumes a particular real value c is $P(X = c) = \int_c^c f(x) dx = 0$, consistent with our previous assertion. Since the area under the graph of f over the interval $[c, d]$ is only a portion of the total area beneath the graph, the probability $P(c \leq X \leq d)$ is always a number between zero and one. Figure A.31 illustrates a probability density function.

A probability density function for a random variable X resembles the density function for a wire of varying density. To obtain the mass of a segment of the wire, we integrate the density of the wire over an interval. To obtain the probability that a random variable has values in a particular interval, we integrate the probability density function over that interval.

EXAMPLE 3 Let $f(x) = 2e^{-2x}$ if $0 \leq x < \infty$ and $f(x) = 0$ for all negative values of x .

- (a) Verify that f is a probability density function.
- (b) The time T in hours until a car passes a spot on a remote road is described by the probability density function f . Find the probability $P(T \leq 1)$ that a hitchhiker at that spot will see a car within one hour.
- (c) Find the probability $P(T = 1)$ that a car passes by the spot after precisely one hour.

Solution

- (a) The function f is continuous except at $x = 0$, and is everywhere nonnegative. Moreover,

$$\int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} 2e^{-2x} dx = \lim_{b \rightarrow \infty} \int_0^b 2e^{-2x} dx = \lim_{b \rightarrow \infty} (1 - e^{-2b}) = 1.$$

So all of the conditions are satisfied and we have shown that f is a probability density function.

- (b) The probability that a car comes after a time lapse between zero and one hour is given by integrating the probability density function over the interval $[0, 1]$. So

$$P(T \leq 1) = \int_0^1 2e^{-2t} dt = -e^{-2t} \Big|_0^1 = 1 - e^{-2} \approx 0.865.$$

This result can be interpreted to mean that if 100 people were to hitchhike at that spot, about 87 of them can expect to see a car within one hour.

- (c) This probability is the integral $\int_1^1 f(t) dt$, which equals zero. We interpret this to mean that a sufficiently accurate measurement of the time until a car comes by the spot would have no possibility of being precisely equal to one hour. It might be very close, perhaps, but it would not be exactly one hour. ■

We can extend the definition to finite intervals. If f is a nonnegative function with at most finitely many discontinuities over the interval $[a, b]$, and its extension F to $(-\infty, \infty)$, obtained by defining F to be 0 outside of $[a, b]$, satisfies the definition for a probability density function, then f is a **probability density function for $[a, b]$** . This means that $\int_a^b f(x) dx = 1$. Similar definitions can be made for the intervals (a, b) , $(a, b]$, and $[a, b)$.

EXAMPLE 4 Show that $f(x) = \frac{4}{27}x^2(3 - x)$ is a probability density function over the interval $[0, 3]$.

Solution The function f is continuous and nonnegative over $[0, 3]$. Also,

$$\int_0^3 \frac{4}{27}x^2(3 - x) dx = \frac{4}{27} \left[x^3 - \frac{1}{4}x^4 \right]_0^3 = \frac{4}{27} \left(27 - \frac{81}{4} \right) = 1.$$

We conclude that f is a probability density function over $[0, 3]$. ■

Exponentially Decreasing Distributions

The distribution in Example 3 is called an *exponentially decreasing probability density function*. These probability density functions always take on the form

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ ce^{-cx} & \text{if } x \geq 0 \end{cases}$$

(see Exercise 23). Exponential density functions can provide models for describing random variables such as the lifetimes of light bulbs, radioactive particles, tooth crowns, and many kinds of electronic components. They also model the amount of time until some specific event occurs, such as the time until a pollinator arrives at a flower, the arrival times of a bus at a stop, the time between individuals joining a queue, the waiting time between phone calls at a help desk, and even the lengths of the phone calls themselves. A graph of an exponential density function is shown in Figure A.32.

Random variables with exponential distributions are *memoryless*. If we think of X as describing the lifetime of some object, then the probability that the object survives for at least $s + t$ hours, given that it has survived t hours, is the same as the initial probability that it survives for at least s hours. For instance, the current age t of a radioactive particle does not change the probability that it will survive for at least another time period of length s . Sometimes the exponential distribution is used as a model when the memoryless

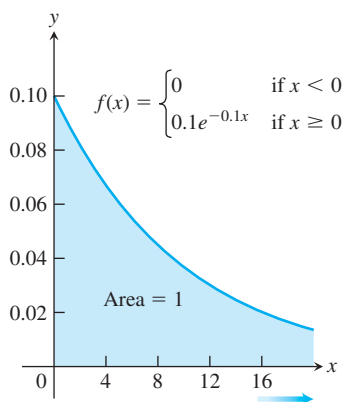


FIGURE A.32 An exponentially decreasing probability density function.

principle is violated because it provides reasonable approximations that are good enough for their intended use. For instance, this might be the case when predicting the lifetime of an artificial hip replacement or heart valve for a particular patient. Here is an application illustrating the exponential distribution.

EXAMPLE 5 An electronics company models the lifetime T in years of a chip they manufacture with the exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ 0.1e^{-0.1t} & \text{if } t \geq 0. \end{cases}$$

Using this model,

- (a) Find the probability $P(T > 2)$ that a chip will last for more than two years.
- (b) Find the probability $P(4 \leq T \leq 5)$ that a chip will fail in the fifth year.
- (c) If 1000 chips are shipped to a customer, how many can be expected to fail within three years?

Solution

- (a) The probability that a chip lasts at least two years is

$$\begin{aligned} P(T > 2) &= \int_2^{\infty} 0.1e^{-0.1t} dt = \lim_{b \rightarrow \infty} \int_2^b 0.1e^{-0.1t} dt \\ &= \lim_{b \rightarrow \infty} [e^{-0.2} - e^{-0.1b}] = e^{-0.2} \approx 0.819. \end{aligned}$$

That is, about 82% of the chips last more than two years.

- (b) The probability is

$$P(4 \leq T \leq 5) = \int_4^5 0.1e^{-0.1t} dt = -e^{-0.1t} \Big|_4^5 = e^{-0.4} - e^{-0.5} \approx 0.064$$

which means that about 6% of the chips fail during the fifth year.

- (c) We want the probability

$$P(0 \leq T \leq 3) = \int_0^3 0.1e^{-0.1t} dt = -e^{-0.1t} \Big|_0^3 = 1 - e^{-0.3} \approx 0.259.$$

We can expect that about 259 of the 1000 chips will fail within three years. ■

Expected Values, Means, and Medians

Suppose the weight in lbs of a steer raised on a cattle ranch is described by a continuous random variable W with probability density function $f(w)$ and that the rancher can sell a steer of weight w for $g(w)$ dollars. How much can the rancher expect to earn for a randomly chosen steer on the ranch?

To answer this question, we consider a small interval $[w_i, w_{i+1}]$ of width Δw_i and note that the probability a steer has weight in this interval is

$$\int_{w_i}^{w_{i+1}} f(w) dw \approx f(w_i) \Delta w_i.$$

The earning on a steer in this interval is approximately $g(w_i)$. The Riemann sum

$$\sum g(w_i) f(w_i) \Delta w_i$$

then approximates the amount the rancher would receive for a steer. We assume that steers have a maximum weight, so f is zero outside some finite interval $[0, b]$. Then taking the limit of the Riemann sum as the width of each interval approaches zero gives the integral

$$\int_{-\infty}^{\infty} g(w) f(w) dw.$$

This integral estimates how much the rancher can expect to earn for a typical steer on the ranch and is the *expected value of the function* g .

The expected values of certain functions of a random variable X have particular importance in probability and statistics. One of the most important of these functions is the expected value of the function $g(x) = x$.

DEFINITION The **expected value** or **mean** of a continuous random variable X with probability density function f is the number

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x) dx.$$

The expected value $E(X)$ can be thought of as a weighted average of the random variable X , where each value of X is weighted by $f(X)$. The mean can also be interpreted as the long-run average value of the random variable X , and it is one measure of the centrality of the random variable X .

EXAMPLE 6 Find the mean of the random variable X with exponential probability density function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ ce^{-cx} & \text{if } x \geq 0. \end{cases}$$

Solution From the definition we have

$$\begin{aligned} \mu &= \int_{-\infty}^{\infty} xf(x) dx = \int_0^{\infty} xce^{-cx} dx \\ &= \lim_{b \rightarrow \infty} \int_0^b xce^{-cx} dx = \lim_{b \rightarrow \infty} \left(-xe^{-cx} \Big|_0^b + \int_0^b e^{-cx} dx \right) \\ &= \lim_{b \rightarrow \infty} \left(-be^{-cb} - \frac{1}{c}e^{-cb} + \frac{1}{c} \right) = \frac{1}{c}. \end{aligned}$$

l'Hôpital's Rule on first term

Therefore, the mean is $\mu = 1/c$. ■

From the result in Example 6, knowing the mean or expected value μ of a random variable X having an exponential density function allows us to write its entire formula.

Exponential Density Function for a Random Variable X with Mean μ

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \mu^{-1}e^{-x/\mu} & \text{if } x \geq 0 \end{cases}$$

EXAMPLE 7 Suppose the time T before a chip fails in Example 5 is modeled instead by the exponential density function with a mean of eight years. Find the probability that a chip will fail within five years.

Solution The exponential density function with mean $\mu = 8$ is

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{8}e^{-t/8} & \text{if } t \geq 0 \end{cases}$$

Then the probability a chip will fail within five years is the definite integral

$$P(0 \leq T \leq 5) = \int_0^5 0.125e^{-0.125t} dt = -e^{-0.125t} \Big|_0^5 = 1 - e^{-0.625} \approx 0.465$$

so about 47% of the chips can be expected to fail within five years. ■

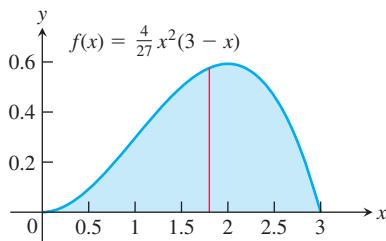


FIGURE A.33 The expected value of a random variable with this probability density function is $\mu = 1.8$ (Example 8).

EXAMPLE 8 Find the expected value for the random variable X with probability density function given by Example 4.

Solution The expected value is

$$\begin{aligned} \mu = E(X) &= \int_0^3 \frac{4}{27} x^3(3-x) dx = \frac{4}{27} \left[\frac{3}{4} x^4 - \frac{1}{5} x^5 \right]_0^3 \\ &= \frac{4}{27} \left(\frac{243}{4} - \frac{243}{5} \right) = 1.8 \end{aligned}$$

From Figure A.33, you can see that this expected value is reasonable because the region beneath the probability density function appears to be balanced about the vertical line $x = 1.8$. That is, the horizontal coordinate of the centroid of a plate described by the region is $\bar{x} = 1.8$. ■

There are other ways to measure the centrality of a random variable with a given probability density function.

DEFINITION The **median** of a continuous random variable X with probability density function f is the number m for which

$$\int_{-\infty}^m f(x) dx = \frac{1}{2} \quad \text{and} \quad \int_m^{\infty} f(x) dx = \frac{1}{2}.$$

The definition of the median means that there is an equal likelihood that the random variable X will be smaller than m or larger than m .

EXAMPLE 9 Find the median of a random variable X with exponential probability density function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ ce^{-cx} & \text{if } x \geq 0. \end{cases}$$

Solution The median m must satisfy

$$\frac{1}{2} = \int_0^m ce^{-cx} dx = -e^{-cx} \Big|_0^m = 1 - e^{-cm}.$$

It follows that

$$e^{-cm} = \frac{1}{2} \quad \text{or} \quad m = \frac{1}{c} \ln 2.$$

Also,

$$\frac{1}{2} = \lim_{b \rightarrow \infty} \int_m^b ce^{-cx} dx = \lim_{b \rightarrow \infty} \left[-e^{-cx} \right]_m^b = \lim_{b \rightarrow \infty} (e^{-cm} - e^{-cb}) = e^{-cm}$$

giving the same value for m . Since $1/c$ is the mean μ of X with an exponential distribution, we conclude that the median is $m = \mu \ln 2$. The mean and median differ because the probability density function is skewed and spreads toward the right. ■

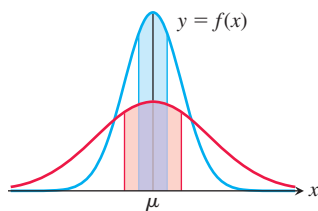


FIGURE A.34 Probability density functions with the same mean can have different spreads in relation to the mean. The blue and red regions under the curves have equal area.

Variance and Standard Deviation

Random variables with exactly the same mean μ but different distributions can behave very differently (see Figure A.34). The *variance* of a random variable X measures how spread out the values of X are in relation to the mean, and we measure this dispersion by the expected value of $(X - \mu)^2$. Since the variance measures the expected square of the difference from the mean, we often work instead with its square root.

DEFINITIONS The **variance** of a random variable X with probability density function f is the expected value of $(X - \mu)^2$:

$$\text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

The **standard deviation** of X is

$$\sigma_X = \sqrt{\text{Var}(X)} = \sqrt{\int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx}.$$

EXAMPLE 10 Find the standard deviation of the random variable T in Example 5, and find the probability that T lies within one standard deviation of the mean.

Solution The probability density function is the exponential density function with mean $\mu = 10$ by Example 6. To find the standard deviation we first calculate the variance integral:

$$\begin{aligned} \int_{-\infty}^{\infty} (t - \mu)^2 f(t) dt &= \int_0^{\infty} (t - 10)^2 (0.1e^{-0.1t}) dt \\ &= \lim_{b \rightarrow \infty} \int_0^b (t - 10)^2 (0.1e^{-0.1t}) dt \\ &= \lim_{b \rightarrow \infty} \left[-(t - 10)^2 - 20(t - 10) \right] e^{-0.1t} \Big|_0^b \\ &\quad + \lim_{b \rightarrow \infty} \int_0^b 20e^{-0.1t} dt \quad \text{Integrating by parts} \\ &= [0 + (-10)^2 + 20(-10)] - 20 \lim_{b \rightarrow \infty} (10e^{-0.1t}) \Big|_0^b \\ &= -100 - 200 \lim_{b \rightarrow \infty} (e^{-0.1b} - 1) = 100. \end{aligned}$$

The standard deviation is the square root of the variance, so $\sigma = 10.0$.

To find the probability that T lies within one standard deviation of the mean, we find the probability $P(\mu - \sigma \leq T \leq \mu + \sigma)$. For this example, we have

$$P(10 - 10 \leq T \leq 10 + 10) = \int_0^{20} 0.1e^{-0.1t} dt = -e^{-0.1t} \Big|_0^{20} = 1 - e^{-2} \approx 0.865$$

This means that about 87% of the chips will fail within twenty years. ■

Uniform Distributions

The **uniform distribution** is very simple, but it occurs commonly in applications. The probability density function for this distribution on the interval $[a, b]$ is

$$f(x) = \frac{1}{b - a}, \quad a \leq x \leq b.$$

If each outcome in the sample space is equally likely to occur, then the random variable X has a uniform distribution. Since f is constant on $[a, b]$, a random variable with a uniform

distribution is just as likely to be in one subinterval of a fixed length as in any other of the same length. The probability that X assumes a value in a subinterval of $[a, b]$ is the length of that subinterval divided by $(b - a)$.

EXAMPLE 11 An anchored arrow is spun around the origin, and the random variable X is the radian angle the arrow makes with the positive x -axis, measured within the interval $[0, 2\pi)$. Assuming there is equal probability for the arrow pointing in any direction, find the probability density function and the probability that the arrow ends up pointing between North and East.

Solution We model the probability density function with the uniform distribution $f(x) = 1/2\pi$, $0 \leq x < 2\pi$, and $f(x) = 0$ elsewhere.

The probability that the arrow ends up pointing between North and East is given by

$$P\left(0 \leq X \leq \frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{1}{2\pi} dx = \frac{1}{4}.$$

Normal Distributions

Numerous applications use the **normal distribution**, which is defined by the probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}.$$

It can be shown that the mean of a random variable X with this probability density function is μ and its standard deviation is σ . In applications the values of μ and σ are often estimated using large sets of data. The function is graphed in Figure A.35, and the graph is sometimes called a *bell curve* because of its shape. Since the curve is symmetric about the mean, the median for X is the same as its mean. It is often observed in practice that many random variables have approximately a normal distribution. Some examples illustrating this phenomenon are the height of a man, the annual rainfall in a certain region, an individual's blood pressure, the serum cholesterol level in the blood, the brain weights in a certain population of adults, and the amount of growth in a given period for a population of sunflower seeds.

The normal probability density function does not have an antiderivative expressible in terms of familiar functions. Once μ and σ are fixed, however, an integral involving the normal probability density function can be computed using numerical integration methods. Usually we use the numerical integration capability of a computer or calculator to estimate the values of these integrals. Such computations show that for any normal distribution, we get the following values for the probability that the random variable X lies within $k = 1, 2, 3$, or 4 standard deviations of the mean:

$$P(\mu - \sigma < X < \mu + \sigma) \approx 0.68269$$

$$P(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95450$$

$$P(\mu - 3\sigma < X < \mu + 3\sigma) \approx 0.99730$$

$$P(\mu - 4\sigma < X < \mu + 4\sigma) \approx 0.99994$$

This means, for instance, that the random variable X will take on a value within two standard deviations of the mean about 95% of the time. About 68% of the time, X will lie within one standard deviation of the mean (see Figure A.36).

EXAMPLE 12 An individual's blood pressure is an important indicator of overall health. A medical study of healthy individuals between 14 and 70 years of age modeled

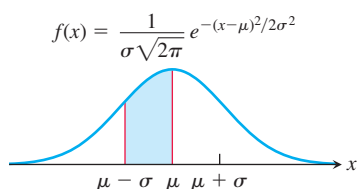


FIGURE A.35 The normal probability density function with mean μ and standard deviation σ .

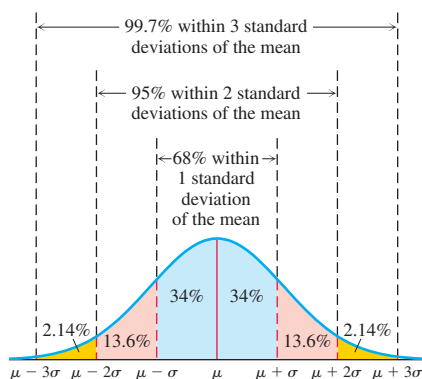


FIGURE A.36 Probabilities of the normal distribution within its standard deviation bands.

their systolic blood pressure using a normal distribution with mean 119.7 mm Hg and standard deviation 10.9 mm Hg.

- (a) Using this model, what percentage of the population has a systolic blood pressure between 140 and 160 mm Hg, the levels set by the American Heart Association for Stage 1 hypertension?
- (b) What percentage has a blood pressure between 160 and 180 mm Hg, the levels set by the American Heart Association for Stage 2 hypertension?
- (c) What percentage has a blood pressure in the normal range of 90–120, as set by the American Heart Association?

Solution

- (a) Since we cannot find an antiderivative, we use a computer to evaluate the probability integral of the normal probability density function with $\mu = 119.7$ and $\sigma = 10.9$:

$$P(140 \leq X \leq 160) = \int_{140}^{160} \frac{1}{10.9\sqrt{2\pi}} e^{-(x-119.7)^2/2(10.9)^2} dx \approx 0.03117.$$

This means that about 3% of the population in the studied age range have Stage 1 hypertension.

- (b) Again we use a computer to calculate the probability that the blood pressure is between 160 and 180 mm Hg:

$$P(160 \leq X \leq 180) = \int_{160}^{180} \frac{1}{10.9\sqrt{2\pi}} e^{-(x-119.7)^2/2(10.9)^2} dx \approx 0.00011.$$

We conclude that about 0.011% of the population has Stage 2 hypertension.

- (c) The probability that the blood pressure falls in the normal range is

$$P(90 \leq X \leq 120) = \int_{90}^{120} \frac{1}{10.9\sqrt{2\pi}} e^{-(x-119.7)^2/2(10.9)^2} dx \approx 0.50776.$$

That is, about 51% of the population has a normal systolic blood pressure. ■

Many national tests are standardized using the normal distribution. The following example illustrates modeling the discrete random variable for scores on a test using the normal distribution function for a continuous random variable.

EXAMPLE 13 The ACT is a standardized test taken by high school students seeking admission to many colleges and universities. The test measures knowledge, skills, and proficiency in the areas of English, math, and science, with scores ranging over the interval $[1, 36]$. Nearly 1.5 million high school students took the test in 2009, and the composite mean score across the academic areas was $\mu = 21.1$ with standard deviation $\sigma = 5.1$.

- (a) What percentage of the population had an ACT score between 18 and 24?
- (b) What is the ranking of a student who scored 27 on the test?
- (c) What is the minimal integer score a student needed to get in order to be in the top 8% of the scoring population?

Solution

- (a) We use a computer to evaluate the probability integral of the normal probability density function with $\mu = 21.1$ and $\sigma = 5.1$:

$$P(18 \leq X \leq 24) = \int_{18}^{24} \frac{1}{5.1\sqrt{2\pi}} e^{-(x-21.1)^2/2(5.1)^2} dx \approx 0.44355.$$

This means that about 44% of the students had an ACT score between 18 and 24.

- (b) Again we use a computer to calculate the probability of a student getting a score lower than 27 on the test:

$$P(1 \leq X < 27) = \int_1^{27} \frac{1}{5.1\sqrt{2\pi}} e^{-(x-21.1)^2/2(5.1)^2} dx \approx 0.87630.$$

We conclude that about 88% of the students scored below a score of 27, so the student ranked in the top 12% of the population.

- (c) We look at how many students had a mark higher than 28:

$$P(28 < X \leq 36) = \int_{28}^{36} \frac{1}{5.1\sqrt{2\pi}} e^{-(x-21.1)^2/2(5.1)^2} dx \approx 0.0863.$$

Since this number gives more than 8% of the students, we look at the next higher integer score:

$$P(29 < X \leq 36) = \int_{29}^{36} \frac{1}{5.1\sqrt{2\pi}} e^{-(x-21.1)^2/2(5.1)^2} dx \approx 0.0595.$$

Therefore, 29 is the lowest integer score a student could get in order to score in the top 8% of the population (and actually scoring here in the top 6%).

The simplest form for a normal distribution of X occurs when its mean is zero and its standard deviation is one. The *standard normal probability density function* f giving mean $\mu = 0$ and standard deviation $\sigma = 1$ is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Note that the substitution $z = (x - \mu)/\sigma$ gives the equivalent integrals

$$\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-((x-\mu)/\sigma)^2/2} dx = \int_\alpha^\beta \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz,$$

where $\alpha = (a - \mu)/\sigma$ and $\beta = (b - \mu)/\sigma$. So we can convert random variable values to the “ z -values” to standardize a normal distribution, and then use the integral on the right-hand side of the last equation to calculate probabilities for the original random variable normal distribution with mean μ and standard deviation σ . In a normal distribution, we know that 95.5% of the population lies within two standard deviations of the mean, so a random variable X converted to a z -value has more than a 95% chance of occurring in the interval $[-2, 2]$.

EXERCISES A.8

Probability Density Functions

In Exercises 1–8, determine which are probability density functions and justify your answer.

1. $f(x) = \frac{1}{18}x$ over $[4, 8]$

2. $f(x) = \frac{1}{2}(2 - x)$ over $[0, 2]$

3. $f(x) = 2^x$ over $\left[0, \frac{\ln(1 + \ln 2)}{\ln 2}\right]$

4. $f(x) = x - 1$ over $[0, 1 + \sqrt{3}]$

5. $f(x) = \begin{cases} \frac{1}{x^2} & x \geq 1 \\ 0 & x < 1 \end{cases}$

6. $f(x) = \begin{cases} \frac{8}{\pi(4 + x^2)}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

7. $f(x) = 2 \cos 2x$ over $\left[0, \frac{\pi}{4}\right]$

8. $f(x) = \frac{1}{x}$ over $(0, e]$

9. Let f be the probability density function for the random variable L in Example 2f. Explain the meaning of each integral.

a. $\int_{25,000}^{32,000} f(l) dl$

b. $\int_{30,000}^{\infty} f(l) dl$

c. $\int_0^{20,000} f(l) dl$

d. $\int_{-\infty}^{15,000} f(l) dl$

10. Let $f(x)$ be the uniform distribution for the random variable X in Example 11. Express the following probabilities as integrals.

- The probability that the arrow points either between South and West or between North and West.
- The probability that the arrow makes an angle of at least 2 radians.

Verify that the functions in Exercises 11–16 are probability density functions for a continuous random variable X over the given interval. Determine the specified probability.

11. $f(x) = xe^{-x}$ over $[0, \infty)$, $P(1 \leq X \leq 3)$

T 12. $f(x) = \frac{\ln x}{x^2}$ over $[1, \infty)$, $P(2 < X < 15)$

13. $f(x) = \frac{3}{2}x(2 - x)$ over $[0, 1]$, $P(0.5 > X)$

T 14. $f(x) = \frac{\sin^2 \pi x}{\pi x^2}$ over $\left[\frac{200}{1059}, \infty\right)$, $P(X < \pi/6)$

15. $f(x) = \begin{cases} \frac{2}{x^3} & x > 1 \\ 0 & x \leq 1 \end{cases}$ over $(-\infty, \infty)$, $P(4 \leq X < 9)$

16. $f(x) = \sin x$ over $[0, \pi/2]$, $P\left(\frac{\pi}{6} < X \leq \frac{\pi}{4}\right)$

In Exercises 17–20, find the value of the constant c so that the given function is a probability density function for a random variable over the specified interval.

17. $f(x) = \frac{1}{6}x$ over $[3, c]$ 18. $f(x) = \frac{1}{x}$ over $[c, c + 1]$

19. $f(x) = 4e^{-2x}$ over $[0, c]$

20. $f(x) = cx\sqrt{25 - x^2}$ over $[0, 5]$

21. Let $f(x) = \frac{c}{1 + x^2}$. Find the value of c so that f is a probability density function. If f is a probability density function for the random variable X , find the probability $P(1 \leq X < 2)$.

22. Find the value of c so that $f(x) = c\sqrt{x}(1 - x)$ is a probability density function for the random variable X over $[0, 1]$, and find the probability $P(0.25 \leq X \leq 0.5)$.

23. Show that if the exponentially decreasing function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ Ae^{-cx} & \text{if } x \geq 0 \end{cases}$$

is a probability density function, then $A = c$.

24. Suppose f is a probability density function for the random variable X with mean μ . Show that its variance satisfies

$$\text{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2.$$

Compute the mean and median for a random variable with the probability density functions in Exercises 25–28.

25. $f(x) = \frac{1}{8}x$ over $[0, 4]$ 26. $f(x) = \frac{1}{9}x^2$ over $[0, 3]$

27. $f(x) = \begin{cases} \frac{2}{x^3} & x \geq 1 \\ 0 & x < 1 \end{cases}$ 28. $f(x) = \begin{cases} \frac{1}{x} & 1 \leq x \leq e \\ 0 & \text{Otherwise} \end{cases}$

Exponential Distributions

29. **Digestion time** The digestion time in hours of a fixed amount of food is exponentially distributed with a mean of 1 hour. What is the probability that the food is digested in less than 30 minutes?

30. **Pollinating flowers** A biologist models the time in minutes until a bee arrives at a flowering plant with an exponential distribution having a mean of 4 minutes. If 1000 flowers are in a field, how many can be expected to be pollinated within 5 minutes?

31. **Lifetime of light bulbs** A manufacturer of light bulbs finds that the mean lifetime of a bulb is 1200 hours. Assume the life of a bulb is exponentially distributed.

- Find the probability that a bulb will last less than its guaranteed lifetime of 1000 hours.
- In a batch of light bulbs, what is the expected time until half the light bulbs in the batch fail?

32. **Lifetime of an electronic component** The life expectancy in years of a component in a microcomputer is exponentially distributed, and 1/3 of the components fail in the first 3 years. The company that manufactures the component offers a 1 year warranty. What is the probability that a component will fail during the warranty period?

33. **Lifetime of an organism** A *hydra* is a small fresh-water animal, and studies have shown that its probability of dying does not increase with the passage of time. The lack of influence of age on mortality rates for this species indicates that an exponential distribution is an appropriate model for the mortality of hydra. A biologist studies a population of 500 hydra and observes that 200 of them die within the first 2 years. How many of the hydra would you expect to die within the first six months?

34. **Car accidents** The number of days that elapse between the beginning of a calendar year and the moment a high-risk driver is involved in an accident is exponentially distributed. Based on historical data, an insurance company expects that 30% of high-risk drivers will be involved in an accident during the first 50 days of the calendar year. In a group of 100 high-risk drivers, how many do you expect to be involved in an accident during the first 80 days of the calendar year?

35. **Customer service time** The mean waiting time to get served after walking into a bakery is 30 seconds. Assume that an exponential density function describes the waiting times.

- What is the probability a customer waits 15 seconds or less?
- What is the probability a customer waits longer than one minute?
- What is the probability a customer waits exactly 5 minutes?
- If 200 customers come to the bakery in a day, how many are likely to be served within three minutes?

36. **Airport waiting time** According to the U.S. Customs and Border Protection Agency, the average airport wait time at Chicago's O'Hare International airport is 16 minutes for a traveler arriving during the hours 7–8 A.M., and 32 minutes for arrival during the hours 4–5 P.M. The wait time is defined as the total processing time from arrival at the airport until the completion of a passenger's security screening. Assume the wait time is exponentially distributed.

- What is the probability of waiting between 10 and 30 minutes for a traveler arriving during the 7–8 A.M. hour?
- What is the probability of waiting more than 25 minutes for a traveler arriving during the 7–8 P.M. hour?

- c. What is the probability of waiting between 35 and 50 minutes for a traveler arriving during the 4–5 P.M. hour?
- d. What is the probability of waiting less than 20 minutes for a traveler arriving during the 4–5 P.M. hour?
- 37. Printer lifetime** The lifetime of a \$200 printer is exponentially distributed with a mean of 2 years. The manufacturer agrees to pay a full refund to a buyer if the printer fails during the first year following its purchase, and a one-half refund if it fails during the second year. If the manufacturer sells 100 printers, how much should it expect to pay in refunds?
- 38. Failure time** The time between failures of a photocopier is exponentially distributed. Half of the copiers at a university require service during the first 2 years of operations. If the university purchased 150 copiers, how many do you expect to require service during the first year of their operation?

T Normal Distributions

- 39. Cholesterol levels** The serum cholesterol levels of children aged 12 to 14 years follows a normal distribution with mean $\mu = 162$ mg/dl and standard deviation $\sigma = 28$ mg/dl. In a population of 1000 of these children, how many would you expect to have serum cholesterol levels between 165 and 193? between 148 and 167?
- 40. Annual rainfall** The annual rainfall in millimeters for San Francisco, California, is approximately a normal random variable with mean 510 mm and standard deviation 120 mm. What is the probability that next year's rainfall will exceed 444 mm?
- 41. Manufacturing time** The assembly time in minutes for a component at an electronic manufacturing plant is normally distributed with a mean of $\mu = 55$ and standard deviation $\sigma = 4$. What is the probability that a component will be made in less than one hour?
- 42. Lifetime of a tire** Assume the random variable L in Example 2f is normally distributed with mean $\mu = 35,000$ kilometers and $\sigma = 6,000$ kilometers.
- a. In a batch of 4000 tires, how many can be expected to last for at least 29,000 kilometers?
- b. What is the minimum number of kilometers you would expect to find as the lifetime for 90% of the tires?
- 43. Height** The average height of American women aged 18–24 is normally distributed with mean $\mu = 166$ cm and $\sigma = 6$ cm.
- a. What percentage of women are taller than 172 cm?
- b. What is the probability a woman is between 155 cm and 163 cm tall?
- 44. Life expectancy** At birth, a French citizen has an average life expectancy of 82 years with a standard deviation of 7 years. If 100 newly born French babies are selected at random, how many would you expect to live between 75 and 85 years? Assume life expectancy is normally distributed.
- 45. Length of pregnancy** A team of medical practitioners determines that in a population of 1000 women with ages ranging from 20 to 35 years, the length of pregnancy from conception to birth is approximately normally distributed with a mean of 266 days and a standard deviation of 16 days. How many of these women would you expect to have a pregnancy lasting from 36 weeks to 40 weeks?
- 46. Brain weights** In a population of 500 adult Swedish men, medical researchers find their brain weights to be approximately normally distributed with mean $\mu = 1400$ gm and standard deviation $\sigma = 100$ gm.
- a. What percentage of brain weights are between 1325 and 1450 gm?
- b. How many men in the population would you expect to have a brain weight exceeding 1480 gm?
- 47. Blood pressure** Diastolic blood pressure in adults is normally distributed with $\mu = 80$ mm Hg and $\sigma = 12$ mm Hg. In a random sample of 300 adults, how many would be expected to have a diastolic blood pressure below 70 mm Hg?
- 48. Albumin levels** Serum albumin in healthy 20-year-old men is normally distributed with $\mu = 4.4$ and $\sigma = 0.2$. How likely is it for a healthy 20-year-old man to have a level in the range 4.3 to 4.45?
- 49. Quality control** A manufacturer of generator shafts finds that it needs to add additional weight to its shafts in order to achieve proper static and dynamic balance. Based on experimental tests, the average weight it needs to add is $\mu = 35$ gm with $\sigma = 9$ gm. Assuming a normal distribution, from 1000 randomly selected shafts, how many would be expected to need an added weight in excess of 40 gm?
- 50. Kilometers driven** A taxicab company in New York City analyzed the daily number of kilometers driven by each of its drivers. It found the average distance was 300 km with a standard deviation of 50 km. Assuming a normal distribution, what prediction can we make about the percentage of drivers who will log in either more than 400 km or less than 250 km?
- 51. Germination of sunflower seeds** The germination rate of a particular seed is the percentage of seeds in the batch which successfully emerge as plants. Assume that the germination rate for a batch of sunflower seeds is 80%, and that among a large population of n seeds the number of successful germinations is normally distributed with mean $\mu = 0.8n$ and $\sigma = 0.4\sqrt{n}$.
- a. In a batch of $n = 2500$ seeds, what is the probability that at least 1960 will successfully germinate?
- b. In a batch of $n = 2500$ seeds, what is the probability that at most 1980 will successfully germinate?
- c. In a batch of $n = 2500$ seeds, what is the probability that between 1940 and 2020 will successfully germinate?
- 52.** Suppose you toss a fair coin n times and record the number of heads that land. Assume that n is large and approximate the discrete random variable X with a continuous random variable that is normally distributed with $\mu = n/2$ and $\sigma = \sqrt{n}/2$. If $n = 400$, find the given probabilities.
- a. $P(190 \leq X < 210)$ b. $P(X < 170)$
 c. $P(X > 220)$ d. $P(X = 300)$

Discrete Random Variables

- 53.** A fair coin is tossed four times and the random variable X assigns the number of tails that appear in each outcome.
- a. Determine the set of possible outcomes.
- b. Find the value of X for each outcome.

- c. Create a probability bar graph for X , as in Figure A.30. What is the probability that at least two heads appear in the four tosses of the coin?
54. You roll a pair of six-sided dice, and the random variable X assigns to each outcome the sum of the number of dots showing on each face, as in Example 2e.
- Find the set of possible outcomes.
 - Create a probability bar graph for X .
 - What is the probability that $X = 8$?
 - What is the probability that $X \leq 5$? $X > 9$?
55. Three people are asked their opinion in a poll about a particular brand of a common product found in grocery stores. They can answer in one of three ways: “Like the product brand” (L), “Dislike the product brand” (D), or “Undecided” (U). For each outcome, the random variable X assigns the number of L’s that appear.
- Find the set of possible outcomes and the range of X .
 - Create a probability bar graph for X .
 - What is the probability that at least two people like the product brand?
 - What is the probability that no more than one person dislikes the product brand?
56. **Spacecraft components** A component of a spacecraft has both a main system and a backup system operating throughout a flight. The probability that both systems fail sometime during the flight is 0.0148. Assuming that each system separately has the same failure rate, what is the probability that the main system fails during the flight?

A.9 The Distributive Law for Vector Cross Products

In this appendix we prove the Distributive Law

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w},$$

which is Property 2 in Section 11.4.

Proof To derive the Distributive Law, we construct $\mathbf{u} \times \mathbf{v}$ a new way. We draw $\mathbf{u}$ and $\mathbf{v}$ from the common point O and construct a plane M perpendicular to $\mathbf{u}$ at O (Figure A.37). We then project $\mathbf{v}$ orthogonally onto M , yielding a vector $\mathbf{v}_1$ with length $|\mathbf{v}|\sin\theta$. We rotate $\mathbf{v}_1$ by 90° about $\mathbf{u}$ in the positive sense to produce a vector $\mathbf{v}_2$. Finally, we multiply $\mathbf{v}_2$ by the length of $\mathbf{u}$. The resulting vector $|\mathbf{u}|\mathbf{v}_2$ is equal to $\mathbf{u} \times \mathbf{v}$ since $\mathbf{v}_2$ has the same direction as $\mathbf{u} \times \mathbf{v}$ by its construction (Figure A.37) and

$$|\mathbf{u}||\mathbf{v}_2| = |\mathbf{u}||\mathbf{v}_1| = |\mathbf{u}||\mathbf{v}|\sin\theta = |\mathbf{u} \times \mathbf{v}|.$$

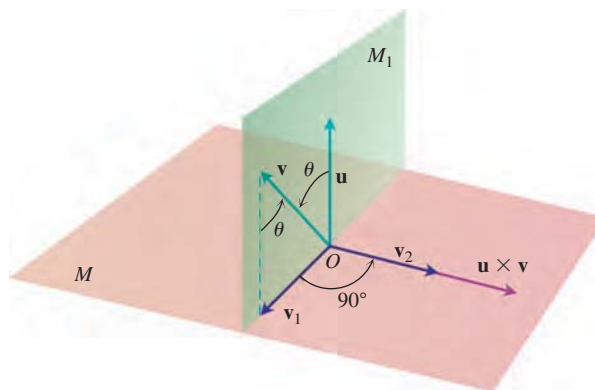


FIGURE A.37 As explained in the text, $\mathbf{u} \times \mathbf{v} = |\mathbf{u}|\mathbf{v}_2$.

Now each of these three operations, namely,

1. projection onto M
2. rotation about $\mathbf{u}$ through 90°
3. multiplication by the scalar $|\mathbf{u}|$,

when applied to a triangle whose plane is not parallel to $\mathbf{u}$, will produce another triangle. If we start with the triangle whose sides are $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{s} = \mathbf{v} + \mathbf{w}$ (Figure A.38) and apply these three steps, we successively obtain the following:

1. A triangle whose sides are $\mathbf{v}_1$, $\mathbf{w}_1$, and $\mathbf{s}_1$ satisfying the vector equation

$$\mathbf{v}_1 + \mathbf{w}_1 = \mathbf{s}_1$$

2. A triangle whose sides are $\mathbf{v}_2$, $\mathbf{w}_2$, and $\mathbf{s}_2$ satisfying the vector equation

$$\mathbf{v}_2 + \mathbf{w}_2 = \mathbf{s}_2$$

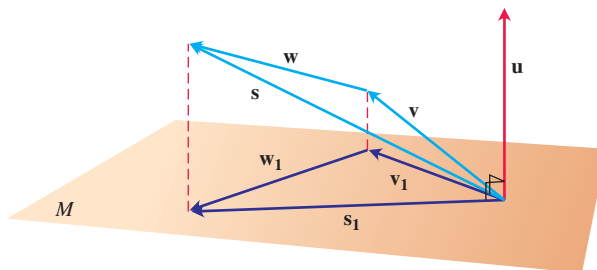


FIGURE A.38 The vectors $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v} + \mathbf{w}$ and their projections onto a plane perpendicular to $\mathbf{u}$.

3. A triangle whose sides are $|\mathbf{u}|\mathbf{v}_2$, $|\mathbf{u}|\mathbf{w}_2$, and $|\mathbf{u}|\mathbf{s}_2$ satisfying the vector equation

$$|\mathbf{u}|\mathbf{v}_2 + |\mathbf{u}|\mathbf{w}_2 = |\mathbf{u}|\mathbf{s}_2.$$

Substituting $|\mathbf{u}|\mathbf{v}_2 = \mathbf{u} \times \mathbf{v}$, $|\mathbf{u}|\mathbf{w}_2 = \mathbf{u} \times \mathbf{w}$, and $|\mathbf{u}|\mathbf{s}_2 = \mathbf{u} \times (\mathbf{v} + \mathbf{w})$ from our discussion above into this last equation gives

$$\mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} + \mathbf{w}),$$

which is the law we wanted to establish. ■

A.10 The Mixed Derivative Theorem and the Increment Theorem

This appendix derives the Mixed Derivative Theorem (Theorem 2, Section 13.3) and the Increment Theorem for Functions of Two Variables (Theorem 3, Section 13.3). Euler first published the Mixed Derivative Theorem in 1734, in a series of papers he wrote on hydrodynamics.

THEOREM 2—The Mixed Derivative Theorem

If $f(x, y)$ and its partial derivatives f_x , f_y , f_{xy} , and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b) , then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Proof The equality of $f_{xy}(a, b)$ and $f_{yx}(a, b)$ can be established by four applications of the Mean Value Theorem (Theorem 4, Section 4.2). By hypothesis, the point (a, b) lies in the interior of a rectangle R in the xy -plane on which f , f_x , f_y , f_{xy} , and f_{yx} are all defined. We let h and k be the numbers such that the point $(a + h, b + k)$ also lies in R , and we consider the difference

$$\Delta = F(a + h) - F(a), \quad (1)$$

where

$$F(x) = f(x, b + k) - f(x, b). \quad (2)$$

We apply the Mean Value Theorem to F , which is continuous because it is differentiable. Then Equation (1) becomes

$$\Delta = hF'(c_1), \quad (3)$$

where c_1 lies between a and $a + h$. From Equation (2),

$$F'(x) = f_x(x, b + k) - f_x(x, b),$$

so Equation (3) becomes

$$\Delta = h[f_x(c_1, b + k) - f_x(c_1, b)]. \quad (4)$$

Now we apply the Mean Value Theorem to the function $g(y) = f_x(c_1, y)$ and have

$$g(b + k) - g(b) = kg'(d_1),$$

or

$$f_x(c_1, b + k) - f_x(c_1, b) = kf_{xy}(c_1, d_1)$$

for some d_1 between b and $b + k$. By substituting this into Equation (4), we get

$$\Delta = hkf_{xy}(c_1, d_1) \quad (5)$$

for some point (c_1, d_1) in the rectangle R' whose vertices are the four points (a, b) , $(a + h, b)$, $(a + h, b + k)$, and $(a, b + k)$. (See Figure A.39.)

By substituting from Equation (2) into Equation (1), we may also write

$$\begin{aligned} \Delta &= f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b) \\ &= [f(a + h, b + k) - f(a, b + k)] - [f(a + h, b) - f(a, b)] \\ &= \phi(b + k) - \phi(b), \end{aligned} \quad (6)$$

where

$$\phi(y) = f(a + h, y) - f(a, y). \quad (7)$$

The Mean Value Theorem applied to Equation (6) now gives

$$\Delta = k\phi'(d_2) \quad (8)$$

for some d_2 between b and $b + k$. By Equation (7),

$$\phi'(y) = f_y(a + h, y) - f_y(a, y). \quad (9)$$

Substituting from Equation (9) into Equation (8) gives

$$\Delta = k[f_y(a + h, d_2) - f_y(a, d_2)].$$

Finally, we apply the Mean Value Theorem to the expression in brackets and get

$$\Delta = khf_{yx}(c_2, d_2) \quad (10)$$

for some c_2 between a and $a + h$.

Together, Equations (5) and (10) show that

$$f_{xy}(c_1, d_1) = f_{yx}(c_2, d_2), \quad (11)$$

where (c_1, d_1) and (c_2, d_2) both lie in the rectangle R' (Figure A.39). Equation (11) is not quite the result we want, since it says only that f_{xy} has the same value at (c_1, d_1) that f_{yx} has at (c_2, d_2) . The numbers h and k in our discussion, however, may be made as small as we wish. The hypothesis that f_{xy} and f_{yx} are both continuous at (a, b) means that $f_{xy}(c_1, d_1) = f_{xy}(a, b) + \varepsilon_1$ and $f_{yx}(c_2, d_2) = f_{yx}(a, b) + \varepsilon_2$, where each of $\varepsilon_1, \varepsilon_2 \rightarrow 0$ as both $h, k \rightarrow 0$. Hence, if we let h and $k \rightarrow 0$, we have $f_{xy}(a, b) = f_{yx}(a, b)$. ■

The equality of $f_{xy}(a, b)$ and $f_{yx}(a, b)$ can be proved with hypotheses weaker than the ones we assumed. For example, it is enough for f , f_x , and f_y to exist in R and for f_{xy} to be continuous at (a, b) . Then f_{yx} will exist at (a, b) and will equal f_{xy} at that point.

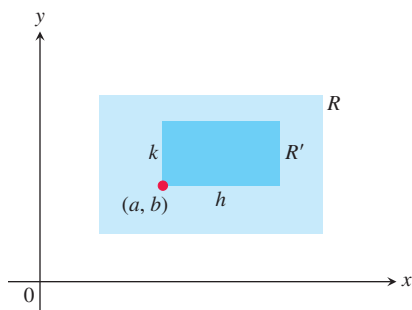


FIGURE A.39 The key to proving $f_{xy}(a, b) = f_{yx}(a, b)$ is that no matter how small R' is, f_{xy} and f_{yx} take on equal values somewhere inside R' (though not necessarily at the same point).

THEOREM 3—The Increment Theorem for Functions of Two Variables

Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0) . Then the change

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

in the value of f that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

in which each of $\varepsilon_1, \varepsilon_2 \rightarrow 0$ as both $\Delta x, \Delta y \rightarrow 0$.

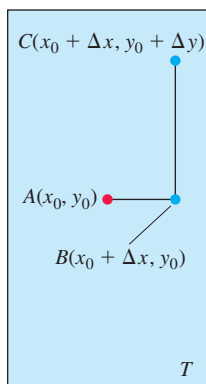


FIGURE A.40 The rectangular region T in the proof of the Increment Theorem. The figure is drawn for Δx and Δy positive, but either increment might be zero or negative.

Proof We work within a rectangle T centered at $A(x_0, y_0)$ and lying within R , and we assume that Δx and Δy are already so small that the line segment joining A to $B(x_0 + \Delta x, y_0)$ and the line segment joining B to $C(x_0 + \Delta x, y_0 + \Delta y)$ lie in the interior of T (Figure A.40).

We may think of Δz as the sum $\Delta z = \Delta z_1 + \Delta z_2$ of two increments, where

$$\Delta z_1 = f(x_0 + \Delta x, y_0) - f(x_0, y_0)$$

is the change in the value of f from A to B and

$$\Delta z_2 = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0)$$

is the change in the value of f from B to C (Figure A.41).

On the closed interval of x -values joining x_0 to $x_0 + \Delta x$, the function $F(x) = f(x, y_0)$ is a differentiable (and hence continuous) function of x , with derivative

$$F'(x) = f_x(x, y_0).$$

By the Mean Value Theorem (Theorem 4, Section 4.2), there is an x -value c between x_0 and $x_0 + \Delta x$ at which

$$F(x_0 + \Delta x) - F(x_0) = F'(c) \Delta x$$

or

$$f(x_0 + \Delta x, y_0) - f(x_0, y_0) = f_x(c, y_0) \Delta x$$

or

$$\Delta z_1 = f_x(c, y_0) \Delta x. \quad (12)$$

Similarly, $G(y) = f(x_0 + \Delta x, y)$ is a differentiable (and hence continuous) function of y on the closed y -interval joining y_0 and $y_0 + \Delta y$, with derivative

$$G'(y) = f_y(x_0 + \Delta x, y).$$

Hence, there is a y -value d between y_0 and $y_0 + \Delta y$ at which

$$G(y_0 + \Delta y) - G(y_0) = G'(d) \Delta y$$

or

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0) = f_y(x_0 + \Delta x, d) \Delta y$$

or

$$\Delta z_2 = f_y(x_0 + \Delta x, d) \Delta y. \quad (13)$$

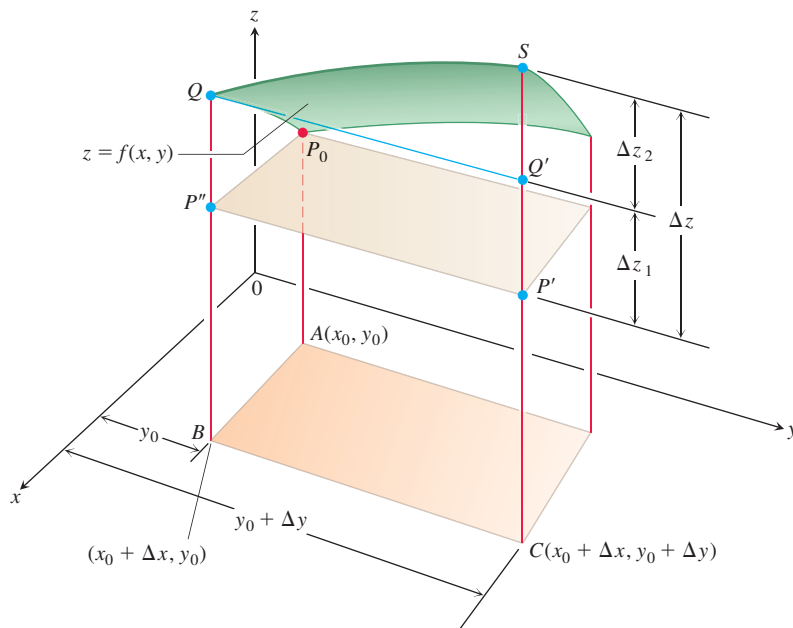


FIGURE A.41 Part of the surface $z = f(x, y)$ near $P_0(x_0, y_0, f(x_0, y_0))$. The points P_0 , P' , and P'' have the same height $z_0 = f(x_0, y_0)$ above the xy -plane. The change in z is $\Delta z = P'S$. The change

$$\Delta z_1 = f(x_0 + \Delta x, y_0) - f(x_0, y_0),$$

shown as $P''Q = P'Q'$, is caused by changing x from x_0 to $x_0 + \Delta x$ while holding y equal to y_0 . Then, with x held equal to $x_0 + \Delta x$,

$$\Delta z_2 = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0)$$

is the change in z caused by changing y_0 from $y_0 + \Delta y$, which is represented by $Q'S$. The total change in z is the sum of Δz_1 and Δz_2 .

Now, as both Δx and $\Delta y \rightarrow 0$, we know that $c \rightarrow x_0$ and $d \rightarrow y_0$. Therefore, since f_x and f_y are continuous at (x_0, y_0) , the quantities

$$\begin{aligned} \varepsilon_1 &= f_x(c, y_0) - f_x(x_0, y_0), \\ \varepsilon_2 &= f_y(x_0 + \Delta x, d) - f_y(x_0, y_0) \end{aligned} \tag{14}$$

both approach zero as both Δx and $\Delta y \rightarrow 0$.

Finally,

$$\begin{aligned} \Delta z &= \Delta z_1 + \Delta z_2 \\ &= f_x(c, y_0) \Delta x + f_y(x_0 + \Delta x, d) \Delta y && \text{From Eqs. (12) and (13)} \\ &= [f_x(x_0, y_0) + \varepsilon_1] \Delta x + [f_y(x_0, y_0) + \varepsilon_2] \Delta y && \text{From Eq. (14)} \\ &= f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y, \end{aligned}$$

where both ε_1 and $\varepsilon_2 \rightarrow 0$ as both Δx and $\Delta y \rightarrow 0$, which is what we set out to prove. ■

Analogous results hold for functions of any finite number of independent variables. Suppose that the first partial derivatives of $w = f(x, y, z)$ are defined throughout an open region containing the point (x_0, y_0, z_0) and that f_x, f_y , and f_z are continuous at (x_0, y_0, z_0) . Then

$$\begin{aligned}\Delta w &= f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0) \\ &= f_x \Delta x + f_y \Delta y + f_z \Delta z + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y + \varepsilon_3 \Delta z,\end{aligned}\quad (15)$$

where $\varepsilon_1, \varepsilon_2$, and $\varepsilon_3 \rightarrow 0$ as $\Delta x, \Delta y$, and $\Delta z \rightarrow 0$.

The partial derivatives f_x, f_y, f_z in Equation (15) are to be evaluated at the point (x_0, y_0, z_0) .

Equation (15) can be proved by treating Δw as the sum of three increments,

$$\Delta w_1 = f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0) \quad (16)$$

$$\Delta w_2 = f(x_0 + \Delta x, y_0 + \Delta y, z_0) - f(x_0 + \Delta x, y_0, z_0) \quad (17)$$

$$\Delta w_3 = f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0 + \Delta x, y_0 + \Delta y, z_0), \quad (18)$$

and applying the Mean Value Theorem to each of these separately. Two coordinates remain constant and only one varies in each of these partial increments $\Delta w_1, \Delta w_2, \Delta w_3$. In Equation (17), for example, only y varies, since x is held equal to $x_0 + \Delta x$ and z is held equal to z_0 . Since $f(x_0 + \Delta x, y, z_0)$ is a continuous function of y with a derivative f_y , it is subject to the Mean Value Theorem, and we have

$$\Delta w_2 = f_y(x_0 + \Delta x, y_1, z_0) \Delta y$$

for some y_1 between y_0 and $y_0 + \Delta y$.

Appendix B

B.1 Determinants

A **matrix** is a rectangular array of numbers. For example,

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix}$$

is a matrix with two rows and three columns. We call A a “2 by 3” matrix. More generally, an “ m by n matrix” is one that has m rows and n columns.

The element in the i th row and j th column of a matrix is represented by a_{ij} . In the example above, we have

$$\begin{aligned} a_{11} &= 2, & a_{12} &= 1, & a_{13} &= 3, \\ a_{21} &= 1, & a_{22} &= 0, & a_{23} &= -2. \end{aligned}$$

If A is an n by n matrix, then we associate A with a number called the **determinant** of A , written sometimes as $\det A$ and sometimes as $|A|$ with vertical bars (which do not mean absolute value). For $n = 1$ and $n = 2$ we have these definitions:

$$\det[a] = a, \quad \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc. \quad (1)$$

For a 3 by 3 matrix, we define

$$\begin{aligned} \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}). \end{aligned} \quad (2)$$

In Equation (2) there are some determinants of 2 by 2 matrices – each of those matrices is obtained by deleting one row and one column of the original 3 by 3 matrix.

For an n by n matrix A , we define the **minor** of the element a_{ij} to be the determinant of the $(n - 1)$ by $(n - 1)$ matrix that remains when the row and the column that contain a_{ij} are deleted from A .

In particular, deleting the row and the column that contain the element a_{11} in the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ and then taking the determinant of the resulting matrix yields}$$

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},$$

There is a simple checkerboard pattern for the signs that correspond to $(-1)^{i+j}$. This pattern can be applied to convert the minor of an element in a matrix to the cofactor of the element. For a determinant of a 3 by 3 matrix the sign pattern is

$$\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array}$$

In the upper left corner, $i = 1, j = 1$, and $(-1)^{1+1} = +1$. In going from any element to an adjacent element in the same row or column, we change i by 1 or j by 1, but not both, so we change the exponent from even to odd or from odd to even, which changes the sign from $+$ to $-$ or from $-$ to $+$.

which is the minor of a_{11} . Similarly, $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and $\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ are the minors of a_{12} and a_{13} , respectively.

Given an n by n matrix A , the **cofactor** of a_{ij} is defined as $(-1)^{i+j}$ times the minor of a_{ij} . We denote the cofactor of a_{ij} by A_{ij} .

We can express Equation (2) using cofactors:

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}. \quad (3)$$

EXAMPLE 1 Given the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 0 \end{bmatrix},$$

- (a) find the minors of a_{11} , a_{12} , and a_{13} .
- (b) find the cofactors A_{11} , A_{12} , and A_{13} .
- (c) use Equation (3) to evaluate $\det A$.

Solution

- (a) To obtain the minor of a_{11} , we delete row 1 and column 1 from the matrix, and then compute the determinant by using Equation (1):

$$\text{minor of } a_{11} = \begin{vmatrix} -1 & -2 \\ 3 & 0 \end{vmatrix} = (-1)(0) - (-2)(3) = 6.$$

Likewise, we obtain

$$\text{minor of } a_{12} = \begin{vmatrix} 3 & -2 \\ 2 & 0 \end{vmatrix} = (3)(0) - (-2)(2) = 4$$

and

$$\text{minor of } a_{13} = \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} = (3)(3) - (-1)(2) = 11.$$

- (b) The cofactor of a_{11} is $(-1)^{1+1} = 1$ times the minor of a_{11} :

$$A_{11} = (-1)^{1+1}(\text{minor of } a_{11}) = (1)(6) = 6.$$

The cofactor of a_{12} is $(-1)^{1+2} = -1$ times the minor of a_{12} :

$$A_{12} = (-1)^{1+2}(\text{minor of } a_{12}) = (-1)(4) = -4$$

and the cofactor of a_{13} is $(-1)^{1+3} = 1$ times the minor of a_{13} :

$$A_{13} = (-1)^{1+3}(\text{minor of } a_{13}) = (1)(11) = 11.$$

(c) To find $\det A$ by using Equation (3), we multiply each element of the first row of A by its cofactor and add:

$$\begin{aligned}\det A &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= (2)(6) + (1)(-4) + (3)(11) \\ &= 12 - 4 + 33 \\ &= 41.\end{aligned}$$

For any integer $n > 3$, the determinant of an n by n matrix is defined to be the sum of the products of the elements in the first row by their cofactors:

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + \cdots + a_{1n}A_{1n}. \quad (4)$$

EXAMPLE 2 Use the cofactors of the elements in the first row of the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 1 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

to evaluate $\det A$.

Solution The cofactors are

$$\begin{aligned}A_{11} &= (-1)^{1+1} \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{vmatrix}, & A_{12} &= (-1)^{1+2} \begin{vmatrix} 2 & 0 & 2 \\ -1 & 1 & -2 \\ 0 & 2 & 1 \end{vmatrix}, \\ A_{13} &= (-1)^{1+3} \begin{vmatrix} 2 & 1 & 2 \\ -1 & 2 & -2 \\ 0 & 1 & 1 \end{vmatrix}, & A_{14} &= (-1)^{1+4} \begin{vmatrix} 2 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}.\end{aligned}$$

These are multiplied by the corresponding elements of A , and then the determinant is obtained by using Equation (4):

$$\begin{aligned}\det A &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + a_{14}A_{14} \\ &= 1 \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{vmatrix} - 2(-1) \begin{vmatrix} 2 & 0 & 2 \\ -1 & 1 & -2 \\ 0 & 2 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & 1 & 2 \\ -1 & 2 & -2 \\ 0 & 1 & 1 \end{vmatrix} + 1(-1) \begin{vmatrix} 2 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}.\end{aligned}$$

Each of the determinants on the right side can now be evaluated using Equation (2):

$$\begin{aligned}\det A &= 1[1(1 + 4) - 0 + 2(4 - 1)] \\ &\quad + 2[2(1 + 4) - 0 + 2(-2 - 0)] \\ &\quad + 3[2(2 + 2) - 1(-1 - 0) + 2(-1 - 0)] \\ &\quad - 1[2(4 - 1) - 1(-2 - 0) + 0] \\ &= 11 + 12 + 21 - 8 = 36.\end{aligned}$$

It becomes very lengthy to compute determinants of large matrices by using the definition. In the mathematical field called **linear algebra**, additional methods are developed to speed up this process. We present a few of those results without proofs.

THEOREM 1—Cofactor Expansion Suppose that A is an n by n matrix, where $n > 2$. For any i , the determinant of A is the sum of the products of the elements in the i th row (or column) with their cofactors.

EXAMPLE 3 Apply Theorem 1 to evaluate the determinant from Example 1 by a cofactor expansion along the third column.

Solution

$$\begin{aligned}
 \det \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 0 \end{bmatrix} &= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} && \text{Expand by cofactors of the third column (Theorem 1).} \\
 &= 3 \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} - 2(-1) \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + 0 \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} && \text{Cofactor of } a_{ij} \text{ is } (-1)^{i+j} \text{ times the minor of } a_{ij}. \\
 &= 3(9 + 2) + 2(6 - 2) + 0(-2 - 3) && \text{Equation (1)} \\
 &= 33 + 8 - 0 = 41
 \end{aligned}$$

According to Theorem 1, it is no accident that the answers in Examples 1 and 3 are the same. If we were to multiply the elements of any row (or column) by the cofactors of the corresponding elements and add the products, we would get the same answer: the value of the determinant.

In Example 3, we do not actually need to compute the cofactor A_{33} since it is multiplied by $a_{33} = 0$. Generally, when calculating the determinant of a matrix containing some zero elements, we may want to choose to expand along a row or column with as many zeros as possible.

In an n by n matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

we refer to the elements $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ as the **main diagonal** of the matrix.

If an n by n matrix has all elements below the main diagonal (or all elements above the main diagonal) equal to zero, then its determinant is easy to calculate as illustrated in our next example.

EXAMPLE 4 Evaluate the determinant of the matrix

$$A = \begin{bmatrix} 7 & -4 & 0 & -1 \\ 0 & -3 & 16 & 8 \\ 0 & 0 & -2 & 12 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Solution

$$\begin{aligned}
 \det \begin{bmatrix} 7 & -4 & 0 & -1 \\ 0 & -3 & 16 & 8 \\ 0 & 0 & -2 & 12 \\ 0 & 0 & 0 & 2 \end{bmatrix} &= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} + a_{41}A_{41} && \text{Expand by cofactors of the first column (Theorem 1).} \\
 &= (7) \begin{vmatrix} -3 & 16 & 8 \\ 0 & -2 & 12 \\ 0 & 0 & 2 \end{vmatrix} + 0 + 0 + 0 && \text{We do not need the values of the cofactors } A_{21}, A_{31}, \text{ and } A_{41} \text{ since each of them is multiplied by 0.} \\
 &= (7) \left(-3 \begin{vmatrix} -2 & 12 \\ 0 & 2 \end{vmatrix} + 0 + 0 \right) && \text{Expand the 3 by 3 determinant by cofactors of the first column (Theorem 1).} \\
 &= (7)(-3)(-2)(2) && \text{Equation (1)} \\
 &= 84
 \end{aligned}$$

The first part of the following theorem generalizes this result. The remaining parts give other methods that can simplify the computation of a determinant.

THEOREM 2—Properties of Determinants

Suppose that A is an n by n matrix, where $n > 2$.

1. If all elements of A below the main diagonal (or all above it) are zero, the determinant of A is the product of the elements on the main diagonal.
2. If each element of a row (or column) of A is multiplied by a constant c and the results are added to a different row (or column), the determinant is not changed.
3. If each element of a row (or column) of A is multiplied by a constant c , the determinant is multiplied by c .
4. If two rows (or columns) of A are interchanged, the determinant just changes its sign.

EXAMPLE 5 We can apply Theorems 1 and 2 to make the evaluation of the determinant from Example 2 easier.

$$\begin{aligned}
 \begin{vmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 1 & -2 \\ 0 & 1 & 2 & 1 \end{vmatrix} &= \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -6 & 0 \\ -1 & 2 & 1 & -2 \\ 0 & 1 & 2 & 1 \end{vmatrix} && \begin{array}{l} \text{We add } -2 \text{ times row 1 to row 2.} \\ \text{By part 2 of Theorem 2 the} \\ \text{determinant does not change.} \end{array} \\
 &= \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -6 & 0 \\ 0 & 0 & 4 & -1 \\ 0 & 1 & 2 & 1 \end{vmatrix} && \begin{array}{l} \text{We add 1 times row 1 to row 3.} \\ \text{By part 2 of Theorem 2 the} \\ \text{determinant does not change.} \end{array} \\
 &= 1 \begin{vmatrix} 5 & -6 & 0 \\ 0 & 4 & -1 \\ 1 & 2 & 1 \end{vmatrix} + 0 + 0 + 0 && \text{Expand by cofactors of the first} \\
 &&& \text{column (Theorem 1).} \\
 &= 5(4 + 2) - (-6)(0 + 1) + 0 = 36 && \text{Equation (2)}
 \end{aligned}$$

The determinant has several important applications, which are studied in a course in linear algebra.

- The absolute value of the determinant gives the volume of the region spanned by the vectors that form the columns (or rows) of the matrix. We saw this in Section 11.4, where we encountered the box product.
- A determinant can be used to indicate whether a system of n linear equations in n unknowns has a unique solution. The answer is yes when the determinant is not equal to zero, and in this case determinants can be used to solve the system.
- We saw in Section 13.7 that the determinant of the Hessian matrix of a function of two variables can help us decide whether a critical point is a saddle point or a local extremum. In Appendix B.2, we will use determinants to extend this to functions of more than two variables.

EXERCISES B.1

Evaluating Determinants

Evaluate the determinants in Exercises 1–14.

$$1. \begin{vmatrix} 2 & 3 & 1 \\ 4 & 5 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

$$2. \begin{vmatrix} 2 & -1 & -2 \\ -1 & 2 & 1 \\ 3 & 0 & -3 \end{vmatrix}$$

$$3. \begin{vmatrix} 5 & 2 & 0 \\ 2 & -1 & 1 \\ 3 & 3 & -1 \end{vmatrix}$$

$$4. \begin{vmatrix} 4 & 0 & -3 \\ 3 & -2 & 2 \\ 1 & 2 & -1 \end{vmatrix}$$

$$5. \begin{vmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -5 \end{vmatrix}$$

$$6. \begin{vmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$7. \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 2 \end{vmatrix}$$

$$8. \begin{vmatrix} 1 & -1 & 2 & 3 \\ 2 & 1 & 2 & 6 \\ 1 & 0 & 2 & 3 \\ -2 & 2 & 0 & -5 \end{vmatrix}$$

$$9. \begin{vmatrix} 0 & -2 & 1 & 0 \\ 3 & 0 & 2 & 1 \\ 1 & 2 & 2 & 3 \\ -1 & 0 & 0 & 2 \end{vmatrix}$$

$$10. \begin{vmatrix} -1 & -1 & 3 & 3 \\ 2 & -2 & 1 & -1 \\ 0 & 1 & 2 & 0 \\ 2 & -1 & -1 & 0 \end{vmatrix}$$

$$11. \begin{vmatrix} -3 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 4 & 1 & -5 & 0 & 0 \\ 0 & 2 & -1 & 4 & 0 \\ 6 & 8 & 3 & 2 & -1 \end{vmatrix}$$

$$12. \begin{vmatrix} -4 & 5 & 6 & 7 & 8 \\ 0 & 2 & 5 & 6 & 7 \\ 0 & 0 & -2 & 5 & 6 \\ 0 & 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & -2 \end{vmatrix}$$

$$13. \begin{vmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 2 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$14. \begin{vmatrix} -2 & 0 & 0 & 1 & 2 \\ 1 & 2 & 1 & 0 & 1 \\ 0 & -2 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 & -1 \\ 1 & 0 & -2 & 0 & 0 \end{vmatrix}$$

B.2 Extreme Values and Saddle Points for Functions of More than Two Variables

Many applications involve functions of multiple independent variables. For instance, to predict energy trends in the United States, an econometric model may use a function $f(p_1, p_2, \dots, p_{50})$ of fifty independent variables representing average prices of gasoline in the fifty states.

It is often important to determine extreme values of such functions. In Section 13.1 we defined a function of n variables whose domain is a set of n -tuples of real numbers $(x_1, x_2, \dots, x_n)$. The domain is a region in n -dimensional space.

We will use open balls to define some important concepts, such as interior and boundary points. An n -dimensional open ball (or open n -ball) of radius r centered at $(a_1, a_2, \dots, a_n)$ is the set consisting of the points $(x_1, x_2, \dots, x_n)$ that satisfy $(x_1 - a_1)^2 + (x_2 - a_2)^2 + \dots + (x_n - a_n)^2 < r^2$.

EXAMPLE 1 Determine whether the following points in 4-dimensional space are in the open 4-ball with radius 5 and center at $(-2, 0, 3, 1)$.

(a) $(1, -2, 1, 0)$

(b) $(0, 1, -1, 5)$

Solution

(a) $(1 - (-2))^2 + (-2 - 0)^2 + (1 - 3)^2 + (0 - 1)^2 = 9 + 4 + 4 + 1 = 18 < 25 = 5^2$; therefore, $(1, -2, 1, 0)$ is in the open 4-ball.

(b) $(0 - (-2))^2 + (1 - 0)^2 + (-1 - 3)^2 + (5 - 1)^2 = 4 + 1 + 16 + 16 = 37 \geq 25 = 5^2$; therefore, $(0, 1, -1, 5)$ is not in the open 4-ball. ■

Using open n -balls we can extend definitions from Section 13.1 to regions in n -dimensional spaces.

DEFINITIONS A point $(x_1, x_2, \dots, x_n)$ in a region D in n -dimensional space is an **interior point** of D if it is the center of an open n -ball of positive radius that lies entirely in D . A point $(x_1, x_2, \dots, x_n)$ is a **boundary point** of D if every open n -ball centered at $(x_1, x_2, \dots, x_n)$ contains points that lie outside D as well as points that lie inside D .

The **interior** of D is the set of interior points of D . The **boundary** of D is the set of boundary points of D .

A region is **open** if it consists entirely of interior points. A region is **closed** if it contains its entire boundary.

A region is **bounded** if it lies inside an open n -ball of finite radius. A region is **unbounded** if it is not bounded.

Extreme Values and Saddle Points of Functions of More than Two Variables

While Section 13.7 discussed extreme values and saddle points for functions of two variables, here we extend this treatment to functions of three or more variables.

DEFINITIONS Let $f(x_1, x_2, \dots, x_n)$ be defined on a set D of n -tuples of real numbers $(x_1, x_2, \dots, x_n)$ including a point $(a_1, a_2, \dots, a_n)$.

$f(a_1, a_2, \dots, a_n)$ is a **local maximum** value of f if $f(a_1, a_2, \dots, a_n) \geq f(x_1, x_2, \dots, x_n)$ for all domain points $(x_1, x_2, \dots, x_n)$ that lie in an open n -ball centered at $(a_1, a_2, \dots, a_n)$.

$f(a_1, a_2, \dots, a_n)$ is a **local minimum** value of f if $f(a_1, a_2, \dots, a_n) \leq f(x_1, x_2, \dots, x_n)$ for all domain points $(x_1, x_2, \dots, x_n)$ that lie in an open n -ball centered at $(a_1, a_2, \dots, a_n)$.

EXAMPLE 2 The function $f(x, y, z) = 10 - x^2 - y^2 - z^2$ is defined on all of three-dimensional space. Figure B.1 shows an assortment of level surfaces of this function. The level set $f(x, y, z) = 10$ is a single point, the origin. At every point (x, y, z) we have $f(x, y, z) = 10 - x^2 - y^2 - z^2 \leq 10 = f(0, 0, 0)$; therefore, f has a local (and absolute) maximum value of 10 at $(0, 0, 0)$. This function has no local minimum values. ■

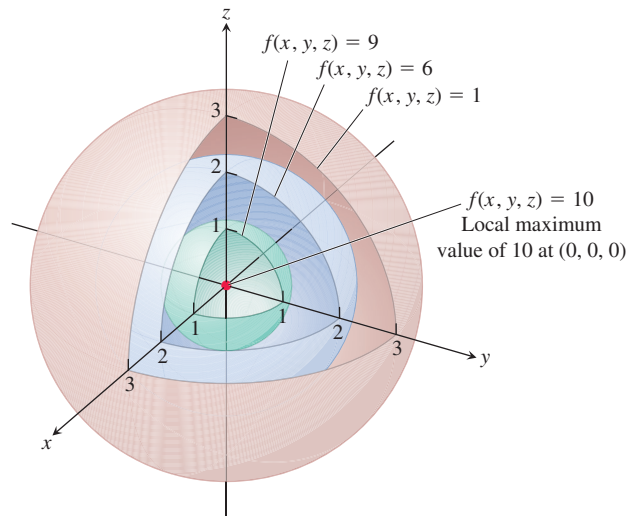


FIGURE B.1 Level surfaces of the function $f(x, y, z) = 10 - x^2 - y^2 - z^2$ in Example 2.

Suppose $f(x_1, x_2, \dots, x_n)$ has a local maximum or minimum value at an interior point $(a_1, a_2, \dots, a_n)$ of its domain and all of its first-order partial derivatives are defined there. The function $g_1(t) = f(t, a_2, \dots, a_n)$ must then have a local extremum at $t = a_1$. Therefore, $g'_1(a_1) = 0$ and consequently

$$\frac{\partial f}{\partial x_1}(a_1, a_2, \dots, a_n) = 0.$$

Likewise, we can use the function $g_2(t) = f(a_1, t, a_3, \dots, a_n)$ to show that

$$\frac{\partial f}{\partial x_2}(a_1, a_2, \dots, a_n) = 0,$$

and continue in the same manner to conclude that

$$\frac{\partial f}{\partial x_i}(a_1, a_2, \dots, a_n) = 0$$

for all $i = 1, 2, \dots, n$.

The above discussion extends Theorem 10 of Section 13.7 to functions of n variables. We present this in a more succinct form by using the gradient notation

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle.$$

THEOREM 3—First Derivative Theorem for Local Extreme Values (General Version) If $f(x_1, x_2, \dots, x_n)$ has a local maximum or minimum value at an interior point $(a_1, a_2, \dots, a_n)$ of its domain and if ∇f exists at that point, then $\nabla f(a_1, a_2, \dots, a_n) = \mathbf{0}$, where $\mathbf{0} = \langle 0, 0, \dots, 0 \rangle$ denotes the zero vector in n -dimensional space.

The following definition of a critical point uses the gradient notation.

DEFINITION An interior point of the domain of f where either $\nabla f = \mathbf{0}$ or ∇f does not exist is called a **critical point** of f .

Note that a vector does not exist whenever one or more of its components does not exist.

The notion of a saddle point generalizes to functions f of n variables.

DEFINITION A critical point $(a_1, a_2, \dots, a_n)$ of a differentiable function $f(x_1, x_2, \dots, x_n)$ is a **saddle point** if every open n -ball centered at the point $(a_1, a_2, \dots, a_n)$ contains points $(x_1, x_2, \dots, x_n)$ with $f(a_1, a_2, \dots, a_n) > f(x_1, x_2, \dots, x_n)$ as well as points $(y_1, y_2, \dots, y_n)$ with $f(a_1, a_2, \dots, a_n) < f(y_1, y_2, \dots, y_n)$.

The Hessian matrix of a function of two variables was introduced in Section 13.7. We now generalize this concept for functions of n variables.

DEFINITIONS The **Hessian matrix** of $f(x_1, x_2, \dots, x_n)$ is the $n \times n$ matrix (containing n rows and n columns)

$$Hf = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{bmatrix}.$$

The **principal minor determinants** of the Hessian matrix are

$$\Delta_1 = f_{x_1x_1},$$

$$\Delta_2 = \begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_2x_1} & f_{x_2x_2} \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} & f_{x_1x_3} \\ f_{x_2x_1} & f_{x_2x_2} & f_{x_2x_3} \\ f_{x_3x_1} & f_{x_3x_2} & f_{x_3x_3} \end{vmatrix},$$

...

$$\Delta_n = |Hf| = \begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{vmatrix}.$$

Theorem 11 in Section 13.7 gave us tools to decide whether a critical point of a function $f(x_1, x_2)$ is a local maximum, local minimum, or a saddle point. The nature of a critical point of a function of more than two variables can be similarly determined using the principal minor determinants of the Hessian matrix.

THEOREM 4—Second Derivative Test for Local Extreme Values (General Version) Suppose that $f(x_1, x_2, \dots, x_n)$ and its first and second partial derivatives are continuous throughout an open n -ball centered at $(a_1, a_2, \dots, a_n)$ and suppose that $\nabla f(a_1, a_2, \dots, a_n) = \mathbf{0}$. Then

- i) f has a local maximum at $(a_1, a_2, \dots, a_n)$ if $(-1)^i \Delta_i > 0$ at the point $(a_1, a_2, \dots, a_n)$ for $i = 1, 2, \dots, n$.
- ii) f has a local minimum at $(a_1, a_2, \dots, a_n)$ if $\Delta_i > 0$ at $(a_1, a_2, \dots, a_n)$ for $i = 1, 2, \dots, n$.
- iii) f has a saddle point at $(a_1, a_2, \dots, a_n)$ if $\Delta_n \neq 0$ at $(a_1, a_2, \dots, a_n)$ and neither of conditions (i) or (ii) hold.
- iv) the test is inconclusive if $\Delta_n = 0$ at $(a_1, a_2, \dots, a_n)$.

For $n = 2$, Theorem 4 reduces to Theorem 11 of Section 13.7 (see Exercise 11).

The principal minor determinants Δ_i keep track of directions along which f has a local maximum and directions along which f has a local minimum. If it has a local maximum (or local minimum) when traveling along each of the n possible axis directions, then f has a local maximum (or local minimum) in an open n -ball.

Applying Theorem 4 to the function $f(x, y, z) = 10 - x^2 - y^2 - z^2$ of Example 2,

we get the Hessian matrix $Hf = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. Its principal minor determinants are

$$\Delta_1 = -2,$$

$$\Delta_2 = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4, \text{ and}$$

$$\Delta_3 = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = -2 \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 0 & -2 \end{vmatrix} + 0 \begin{vmatrix} 0 & -2 \\ 0 & 0 \end{vmatrix} = -8.$$

At the critical point $(0, 0, 0)$, we have $\Delta_1 < 0$, $\Delta_2 > 0$, and $\Delta_3 < 0$, so Theorem 4 confirms the conclusion we reached in Example 2: f has a local maximum at $(0, 0, 0)$.

EXAMPLE 3 Find the local extreme values and saddle points of the function

$$f(x, y, z) = 2x^3y + 3y^2 + 6yz + 6z^2 - 6xy.$$

Solution The function f is differentiable, therefore its critical points are points at which $\nabla f = \mathbf{0}$. Setting each component of the gradient equal to zero yields the following equations:

$$f_x = 6x^2y - 6y = 6y(x^2 - 1) = 0$$

$$f_y = 2x^3 + 6y + 6z - 6x = 0$$

$$f_z = 6y + 12z = 0.$$

The equation $f_x = 0$ can be satisfied in three possible ways: (i) if $y = 0$, (ii) if $x = 1$, or (iii) if $x = -1$. Let's consider these possibilities one at a time.

(i) If $y = 0$, then the equation $f_z = 0$ yields $z = 0$, and then the equation $f_y = 0$ leads to

$$2x^3 - 6x = 2x(x^2 - 3) = 0.$$

This equation has three solutions: $x = 0, \pm\sqrt{3}$, so we obtain critical points $(0, 0, 0)$, $(\sqrt{3}, 0, 0)$, and $(-\sqrt{3}, 0, 0)$.

(ii) If $x = 1$, then the equation $f_y = 0$ becomes $6y + 6z - 4 = 0$. Together with the equation $f_z = 6y + 12z = 0$, this forms a system of two linear equations, which can be solved to obtain $y = 4/3$ and $z = -2/3$. The resulting critical point is $(1, 4/3, -2/3)$.

(iii) If $x = -1$, then similarly to case (ii) we solve the remaining two equations, which yields $y = -4/3$ and $z = 2/3$ and the final critical point is $(-1, -4/3, 2/3)$.

The Hessian matrix of f is

$$Hf = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} 12xy & 6x^2 - 6 & 0 \\ 6x^2 - 6 & 6 & 6 \\ 0 & 6 & 12 \end{bmatrix},$$

and the principal minor determinants are

$$\Delta_1 = 12xy$$

$$\Delta_2 = \begin{vmatrix} 12xy & 6x^2 - 6 \\ 6x^2 - 6 & 6 \end{vmatrix} = 72xy - (6x^2 - 6)^2$$

$$\begin{aligned} \Delta_3 &= \begin{vmatrix} 12xy & 6x^2 - 6 & 0 \\ 6x^2 - 6 & 6 & 6 \\ 0 & 6 & 12 \end{vmatrix} = 12xy \begin{vmatrix} 6 & 6 \\ 6 & 12 \end{vmatrix} - (6x^2 - 6) \begin{vmatrix} 6x^2 - 6 & 6 \\ 0 & 12 \end{vmatrix} + 0 \\ &= 12xy \cdot 36 - (6x^2 - 6)^2(12). \end{aligned}$$

We list the values of these determinants at the critical points in the following table.

Critical Point	Δ_1	Δ_2	Δ_3
$(0, 0, 0)$	0	-36	-432
$(\sqrt{3}, 0, 0)$	0	-144	-1728
$(-\sqrt{3}, 0, 0)$	0	-144	-1728
$(1, \frac{4}{3}, -\frac{2}{3})$	16	96	576
$(-1, -\frac{4}{3}, \frac{2}{3})$	16	96	576

According to Theorem 4,

- (i) f has a local maximum at a critical point if $\Delta_1 < 0$, $\Delta_2 > 0$, and $\Delta_3 < 0$. This pattern is not observed at any of our critical points; therefore, f has no local maximum values.
- (ii) f has a local minimum at a critical point if $\Delta_1 > 0$, $\Delta_2 > 0$, and $\Delta_3 > 0$. The signs of the principal minor determinants listed in the table follow this pattern for the last two critical points, so f has local minimum values at $(1, 4/3, -2/3)$ and at $(-1, -4/3, 2/3)$.
- (iii) f has a saddle point at a critical point if $\Delta_3 \neq 0$, but neither of the conditions (i) or (ii) hold. Based on our table, this is the case at each of the first three critical points. We conclude that f has saddle points at $(0, 0, 0)$, $(\sqrt{3}, 0, 0)$, and $(-\sqrt{3}, 0, 0)$.

EXERCISES B.2

Finding Local Extrema and Saddle Points

Find all the local maxima, local minima, and saddle points of the functions in Exercises 1–10.

1. $f(x, y, z) = x^3y - 3xy + 3yz - 3y + 6z$

2. $f(x, y, z) = xy + x^2z^2 + x^2 + y^2 + z^2$

3. $f(x, y, z) = x^3 - 3x + y^2 + 2y + 2z^2 - 8z + 6$

4. $f(x, y, z) = \frac{x^3}{8} - \frac{x}{2} + xy - y^2 - z^2$

5. $f(x, y, z) = \frac{12z}{x^2 + y^2 + z^2 + 4}$

6. $f(x, y, z) = x^2 + 4y^2 + xz - yz + 4$

7. $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2 + xy + zw - y + 2z$

8. $f(x, y, z, w) = 3x + yz - 4z - w - x^2 - y^2 - z^2 - w^2$

9. $f(x, y, z, v, w) = x^2 + 2y^2 - z^2 + v^2 + w^2 + x - 3v + xv$

10. $f(x, y, z, v, w) = x + z + yv - 4w - 3x^2 - 2y^2 - z^2 - v^2 - w^2$

Theory

11. Show that for $n = 2$ Theorem 4 reduces to Theorem 11 of Section 13.7.

B.3 The Method of Gradient Descent

In Section 13.7 and in Appendix B.2 we discussed methods allowing us to analytically determine local maxima, local minima, and saddle points of a differentiable function f of n variables. While these methods apply for any integer $n \geq 2$, they may not be practical when n is very large. Furthermore, there are important applications where we can evaluate (or approximate) f and its partial derivative values without having an explicit formula for f . These are among the reasons to numerically search for local extrema of a function of n variables.

In this appendix, we will refer to values of a function of n variables $f(x_1, x_2, \dots, x_n)$ using the notation $f(\mathbf{x})$ where $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$ is the position vector of the point $(x_1, x_2, \dots, x_n)$. The directional derivative of f at that point in the direction of a unit vector $\mathbf{u}$ can be expressed as

$$D_{\mathbf{u}}f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}.$$

In Section 13.5, where the notion of the directional derivative was introduced, we noticed that the function f increases most rapidly in the direction of ∇f while it decreases most rapidly in the direction of $-\nabla f$. This leads to the **method of gradient descent** (also known as the **method of steepest descent**) for numerically approximating a local minimum for a function f of n variables.

The initial approximation $\mathbf{x}_0$ can be chosen to be the best guess for where a minimum might be. (Alternatively a randomly chosen point can be used for $\mathbf{x}_0$.) We then take a step in the direction of $-\nabla f(\mathbf{x}_0)$ to yield the next approximation

$$\mathbf{x}_1 = \mathbf{x}_0 - h_0 \nabla f(\mathbf{x}_0),$$

where the positive quantity h_0 controls the size of the step. We obtain subsequent approximations $\mathbf{x}_2, \mathbf{x}_3, \dots$ in the same manner.

Method of Gradient Descent

1. Start with a first approximation to a local minimum of $f(\mathbf{x}_0)$.
2. Use the first approximation to obtain a second, the second to obtain a third, and so on, using the formula

$$\mathbf{x}_{k+1} = \mathbf{x}_k - h_k \nabla f(\mathbf{x}_k) \quad (1)$$

with $h_k > 0$.

In applications, including those in the field of machine learning, the method of gradient descent can be applied to approximate a local minimum of a function involving a large number of independent variables. To gain insight into this method, consider Figure B.2, where we show the method being applied to a function of two variables $f(\mathbf{x}) = f(x, y)$ (we take $\mathbf{x} = \langle x, y \rangle$ to be the position vector of the point (x, y)). Notice that the vector $-\nabla f(\mathbf{x}_0)$ is perpendicular to the level curve of f passing through the point $\mathbf{x}_0$, which consists of all points $\mathbf{x}$ in the plane that satisfy the equation $f(\mathbf{x}) = f(\mathbf{x}_0)$. Adding $-\nabla f(\mathbf{x}_0)$ to $\mathbf{x}_0$ gives us our next approximation, $\mathbf{x}_1$. Once again, $-\nabla f(\mathbf{x}_1)$ is perpendicular to the level curve through that point. The process then continues in a similar manner. In addition to level curves, Figure B.2 shows the surface $z = f(\mathbf{x})$, which helps visualize the descent of the values $f(\mathbf{x}_0), f(\mathbf{x}_1), f(\mathbf{x}_2), \dots$. These values appear to be approaching a minimum value of f .

In Figure B.2, we used a constant value $h = 1$ for all h_k . Figure B.3 illustrates how different choices of h value could affect the method's performance. When h is small (Figure B.3a), the method is forced to take more steps. It is often more efficient to take larger steps (Figure B.3b, same h as in Figure B.2). If the value of h is too large (Figure B.3c),

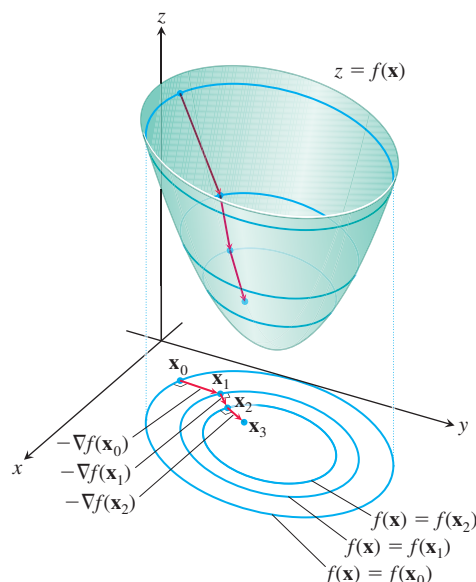


FIGURE B.2 The method of gradient descent is applied to a function f of two variables using a constant $h = 1$. The level curves and the corresponding surface plot show how the value of f decreases at each step.

then the method may not converge. To understand why the method may fail when h is too large, recall that while at $\mathbf{x}_k$ the function f decreases most rapidly in the direction of $-\nabla f(\mathbf{x}_k)$, the function may start increasing if we continue in this direction and go too far from $\mathbf{x}_k$.

Choosing a desirable value for the quantity h (which in machine learning is sometimes called the *learning rate*) that is neither too small nor too large could be challenging. We consider here only the case where the gradient descent method uses a constant, pre-chosen value $h_k = h$. More advanced methods focus on identifying desirable h_k values (which may not be constant) – one of those would involve performing a *line search* (see Exercise 5).

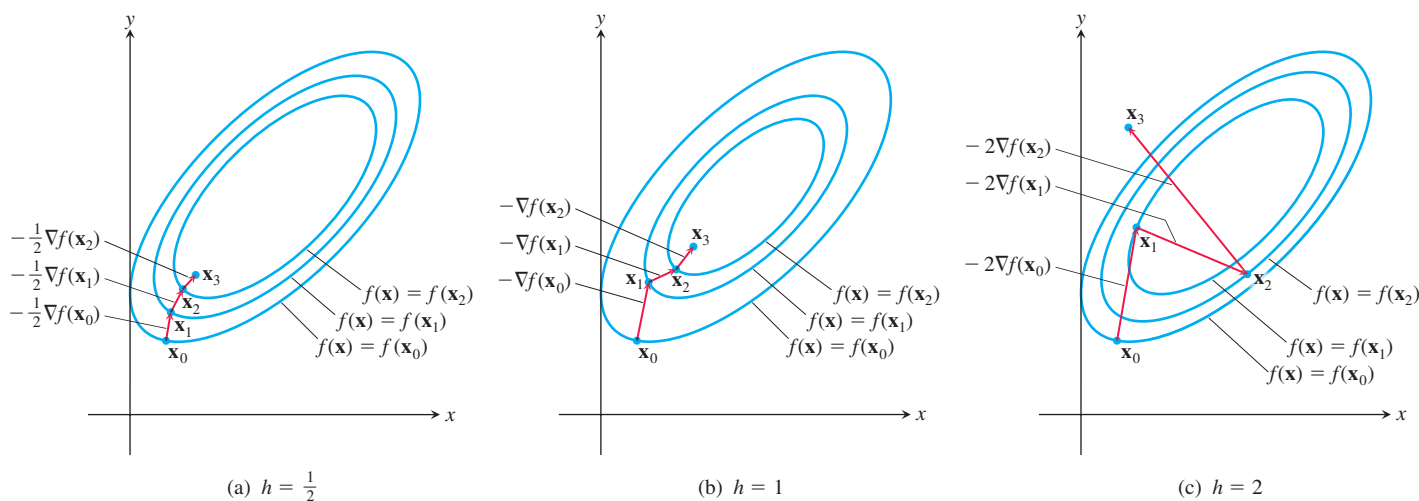


FIGURE B.3 Different choices of h affect the performance of the method of gradient descent.

EXAMPLE 1 Apply the method of gradient descent to $f(x, y, z) = 2x^3y + 3y^2 + 6yz + 6z^2 - 6xy$ using the initial approximation $\mathbf{x}_0 = \langle 2, 1, -1 \rangle$ and taking $h_k = h = 0.1$. List the components of $\mathbf{x}_k$ and $f(\mathbf{x}_k)$ for $k = 0, 1, 2, \dots, 10$.

Solution At the initial approximation $\mathbf{x}_0 = \langle 2, 1, -1 \rangle$, we have $f(\mathbf{x}_0) = f(2, 1, -1) = 7$.

The gradient of the function f is

$$\nabla f = \langle 6x^2y - 6y, 2x^3 + 6y + 6z - 6x, 6y + 12z \rangle.$$

Evaluating the gradient at the initial approximation yields $\nabla f(\mathbf{x}_0) = \langle 18, 4, -6 \rangle$.

We now apply Equation (1) to obtain the second approximation

$$\begin{aligned}\mathbf{x}_1 &= \mathbf{x}_0 - h_0 \nabla f(\mathbf{x}_0) \\ &= \langle 2, 1, -1 \rangle - (0.1)\langle 18, 4, -6 \rangle \\ &= \langle 0.2, 0.6, -0.4 \rangle.\end{aligned}$$

At this point, $f(\mathbf{x}_1) \approx -0.11$.

Subsequent approximations $\mathbf{x}_2, \mathbf{x}_3, \dots$ are then obtained using technology by repeatedly applying Equation (1). These approximations, along with the corresponding f values, are listed in the table below.

k	x component of $\mathbf{x}_k$	y component of $\mathbf{x}_k$	z component of $\mathbf{x}_k$	$f(\mathbf{x}_k)$
0	2	1	-1	7
1	0.2000000000000001	0.6000000000000001	-0.3999999999999999	-0.1104000000000005
2	0.5456000000000001	0.5984000000000001	-0.2800000000000001	-1.22520965376901
3	0.7977612025856	0.702237228236801	-0.30304	-1.8946600083389
4	0.950951612242085	0.839832907434822	-0.360734336942082	-2.26841318086511
5	0.999170298397954	0.95095391811045	-0.431752877072477	-2.43587094205513
6	1.00011671520043	1.03943288055905	-0.484221775451776	-2.5295411401069
7	0.999971125564279	1.1063062093209	-0.526815373245074	-2.58519263199985
8	1.0000094577719	1.15861170717517	-0.558420650943528	-2.61825804593691
9	0.999996308247422	1.19849707338251	-0.583482894116396	-2.63790410815758
10	1.00000161770321	1.22948856581467	-0.602401665206229	-2.64957702176556

The function f in Example 1 was analyzed in Example 3 of Appendix B.2 and was shown to have a local minimum $f(1, 4/3, -2/3) = -8/3$. The sequence of approximations generated in Example 1 appears to converge to this local minimum.

The method of gradient descent is designed to approximate a local minimum of a function of many variables. If we are interested in approximating a local *maximum* instead, then we apply the **method of gradient ascent**

$$\mathbf{x}_{k+1} = \mathbf{x}_k + h_k \nabla f(\mathbf{x}_k),$$

in which we follow the direction of the gradient ∇f , rather than $-\nabla f$.

EXERCISES B.3

You may need to use appropriate technology (such as a calculator or a computer).

Gradient Descent

1. Apply the method of gradient descent to $f(x, y, z) = \frac{12z}{x^2 + y^2 + z^2 + 4}$ using the initial approximation $\mathbf{x}_0 = \langle 1, 1, -1 \rangle$ and the given constant value $h_k = h = 1$. List the components of $\mathbf{x}_k$ and $f(\mathbf{x}_k)$ for $k = 0, 1, 2, \dots, 5$.
2. Apply the method of gradient descent to $f(x, y, z) = x^3 - 3x + y^2 + 2y + 2z^2 - 8z + 6$ using the initial approximation $\mathbf{x}_0 = \langle 0, 0, 1 \rangle$ and the given constant value $h_k = h = 0.2$. List the components of $\mathbf{x}_k$ and $f(\mathbf{x}_k)$ for $k = 0, 1, 2, \dots, 5$.
3. Apply the method of gradient descent to $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2 + xy + zw - y + 2z$ using the initial approximation $\mathbf{x}_0 = \langle -1, 1, -1, 1 \rangle$ and the given constant value $h_k = h = 0.5$. List the components of $\mathbf{x}_k$ and $f(\mathbf{x}_k)$ for $k = 0, 1, 2, \dots, 5$.

4. Apply the method of gradient descent to $f(x, y, z, v, w) = -x - z - yv + 4w + 3x^2 + 2y^2 + z^2 + v^2 + w^2$ using the initial approximation $\mathbf{x}_0 = \langle 1, 1, 1, 1, 1 \rangle$ and the given constant value $h_k = h = 0.2$. List the components of $\mathbf{x}_k$ and $f(\mathbf{x}_k)$ for $k = 0, 1, 2, \dots, 5$.

Line Search

5. In Example 1, we applied the method of gradient descent to the function $f(x, y, z) = 2x^3y + 3y^2 + 6yz + 6z^2 - 6xy$ with the initial approximation $\mathbf{x}_0 = \langle 2, 1, -1 \rangle$. Plot the function $g(h_0) = f(\mathbf{x}_0 - h_0 \nabla f(\mathbf{x}_0))$ to approximate a positive value of h_0 that minimizes $g(h_0)$. (This process, known as the line search, can then be repeatedly used at subsequent steps of the gradient descent method to obtain the values $h_1, h_2, \dots$. Using these values, rather than a constant h , can lead to a faster convergence to a minimum.)

ANSWERS TO ODD-NUMBERED EXERCISES

Appendix B

APPENDIX B.1, p. B-6

1. -5 3. 0 5. 100 7. 1 9. 22
11. -60 13. -20

APPENDIX B.2, p. B-11

1. $f\left(1, -2, \frac{5}{3}\right) = 10$, saddle point; $f\left(-1, -2, \frac{1}{3}\right) = 2$, saddle point
3. $f(1, -1, 2) = -5$, local minimum; $f(-1, -1, 2) = -1$, saddle point
5. $f(0, 0, 2) = 3$, local maximum; $f(0, 0, -2) = -3$, local minimum
7. $f\left(-\frac{1}{3}, \frac{2}{3}, -\frac{4}{3}, \frac{2}{3}\right) = -\frac{5}{3}$, local minimum
9. $f\left(-\frac{5}{3}, 0, 0, \frac{7}{3}, 0\right) = -\frac{13}{3}$, saddle point

APPENDIX B.3, p. B-15

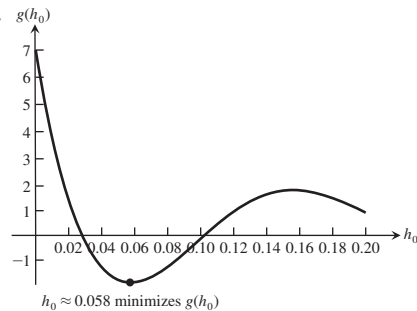
1.

k	x component of $\mathbf{x}_k$	y component of $\mathbf{x}_k$	z component of $\mathbf{x}_k$	$f(\mathbf{x}_k)$
0	1	1	-1	-1.714286
1	0.510204	0.510204	-2.22449	-2.81909
2	0.20641	0.20641	-2.167243	-2.961338
3	0.067207	0.067207	-2.072064	-2.994859
4	0.018721	0.018721	-2.022551	-2.999552
5	0.004841	0.004841	-2.006053	-2.999969

3.

k	x component of $\mathbf{x}_k$	y component of $\mathbf{x}_k$	z component of $\mathbf{x}_k$	w component of $\mathbf{x}_k$	$f(\mathbf{x}_k)$
0	-1	1	-1	1	-1
1	-0.5	1	-1.5	0.5	-1.5
2	-0.5	0.75	-1.25	0.75	-1.625
3	-0.375	0.75	-1.375	0.625	-1.65625
4	-0.375	0.6875	-1.3125	0.6875	-1.664062
5	-0.34375	0.6875	-1.34375	0.65625	-1.666016

5.



ANSWERS TO ODD-NUMBERED EXERCISES

Chapter 1

SECTION 1.1, pp. 31–33

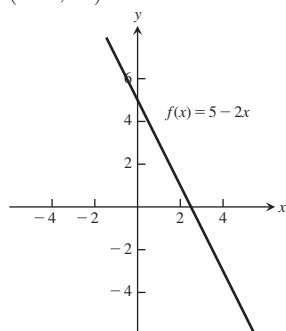
1. $D: (-\infty, \infty), R: [1, \infty)$
 3. $D: [-2, \infty), R: [0, \infty)$
 5. $D: (-\infty, 3) \cup (3, \infty), R: (-\infty, 0) \cup (0, \infty)$
 7. (a) Not a function of x because some values of x have two values of y
 (b) A function of x because for every x there is only one possible y

9. $A = \frac{\sqrt{3}}{4}x^2, p = 3x$

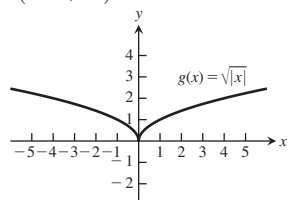
11. $x = \frac{d}{\sqrt{3}}, A = 2d^2, V = \frac{d^3}{3\sqrt{3}}$

13. $L = \frac{\sqrt{20x^2 - 20x + 25}}{4}$

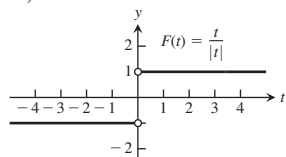
15. $(-\infty, \infty)$



17. $(-\infty, \infty)$

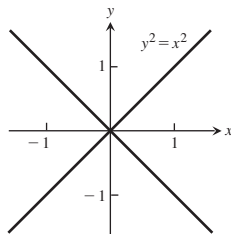
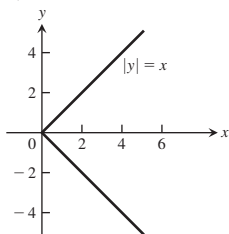


19. $(-\infty, 0) \cup (0, \infty)$

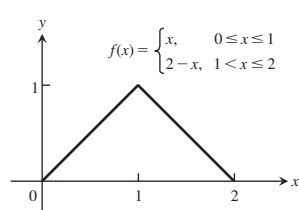


21. $(-\infty, -5) \cup (-5, -3] \cup [3, 5) \cup (5, \infty)$

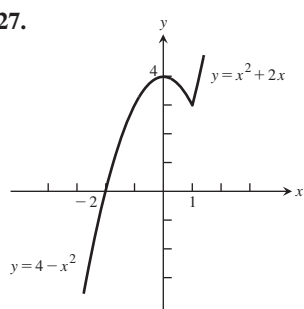
23. (a) For each positive value of x , there are two values of y .
 (b) For each value of $x \neq 0$, there are two values of y .



25.



27.



29. (a) $f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ -x + 2, & 1 < x \leq 2 \end{cases}$

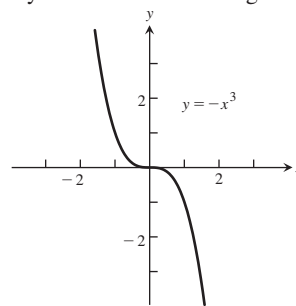
(b) $f(x) = \begin{cases} 2, & 0 \leq x < 1 \\ 0, & 1 \leq x < 2 \\ 2, & 2 \leq x < 3 \\ 0, & 3 \leq x \leq 4 \end{cases}$

31. (a) $f(x) = \begin{cases} -x, & -1 \leq x < 0 \\ 1, & 0 < x \leq 1 \\ -\frac{1}{2}x + \frac{3}{2}, & 1 < x < 3 \end{cases}$

(b) $f(x) = \begin{cases} \frac{1}{2}x, & -2 \leq x \leq 0 \\ -2x + 2, & 0 < x \leq 1 \\ -1, & 1 < x \leq 3 \end{cases}$

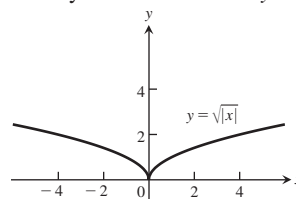
33. (a) $0 \leq x < 1$ (b) $-1 < x \leq 0$ 35. Yes

37. Symmetric about the origin



Dec. $-\infty < x < \infty$

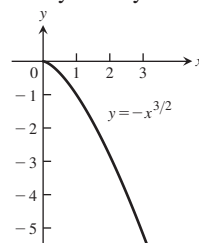
41. Symmetric about the y-axis



Dec. $-\infty < x \leq 0$;

Inc. $0 \leq x < \infty$

45. No symmetry



Dec. $0 \leq x < \infty$

47. Even 49. Even 51. Odd 53. Even

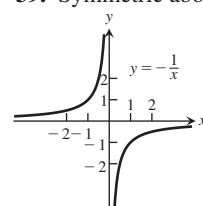
55. Neither 57. Neither 59. Odd 61. Even

63. $t = 180$ 65. $s = 2.4$ 67. $V = x(14 - 2x)(22 - 2x)$

69. (a) h (b) f (c) g 71. (a) $(-2, 0) \cup (4, \infty)$

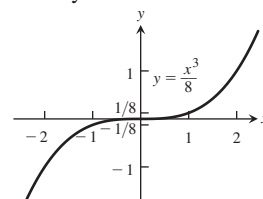
75. $C = 5(2 + \sqrt{2})h$

39. Symmetric about the origin



Inc. $-\infty < x < 0$ and
 $0 < x < \infty$

43. Symmetric about the origin



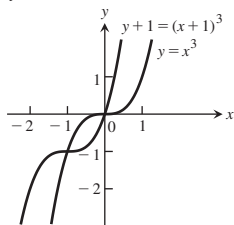
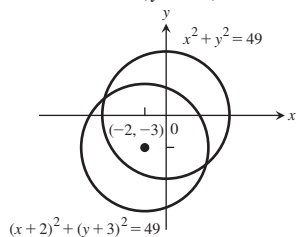
Inc. $-\infty < x < \infty$

SECTION 1.2, pp. 38–41

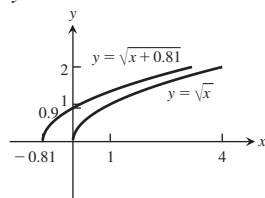
1. $D_f: -\infty < x < \infty, D_g: x \geq 1, D_{f \circ g} = D_{f \circ g} = D_g$
 3. $D_f: -\infty < x < \infty, D_g: -\infty < x < \infty, D_{f \circ g}: -\infty < x < \infty, D_{g \circ f}: -\infty < x < \infty$
 5. (a) 2 (b) 22 (c) $x^2 + 2$ (d) $x^2 + 10x + 22$ (e) 5
 (f) -2 (g) $x + 10$ (h) $x^4 - 6x^2 + 6$
 7. $13 - 3x$ 9. $\sqrt{4x+1}$
 11. (a) $f(g(x))$ (b) $j(g(x))$ (c) $g(g(x))$ (d) $j(j(x))$
 (e) $g(h(f(x)))$ (f) $h(j(f(x)))$
 13.

$g(x)$	$f(x)$	$(f \circ g)(x)$
(a) $x - 7$	$\sqrt{x}$	$\sqrt{x - 7}$
(b) $x + 2$	$3x$	$3x + 6$
(c) x^2	$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
(d) $\frac{x}{x - 1}$	$\frac{x}{x - 1}$	x
(e) $\frac{1}{x - 1}$	$1 + \frac{1}{x}$	x
(f) $\frac{1}{x}$	$\frac{1}{x}$	x

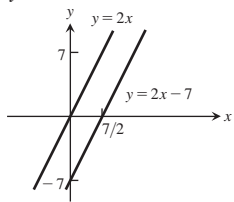
 15. (a) 1 (b) 2 (c) -2 (d) 0 (e) -1 (f) 0
 17. (a) $f(g(x)) = \sqrt{\frac{1}{x} + 1}, g(f(x)) = \frac{1}{\sqrt{x + 1}}$
 (b) $D_{f \circ g} = (-\infty, -1] \cup (0, \infty), D_{g \circ f} = (-1, \infty)$
 19. $g(x) = \frac{2x}{x - 1}$ 21. $V(t) = 4t^2 - 8t + 6$
 23. (a) $y = -(x + 7)^2$ (b) $y = -(x - 4)^2$
 25. (a) Position 4 (b) Position 1 (c) Position 2 (d) Position 3
 27. $(x + 2)^2 + (y + 3)^2 = 49$ 29. $y + 1 = (x + 1)^3$



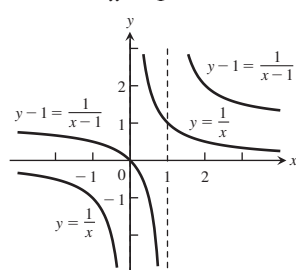
31. $y = \sqrt{x + 0.81}$



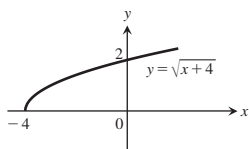
33. $y = 2x$



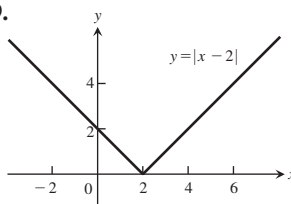
35. $y - 1 = \frac{1}{x - 1}$



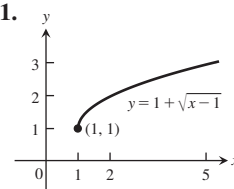
37.



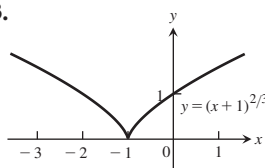
39.



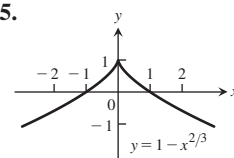
41.



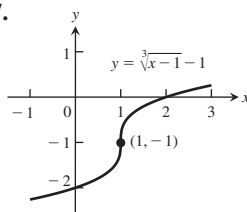
43.



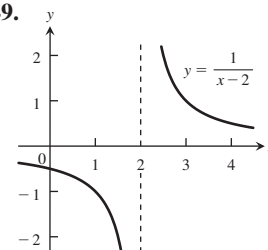
45.



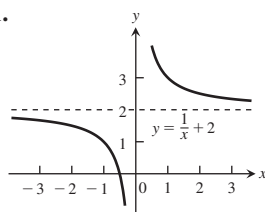
47.



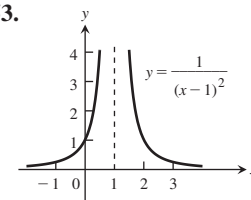
49.



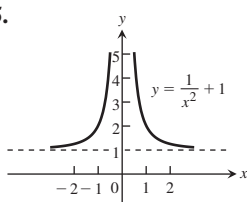
51.



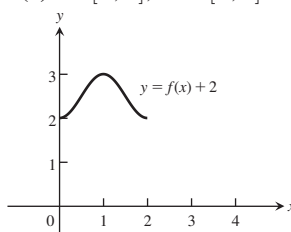
53.



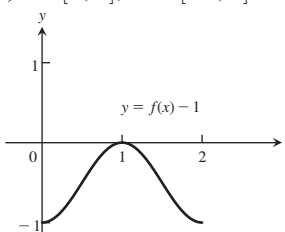
55.



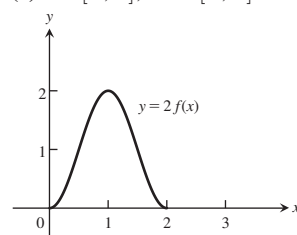
57. (a) $D: [0, 2], R: [2, 3]$



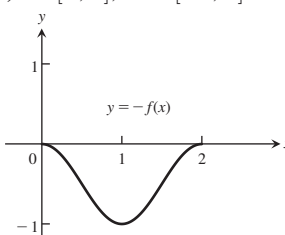
(b) $D: [0, 2], R: [-1, 0]$



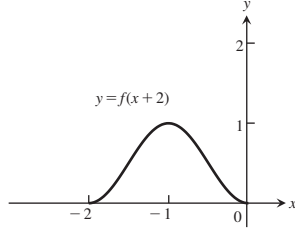
(c) $D: [0, 2], R: [0, 2]$



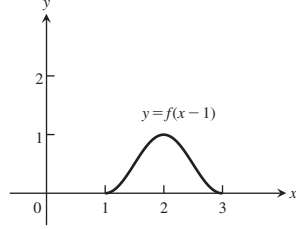
(d) $D: [0, 2], R: [-1, 0]$



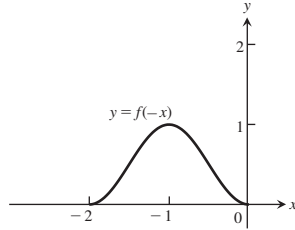
(e) $D: [-2, 0], \quad R: [0, 1]$



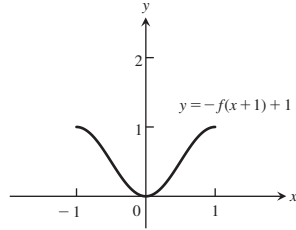
(f) $D: [1, 3], \quad R: [0, 1]$



(g) $D: [-2, 0], \quad R: [0, 1]$



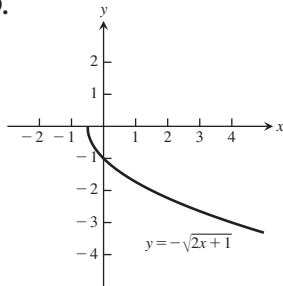
(h) $D: [-1, 1], \quad R: [0, 1]$



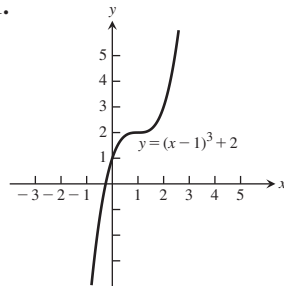
59. $y = 3x^2 - 3$ 61. $y = \frac{1}{2} + \frac{1}{2x^2}$ 63. $y = \sqrt{4x+1}$

65. $y = \sqrt{4 - \frac{x^2}{4}}$ 67. $y = 1 - 27x^3$

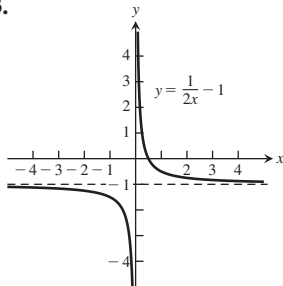
69.



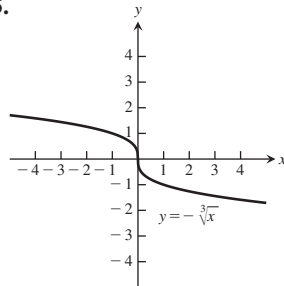
71.



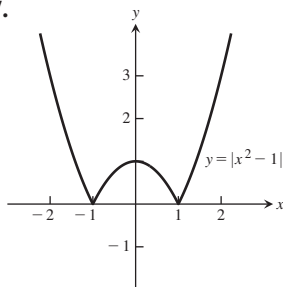
73.



75.



77.



79. (a) Odd (b) Odd (c) Odd (d) Even (e) Even
(f) Even (g) Even (h) Even (i) Odd

SECTION 1.3, pp. 47-49

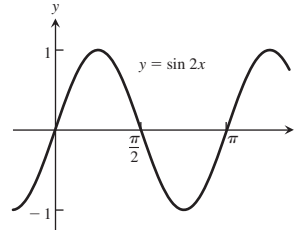
1. (a) 8π m (b) $\frac{55\pi}{9}$ m 3. 8.4 cm (or 84 mm)

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
$\sin \theta$	0	$-\frac{\sqrt{3}}{2}$	0	1	$\frac{1}{\sqrt{2}}$
$\cos \theta$	-1	$-\frac{1}{2}$	1	0	$-\frac{1}{\sqrt{2}}$
$\tan \theta$	0	$\sqrt{3}$	0	UND	-1
$\cot \theta$	UND	$\frac{1}{\sqrt{3}}$	UND	0	-1
$\sec \theta$	-1	-2	1	UND	$-\sqrt{2}$
$\csc \theta$	UND	$-\frac{2}{\sqrt{3}}$	UND	1	$\sqrt{2}$

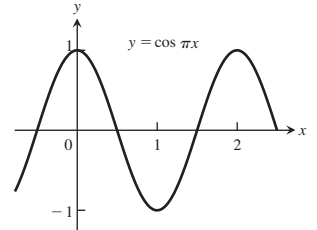
7. $\cos x = -4/5, \tan x = -3/4$

9. $\sin x = -\frac{\sqrt{8}}{3}, \tan x = -\sqrt{8}$

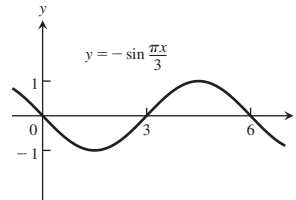
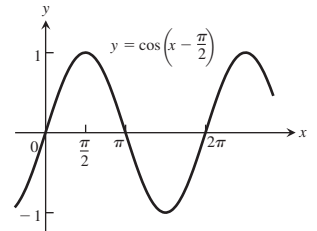
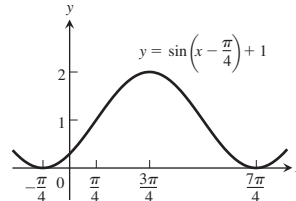
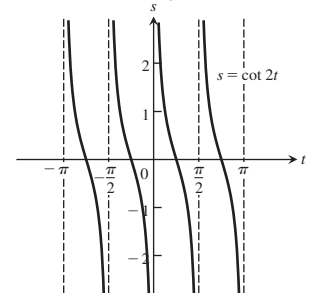
11. $\sin x = -\frac{1}{\sqrt{5}}, \cos x = -\frac{2}{\sqrt{5}}$

13. Period π 

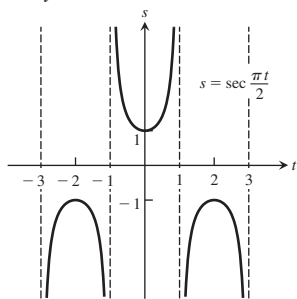
15. Period 2



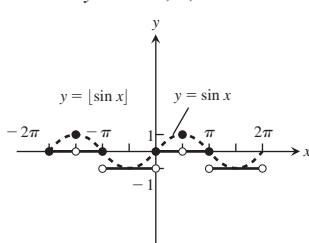
17. Period 6

19. Period 2π 21. Period 2π 23. Period $\pi/2$, symmetric about the origin

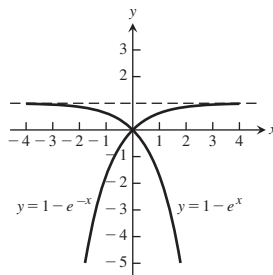
25. Period 4, symmetric about the y-axis



29. $D: (-\infty, \infty)$,
 $R: y = -1, 0, 1$

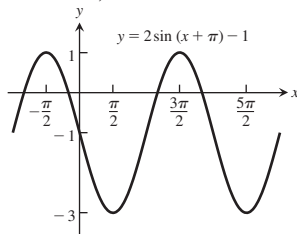


9.



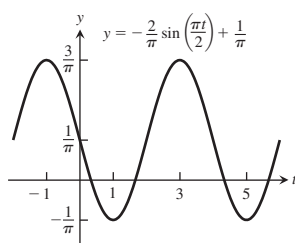
39. $-\cos x$ 41. $-\cos x$ 43. $\frac{\sqrt{6} + \sqrt{2}}{4}$ 45. $\frac{\sqrt{2} + \sqrt{6}}{4}$
47. $\frac{2 + \sqrt{2}}{4}$ 49. $\frac{2 - \sqrt{3}}{4}$ 51. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
53. $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$ 59. $c = \sqrt{7} \approx 2.65$ 63. $a \approx 1.464$
65. $r = \frac{\alpha \sin(\theta)}{1 - \sin(\theta)}$

67. $A = 2, B = 2\pi,$
 $C = -\pi, D = -1$



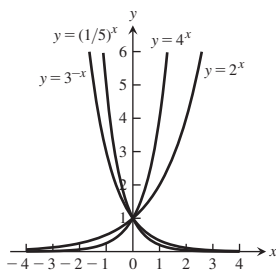
69. $A = -\frac{2}{\pi}, B = 4,$

$$C = 0, D = \frac{1}{\pi}$$

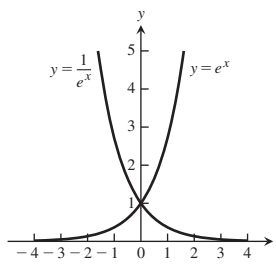


SECTION 1.4, pp. 53–54

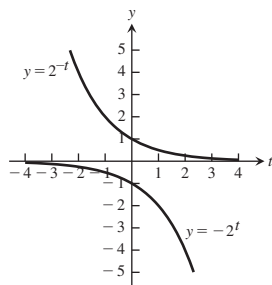
1.



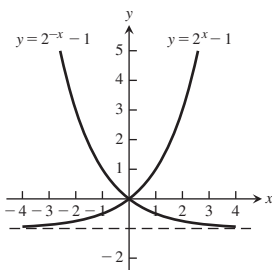
5.



3.



7.

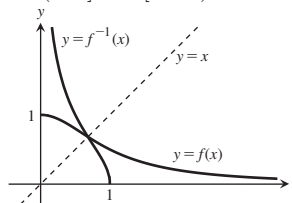


11. $16^{1/4} = 2$ 13. $4^{1/2} = 2$ 15. 5 17. $14\sqrt{3}$ 19. 4
21. $D: -\infty < x < \infty; R: 0 < y < 1/2$
23. $D: -\infty < t < \infty; R: 1 < y < \infty$
25. $x \approx 2.3219$ 27. $x \approx -0.6309$ 29. After 19 years
31. (a) $A(t) = 6.6\left(\frac{1}{2}\right)^{t/14}$ (b) About 38 days later
33. ≈ 11.433 years, or when interest is paid
35. $2^{48} \approx 2.815 \times 10^{14}$

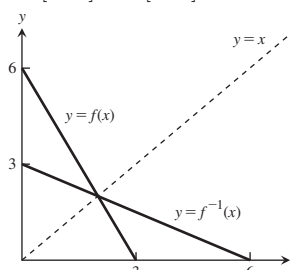
SECTION 1.5, pp. 64–66

1. One-to-one 3. Not one-to-one 5. One-to-one
7. Not one-to-one 9. One-to-one

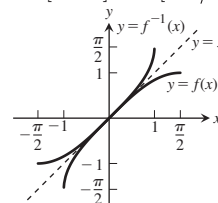
11. $D: (0, 1]$ $R: [0, \infty)$



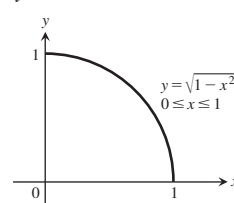
15. $D: [0, 6]$ $R: [0, 3]$



13. $D: [-1, 1]$ $R: [-\pi/2, \pi/2]$

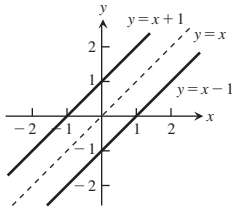


17. Symmetric about the line
 $y = x$



19. $f^{-1}(x) = \sqrt{x-1}$ 21. $f^{-1}(x) = \sqrt[3]{x+1}$
23. $f^{-1}(x) = \sqrt{x} - 1$
25. $f^{-1}(x) = \sqrt[5]{x}; D: -\infty < x < \infty; R: -\infty < y < \infty$
27. $f^{-1}(x) = \sqrt[3]{x-1}; D: -\infty < x < \infty; R: -\infty < y < \infty$
29. $f^{-1}(x) = \frac{1}{\sqrt{x}}; D: x > 0; R: y > 0$
31. $f^{-1}(x) = \frac{2x+3}{x-1}; D: -\infty < x < \infty, x \neq 1;$
 $R: -\infty < y < \infty, y \neq 2$
33. $f^{-1}(x) = 1 - \sqrt{x+1}; D: -1 \leq x < \infty; R: -\infty < y \leq 1$
35. $f^{-1}(x) = \frac{2x+b}{x-1};$
 $D: -\infty < x < \infty, x \neq 1, R: -\infty < y < \infty, y \neq 2$
37. (a) $f^{-1}(x) = \frac{1}{m}x$
(b) The graph of f^{-1} is the line through the origin with slope $1/m$.

39. (a) $f^{-1}(x) = x - 1$



(b) $f^{-1}(x) = x - b$. The graph of f^{-1} is a line parallel to the graph of f . The graphs of f and f^{-1} lie on opposite sides of the line $y = x$ and are equidistant from that line.

(c) Their graphs will be parallel to one another and lie on opposite sides of the line $y = x$ equidistant from that line.

41. (a) $\ln 3 - 2 \ln 2$ (b) $2(\ln 2 - \ln 3)$ (c) $-\ln 2$

(d) $\frac{2}{3} \ln 3$ (e) $\ln 3 + \frac{1}{2} \ln 2$ (f) $\frac{1}{2}(3 \ln 3 - \ln 2)$

43. (a) $\ln 5$ (b) $\ln(x - 3)$ (c) $\ln\left(\frac{2t^2}{b}\right)$

45. (a) 7.2 (b) $\frac{1}{x^2}$ (c) $\frac{x}{y}$ 47. (a) 1 (b) 1 (c) $-x^2 - y^2$

49. e^{2t+4} 51. $e^{5t} + b$ 53. $y = 2xe^x + 1$

55. (a) $k = \ln 2$ (b) $k = (1/10) \ln 2$ (c) $k = 1000 \ln a$

57. (a) $t = -10 \ln 3$ (b) $t = -\frac{\ln 2}{k}$ (c) $t = \frac{\ln 4}{\ln 2}$

59. $t = 4(\ln|x|)^2$ 61. $t = \ln 3$ 63. $t = e^2/(e^2 - 1)$

65. (a) 7 (b) $\sqrt{2}$ (c) 75 (d) 2 (e) 0.5 (f) -1

67. (a) $\sqrt{x}$ (b) x^2 (c) $\sin x$ 69. (a) $\frac{\ln 3}{\ln 2}$ (b) 3 (c) 2

71. (a) $-\pi/6$ (b) $\pi/4$ (c) $-\pi/3$ 73. (a) π (b) $\pi/2$

75. Yes, $g(x)$ is also one-to-one.

77. Yes, $f \circ g$ is also one-to-one.

79. (a) $f^{-1}(x) = \log_2\left(\frac{x}{100 - x}\right)$ (b) $f^{-1}(x) = \log_{1.1}\left(\frac{x}{50 - x}\right)$

(c) $f^{-1}(x) = \ln\left(\frac{x+1}{1-x}\right)$ (d) $f^{-1}(x) = e^{\frac{2x}{x+1}}$

81. (a) $y = \ln x - 3$ (b) $y = \ln(x - 1)$

(c) $y = 3 + \ln(x + 1)$ (d) $y = \ln(x - 2) - 4$

(e) $y = \ln(-x)$ (f) $y = e^x$

83. ≈ -0.7667 85. (a) Amount $= 8\left(\frac{1}{2}\right)^{t/12}$ (b) 36 hours

87. ≈ 44.081 years

PRACTICE EXERCISES, pp. 67–69

1. $A = \pi r^2$, $C = 2\pi r$, $A = \frac{C^2}{4\pi}$ 3. $x = \tan \theta$, $y = \tan^2 \theta$

5. Origin 7. Neither 9. Even 11. Even

13. Odd 15. Neither

17. (a) Even (b) Odd (c) Odd (d) Even (e) Even

19. (a) Domain: all reals (b) Range: $[-2, \infty)$

21. (a) Domain: $[-4, 4]$ (b) Range: $[0, 4]$

23. (a) Domain: all reals (b) Range: $(-3, \infty)$

25. (a) Domain: all reals (b) Range: $[-3, 1]$

27. (a) Domain: $(3, \infty)$ (b) Range: all reals

29. (a) Domain: $(-\infty, -1]$ and $[3, \infty)$ (b) Range: $(-\infty, 5]$

31. (a) Domain: $(-\infty, 0)$ and $(0, \infty)$ (b) Range: $[-4, 4]$

33. (a) Increasing (b) Neither (c) Decreasing (d) Increasing

35. (a) Domain: $[-4, 4]$ (b) Range: $[0, 2]$

37. $f(x) = \begin{cases} 1 - x, & 0 \leq x < 1 \\ 2 - x, & 1 \leq x \leq 2 \end{cases}$

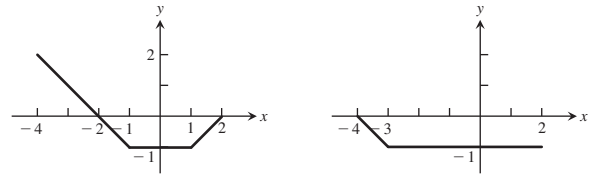
39. (a) 1 (b) $\frac{1}{\sqrt{2.5}} = \sqrt{\frac{2}{5}}$ (c) $x, x \neq 0$ (d) $\frac{1}{\sqrt{1/\sqrt{x+2}+2}}$

41. (a) $(f \circ g)(x) = -x, x \geq -2, (g \circ f)(x) = \sqrt{4 - x^2}$

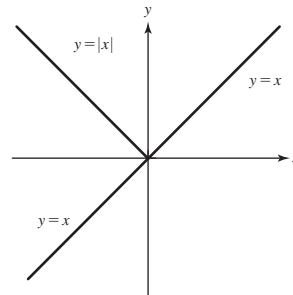
(b) Domain $(f \circ g)$: $[-2, \infty)$, domain $(g \circ f)$: $[-2, 2]$

(c) Range $(f \circ g)$: $(-\infty, 2]$, range $(g \circ f)$: $[0, 2]$

43.



45. Replace the portion for $x < 0$ with the mirror image of the portion for $x > 0$ to make the new graph symmetric with respect to the y -axis.



47. Reflects the portion for $y < 0$ across the x -axis

49. Reflects the portion for $y < 0$ across the x -axis

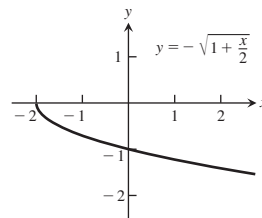
51. Adds the mirror image of the portion for $x > 0$ to make the new graph symmetric with respect to the y -axis

53. (a) $y = g(x - 3) + \frac{1}{2}$ (b) $y = g\left(x + \frac{2}{3}\right) - 2$

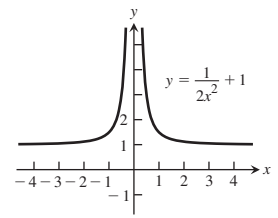
(c) $y = g(-x)$ (d) $y = -g(x)$

(e) $y = 5g(x)$ (f) $y = g(5x)$

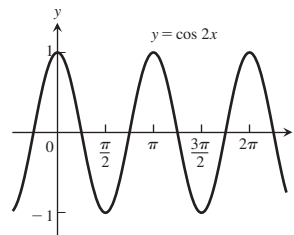
55.



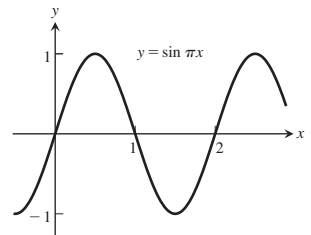
57.



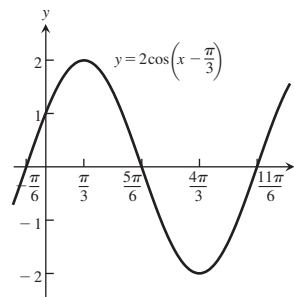
59. Period π



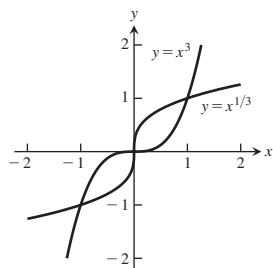
61. Period 2



63.



65. (a) $a = 1$ $b = \sqrt{3}$ (b) $a = 2\sqrt{3}/3$ $c = 4\sqrt{3}/3$
67. (a) $a = \frac{b}{\tan B}$ (b) $c = \frac{a}{\sin A}$
69. ≈ 16.98 m 71. (b) 4π
73. (a) Domain: $-\infty < x < \infty$ (b) Domain: $x > 0$
75. (a) Domain: $-3 \leq x \leq 3$ (b) Domain: $0 \leq x \leq 4$
77. $(f \circ g)(x) = \ln(4 - x^2)$ and domain: $-2 < x < 2$;
 $(g \circ f)(x) = 4 - (\ln x)^2$ and domain: $x > 0$;
 $(f \circ f)(x) = \ln(\ln x)$ and domain: $x > 1$;
 $(g \circ g)(x) = -x^4 + 8x^2 - 12$ and domain: $-\infty < x < \infty$.
83. (a) D: $(-\infty, \infty)$ R: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (b) D: $[-1, 1]$ R: $[-1, 1]$
85. (a) No (b) Yes
87. (a) $f(g(x)) = (\sqrt[3]{x})^3 = x$, $g(f(x)) = \sqrt[3]{x^3} = x$



ADDITIONAL AND ADVANCED EXERCISES, pp. 69–71

1. Yes. For instance: $f(x) = 1/x$ and $g(x) = 1/x$, or $f(x) = 2x$ and $g(x) = x/2$, or $f(x) = e^x$ and $g(x) = \ln x$.
3. If $f(x)$ is odd, then $g(x) = f(x) - 2$ is not odd. Nor is $g(x)$ even, unless $f(x) = 0$ for all x . If f is even, then $g(x) = f(x) - 2$ is also even.
- 5.
-
19. (a) Domain: all reals. Range: If $a > 0$, then (d, ∞) ; if $a < 0$, then $(-\infty, d)$.
 (b) Domain: (c, ∞) , range: all reals
21. (a) $y = 100,000 - 10,000x$, $0 \leq x \leq 10$
 (b) After 4.5 years
23. After $\frac{\ln(10/3)}{\ln 1.08} \approx 15.6439$ years. (If the bank only pays interest at the end of the year, it will take 16 years.)
25. $x = 2$, $x = 1$ 27. $1/2$

Chapter 2

SECTION 2.1, pp. 77–79

1. (a) 19 (b) 1
3. (a) $-\frac{4}{\pi}$ (b) $-\frac{3\sqrt{3}}{\pi}$ 5. 1
7. (a) 4 (b) $y = 4x - 9$
9. (a) 2 (b) $y = 2x - 7$
11. (a) 12 (b) $y = 12x - 16$
13. (a) -9 (b) $y = -9x - 2$
15. (a) $-1/4$ (b) $y = -x/4 - 1$
17. (a) $1/4$ (b) $y = x/4 + 1$

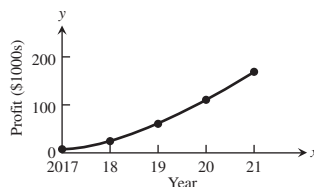
19. Your estimates may not completely agree with these.

PQ_1	PQ_2	PQ_3	PQ_4
43	46	49	50

The appropriate units are m/s.

- (b) ≈ 50 m/s or 180 km/h

21. (a)



- (b) $\approx \$56,000/\text{year}$

- (c) $\approx \$42,000/\text{year}$

23. (a) 0.414213, 0.449489, $(\sqrt{1+h}-1)/h$ (b) $g(x) = \sqrt{x}$

$1+h$	1.1	1.01	1.001	1.0001	1.00001	1.000001
$\sqrt{1+h}$	1.04880	1.004987	1.0004998	1.0000499	1.0000005	1.00000005
$(\sqrt{1+h}-1)/h$	0.4880	0.4987	0.4998	0.499	0.5	0.5

- (c) 0.5

25. (a) 15 km/h, 3.3 km/h, 10 km/h (b) 10 km/h, 0 km/h, 4 km/h
 (c) 20 km/h when $t = 3.5$ hr

SECTION 2.2, pp. 87–90

1. (a) Does not exist. As x approaches 1 from the right, $g(x)$ approaches 0. As x approaches 1 from the left, $g(x)$ approaches 1. There is no single number L that all the values $g(x)$ get arbitrarily close to as $x \rightarrow 1$.
 (b) 1 (c) 0 (d) $1/2$
3. (a) True (b) True (c) False (d) False (e) False
 (f) True (g) True (h) False (i) True (j) True (k) False
5. As x approaches 0 from the left, $x/|x|$ approaches -1 . As x approaches 0 from the right, $x/|x|$ approaches 1. There is no single number L that the function values all get arbitrarily close to as $x \rightarrow 0$.
7. Nothing can be said. 9. No; no; no 11. -4 13. -8
15. 3 17. $-25/2$ 19. 16 21. $3/2$ 23. $1/10$ 25. -7 27. $3/2$
29. $-1/2$ 31. -1 33. $4/3$ 35. $1/6$ 37. 4 39. $1/2$ 41. $3/2$
43. -1 45. 1 47. $1/3$ 49. $\sqrt{4-\pi}$
51. (a) Quotient Rule (b) Difference and Power Rules
 (c) Sum and Constant Multiple Rules
53. (a) -10 (b) -20 (c) -1 (d) $5/7$
55. (a) 4 (b) -21 (c) -12 (d) $-7/3$
57. 2 59. 3 61. $1/(2\sqrt{7})$ 63. $\sqrt{5}$
65. (a) The limit is 1.
67. (a) $f(x) = (x^2 - 9)/(x + 3)$

x	-3.1	-3.01	-3.001	-3.0001	-3.00001	-3.000001
$f(x)$	-6.1	-6.01	-6.001	-6.0001	-6.00001	-6.000001

x	-2.9	-2.99	-2.999	-2.9999	-2.99999	-2.999999
$f(x)$	-5.9	-5.99	-5.999	-5.9999	-5.99999	-5.999999

- (c) $\lim_{x \rightarrow -3} f(x) = -6$

69. (a) $G(x) = (x + 6)/(x^2 + 4x - 12)$

x	-5.9	-5.99	-5.999	-5.9999
$G(x)$	-.126582	-.1251564	-.1250156	-.1250015

	-5.99999	-5.999999
	-.1250001	-.1250000

x	-6.1	-6.01	-6.001	-6.0001
$G(x)$	-.123456	-.124843	-.124984	-.124998

-6.00001	-6.000001
-.124999	-.124999

(c) $\lim_{x \rightarrow -6} G(x) = -1/8 = -0.125$

71. (a) $f(x) = (x^2 - 1)/(|x| - 1)$

x	-1.1	-1.01	-1.001	-1.0001	-1.00001	-1.000001
$f(x)$	2.1	2.01	2.001	2.0001	2.00001	2.000001

x	-.9	-.99	-.999	-.9999	-.99999	-.999999
$f(x)$	1.9	1.99	1.999	1.9999	1.99999	1.999999

(c) $\lim_{x \rightarrow -1} f(x) = 2$

73. (a) $g(\theta) = (\sin \theta)/\theta$

θ	.1	.01	.001	.0001	.00001	.000001
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

θ	-.1	-.01	-.001	-.0001	-.00001	-.000001
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

$\lim_{\theta \rightarrow 0} g(\theta) = 1$

75. (a) $f(x) = x^{1/(1-x)}$

x	.9	.99	.999	.9999	.99999	.999999
$f(x)$	.348678	.366032	.367695	.367861	.367877	.367879

x	1.1	1.01	1.001	1.0001	1.00001	1.000001
$f(x)$	.385543	.369711	.368063	.367897	.367881	.367878

$\lim_{x \rightarrow 1} f(x) \approx 0.36788$

77. $c = 0, 1, -1$; the limit is 0 at $c = 0$, and 1 at $c = 1, -1$.

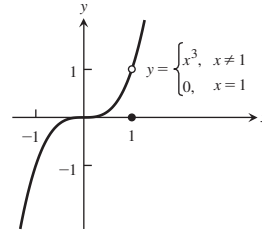
79. 7 81. (a) 5 (b) 5 83. (a) 0 (b) 0

SECTION 2.3, pp. 95–98

1. $\delta = 2$, or a smaller positive value
3. $\delta = 1/2$, or a smaller positive value
5. $\delta = 1/18$, or a smaller positive value
7. $\delta = 0.1$, or a smaller positive value
9. $\delta = 7/16$, or a smaller positive value
11. $\delta = \sqrt{5} - 2$, or a smaller positive value
13. $\delta = 0.36$, or a smaller positive value
15. $(3.99, 4.01)$, $\delta = 0.01$, or a smaller positive value
17. $(-0.19, 0.21)$, $\delta = 0.19$, or a smaller positive value
19. $(3, 15)$, $\delta = 5$, or a smaller positive value
21. $(10/3, 5)$, $\delta = 2/3$, or a smaller positive value
23. $(-\sqrt{4.5}, -\sqrt{3.5})$, $\delta = \sqrt{4.5} - 2 \approx 0.12$, or a smaller positive value
25. $(\sqrt{15}, \sqrt{17})$, $\delta = \sqrt{17} - 4 \approx 0.12$, or a smaller positive value
27. $(2 - \frac{0.03}{m}, 2 + \frac{0.03}{m})$, $\delta = \frac{0.03}{m}$, or a smaller positive value
29. $(\frac{1}{2} - \frac{c}{m}, \frac{1}{2} + \frac{c}{m})$, $\delta = \frac{c}{m}$, or a smaller positive value
31. $L = -3$, $\delta = 0.01$, or a smaller positive value
33. $L = 4$, $\delta = 0.05$, or a smaller positive value
35. $L = 4$, $\delta = 0.75$, or a smaller positive value
55. $[8.7332, 8.7476]$. To be safe, the left endpoint was rounded up and the right endpoint was rounded down.
59. The limit does not exist as x approaches 3.

SECTION 2.4, pp. 104–106

1. (a) True (b) True (c) False
(d) True (e) True (f) True
(g) False (h) False (i) True
(j) False (k) True (l) False
3. 2, 1 (b) No, $\lim_{x \rightarrow 2^+} f(x) \neq \lim_{x \rightarrow 2^-} f(x)$
(c) 3, 3 (d) Yes, 3
5. (a) No (b) Yes, 0 (c) No
7. (a)

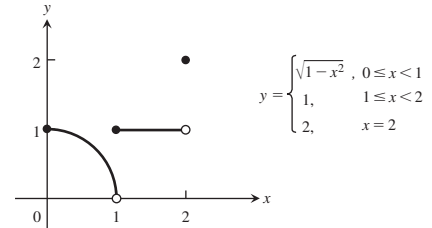


(b) 1, 1

(c) Yes, 1

9. (a) $D: 0 \leq x \leq 2$, $R: 0 < y \leq 1$ and $y = 2$

(b) $[0, 1) \cup (1, 2]$ (c) $x = 2$ (d) $x = 0$

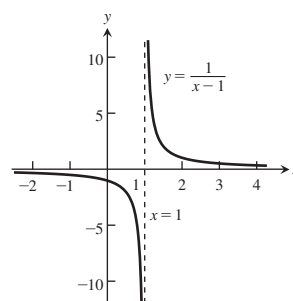


11. $\sqrt{3}$ 13. 1 15. $2/\sqrt{5}$ 17. (a) 1 (b) -1
19. (a) 1 (b) -1 21. (a) 1 (b) $2/3$ 23. 1
25. $3/4$ 27. 2 29. $1/2$ 31. 2 33. 0 35. 1
37. $1/2$ 39. 0 41. $3/8$ 43. 3 45. 0
51. $\delta = \varepsilon^2$, or a smaller positive value, $\lim_{x \rightarrow 5^+} \sqrt{x-5} = 0$
55. (a) 400 (b) 399 (c) The limit does not exist.

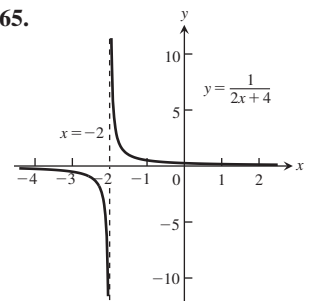
SECTION 2.5, pp. 117–120

1. (a) 0 (b) -2 (c) 2 (d) Does not exist (e) -1
(f) ∞ (g) Does not exist (h) 1 (i) 0
3. (a) -3 (b) -3 5. (a) $1/2$ (b) $1/2$ 7. (a) $-5/3$
(b) $-5/3$ 9. 0 11. -1 13. (a) $2/5$ (b) $2/5$
15. (a) 0 (b) 0 17. (a) 7 (b) 7 19. (a) 0 (b) 0
21. (a) ∞ (b) ∞ 23. 2 25. ∞ 27. 0 29. 1
31. ∞ 33. 1 35. $1/2$ 37. ∞ 39. $-\infty$
41. $-\infty$ 43. ∞ 45. (a) ∞ (b) $-\infty$ 47. ∞
49. ∞ 51. $-\infty$ 53. (a) ∞ (b) $-\infty$ (c) $-\infty$ (d) ∞
55. (a) $-\infty$ (b) ∞ (c) 0 (d) $3/2$
57. (a) $-\infty$ (b) $1/4$ (c) $1/4$ (d) $1/4$ (e) It will be $-\infty$.
59. (a) $-\infty$ (b) ∞ 61. (a) ∞ (b) ∞ (c) ∞ (d) ∞

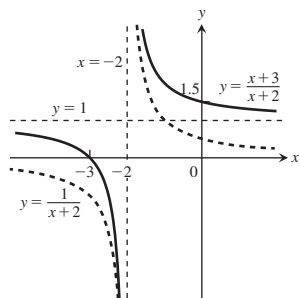
63.



65.



67.

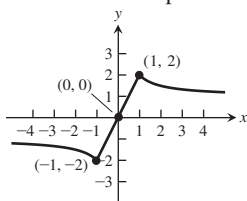

69. Domain: $(-\infty, \infty)$; horizontal asymptote: $y = 7$

71. Domain: $(-\infty, \infty)$; horizontal asymptotes: $y = -1, y = 4$

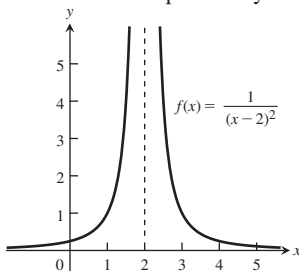
73. Domain: $(-\infty, 0)$ and $(0, \infty)$; horizontal asymptotes:

 $y = -1, y = 1$; vertical asymptote: $x = 0$

75. Here is one possibility.



79. Here is one possibility.


85. At most one 87. 0 89. $-3/4$ 91. $5/2$

99. (a) For every positive real number B there exists a corresponding number $\delta > 0$ such that for all x ,

$$c - \delta < x < c \Rightarrow f(x) > B.$$

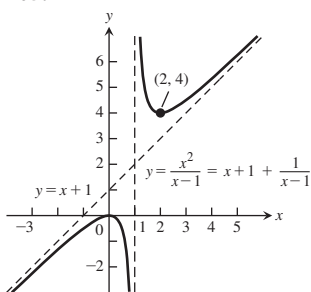
(b) For every negative real number $-B$ there exists a corresponding number $\delta > 0$ such that for all x ,

$$c < x < c + \delta \Rightarrow f(x) < -B.$$

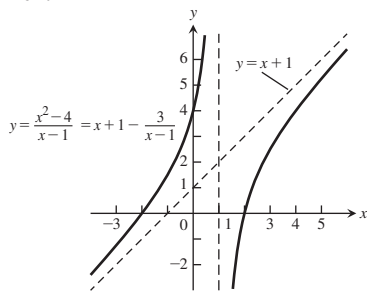
(c) For every negative real number $-B$ there exists a corresponding number $\delta > 0$ such that for all x ,

$$c - \delta < x < c \Rightarrow f(x) < -B.$$

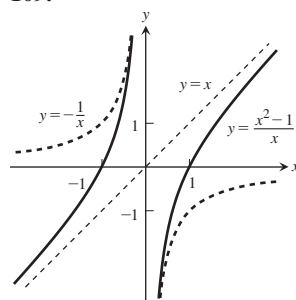
105.



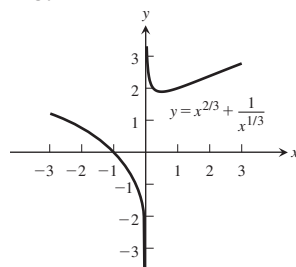
107.



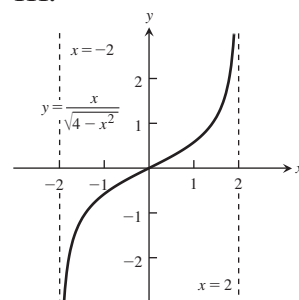
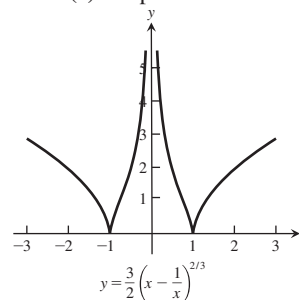
109.



113.



111.


115. (a) $y \rightarrow \infty$ (b) $y \rightarrow \infty$
(c) cusps at ± 1


SECTION 2.6, pp. 130–132

1. No; not defined at $x = 2$

3. Continuous

5. (a) Yes (b) Yes (c) Yes (d) Yes

7. (a) No (b) No 9. 0

11. 1, nonremovable; 0, removable 13. All x except $x = 2$

15. All x except $x = 3, x = 1$ 17. All x

19. All x except $x = 0$ 21. All x except $n\pi/2, n$ any integer

23. All x except $n\pi/2, n$ an odd integer 25. All $x \geq -3/2$

27. All x 29. All x 31. All x except $x = 1$

33. 0; continuous at $x = \pi$ 35. 1; continuous at $y = 1$

37. $\sqrt{2}/2$; continuous at $t = 0$ 39. 1; continuous at $x = 0$

41. -1 43. 0 45. $g(3) = 6$ 47. $f(1) = 3/2$

49. $a = 4/3$ 51. $a = -2, 3$ 53. $a = 5/2, b = -1/2$

77. $x \approx 1.8794, -1.5321, -0.3473$ 79. $x \approx 1.7549$

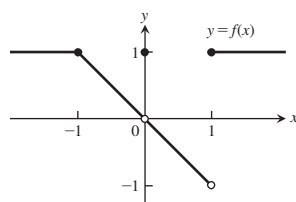
81. $x \approx 3.5156$ 83. $x \approx 0.7391$

PRACTICE EXERCISES, pp. 133–134

1. At $x = -1$: $\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^+} f(x) = 1$, so $\lim_{x \rightarrow -1} f(x) = 1 = f(-1)$; continuous at $x = -1$

At $x = 0$: $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = 0$, so $\lim_{x \rightarrow 0} f(x) = 0$.

However, $f(0) \neq 0$, so f is discontinuous at $x = 0$. The discontinuity can be removed by redefining $f(0)$ to be 0.

At $x = 1$: $\lim_{x \rightarrow 1^-} f(x) = -1$ and $\lim_{x \rightarrow 1^+} f(x) = 1$, so $\lim_{x \rightarrow 1} f(x)$ does not exist. The function is discontinuous at $x = 1$, and the discontinuity is not removable.


3. (a) -21 (b) 49 (c) 0 (d) 1 (e) 1 (f) 7
 (g) -7 (h) $-\frac{1}{7}$ 5. 4
 7. (a) $(-\infty, \infty)$ (b) $[0, \infty)$ (c) $(-\infty, 0)$ and $(0, \infty)$ (d) $[0, \infty)$
 9. (a) Does not exist (b) 0 11. $\frac{1}{2}$ 13. $2x$ 15. $-\frac{1}{4}$
 17. $2/3$ 19. $2/\pi$ 21. 1 23. 4 25. $-\infty$
 27. 0 29. 2 31. 0
 35. No in both cases, because $\lim_{x \rightarrow 1} f(x)$ does not exist, and $\lim_{x \rightarrow -1} f(x)$ does not exist.
 37. Yes, f does have a continuous extension, to $a = 1$ with $f(1) = 4/3$.
 39. No 41. $2/5$ 43. 0 45. $-\infty$ 47. 0 49. 1
 51. 1 53. $-\pi/2$ 55. (a) $x = 3$ (b) $x = 1$ (c) $x = -4$
 57. Domain: $[-4, 2)$ and $(2, 4]$, Range: $(-\infty, \infty)$

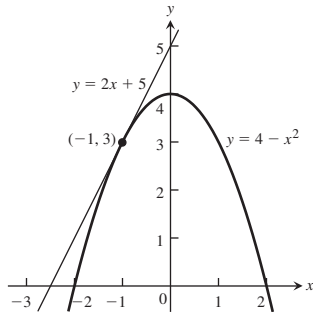
ADDITIONAL AND ADVANCED EXERCISES, pp. 134–137

3. 0; the left-hand limit was taken because the function is undefined for $v > c$.
 5. $17.5 < t < 22.5$; within 2.5°C
 13. (a) B (b) A (c) A (d) A
 21. (a) $\lim_{a \rightarrow 0} r_+(a) = 0.5$, $\lim_{a \rightarrow -1^+} r_+(a) = 1$
 (b) $\lim_{a \rightarrow 0} r_-(a)$ does not exist, $\lim_{a \rightarrow -1^+} r_-(a) = 1$
 25. 0 27. 1 29. 4 31. -1 33. $1/2$ 35. 4
 37. 1 39. To make g continuous at the origin, define $g(0) = 1$.
 41. $y = 2x$ 43. $y = x$, $y = -x$ 47. $-4/3$
 49. (a) Domain: $\{0, 1, 1/2, 1/3, 1/4, \dots\}$
 (b) The domain intersects any interval (a, b) containing 0 .
 (c) 0
 51. (a) Domain: $(-\infty, -1/\pi] \cup [-1/(2\pi), -1/(3\pi)] \cup [-1/(4\pi), -1/(5\pi)] \cup \dots \cup [1/(5\pi), 1/(4\pi)] \cup [1/(3\pi), 1/(2\pi)] \cup [1/\pi, \infty)$
 (b) The domain intersects any interval (a, b) containing 0 .
 (c) 0

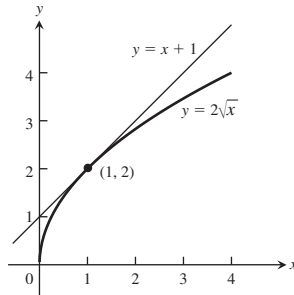
Chapter 3

SECTION 3.1, pp. 140–142

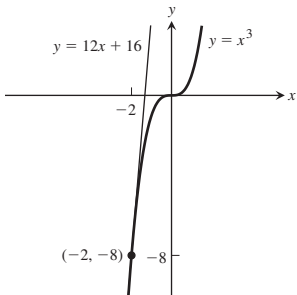
1. $P_1: m_1 = 1$, $P_2: m_2 = 5$
 3. $P_1: m_1 = 5/2$, $P_2: m_2 = -1/2$
 5. $y = 2x + 5$



7. $y = x + 1$



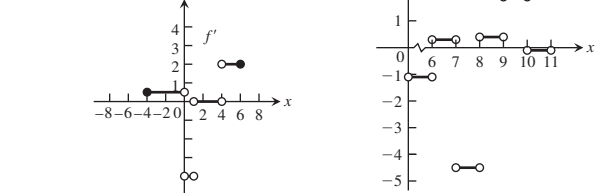
9. $y = 12x + 16$



11. $m = 4$, $y - 5 = 4(x - 2)$
 13. $m = -2$, $y - 3 = -2(x - 3)$
 15. $m = 12$, $y - 8 = 12(t - 2)$
 17. $m = \frac{1}{4}$, $y - 2 = \frac{1}{4}(x - 4)$
 19. $m = -1$
 21. $m = -1/4$
 23. (a) It is the rate of change of the number of cells when $t = 5$.
 The units are the number of cells per hour.
 (b) $P'(3)$ because the slope of the curve is greater there.
 (c) $51.72 \approx 52$ cells/h
 25. $(-2, -5)$ 27. $y = -(x + 1)$, $y = -(x - 3)$
 29. 19.6 m/s 31. 6π 35. Yes 37. No
 39. (a) Nowhere
 41. (a) At $x = 0$
 43. (a) Nowhere
 45. (a) At $x = 1$
 47. (a) At $x = 0$

SECTION 3.2, pp. 147–151

1. $-2x, 6, 0, -2$ 3. $-\frac{2}{t^3}, 2, -\frac{1}{4}, -\frac{2}{3\sqrt{3}}$
 5. $\frac{3}{2\sqrt{3\theta}}, \frac{3}{2\sqrt{3}}, \frac{1}{2}, \frac{3}{2\sqrt{2}}$ 7. $6x^2$ 9. $\frac{1}{(2t + 1)^2}$
 11. $\frac{3}{2}q^{1/2}$ 13. $1 - \frac{9}{x^2}, 0$ 15. $3t^2 - 2t, 5$
 17. $\frac{-4}{(x - 2)\sqrt{x - 2}}$, $y - 4 = -\frac{1}{2}(x - 6)$ 19. 6
 21. $1/8$ 23. $\frac{-1}{(x + 2)^2}$ 25. $\frac{-1}{(x - 1)^2}$ 27. (b) 29. (d)
 31. (a) $x = 0, 1, 4$ 33.



35. (a) i) 0.77°C/h ii) 1.43°C/h
 iii) 0°C/h iv) -1.88°C/h
 (b) 3.63°C/h at 12 P.M., -4°C/h at 6 P.M.
 (c)

37. Since $\lim_{h \rightarrow 0^+} \frac{f(0 + h) - f(0)}{h} = 1$
 while $\lim_{h \rightarrow 0^-} \frac{f(0 + h) - f(0)}{h} = 0$,
 $f'(0) = \lim_{h \rightarrow 0} \frac{f(0 + h) - f(0)}{h}$ does not exist and $f(x)$ is
 not differentiable at $x = 0$.

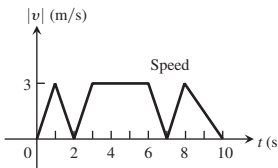
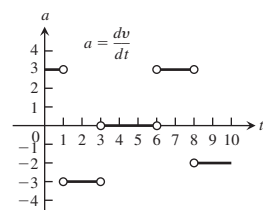
39. Since $\lim_{h \rightarrow 0^+} \frac{f(1+h) - f(1)}{h} = 2$ while
 $\lim_{h \rightarrow 0^-} \frac{f(1+h) - f(1)}{h} = \frac{1}{2}$, $f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$
 does not exist and $f(x)$ is not differentiable at $x = 1$.
41. Since $f(x)$ is not continuous at $x = 0$, $f(x)$ is not differentiable at $x = 0$.
43. Since $\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = 3$ while
 $\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = 0$, f is not differentiable at $x = 0$.
45. (a) $-3 \leq x \leq 2$ (b) None (c) None
47. (a) $-3 \leq x < 0, 0 < x \leq 3$ (b) None (c) $x = 0$
49. (a) $-1 \leq x < 0, 0 < x \leq 2$ (b) $x = 0$ (c) None

SECTION 3.3, pp. 159–161

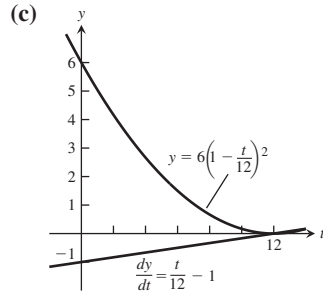
1. $\frac{dy}{dx} = -2x$, $\frac{d^2y}{dx^2} = -2$
3. $\frac{ds}{dt} = 15t^2 - 15t^4$, $\frac{d^2s}{dt^2} = 30t - 60t^3$
5. $\frac{dy}{dx} = 4x^2 - 1 + 2e^x$, $\frac{d^2y}{dx^2} = 8x + 2e^x$
7. $\frac{dw}{dz} = -\frac{6}{z^3} + \frac{1}{z^2}$, $\frac{d^2w}{dz^2} = \frac{18}{z^4} - \frac{2}{z^3}$
9. $\frac{dy}{dx} = 12x - 10 + 10x^{-3}$, $\frac{d^2y}{dx^2} = 12 - 30x^{-4}$
11. $\frac{dr}{ds} = \frac{-2}{3s^3} + \frac{5}{2s^2}$, $\frac{d^2r}{ds^2} = \frac{2}{s^4} - \frac{5}{s^3}$
13. $y' = -5x^4 + 12x^2 - 2x - 3$
15. $y' = 3x^2 + 10x + 2 - \frac{1}{x^2}$
17. $y' = \frac{-19}{(3x-2)^2}$
19. $g'(x) = \frac{x^2 + x + 4}{(x+0.5)^2}$
21. $\frac{dv}{dt} = \frac{t^2 - 2t - 1}{(1+t^2)^2}$
23. $f'(s) = \frac{1}{\sqrt{s}(\sqrt{s}+1)^2}$
25. $v' = -\frac{1}{x^2} + 2x^{-3/2}$
27. $y' = \frac{-4x^3 - 3x^2 + 1}{(x^2-1)^2(x^2+x+1)^2}$
29. $y' = -2e^{-x} + 3e^{3x}$
31. $y' = 3x^2e^x + x^3e^x$
33. $y' = \frac{9}{4}x^{5/4} - 2e^{-2x}$
35. $\frac{ds}{dt} = 3t^{1/2}$
37. $y' = \frac{2}{7x^{5/7}} - ex^{e-1}$
39. $\frac{dr}{ds} = \frac{se^s - e^s}{s^2}$
41. $y' = 2x^3 - 3x - 1$, $y'' = 6x^2 - 3$, $y''' = 12x$, $y^{(4)} = 12$,
 $y^{(n)} = 0$ for $n \geq 5$
43. $y' = 3x^2 + 8x + 1$, $y'' = 6x + 8$, $y''' = 6$, $y^{(n)} = 0$
 for $n \geq 4$
45. $y' = 2x - 7x^{-2}$, $y'' = 2 + 14x^{-3}$
47. $\frac{dr}{d\theta} = 3\theta^{-4}$, $\frac{d^2r}{d\theta^2} = -12\theta^{-5}$
49. $\frac{dw}{dz} = -z^{-2} - 1$, $\frac{d^2w}{dz^2} = 2z^{-3}$
51. $\frac{dw}{dz} = 6ze^{2z}(1+z)$, $\frac{d^2w}{dz^2} = 6e^{2z}(1+4z+2z^2)$
53. (a) 13 (b) -7 (c) 7/25 (d) 20
55. (a) $y = -\frac{x}{8} + \frac{5}{4}$ (b) $m = -4$ at $(0, 1)$
 (c) $y = 8x - 15$, $y = 8x + 17$
57. $y = 4x$, $y = 2$ 59. $a = 1, b = 1, c = 0$ 61. $(2, 4)$

63. $(0, 0), (4, 2)$ 65. $y = -16x + 24$
67. (a) $y = 2x + 2$ (c) $(2, 6)$
69. 50 71. $a = -3$
73. $P'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1$
75. The Product Rule is then the Constant Multiple Rule, so the latter is a special case of the Product Rule.
77. (a) $\frac{d}{dx}(uvw) = uvw' + uv'w + u'vw$
 (b) $\frac{d}{dx}(u_1u_2u_3u_4) = u_1u_2u_3u_4' + u_1u_2u_3'u_4 + u_1u_2'u_3u_4 + u_1'u_2u_3u_4$
 (c) $\frac{d}{dx}(u_1 \cdots u_n) = u_1u_2 \cdots u_{n-1}u_n' + u_1u_2 \cdots u_{n-2}u_{n-1}'u_n + \cdots + u_1'u_2 \cdots u_n$
79. $\frac{dP}{dV} = -\frac{nRT}{(V-nb)^2} + \frac{2an^2}{V^3}$

SECTION 3.4, pp. 167–170

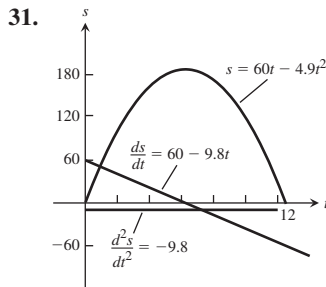
1. (a) -2 m, -1 m/s
 (b) 3 m/s, 1 m/s; 2 m/s², 2 m/s²
 (c) Changes direction at $t = 3/2$ s
3. (a) -9 m, -3 m/s
 (b) 3 m/s, 12 m/s; 6 m/s², -12 m/s²
 (c) No change in direction
5. (a) -20 m, -5 m/s
 (b) 45 m/s, $(1/5)$ m/s; 140 m/s², $(4/25)$ m/s²
 (c) No change in direction
7. (a) $a(1) = -6$ m/s², $a(3) = 6$ m/s²
 (b) $v(2) = 3$ m/s (c) 6 m
9. Mars: ≈ 7.5 s, Jupiter: ≈ 1.2 s
11. $g_s = 0.75$ m/s²
13. (a) $v = -9.8t$, $|v| = 9.8t$ m/s, $a = -9.8$ m/s²
 (b) $t \approx 3.4$ s (c) $v \approx -33.1$ m/s
15. (a) $t = 2, t = 7$ (b) $3 \leq t \leq 6$
 (c) (d)
- 
- 
17. (a) 57 m/s (b) 2 s (c) 8 s, 0 m/s
 (d) 10.8 s, 27 m/s (e) 2.8 s
 (f) Greatest acceleration happens 2 s after launch
 (g) Constant acceleration between 2 and 10.8 s, -10 m/s²
19. C = position, A = velocity, B = acceleration
21. (a) $\$110$ /machine
 (b) $\$80$
 (c) $\$79.90$
23. (a) $b'(0) = 10^4$ bacteria/h
 (b) $b'(5) = 0$ bacteria/h
 (c) $b'(10) = -10^4$ bacteria/h
25. (a) $\frac{dy}{dt} = \frac{t}{12} - 1$

- (b) The largest value of $\frac{dy}{dt}$ is 0 m/h when $t = 12$ and the smallest value of $\frac{dy}{dt}$ is -1 m/h when $t = 0$.

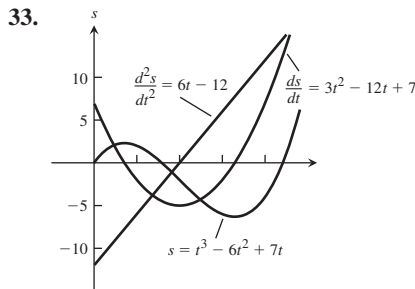


27. 0.846 m, 1.482 m, additional meters to stop car for 1 km/h speed increase

29. $t = 25$ s, $D = \frac{6250}{9}$ m



- (a) $v = 0$ when $t = 6.12$ s.
 (b) $v > 0$ when $0 \leq t < 6.12 \Rightarrow$ the object moves up; $v < 0$ when $6.12 < t \leq 12.24 \Rightarrow$ the object moves down.
 (c) The object changes direction at $t = 6.12$ s.
 (d) The object speeds up on $(6.12, 12.24]$ and slows down on $[0, 6.12)$.
 (e) The object is moving fastest at the endpoints $t = 0$ and $t = 12.24$ when it is traveling 60 m/s. It's moving slowest at $t = 6.12$ when the speed is 0.
 (f) When $t = 6.12$, the object is $s = 183.6$ m from the origin and farthest away.



- (a) $v = 0$ when $t = \frac{6 \pm \sqrt{15}}{3}$ s
 (b) $v < 0$ when $\frac{6 - \sqrt{15}}{3} < t < \frac{6 + \sqrt{15}}{3} \Rightarrow$ the object moves left; $v > 0$ when $0 \leq t < \frac{6 - \sqrt{15}}{3}$ or $\frac{6 + \sqrt{15}}{3} < t \leq 4 \Rightarrow$ the object moves right.
 (c) The object changes direction at $t = \frac{6 \pm \sqrt{15}}{3}$ s.

- (d) The object speeds up on $\left(\frac{6 - \sqrt{15}}{3}, 2\right) \cup \left(\frac{6 + \sqrt{15}}{3}, 4\right]$ and slows down on $\left[0, \frac{6 - \sqrt{15}}{3}\right) \cup \left(2, \frac{6 + \sqrt{15}}{3}\right)$.

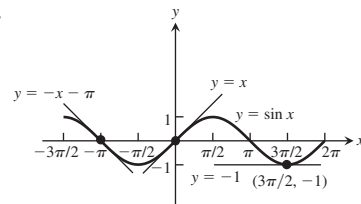
- (e) The object is moving fastest at $t = 0$ and $t = 4$ when it is moving 7 units/second and slowest at $t = \frac{6 \pm \sqrt{15}}{3}$ s.

- (f) When $t = \frac{6 + \sqrt{15}}{3}$ the object is at position $s \approx -6.303$ units and farthest from the origin.

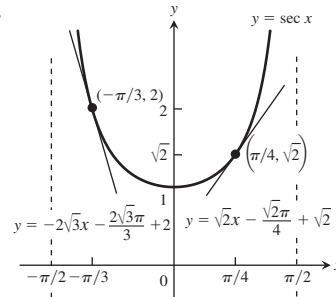
SECTION 3.5, pp. 174–176

1. $-10 - 3 \sin x$ 3. $2x \cos x - x^2 \sin x$
 5. $-\csc x \cot x - \frac{2}{\sqrt{x}} - \frac{7}{e^x}$ 7. $\sin x \sec^2 x + \sin x$
 9. $(e^{-x} \sec x)(1 - x + x \tan x)$ 11. $\frac{-\csc^2 x}{(1 + \cot x)^2}$
 13. $4 \tan x \sec x - \csc^2 x$ 15. 0
 17. $3x^2 \sin x \cos x + x^3 \cos^2 x - x^3 \sin^2 x$
 19. $\sec^2 t + e^{-t}$ 21. $\frac{-2 \csc t \cot t}{(1 - \csc t)^2}$ 23. $-\theta(\theta \cos \theta + 2 \sin \theta)$
 25. $\sec \theta \csc \theta (\tan \theta - \cot \theta) = \sec^2 \theta - \csc^2 \theta$
 27. $\sec^2 q$ 29. $\sec^2 q$
 31. $\frac{q^3 \cos q - q^2 \sin q - q \cos q - \sin q}{(q^2 - 1)^2}$
 33. (a) $2 \csc^3 x - \csc x$ (b) $2 \sec^3 x - \sec x$

35.



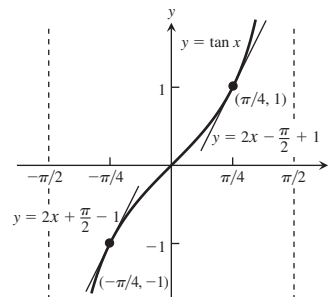
37.



39. Yes, at $x = \pi$ 41. No

43. Yes, at $x = 0, \pi$, and 2π

45. $\left(-\frac{\pi}{4}, -1\right); \left(\frac{\pi}{4}, 1\right)$



47. (a) $y = -x + \pi/2 + 2$ (b) $y = 4 - \sqrt{3}$
 49. $-\sqrt{2}$ m/s, $\sqrt{2}$ m/s, $\sqrt{2}$ m/s², $\sqrt{2}$ m/s³ 51. $c = 9$
 53. (a) $\sin x$ (b) $3 \cos x - \sin x$ (c) $73 \sin x + x \cos x$
 55. (a) i) 10 cm ii) 5 cm iii) $-5\sqrt{2} \approx -7.1$ cm
 (b) i) 0 cm/s ii) $-5\sqrt{3} \approx -8.7$ cm/s
 iii) $-5\sqrt{2} \approx -7.1$ cm/s

SECTION 3.6, pp. 181–184

1. $12x^3$ 3. $3 \cos(3x + 1)$ 5. $\frac{\cos x}{2\sqrt{\sin x}}$
 7. $2\pi x \sec^2(\pi x^2)$
 9. With $u = (2x + 1)$, $y = u^5$: $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = 5u^4 \cdot 2 = 10(2x + 1)^4$
 11. With $u = (1 - (x/7))$, $y = u^{-7}$: $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = -7u^{-8} \cdot \left(-\frac{1}{7}\right) = \left(1 - \frac{x}{7}\right)^{-8}$
 13. With $u = ((x^2/8) + x - (1/x))$, $y = u^4$: $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = 4u^3 \cdot \left(\frac{x}{4} + 1 + \frac{1}{x^2}\right) = 4\left(\frac{x^2}{8} + x - \frac{1}{x}\right)^3 \left(\frac{x}{4} + 1 + \frac{1}{x^2}\right)$
 15. With $u = \tan x$, $y = \sec u$: $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (\sec u \tan u)(\sec^2 x) = \sec(\tan x) \tan(\tan x) \sec^2 x$
 17. With $u = \tan x$, $y = u^3$: $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = 3u^2 \sec^2 x = 3 \tan^2 x (\sec^2 x)$
 19. $y = e^u$, $u = -5x$: $\frac{dy}{dx} = -5e^{-5x}$
 21. $y = e^u$, $u = 5 - 7x$: $\frac{dy}{dx} = -7e^{(5-7x)}$
 23. $-\frac{1}{2\sqrt{3-t}}$ 25. $\frac{4}{\pi}(\cos 3t - \sin 5t)$ 27. $\frac{\csc \theta}{\cot \theta + \csc \theta}$
 29. $2x \sin^4 x + 4x^2 \sin^3 x \cos x + \cos^{-2} x + 2x \cos^{-3} x \sin x$
 31. $(3x - 2)^5 - \frac{1}{x^3(4 - \frac{1}{2x^2})^2}$ 33. $\frac{(4x + 3)^3(4x + 7)}{(x + 1)^4}$
 35. $(1 - x)e^{-x} + 3x^2e^{x^3}$ 37. $\left(\frac{5}{2}x^2 - 3x + 3\right)e^{5x/2}$
 39. $\sqrt{x} \sec^2(2\sqrt{x}) + \tan(2\sqrt{x})$ 41. $\frac{x \sec x \tan x + \sec x}{2\sqrt{7} + x \sec x}$
 43. $\frac{2 \sin \theta}{(1 + \cos \theta)^2}$ 45. $-2 \sin(\theta^2) \sin 2\theta + 2\theta \cos(2\theta) \cos(\theta^2)$
 47. $\left(\frac{t+2}{2(t+1)^{3/2}}\right) \cos\left(\frac{t}{\sqrt{t+1}}\right)$ 49. $2\theta e^{-\theta^2} \sin(e^{-\theta^2})$
 51. $2\pi \sin(\pi t - 2) \cos(\pi t - 2)$ 53. $\frac{8 \sin(2t)}{(1 + \cos 2t)^5}$
 55. $10t^{10} \tan^9 t \sec^2 t + 10t^9 \tan^{10} t$
 57. $\frac{dy}{dt} = -2\pi \sin(\pi t - 1) \cdot \cos(\pi t - 1) \cdot e^{\cos^2(\pi t - 1)}$
 59. $\frac{-3t^6(t^2 + 4)}{(t^3 - 4t)^4}$ 61. $-2 \cos(\cos(2t - 5))(\sin(2t - 5))$
 63. $\left(1 + \tan^4\left(\frac{t}{12}\right)\right)^2 \left(\tan^3\left(\frac{t}{12}\right) \sec^2\left(\frac{t}{12}\right)\right)$
 65. $-\frac{t \sin(t^2)}{\sqrt{1 + \cos(t^2)}}$

67. $6 \tan(\sin^3 t) \sec^2(\sin^3 t) \sin^2 t \cos t$
 69. $3(2t^2 - 5)^3(18t^2 - 5)$ 71. $\frac{6}{x^3}\left(1 + \frac{1}{x}\right)\left(1 + \frac{2}{x}\right)$
 73. $2 \csc^2(3x - 1) \cot(3x - 1)$ 75. $16(2x + 1)^2(5x + 1)$
 77. $2(2x^2 + 1)e^{x^2}$
 79. $f'(x) = 0$ for $x = 1, 4$; $f''(x) = 0$ for $x = 2, 4$ 81. $5/2$
 83. $-\pi/4$ 85. 0 87. -5
 89. (a) $2/3$ (b) $2\pi + 5$ (c) $15 - 8\pi$ (d) $37/6$
 (e) -1 (f) $\sqrt{2}/24$ (g) $5/32$ (h) $-5/(3\sqrt{17})$
 91. 5 93. (a) 1 (b) 1
 95. $y = 1 - 4x$ 97. (a) $y = \pi x + 2 - \pi$ (b) $\pi/2$
 99. It multiplies the velocity, acceleration, and jerk by 2, 4, and 8, respectively.
 101. $v(6) = \frac{2}{5}$ m/s, $a(6) = -\frac{4}{125}$ m/s²

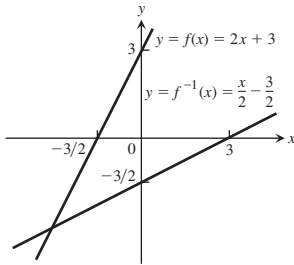
SECTION 3.7, pp. 187–189

1. $\frac{-2xy - y^2}{x^2 + 2xy}$ 3. $\frac{1 - 2y}{2x + 2y - 1}$
 5. $\frac{-2x^3 + 3x^2y - xy^2 + x}{x^2y - x^3 + y}$ 7. $\frac{1}{y(x + 1)^2}$
 9. $\cos y \cot y$ 11. $\frac{-\cos^2(xy) - y}{x}$
 13. $\frac{-y^2}{y \sin\left(\frac{1}{y}\right) - \cos\left(\frac{1}{y}\right) + xy}$ 15. $\frac{2e^{2x} - \cos(x + 3y)}{3 \cos(x + 3y)}$
 17. $-\frac{\sqrt{r}}{\sqrt{\theta}}$ 19. $\frac{-r}{\theta}$ 21. $y' = -\frac{x}{y}$, $y'' = \frac{-y^2 - x^2}{y^3}$
 23. $\frac{dy}{dx} = \frac{xe^{x^2} + 1}{y}$, $\frac{d^2y}{dx^2} = \frac{(2x^2y^2 + y^2 - 2x)e^{x^2} - x^2e^{2x^2} - 1}{y^3}$
 25. $y' = \frac{\sqrt{y}}{\sqrt{y} + 1}$, $y'' = \frac{1}{2(\sqrt{y} + 1)^3}$
 27. $y' = \frac{3x^2}{1 - \cos y}$, $y'' = \frac{6x(1 - \cos y)^2 - 9x^4 \sin y}{(1 - \cos y)^3}$
 29. -2 31. $(-2, 1)$: $m = -1$, $(-2, -1)$: $m = 1$
 33. (a) $y = \frac{7}{4}x - \frac{1}{2}$ (b) $y = -\frac{4}{7}x + \frac{29}{7}$
 35. (a) $y = 3x + 6$ (b) $y = -\frac{1}{3}x + \frac{8}{3}$
 37. (a) $y = \frac{6}{7}x + \frac{6}{7}$ (b) $y = -\frac{7}{6}x - \frac{7}{6}$
 39. (a) $y = -\frac{\pi}{2}x + \pi$ (b) $y = \frac{2}{\pi}x - \frac{2}{\pi} + \frac{\pi}{2}$
 41. (a) $y = 2\pi x - 2\pi$ (b) $y = -\frac{x}{2\pi} + \frac{1}{2\pi}$
 43. Points: $(-\sqrt{7}, 0)$ and $(\sqrt{7}, 0)$, Slope: -2
 45. $m = -1$ at $\left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)$, $m = \sqrt{3}$ at $\left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right)$
 47. $(-3, 2)$: $m = -\frac{27}{8}$; $(-3, -2)$: $m = \frac{27}{8}$; $(3, 2)$: $m = \frac{27}{8}$;
 $(3, -2)$: $m = -\frac{27}{8}$
 49. $(3, -1)$
 55. $\frac{dy}{dx} = -\frac{y^3 + 2xy}{x^2 + 3xy^2}$, $\frac{dx}{dy} = -\frac{x^2 + 3xy^2}{y^3 + 2xy}$, $\frac{dx}{dy} = \frac{1}{dy/dx}$

SECTION 3.8, pp. 198–200

1. (a) $f^{-1}(x) = \frac{x}{2} - \frac{3}{2}$

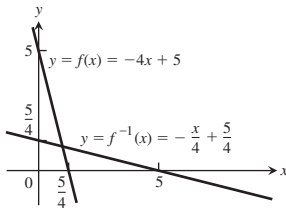
(b)



(c) 2, 1/2

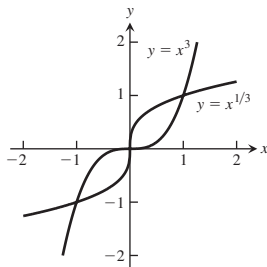
3. (a) $f^{-1}(x) = -\frac{x}{4} + \frac{5}{4}$

(b)



(c) -4, -1/4

5. (b)

(c) Slope of f at $(1, 1)$: 3; slope of g at $(1, 1)$: $1/3$; slope of f at $(-1, -1)$: 3; slope of g at $(-1, -1)$: $1/3$ (d) $y = 0$ is tangent to $y = x^3$ at $x = 0$; $x = 0$ is tangent to $y = \sqrt[3]{x}$ at $x = 0$.

7. $1/9$ 9. 3 11. (a) 2 (b) $1/3$ (c) 3

13. (a) 2 (b) 7 (c) $3/4$ 15. $\frac{1}{x} + 1$ 17. $2/t$

19. $-1/x$ 21. $\frac{1}{\theta + 1} - e^\theta$ 23. $3/x$

25. $2(\ln t) + (\ln t)^2$ 27. $x^3 \ln x$ 29. $\frac{1 - \ln t}{t^2}$

31. $\frac{1}{x(1 + \ln x)^2}$ 33. $\frac{1}{x \ln x}$ 35. $2 \cos(\ln \theta)$

37. $-\frac{3x + 2}{2x(x + 1)}$ 39. $\frac{2}{t(1 - \ln t)^2}$ 41. $\frac{\tan(\ln \theta)}{\theta}$

43. $\frac{10x}{x^2 + 1} + \frac{1}{2(1 - x)}$

45. $\left(\frac{1}{2}\right)\sqrt{x(x+1)}\left(\frac{1}{x} + \frac{1}{x+1}\right) = \frac{2x+1}{2\sqrt{x(x+1)}}$

47. $\left(\frac{1}{2}\right)\sqrt{\frac{t}{t+1}}\left(\frac{1}{t} - \frac{1}{t+1}\right) = \frac{1}{2\sqrt{t(t+1)^{3/2}}}$

49. $\sqrt{\theta + 3}(\sin \theta)\left(\frac{1}{2(\theta + 3)} + \cot \theta\right)$

51. $t(t+1)(t+2)\left[\frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2}\right] = 3t^2 + 6t + 2$

53. $\frac{\theta + 5}{\theta \cos \theta}\left[\frac{1}{\theta + 5} - \frac{1}{\theta} + \tan \theta\right]$

55. $\frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left[\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}\right]$

57. $\frac{1}{3}\sqrt[3]{\frac{x(x-2)}{x^2+1}}\left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right)$ 59. $-2 \tan \theta$

61. $\frac{1-t}{t}$

63. $1/(1 + e^\theta)$

65. $e^{\cos t}(1 - t \sin t)$

67. $\frac{ye^y \cos x}{1 - ye^y \sin x}$

69. $\frac{dy}{dx} = \frac{y^2 - xy \ln y}{x^2 - xy \ln x}$

71. $2^x \ln 2$

73. $\left(\frac{\ln 5}{2\sqrt{s}}\right)5^{\sqrt{s}}$

75. $\pi x^{(\pi-1)}$

77. $\frac{1}{\theta \ln 2}$

79. $\frac{3}{x \ln 4}$

81. $\frac{2(\ln r)}{r(\ln 2)(\ln 4)}$

83. $\frac{-2}{(x+1)(x-1)}$

85. $\sin(\log_7 \theta) + \frac{1}{\ln 7} \cos(\log_7 \theta)$

87. $\frac{1}{\ln 5}$

89. $\frac{1}{t}(\log_2 3)3^{\log_2 t}$

91. $\frac{1}{t}$

93. $(x+1)^x\left(\frac{x}{x+1} + \ln(x+1)\right)$

95. $(\sqrt{t})^t\left(\frac{\ln t}{2} + \frac{1}{2}\right)$

97. $(\sin x)^x(\ln \sin x + x \cot x)$

99. $(x^{\ln x})\left(\frac{\ln x^2}{x}\right)$

101. $\frac{3y - xy \ln y}{x^2 - x}$

103. $\frac{1 - xy \ln y}{x^2(1 + \ln y)}$

SECTION 3.9, pp. 204–206

1. (a) $\pi/4$ (b) $-\pi/3$ (c) $\pi/6$

3. (a) $-\pi/6$ (b) $\pi/4$ (c) $-\pi/3$

5. (a) $\pi/3$ (b) $3\pi/4$ (c) $\pi/6$

7. (a) $3\pi/4$ (b) $\pi/6$ (c) $2\pi/3$

9. $1/\sqrt{2}$ 11. $-1/\sqrt{3}$ 13. $\pi/2$ 15. $\pi/2$ 17. $\pi/2$

19. 0 21. $\frac{-2x}{\sqrt{1-x^4}}$ 23. $\frac{\sqrt{2}}{\sqrt{1-2t^2}}$ 25. $\frac{1}{[2s+1]\sqrt{s^2+s}}$

27. $\frac{-2x}{(x^2+1)\sqrt{x^4+2x^2}}$ 29. $\frac{-1}{\sqrt{1-t^2}}$ 31. $\frac{-1}{2\sqrt{t}(1+t)}$

33. $\frac{1}{(\tan^{-1} x)(1+x^2)}$ 35. $\frac{-e^t}{|e^t|\sqrt{(e^t)^2-1}} = \frac{-1}{\sqrt{e^{2t}-1}}$

37. $\frac{-2s^2}{\sqrt{1-s^2}}$ 39. 0 41. $\sin^{-1} x$

43. $\frac{1}{2\sqrt{\arcsin x}\sqrt{1-x^2}}$ 45. $-\sin(x - \arccos x)\left(1 + \frac{1}{\sqrt{1-x^2}}\right)$

47. $\frac{-9x^2}{x^6+1}(\operatorname{arccot}(x^3))^2$ 49. 0 51. $\frac{8\sqrt{2}}{4+3\pi}$

57. (a) Defined; there is an angle whose tangent is 2.

(b) Not defined; there is no angle whose cosine is 2.

59. (a) Not defined; no angle has secant 0.

(b) Not defined; no angle has sine $\sqrt{2}$.69. (a) Domain: all real numbers except those having the form $\frac{\pi}{2} + k\pi$ where k is an integer; range: $-\pi/2 < y < \pi/2$ (b) Domain: $-\infty < x < \infty$; range: $-\infty < y < \infty$ 71. (a) Domain: $-\infty < x < \infty$; range: $0 \leq y \leq \pi$ (b) Domain: $-1 \leq x \leq 1$; range: $-1 \leq y \leq 1$

73. The graphs are identical.

SECTION 3.10, pp. 211–214

1. $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ 3. 10 5. -6 7. $-3/2$ 9. $31/13$
11. (a) $-180 \text{ m}^2/\text{min}$ (b) $-135 \text{ m}^3/\text{min}$
13. (a) $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$ (b) $\frac{dV}{dt} = 2\pi hr \frac{dr}{dt}$
 (c) $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi hr \frac{dr}{dt}$
15. (a) 1 volt/s (b) $-\frac{1}{3}$ amp/s (c) $\frac{dR}{dt} = \frac{1}{I} \left(\frac{dV}{dt} - \frac{V}{I} \frac{dI}{dt} \right)$
 (d) $3/2$ ohms/s, R is increasing.
17. (a) $\frac{ds}{dt} = \frac{x}{\sqrt{x^2 + y^2}} \frac{dx}{dt}$
 (b) $\frac{ds}{dt} = \frac{x}{\sqrt{x^2 + y^2}} \frac{dx}{dt} + \frac{y}{\sqrt{x^2 + y^2}} \frac{dy}{dt}$
 (c) $\frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt}$
19. (a) $\frac{dA}{dt} = \frac{1}{2} ab \cos \theta \frac{d\theta}{dt}$
 (b) $\frac{dA}{dt} = \frac{1}{2} ab \cos \theta \frac{d\theta}{dt} + \frac{1}{2} b \sin \theta \frac{da}{dt}$
 (c) $\frac{dA}{dt} = \frac{1}{2} ab \cos \theta \frac{d\theta}{dt} + \frac{1}{2} b \sin \theta \frac{da}{dt} + \frac{1}{2} a \sin \theta \frac{db}{dt}$
21. (a) $14 \text{ cm}^2/\text{s}$, increasing (b) 0 cm/s , constant
 (c) $-14/13 \text{ cm/s}$, decreasing
23. (a) -3.6 m/s (b) $-5.355 \text{ m}^2/\text{s}$ (c) -1 rad/s
25. 6 m/s
27. (a) $\frac{dh}{dt} = \frac{1125}{32\pi} \approx 11.19 \text{ cm/min}$
 (b) $\frac{dr}{dt} = \frac{375}{8\pi} \approx 14.92 \text{ cm/min}$
29. (a) $-\frac{1}{24\pi} \text{ m/min}$ (b) $r = \sqrt{26y - y^2} \text{ m}$
 (c) $\frac{dr}{dt} = -\frac{5}{288\pi} \text{ m/min}$
31. 1 m/min , $40\pi \text{ m}^2/\text{min}$ 33. 3.16 m/s
35. Increasing at $466/1681 \text{ L/min}^2$ 37. -5 m/s
39. -441 m/s
41. $\frac{5}{72\pi} \text{ cm/min}$, $\frac{10}{3} \text{ cm}^2/\text{min}$
43. (a) $-10/\sqrt{13} \approx -2.774 \text{ m/s}$
 (b) $d\theta_1/dt = 5/39 \text{ rad/s}$, $d\theta_2/dt = -5/39 \text{ rad/s}$
 (c) $d\theta_1/dt = 1/6 \text{ rad/s}$, $d\theta_2/dt = -1/6 \text{ rad/s}$
45. -5.5 deg/min
47. $12\pi \text{ km/min}$

SECTION 3.11, pp. 224–226

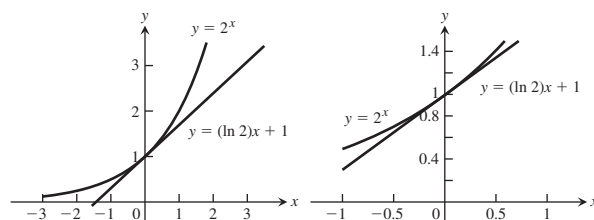
1. $L(x) = 10x - 13$ 3. $L(x) = 2$ 5. $L(x) = x - \pi$
 7. $2x$ 9. $-x - 5$ 11. $\frac{1}{12}x + \frac{4}{3}$ 13. $1 - x$
15. $f(0) = 1$. Also, $f'(x) = k(1+x)^{k-1}$, so $f'(0) = k$. This means the linearization at $x = 0$ is $L(x) = 1 + kx$.

17. (a) 1.01 (b) 1.003

19. $\left(3x^2 - \frac{3}{2\sqrt{x}}\right) dx$ 21. $\frac{2-2x^2}{(1+x^2)^2} dx$
23. $\frac{1-y}{3\sqrt{y+x}} dx$ 25. $\frac{5}{2\sqrt{x}} \cos(5\sqrt{x}) dx$
27. $(4x^2) \sec^2\left(\frac{x^3}{3}\right) dx$
29. $\frac{3}{\sqrt{x}} (\csc(1-2\sqrt{x}) \cot(1-2\sqrt{x})) dx$
31. $\frac{1}{2\sqrt{x}} \cdot e^{\sqrt{x}} dx$ 33. $\frac{2x}{1+x^2} dx$ 35. $\frac{2xe^{x^2}}{1+e^{2x^2}} dx$
37. $\frac{-1}{\sqrt{e^{-2x}-1}} dx$
39. (a) 0.41 (b) 0.4 (c) 0.01
41. (a) 0.231 (b) 0.2 (c) 0.031
43. (a) $-1/3$ (b) $-2/5$ (c) $1/15$
45. $dV = 4\pi r_0^2 dr$ 47. $dS = 12x_0 dx$
49. $dV = 2\pi r_0 h dr$
51. (a) $0.08\pi \text{ m}^2$ (b) 2%
53. $dV \approx 565.5 \text{ cm}^3$ 55. (a) 2% (b) 4%
57. $\frac{1}{3}\%$ 59. 3%
61. The ratio equals 37.52, so a change in the acceleration of gravity on the moon has about 38 times the effect that a change of the same magnitude has on Earth.
63. Increase $V \approx 40\%$
65. (a) i) $b_0 = f(a)$ ii) $b_1 = f'(a)$ iii) $b_2 = \frac{f''(a)}{2}$
 (b) $Q(x) = 1 + x + x^2$
 (d) $Q(x) = 1 - (x-1) + (x-1)^2$
 (e) $Q(x) = 1 + \frac{x}{2} - \frac{x^2}{8}$
 (f) The linearization of any differentiable function $u(x)$ at $x = a$ is $L(x) = u(a) + u'(a)(x-a) = b_0 + b_1(x-a)$, where b_0 and b_1 are the coefficients of the constant and linear terms of the quadratic approximation. Thus, the linearization for $f(x)$ at $x = 0$ is $1 + x$; the linearization for $g(x)$ at $x = 1$ is $1 - (x-1)$ or $2 - x$; and the linearization for $h(x)$ at $x = 0$ is $1 + \frac{x}{2}$.

67. (a) $L(x) = x \ln 2 + 1 \approx 0.69x + 1$

(b)



PRACTICE EXERCISES, pp. 227–231

1. $5x^4 - 0.25x + 0.25$ 3. $3x(x-2)$
5. $2(x+1)(2x^2+4x+1)$
7. $3(\theta^2 + \sec \theta + 1)^2(2\theta + \sec \theta \tan \theta)$
9. $\frac{1}{2\sqrt{t}(1+\sqrt{t})^2}$ 11. $2\sec^2 x \tan x$

13. $8 \cos^3(1 - 2t) \sin(1 - 2t)$ 15. $5(\sec t)(\sec t + \tan t)^5$

17. $\frac{\theta \cos \theta + \sin \theta}{\sqrt{2\theta \sin \theta}}$ 19. $\frac{\cos \sqrt{2\theta}}{\sqrt{2\theta}}$

21. $x \csc\left(\frac{2}{x}\right) + \csc\left(\frac{2}{x}\right) \cot\left(\frac{2}{x}\right)$

23. $\frac{1}{2}x^{1/2} \sec(2x)^2 [16 \tan(2x)^2 - x^{-2}]$ 25. $-10x \csc^2(x^2)$

27. $8x^3 \sin(2x^2) \cos(2x^2) + 2x \sin^2(2x^2)$ 29. $\frac{-(t+1)}{8t^3}$

31. $\frac{1-x}{(x+1)^3}$ 33. $\frac{-1}{2x^2\left(1+\frac{1}{x}\right)^{1/2}}$ 35. $\frac{-2 \sin \theta}{(\cos \theta - 1)^2}$

37. $3\sqrt{2x+1}$ 39. $-9\left[\frac{5x + \cos 2x}{(5x^2 + \sin 2x)^{5/2}}\right]$ 41. $-2e^{-x/5}$

43. xe^{4x}

45. $\frac{2 \sin \theta \cos \theta}{\sin^2 \theta} = 2 \cot \theta$ 47. $\frac{2}{(\ln 2)x}$ 49. $-8^{-t}(\ln 8)$

51. $18x^{2.6}$ 53. $(x+2)^{x+2}[\ln(x+2) + 1]$ 55. $-\frac{1}{\sqrt{1-u^2}}$

57. $\frac{-1}{\sqrt{1-x^2} \arccos x}$ 59. $\tan^{-1}(t) + \frac{t}{1+t^2} - \frac{1}{2t}$

61. $\frac{1-z}{\sqrt{z^2-1}} + \operatorname{arcsec} z$ 63. -1 65. $-\frac{y+2}{x+3}$

67. $\frac{-3x^2 - 4y + 2}{4x - 4y^{1/3}}$ 69. $-\frac{y}{x}$ 71. $\frac{1}{2y(x+1)^2}$

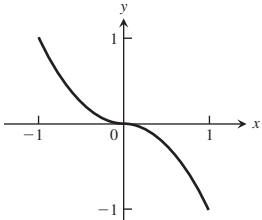
73. $-1/2$ 75. y/x 77. $-\frac{2e^{-\arctan x}}{1+x^2}$

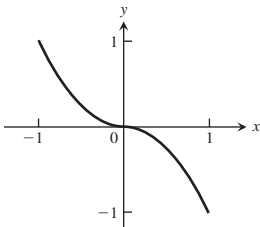
79. $\frac{dp}{dq} = \frac{6q-4p}{3p^2+4q}$ 81. $\frac{dr}{ds} = (2r-1)(\tan 2s)$

83. (a) $\frac{d^2y}{dx^2} = \frac{-2xy^3 - 2x^4}{y^5}$ (b) $\frac{d^2y}{dx^2} = \frac{-2xy^2 - 1}{x^4y^3}$

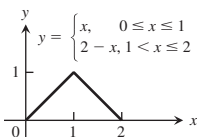
85. (a) 7 (b) -2 (c) 5/12 (d) 1/4 (e) 12 (f) 9/12 (g) 3/4

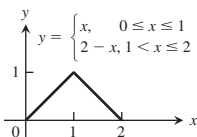
87. 0 89. $\frac{3\sqrt{2}e^{\sqrt{3/2}}}{4} \cos(e^{\sqrt{3/2}})$ 91. $-\frac{1}{2}$ 93. $\frac{-2}{(2t+1)^2}$

95. (a)  (b) Yes (c) Yes



$$f(x) = \begin{cases} x^2, & -1 \leq x < 0 \\ -x^2, & 0 \leq x < 1 \end{cases}$$

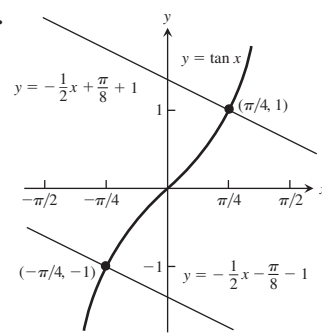
97. (a)  (b) Yes (c) No



99. $\left(\frac{5}{2}, \frac{9}{4}\right)$ and $\left(\frac{3}{2}, -\frac{1}{4}\right)$ 101. $(-1, 27)$ and $(2, 0)$

103. (a) $(-2, 16), (3, 11)$ (b) $(0, 20), (1, 7)$

105.



107. $\frac{1}{4}$ 109. 4

111. Tangent: $y = -\frac{1}{4}x + \frac{9}{4}$, normal: $y = 4x - 2$

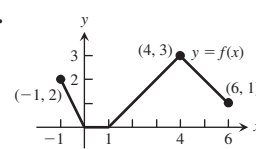
113. Tangent: $y = 2x - 4$, normal: $y = -\frac{1}{2}x + \frac{7}{2}$

115. Tangent: $y = -\frac{5}{4}x + 6$, normal: $y = \frac{4}{5}x - \frac{11}{5}$

117. $(1, 1): m = -\frac{1}{2}; (1, -1): m$ not defined

119. $B = \text{graph of } f, A = \text{graph of } f'$

121.



123. $\frac{2(x^2+1)}{\sqrt{\cos 2x}} \left[\frac{2x}{x^2+1} + \tan 2x \right]$

125. $5 \left[\frac{(t+1)(t-1)}{(t-2)(t+3)} \right]^5 \left[\frac{1}{t+1} + \frac{1}{t-1} - \frac{1}{t-2} - \frac{1}{t+3} \right]$

127. $\frac{1}{\sqrt{\theta}} (\sin \theta)^{\sqrt{\theta}} \left(\frac{\ln \sin \theta}{2} + \theta \cot \theta \right)$

129. (a) $\frac{dS}{dt} = (4\pi r + 2\pi h) \frac{dr}{dt}$ (b) $\frac{dS}{dt} = 2\pi r \frac{dh}{dt}$

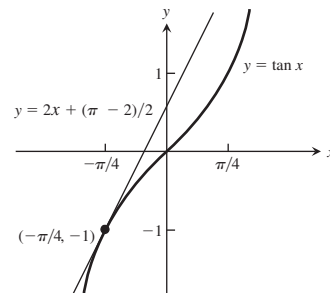
(c) $\frac{dS}{dt} = (4\pi r + 2\pi h) \frac{dr}{dt} + 2\pi r \frac{dh}{dt}$ (d) $\frac{dr}{dt} = -\frac{r}{2r+h} \frac{dh}{dt}$

131. $-40 \text{ m}^2/\text{s}$ 133. 0.02 ohm/s 135. 2 m/s

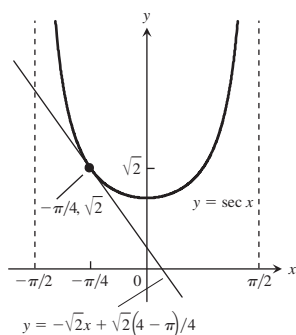
137. (a) $r = \frac{2}{5}h$ (b) $-\frac{5}{16\pi} \text{ m/min}$

139. (a) $\frac{3}{5} \text{ km/s}$ or 600 m/s (b) $\frac{18}{\pi} \text{ rpm}$

141. (a) $L(x) = 2x + \frac{\pi-2}{2}$



(b) $L(x) = -\sqrt{2}x + \frac{\sqrt{2}(4 - \pi)}{4}$



143. $L(x) = 1.5x + 0.5$ 145. $dS = \frac{\pi r h_0}{\sqrt{r^2 + h_0^2}} dh$

147. (a) 4% (b) 8% (c) 12%

ADDITIONAL AND ADVANCED EXERCISES, pp. 231–234

1. (a) $\sin 2\theta = 2 \sin \theta \cos \theta$; $2 \cos 2\theta = 2 \sin \theta(-\sin \theta) + \cos \theta(2 \cos \theta)$; $2 \cos 2\theta = -2 \sin^2 \theta + 2 \cos^2 \theta$; $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$

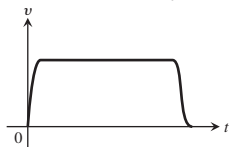
(b) $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$; $-2 \sin 2\theta = 2 \cos \theta(-\sin \theta) - 2 \sin \theta(\cos \theta)$; $\sin 2\theta = \cos \theta \sin \theta + \sin \theta \cos \theta$; $\sin 2\theta = 2 \sin \theta \cos \theta$

3. (a) $a = 1, b = 0, c = -\frac{1}{2}$ (b) $b = \cos a, c = \sin a$

5. $h = -4, k = \frac{9}{2}, a = \frac{5\sqrt{5}}{2}$

7. (a) 0.09y (b) Increasing at 1% per year

9. Answers will vary. Here is one possibility.



11. (a) 2 s, 19.6 m/s (b) 12.25 s, 120 m

15. $y' = \frac{x^{y-1}y^2 - ye^x(x^y + 1)\ln y}{e^x(x^y + 1) - x^y y \ln x}$

17. (a) $m = -\frac{b}{\pi}$ (b) $m = -1, b = \pi$

19. (a) $a = \frac{3}{4}, b = \frac{9}{4}$ 21. f odd $\Rightarrow f'$ is even

25. h' is defined but not continuous at $x = 0$; k' is defined and continuous at $x = 0$.

27. $\frac{-7}{75}$ rad/s

31. (a) 0.248 m (b) 0.02015 s
(c) It will lose about 28.3 min/day.

Chapter 4

SECTION 4.1, pp. 241–243

1. Absolute minimum at $x = c_2$; absolute maximum at $x = b$

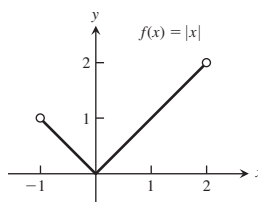
3. Absolute maximum at $x = c$; no absolute minimum

5. Absolute minimum at $x = a$; absolute maximum at $x = c$

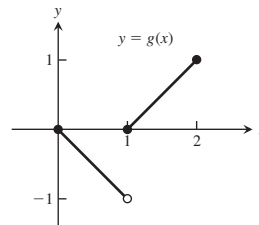
7. No absolute minimum; no absolute maximum

9. Absolute maximum at $(0, 5)$ 11. (c) 13. (d)

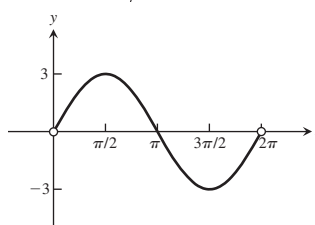
15. Absolute minimum at $x = 0$; no absolute maximum



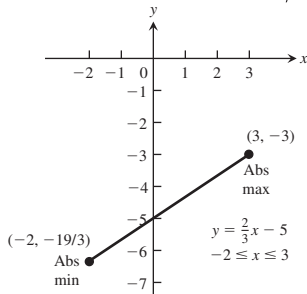
17. Absolute maximum at $x = 2$; no absolute minimum



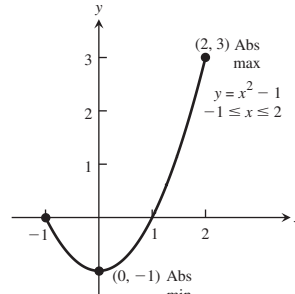
19. Absolute maximum at $x = \pi/2$; absolute minimum at $x = 3\pi/2$



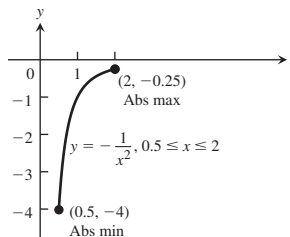
21. Absolute maximum: -3 ; absolute minimum: $-19/3$



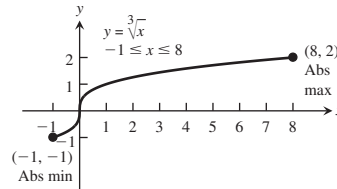
23. Absolute maximum: 3; absolute minimum: -1



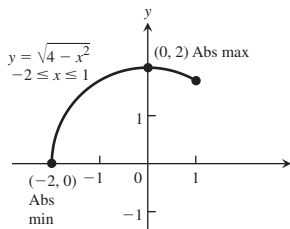
25. Absolute maximum: -0.25 ; absolute minimum: -4



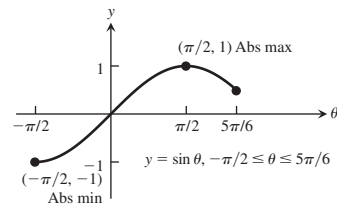
27. Absolute maximum: 2; absolute minimum: -1



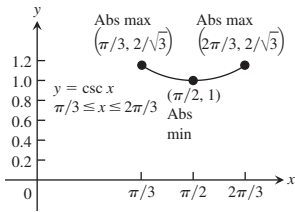
29. Absolute maximum: 2; absolute minimum: 0



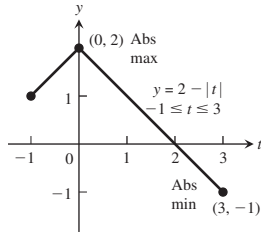
31. Absolute maximum: 1; absolute minimum: -1



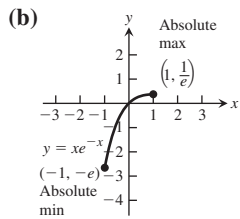
33. Absolute maximum: $2/\sqrt{3}$; absolute minimum: 1



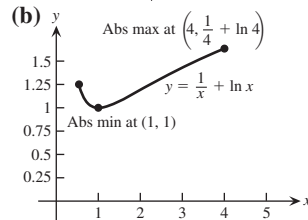
35. Absolute maximum: 2; absolute minimum: -1



37. (a) Absolute maximum is $1/e$ at $x = 1$; absolute minimum is $-e$ at $x = -1$.



39. (a) Absolute maximum value is $(1/4) + \ln 4$ at $x = 4$; absolute minimum value is 1 at $x = 1$; local maximum at $(1/2, 2 - \ln 2)$.



41. Increasing on $(0, 8)$, decreasing on $(-1, 0)$; absolute maximum: 16 at $x = 8$; absolute minimum: 0 at $x = 0$

43. Increasing on $(-32, 1)$; absolute maximum: 1 at $\theta = 1$; absolute minimum: -8 at $\theta = -32$

45. $x = 3$

47. $x = 1, x = 4$

49. $x = 1$

51. $x = 0, x = 4$

53. $x = 0, x = 1$

55. $x = 2, x = -4$

57. (a) No

- (b) The derivative is defined and nonzero for $x \neq 2$. Also, $f(2) = 0$ and $f(x) > 0$ for all $x \neq 2$.

- (c) No, because $(-\infty, \infty)$ is not a closed interval.

- (d) The answers are the same as parts (a) and (b), with 2 replaced by a .

59. y is increasing on $(-\infty, \infty)$ so has no extrema.

61. y is decreasing on $(-\infty, \infty)$ so has no extrema.

63. Yes

65. g assumes a local maximum at $-c$.

67. (a) Maximum value is 144 at $x = 2$.

- (b) The largest volume of the box is 144 cubic units, and it occurs when $x = 2$.

69. $\frac{v_0^2}{2g} + s_0$

71. Maximum value is 11 at $x = 5$; minimum value is 5 on the interval $[-3, 2]$; local maximum at $(-5, 9)$.

73. Maximum value is 5 on the interval $[3, \infty)$; minimum value is -5 on the interval $(-\infty, -2]$.

SECTION 4.2, pp. 248–249

1. $1/2$ 3. 1 5. $\pm\sqrt{1 - \frac{4}{\pi^2}} \approx \pm 0.771$

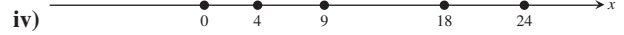
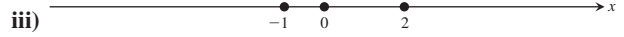
7. $\frac{1}{3}(1 + \sqrt{7}) \approx 1.22, \frac{1}{3}(1 - \sqrt{7}) \approx -0.549$

9. Does not; f is not differentiable at the interior domain point $x = 0$.

11. Does

13. Does not; f is not differentiable at $x = -1$.

17. (a)



29. Yes 31. (a) 4 (b) 3 (c) 3

33. (a) $\frac{x^2}{2} + C$ (b) $\frac{x^3}{3} + C$ (c) $\frac{x^4}{4} + C$

35. (a) $\frac{1}{x} + C$ (b) $x + \frac{1}{x} + C$ (c) $5x - \frac{1}{x} + C$

37. (a) $-\frac{1}{2}\cos 2t + C$ (b) $2\sin \frac{t}{2} + C$

- (c) $-\frac{1}{2}\cos 2t + 2\sin \frac{t}{2} + C$

39. $f(x) = x^2 - x$ 41. $f(x) = 1 + \frac{e^{2x}}{2}$

43. $s = 4.9t^2 + 5t + 10$ 45. $s = \frac{1 - \cos(\pi t)}{\pi}$

47. $s = e^t + 19t + 4$ 49. $s = \sin(2t) - 3$

51. If $T(t)$ is the temperature of the thermometer at time t , then $T(0) = -19^\circ\text{C}$ and $T(14) = 100^\circ\text{C}$. From the Mean Value Theorem, there exists a $0 < t_0 < 14$ such that

$$\frac{T(14) - T(0)}{14 - 0} = 8.5^\circ\text{C/s} = T'(t_0), \text{ the rate at which the}$$

temperature was changing at $t = t_0$ as measured by the rising mercury on the thermometer.

53. Because its average speed was approximately 7.667 knots, and by the Mean Value Theorem, it must have been going that speed at least once during the trip.

57. The conclusion of the Mean Value Theorem yields

$$\frac{\frac{1}{b} - \frac{1}{a}}{\frac{b}{a} - 1} = -\frac{1}{c^2} \Rightarrow c^2 \left(\frac{a-b}{ab} \right) = a-b \Rightarrow c = \sqrt{ab}.$$

61. $f(x)$ must be zero at least once between a and b by the Intermediate Value Theorem. Now suppose that $f(x)$ is zero twice between a and b . Then, by the Mean Value Theorem, $f'(x)$ would have to be zero at least once between the two zeros of $f(x)$, but this can't be true since we are given that $f'(x) \neq 0$ on this interval. Therefore, $f(x)$ is zero once and only once between a and b .

71. $1.09999 \leq f(0.1) \leq 1.1$

SECTION 4.3, pp. 253–255

1. (a) 0, 1

- (b) Increasing on $(-\infty, 0)$ and $(1, \infty)$; decreasing on $(0, 1)$

- (c) Local maximum at $x = 0$; local minimum at $x = 1$

3. (a) -2, 1

- (b) Increasing on $(-2, 1)$ and $(1, \infty)$; decreasing on $(-\infty, -2)$

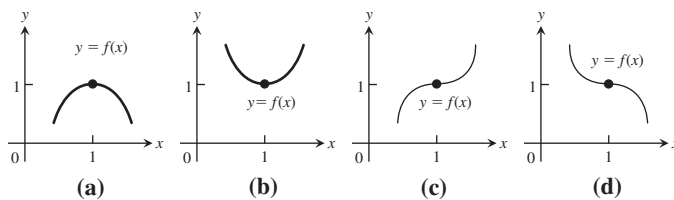
- (c) No local maximum; local minimum at $x = -2$

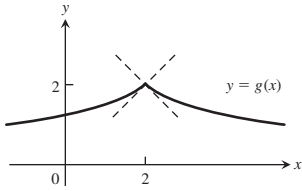
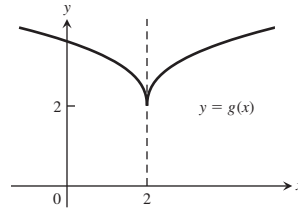
5. (a) Critical point at $x = 1$

- (b) Decreasing on $(-\infty, 1)$, increasing on $(1, \infty)$

- (c) Local (and absolute) minimum at $x = 1$

7. (a) 0, 1
 (b) Increasing on $(-\infty, -2)$ and $(1, \infty)$; decreasing on $(-2, 0)$ and $(0, 1)$
 (c) Local minimum at $x = 1$
9. (a) $-2, 2$
 (b) Increasing on $(-\infty, -2)$ and $(2, \infty)$; decreasing on $(-2, 0)$ and $(0, 2)$
 (c) Local maximum at $x = -2$; local minimum at $x = 2$
11. (a) $-2, 0$
 (b) Increasing on $(-\infty, -2)$ and $(0, \infty)$; decreasing on $(-2, 0)$
 (c) Local maximum at $x = -2$; local minimum at $x = 0$
13. (a) $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}$
 (b) Increasing on $(\frac{2\pi}{3}, \frac{4\pi}{3})$; decreasing on $(0, \frac{\pi}{2})$, $(\frac{\pi}{2}, \frac{2\pi}{3})$, and $(\frac{4\pi}{3}, 2\pi)$
 (c) Local maximum at $x = 0$ and $x = \frac{4\pi}{3}$; local minimum at $x = \frac{2\pi}{3}$ and $x = 2\pi$
15. (a) Increasing on $(-2, 0)$ and $(2, 4)$; decreasing on $(-4, -2)$ and $(0, 2)$
 (b) Absolute maximum at $(-4, 2)$; local maximum at $(0, 1)$ and $(4, -1)$; absolute minimum at $(2, -3)$; local minimum at $(-2, 0)$
17. (a) Increasing on $(-4, -1)$, $(1/2, 2)$, and $(2, 4)$; decreasing on $(-1, 1/2)$
 (b) Absolute maximum at $(4, 3)$; local maximum at $(-1, 2)$ and $(2, 1)$; no absolute minimum; local minimum at $(-4, -1)$ and $(1/2, -1)$
19. (a) Increasing on $(-\infty, -1.5)$; decreasing on $(-1.5, \infty)$
 (b) Local maximum: 5.25 at $t = -1.5$
21. (a) Decreasing on $(-\infty, 0)$; increasing on $(0, 4/3)$; decreasing on $(4/3, \infty)$
 (b) Local minimum at $x = 0(0, 0)$; local maximum at $x = 4/3(4/3, 32/27)$
23. (a) Decreasing on $(-\infty, 0)$; increasing on $(0, 1/2)$; decreasing on $(1/2, \infty)$
 (b) Local minimum at $\theta = 0(0, 0)$; local maximum at $\theta = 1/2(1/2, 1/4)$
25. (a) Increasing on $(-\infty, \infty)$; never decreasing
 (b) No local extrema
27. (a) Increasing on $(-2, 0)$ and $(2, \infty)$; decreasing on $(-\infty, -2)$ and $(0, 2)$
 (b) Local maximum: 16 at $x = 0$; local minimum: 0 at $x = \pm 2$
29. (a) Increasing on $(-\infty, -1)$; decreasing on $(-1, 0)$; increasing on $(0, 1)$; decreasing on $(1, \infty)$
 (b) Local maximum: 0.5 at $x = \pm 1$; local minimum: 0 at $x = 0$
31. (a) Increasing on $(10, \infty)$; decreasing on $(1, 10)$
 (b) Local maximum: 1 at $x = 1$; local minimum: -8 at $x = 10$
33. (a) Decreasing on $(-2\sqrt{2}, -2)$; increasing on $(-2, 2)$; decreasing on $(2, 2\sqrt{2})$
 (b) Local minima: $g(-2) = -4$, $g(2\sqrt{2}) = 0$; local maxima: $g(-2\sqrt{2}) = 0$, $g(2) = 4$
35. (a) Increasing on $(-\infty, 1)$; decreasing when $1 < x < 2$; decreasing when $2 < x < 3$; discontinuous at $x = 2$; increasing on $(3, \infty)$
 (b) Local minimum at $x = 3(3, 6)$; local maximum at $x = 1(1, 2)$
37. (a) Increasing on $(-2, 0)$ and $(0, \infty)$; decreasing on $(-\infty, -2)$
 (b) Local minimum: $-6\sqrt{2}$ at $x = -2$
39. (a) Increasing on $(-\infty, -2/\sqrt{7})$ and $(2/\sqrt{7}, \infty)$; decreasing on $(-2/\sqrt{7}, 0)$ and $(0, 2/\sqrt{7})$
 (b) Local maximum: $24\sqrt[3]{2}/7^{7/6} \approx 3.12$ at $x = -2/\sqrt{7}$; local minimum: $-24\sqrt[3]{2}/7^{7/6} \approx -3.12$ at $x = 2/\sqrt{7}$
41. (a) Increasing on $((1/3)\ln(1/2), \infty)$, decreasing on $(-\infty, (1/3)\ln(1/2))$
 (b) Local minimum is $\frac{3}{2^{2/3}}$ at $x = (1/3)\ln(1/2)$; no local maximum
43. (a) Increasing on (e^{-1}, ∞) ; decreasing on $(0, e^{-1})$
 (b) A local minimum is $-e^{-1}$ at $x = e^{-1}$; no local maximum
45. (a) Increasing on $(0, e^{-2})$ and $(1, \infty)$; decreasing on $(e^{-2}, 1)$
 (b) Local maximum is $4/e^2$ at $x = e^{-2}$
47. (a) Local maximum: 1 at $x = 1$; local minimum: 0 at $x = 2$
 (b) Absolute maximum: 1 at $x = 1$; no absolute minimum
49. (a) Local maximum: 1 at $x = 1$; local minimum: 0 at $x = 2$
 (b) No absolute maximum; absolute minimum: 0 at $x = 2$
51. (a) Local maxima: -9 at $t = -3$ and 16 at $t = 2$; local minimum: -16 at $t = -2$
 (b) Absolute maximum: 16 at $t = 2$; no absolute minimum
53. (a) Local minimum: 0 at $x = 0$
 (b) No absolute maximum; absolute minimum: 0 at $x = 0$
55. (a) Local maximum: 5 at $x = 0$; local minimum: 0 at $x = -5$ and $x = 5$
 (b) Absolute maximum: 5 at $x = 0$; absolute minimum: 0 at $x = -5$ and $x = 5$
57. (a) Local maximum: 2 at $x = 0$;
 local minimum: $\frac{\sqrt{3}}{4\sqrt{3}-6}$ at $x = 2 - \sqrt{3}$
 (b) No absolute maximum; absolute minimum at $x = 2 - \sqrt{3}$
59. (a) Local maximum: 1 at $x = \pi/4$;
 local maximum: 0 at $x = \pi$;
 local minimum: 0 at $x = 0$;
 local minimum: -1 at $x = 3\pi/4$
61. (a) Local maximum: 2 at $x = \pi/6$;
 local maximum: $\sqrt{3}$ at $x = 2\pi$;
 local minimum: -2 at $x = 7\pi/6$;
 local minimum: $\sqrt{3}$ at $x = 0$
63. (a) Local minimum: $(\pi/3) - \sqrt{3}$ at $x = 2\pi/3$;
 local maximum: 0 at $x = 0$;
 local maximum: π at $x = 2\pi$
65. (a) Local minimum: 0 at $x = \pi/4$
67. Local minimum at $x = 1$; no local maximum
69. (a) Increasing on $(-1, 2)$;
 decreasing on $(-2, -1)$
 (b) The sign of f cannot be determined solely from the graph of f' .
71. Local maximum: 3 at $\theta = 0$;
 local minimum: -3 at $\theta = 2\pi$
- 73.



75. (a)**(b)****79.** $a = -2, b = 4$

81. (a) Absolute minimum occurs at $x = \pi/3$ with $f(\pi/3) = -\ln 2$, and the absolute maximum occurs at $x = 0$ with $f(0) = 0$.

(b) Absolute minimum occurs at $x = 1/2$ and $x = 2$ with $f(1/2) = f(2) = \cos(\ln 2)$, and the absolute maximum occurs at $x = 1$ with $f(1) = 1$.

83. Minimum of $2 - 2 \ln 2 \approx 0.613706$ at $x = \ln 2$; maximum of 1 at $x = 0$

85. Absolute maximum value of $1/2e$ assumed at $x = 1/\sqrt{e}$

89. Increasing; $\frac{df^{-1}}{dx} = \frac{1}{9}x^{-2/3}$

91. Decreasing; $\frac{df^{-1}}{dx} = -\frac{1}{3}x^{-2/3}$

SECTION 4.4, pp. 264–268

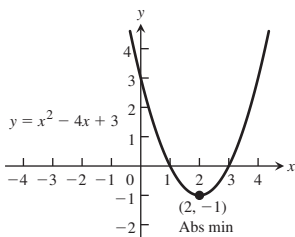
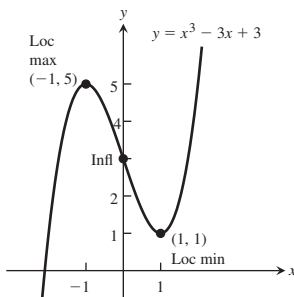
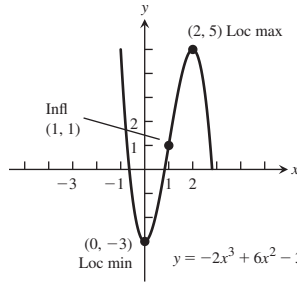
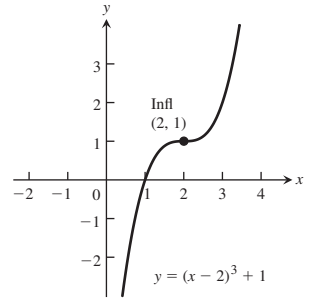
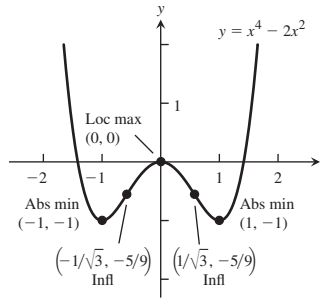
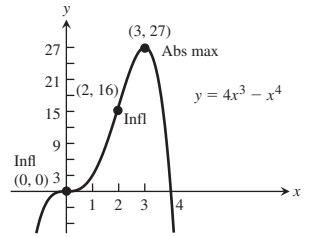
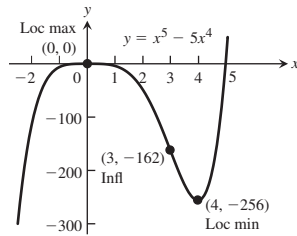
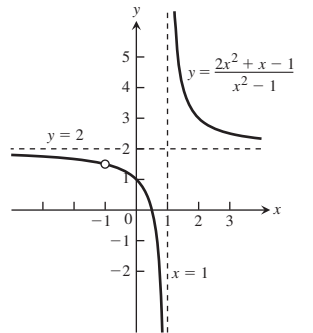
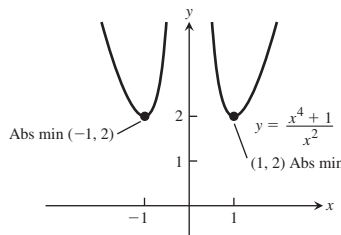
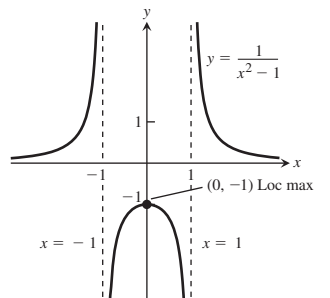
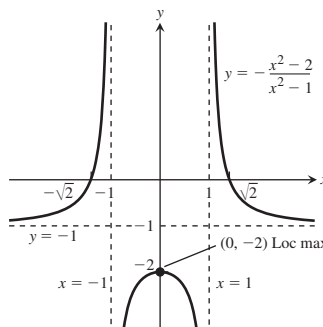
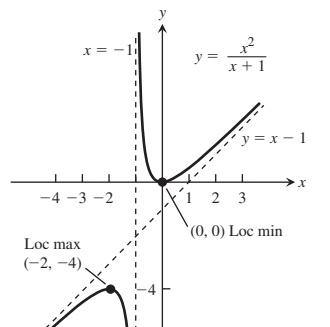
1. Local maximum: $3/2$ at $x = -1$; local minimum: -3 at $x = 2$; point of inflection at $(1/2, -3/4)$; rising on $(-\infty, -1)$ and $(2, \infty)$; falling on $(-1, 2)$; concave up on $(1/2, \infty)$; concave down on $(-\infty, 1/2)$

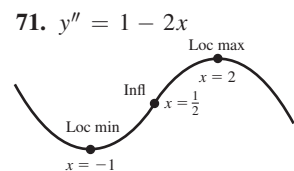
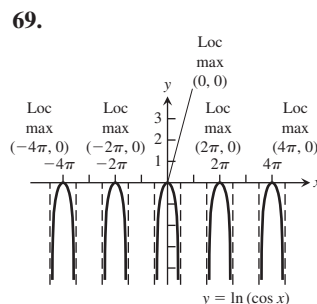
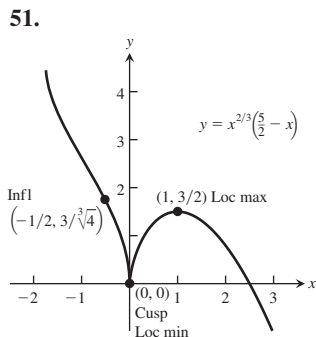
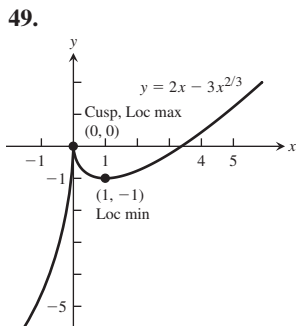
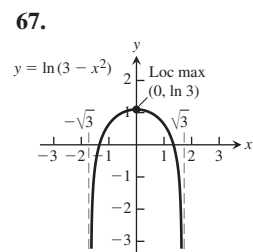
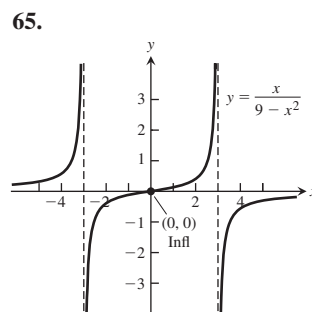
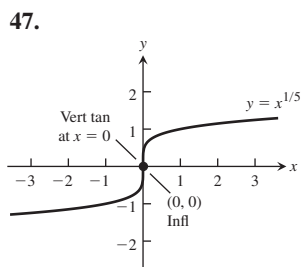
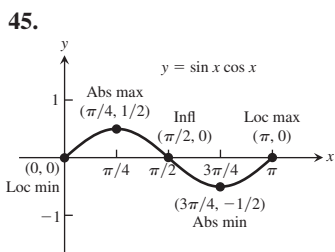
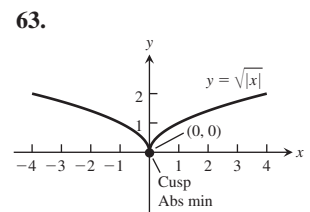
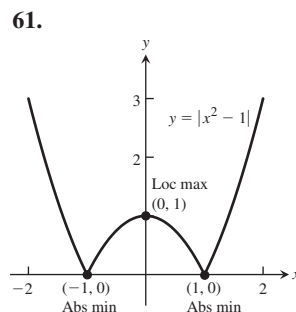
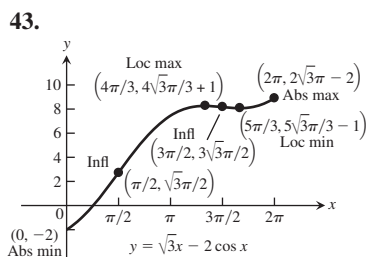
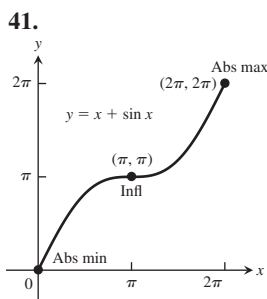
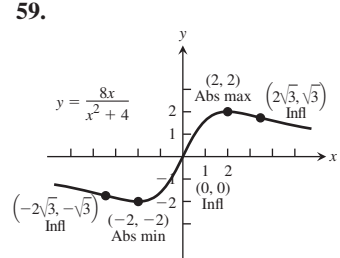
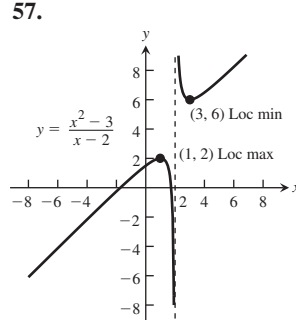
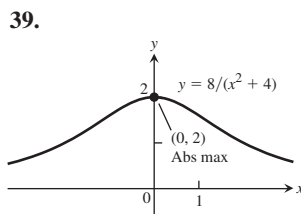
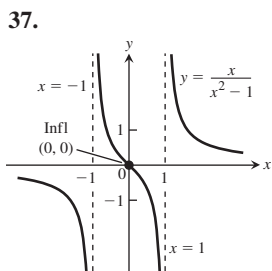
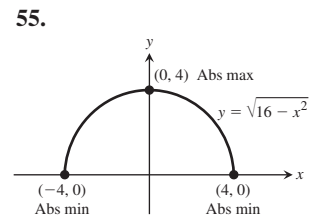
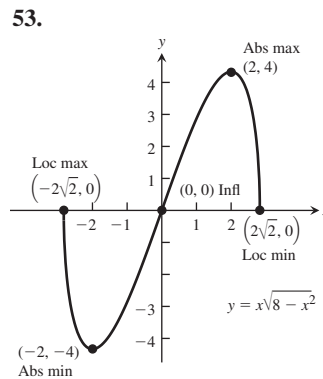
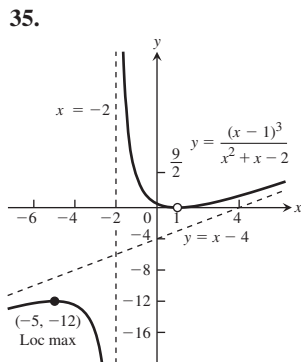
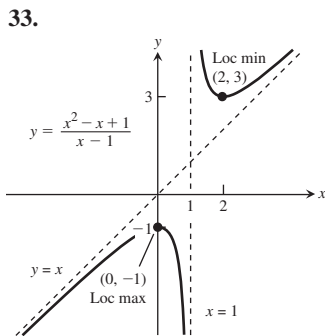
3. Local maximum: $3/4$ at $x = 0$; local minimum: 0 at $x = \pm 1$; points of inflection at $(-\sqrt{3}, \frac{3\sqrt{3}}{4})$ and $(\sqrt{3}, \frac{3\sqrt{3}}{4})$; rising on $(-1, 0)$ and $(1, \infty)$; falling on $(-\infty, -1)$ and $(0, 1)$; concave up on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$; concave down on $(-\sqrt{3}, -1)$, $(-1, 1)$, and $(1, \sqrt{3})$

5. Local maxima: $\frac{-2\pi}{3} + \frac{\sqrt{3}}{2}$ at $x = -2\pi/3, \frac{\pi}{3} + \frac{\sqrt{3}}{2}$ at

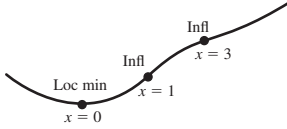
$x = \pi/3$; local minima: $-\frac{\pi}{3} - \frac{\sqrt{3}}{2}$ at $x = -\pi/3, \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$ at $x = 2\pi/3$; points of inflection at $(-\pi/2, -\pi/2)$, $(0, 0)$, and $(\pi/2, \pi/2)$; rising on $(-\pi/3, \pi/3)$; falling on $(-2\pi/3, -\pi/3)$ and $(\pi/3, 2\pi/3)$; concave up on $(-\pi/2, 0)$ and $(\pi/2, 2\pi/3)$; concave down on $(-2\pi/3, -\pi/2)$ and $(0, \pi/2)$

7. Local maxima: 1 at $x = -\pi/2$ and $x = \pi/2$, 0 at $x = -2\pi$ and $x = 2\pi$; local minima: -1 at $x = -3\pi/2$ and $x = 3\pi/2$, 0 at $x = 0$; points of inflection at $(-\pi, 0)$ and $(\pi, 0)$; rising on $(-3\pi/2, -\pi/2)$, $(0, \pi/2)$, and $(3\pi/2, 2\pi)$; falling on $(-2\pi, -3\pi/2)$, $(-\pi/2, 0)$, and $(\pi/2, 3\pi/2)$; concave up on $(-2\pi, -\pi)$ and $(\pi, 2\pi)$; concave down on $(-\pi, 0)$ and $(0, \pi)$

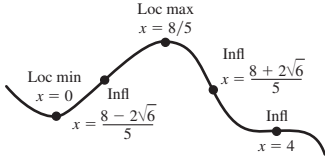
9.**11.****13.****15.****17.****19.****21.****23.****25.****27.****29.****31.**



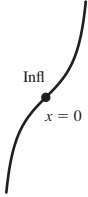
73. $y'' = 3(x-3)(x-1)$



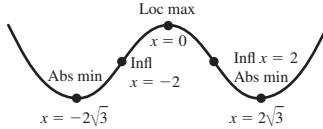
77. $y'' = 4(4-x)(5x^2-16x+8)$



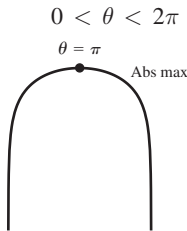
79. $y'' = 2\sec^2 x \tan x$



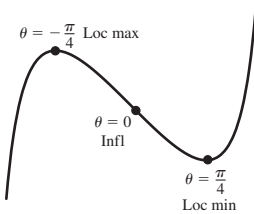
75. $y'' = 3(x-2)(x+2)$



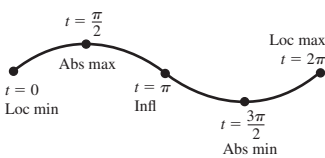
81. $y'' = -\frac{1}{2}\csc^2 \frac{\theta}{2}$



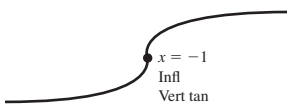
83. $y'' = 2 \tan \theta \sec^2 \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$



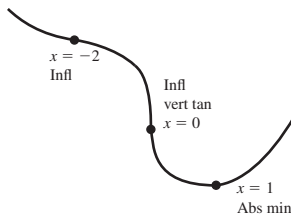
85. $y'' = -\sin t, 0 \leq t \leq 2\pi$



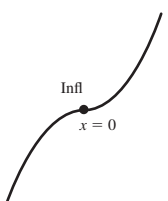
87. $y'' = -\frac{2}{3}(x+1)^{-5/3}$



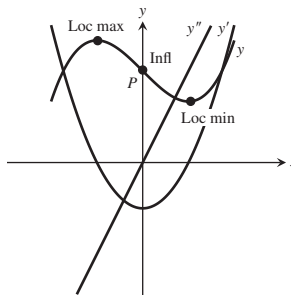
89. $y'' = \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-5/3}$



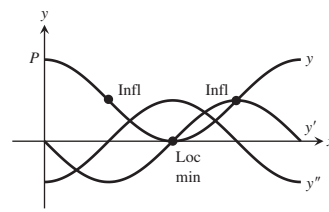
91. $y'' = \begin{cases} -2, & x < 0 \\ 2, & x > 0 \end{cases}$



93.



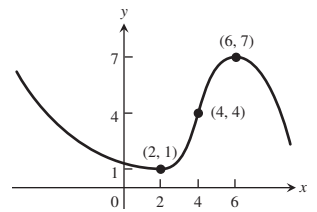
95.



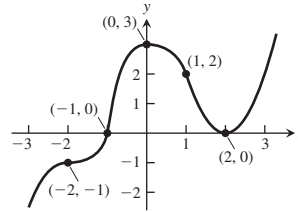
97.

Point	y'	y''
P	-	+
Q	+	0
R	+	-
S	0	-
T	-	-

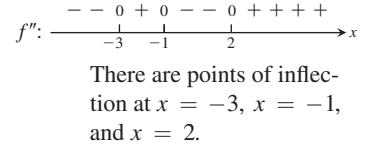
99.



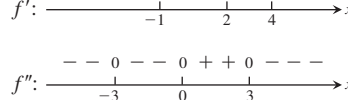
101.



103.



105.



There are local maxima at $x = -1$ and $x = 4$. There is a local minimum at $x = 2$. There are points of inflection at $x = 0$ and $x = 3$.

107. Concave up on $(-2, 0)$ and $(1, 2)$; concave down on $(0, 1)$

109. (a) Toward origin: $0 \leq t < 2$ and $6 \leq t \leq 10$; away from origin: $2 \leq t \leq 6$ and $10 \leq t \leq 15$

(b) $t = 2, t = 6, t = 10$

(c) $t = 5, t = 7, t = 13$

(d) Positive: $5 \leq t \leq 7, 13 \leq t \leq 15$; negative: $0 \leq t \leq 5, 7 \leq t \leq 13$

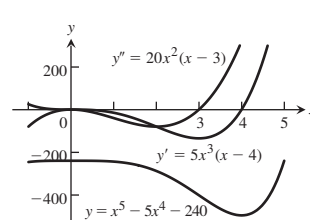
111. ≈ 60 thousand units

113. Local minimum at $x = 2$; inflection points at $x = 1$ and $x = 5/3$

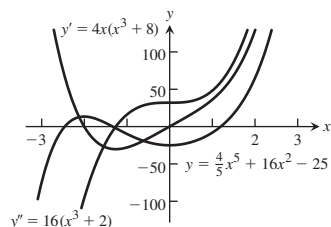
117. $b = -3$ 121. $x = -1, x = 2$

123. $a = 1, b = 3, c = 9$

125. The zeros of $y' = 0$ and $y'' = 0$ are extrema and points of inflection, respectively. Inflection at $x = 3$; local maximum at $x = 0$; local minimum at $x = 4$

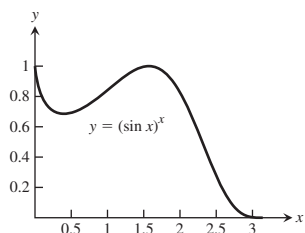


127. The zeros of $y' = 0$ and $y'' = 0$ are extrema and points of inflection, respectively. Inflection at $x = -\sqrt[3]{2}$; local maximum at $x = -2$; local minimum at $x = 0$.



SECTION 4.5, pp. 275–276

1. $-1/4$ 3. $5/7$ 5. $1/2$ 7. $1/4$ 9. $-23/7$
 11. $5/7$ 13. 0 15. -16 17. -2 19. $\sqrt{3}/2$
 21. $1/4$ 23. 2 25. 3 27. -1 29. $\ln 3$ 31. $\frac{1}{\ln 2}$
 33. $\ln 2$ 35. 1 37. $1/2$ 39. $\ln 2$ 41. $-\infty$
 43. $-1/2$ 45. -1 47. 1 49. 0 51. 2 53. $1/e$
 55. 1 57. $1/e$ 59. $e^{1/2}$ 61. 1 63. e^3 65. 0
 67. 1 69. 3 71. 1 73. 0 75. ∞
 77. (b) is correct. 79. (d) is correct. 81. $c = \frac{27}{10}$
 83. (b) $-\frac{1}{2}$ 85. -1 89. (a) $y = 1$ (b) $y = 0, y = \frac{3}{2}$
 91. (a) We should assign the value 1 to $f(x) = (\sin x)^x$ to make it continuous at $x = 0$.

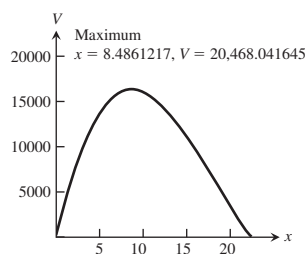


- (c) The maximum value of $f(x)$ is close to 1 near the point $x \approx 1.55$ (see the graph in part (a)).

SECTION 4.6, pp. 282–289

1. 16 cm, 4 cm by 4 cm
 3. (a) $(x, 1 - x)$ (b) $A(x) = 2x(1 - x)$
 (b) $\frac{1}{2}$ square units, 1 by $\frac{1}{2}$
 5. $14 \times 35 \times 5$ cm, 2450 cm^3
 7. $80,000 \text{ m}^2$; 400 m by 200 m
 9. (a) The optimum dimensions of the tank are 2 m on the base edges and 1 m deep.
 (b) Minimizing the surface area of the tank minimizes its weight for a given wall thickness. The thickness of the steel walls would likely be determined by other considerations such as structural requirements.
 11. 45×22.5 cm 13. $\frac{\pi}{2}$ 15. $h:r = 8:\pi$

17. (a) $V(x) = 2x(60 - 2x)(45 - 2x)$ (b) Domain: $(0, 22.5)$



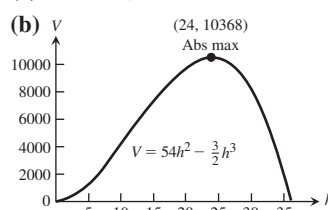
- (c) Maximum volume $\approx 20,468 \text{ cm}^3$ when $x \approx 8.49$ cm
 (d) $V'(x) = 24x^2 - 840x + 5400$, so the critical point is at

$$x = \frac{35 - 5\sqrt{13}}{2}, \text{ which confirms the result in part (c).}$$

- (e) $x = 12.5$ cm or $x = 5$ cm

19. Radius = $10\sqrt{2/3}$ cm, height = $20/\sqrt{3}$ cm,
 volume = $4000\pi/(3\sqrt{3}) \text{ cm}^3$

21. (a) $h = 24, w = 18$



23. If r is the radius of the hemisphere, h the height of the cylinder, and V the volume, then $r = \left(\frac{3V}{8\pi}\right)^{1/3}$ and $h = \left(\frac{3V}{\pi}\right)^{1/3}$.

25. (b) $x = 16.2$ (c) $L \approx 28$ cm

27. Radius = $\sqrt{2}$ m, height = 1 m, volume = $\frac{2\pi}{3} \text{ m}^3$

29. 1 31. $\frac{9b}{9 + \sqrt{3}\pi}$ m, triangle; $\frac{b\sqrt{3}\pi}{9 + \sqrt{3}\pi}$ m, circle

33. $5 + 2\sqrt{3}$ m 35. $\frac{3}{2} \times 2$ 37. (a) 16 (b) -1

39. $r = \frac{2\sqrt{2}}{3}$ $h = \frac{4}{3}$

41. (a) $4 + \ln 2$ (b) $\left(\frac{1}{2}\right)(4 + \ln 2)$ 43. Area 8 when $a = 2$

45. (a) $v(0) = 29.4 \text{ m/s}$ (b) 78.4 m at $t = 3$ s
 (c) Velocity when $s = 0$ is $v(7) = -39.2 \text{ m/s}$.

47. ≈ 9.58 m

49. (a) 4 m

51. (a) $10\sqrt{3} \times 10\sqrt{6}$ cm

53. (a) $10\pi \approx 31.42 \text{ cm/s}$; when $t = 0.5$ s, 1.5 s, 2.5 s,
 3.5 s; $s = 0$, acceleration is 0.

- (b) 10 cm from rest position; speed is 0.

55. (a) $s = ((12 - 12t)^2 + 64t^2)^{1/2}$

- (b) -12 knots; 8 knots (c) No

- (e) $4\sqrt{13}$; this limit is the square root of the sum of the squares of the individual speeds.

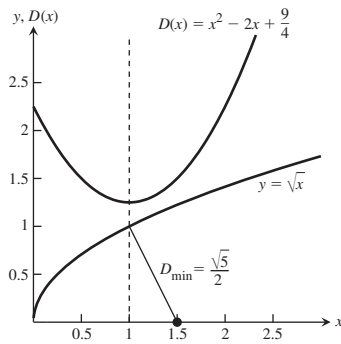
57. $x = \frac{a}{2}, v = \frac{ka^2}{4}$ 59. $\frac{c}{2} + 50$

61. (a) $\sqrt{\frac{2km}{h}}$ (b) $\sqrt{\frac{2km}{h}}$ 65. $2 \times 2 \times \frac{3}{2}$ m, \$720

67. $M = \frac{C}{2}$ 73. (a) $y = -1$

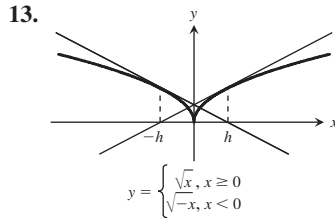
75. (a) The minimum distance is $\frac{\sqrt{5}}{2}$.

- (b) The minimum distance is from the point $(3/2, 0)$ to the point $(1, 1)$ on the graph of $y = \sqrt{x}$, and this occurs at the value $x = 1$, where $D(x)$, the distance squared, has its minimum value.



SECTION 4.7, pp. 292–294

1. $x_2 = -\frac{5}{3}, \frac{13}{21}$ 3. $x_2 = -\frac{51}{31}, \frac{5763}{4945}$ 5. $x_2 = \frac{2387}{2000}$
 7. x_2 is approximately 2.20794 9. x_2 is approximately 0.68394
 11. x_1 , and all later approximations will equal x_0 .



15. The points of intersection of $y = x^3$ and $y = 3x + 1$ or of $y = x^3 - 3x$ and $y = 1$ have the same x -values as the roots of part (i) or the solutions of part (iv).
 17. $x \approx 1.165561185$, $y \approx 2.33112237$
 19. (a) Two (b) 0.35003501505249 and -1.0261731615301
 21. ± 1.3065629648764 , ± 0.5411961001462 23. $x \approx 0.45$
 25. 0.8192 27. 0, 0.53485 29. The root is 1.17951.
 31. (a) For $x_0 = -2$ or $x_0 = -0.8$, $x_i \rightarrow -1$ as i gets large.
 (b) For $x_0 = -0.5$ or $x_0 = 0.25$, $x_i \rightarrow 0$ as i gets large.
 (c) For $x_0 = 0.8$ or $x_0 = 2$, $x_i \rightarrow 1$ as i gets large.
 (d) For $x_0 = -\sqrt{21}/7$ or $x_0 = \sqrt{21}/7$, Newton's method does not converge. The values of x_i alternate between $-\sqrt{21}/7$ and $\sqrt{21}/7$ as i increases.

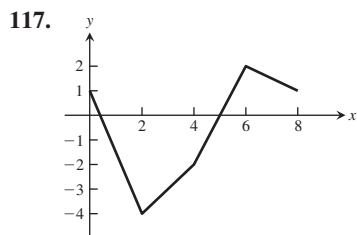
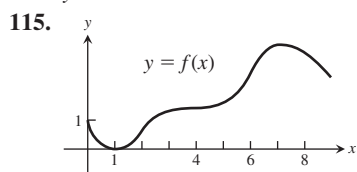
33. Answers will vary with machine speed.

SECTION 4.8, pp. 300–304

1. (a) x^2 (b) $\frac{x^3}{3}$ (c) $\frac{x^3}{3} - x^2 + x$
 3. (a) x^{-3} (b) $-\frac{1}{3}x^{-3}$ (c) $-\frac{1}{3}x^{-3} + x^2 + 3x$
 5. (a) $-\frac{1}{x}$ (b) $-\frac{5}{x}$ (c) $2x + \frac{5}{x}$
 7. (a) $\sqrt{x^3}$ (b) $\sqrt{x}$ (c) $\frac{2\sqrt{x^3}}{3} + 2\sqrt{x}$

9. (a) $x^{2/3}$ (b) $x^{1/3}$ (c) $x^{-1/3}$
 11. (a) $\ln x$ (b) $7 \ln x$ (c) $x - 5 \ln x$
 13. (a) $\cos(\pi x)$ (b) $-3 \cos x$ (c) $-\frac{1}{\pi} \cos(\pi x) + \cos(3x)$
 15. (a) $\tan x$ (b) $2 \tan\left(\frac{x}{3}\right)$ (c) $-\frac{2}{3} \tan\left(\frac{3x}{2}\right)$
 17. (a) $-\csc x$ (b) $\frac{1}{5} \csc(5x)$ (c) $2 \csc\left(\frac{\pi x}{2}\right)$
 19. (a) $\frac{1}{3}e^{3x}$ (b) $-e^{-x}$ (c) $2e^{x/2}$
 21. (a) $\frac{1}{\ln 3}3^x$ (b) $\frac{-1}{\ln 2}2^{-x}$ (c) $\frac{1}{\ln(5/3)}\left(\frac{5}{3}\right)^x$
 23. (a) $2 \arcsin x$ (b) $\frac{1}{2} \arctan x$ (c) $\frac{1}{2} \arctan 2x$
 25. $\frac{x^2}{2} + x + C$ 27. $t^3 + \frac{t^2}{4} + C$ 29. $\frac{x^4}{2} - \frac{5x^2}{2} + 7x + C$
 31. $-\frac{1}{x} - \frac{x^3}{3} - \frac{x}{3} + C$ 33. $\frac{3}{2}x^{2/3} + C$
 35. $\frac{2}{3}x^{3/2} + \frac{3}{4}x^{4/3} + C$ 37. $4y^2 - \frac{8}{3}y^{3/4} + C$
 39. $x^2 + \frac{2}{x} + C$ 41. $2\sqrt{t} - \frac{2}{\sqrt{t}} + C$
 43. $-2 \sin t + C$ 45. $-21 \cos \frac{\theta}{3} + C$
 47. $3 \cot x + C$ 49. $-\frac{1}{2} \csc \theta + C$
 51. $\frac{1}{3}e^{3x} - 5e^{-x} + C$ 53. $-e^{-x} + \frac{4^x}{\ln 4} + C$
 55. $4 \sec x - 2 \tan x + C$ 57. $-\frac{1}{2} \cos 2x + \cot x + C$
 59. $\frac{t}{2} + \frac{\sin 4t}{8} + C$ 61. $\ln|x| - 5 \arctan x + C$
 63. $\frac{3x(\sqrt{3}+1)}{\sqrt{3}+1} + C$ 65. $\tan \theta + C$ 67. $-\cot x - x + C$
 69. $-\cos \theta + \theta + C$
 83. (a) Wrong: $\frac{d}{dx}\left(\frac{x^2}{2} \sin x + C\right) = \frac{2x}{2} \sin x + \frac{x^2}{2} \cos x = x \sin x + \frac{x^2}{2} \cos x$
 (b) Wrong: $\frac{d}{dx}(-x \cos x + C) = -\cos x + x \sin x$
 (c) Right: $\frac{d}{dx}(-x \cos x + \sin x + C) = -\cos x + x \sin x + \cos x = x \sin x$
 85. (a) Wrong: $\frac{d}{dx}\left(\frac{(2x+1)^3}{3} + C\right) = \frac{3(2x+1)^2(2)}{3} = 2(2x+1)^2$
 (b) Wrong: $\frac{d}{dx}((2x+1)^3 + C) = 3(2x+1)^2(2) = 6(2x+1)^2$
 (c) Right: $\frac{d}{dx}((2x+1)^3 + C) = 6(2x+1)^2$
 87. Right 89. (b) 91. $y = x^2 - 7x + 10$
 93. $y = -\frac{1}{x} + \frac{x^2}{2} - \frac{1}{2}$ 95. $y = 9x^{1/3} + 4$

97. $s = t + \sin t + 4$ 99. $r = \cos(\pi\theta) - 1$
 101. $v = \frac{1}{2}\sec t + \frac{1}{2}$ 103. $v = 3\operatorname{arccsc} t - \pi$
 105. $y = x^2 - x^3 + 4x + 1$ 107. $r = \frac{1}{t} + 2t - 2$
 109. $y = x^3 - 4x^2 + 5$ 111. $y = -\sin t + \cos t + t^3 - 1$
 113. $y = 2x^{3/2} - 50$

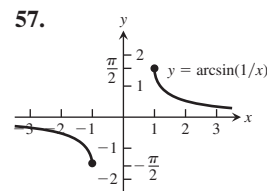
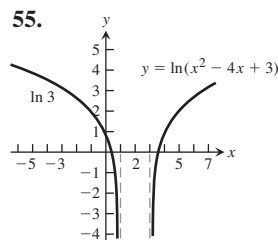
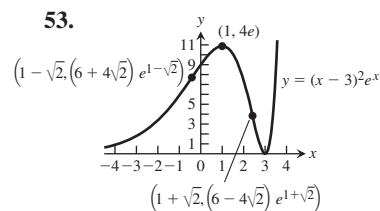
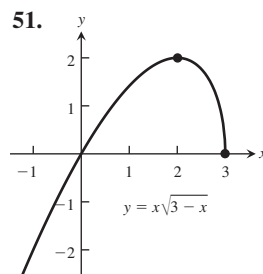
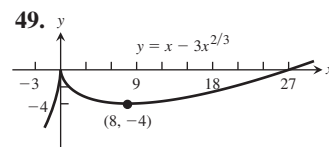
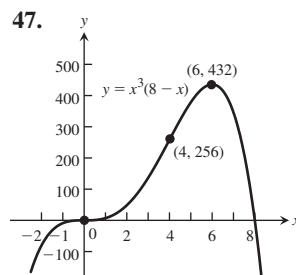
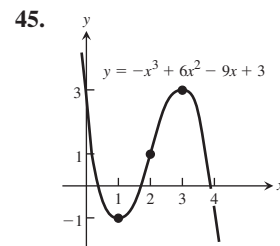
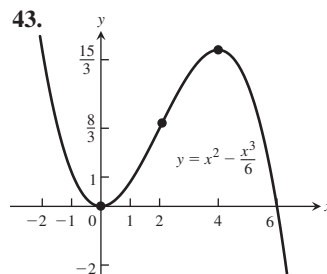


119. $y = x - x^{4/3} + \frac{1}{2}$
 121. $y = -\sin x - \cos x - 2$
 123. (a) (i) 33.2 units, (ii) 33.2 units, (iii) 33.2 units
 (b) True
 125. $t = 88/k, k = 16$
 127. (a) $v = 10t^{3/2} - 6t^{1/2}$ (b) $s = 4t^{5/2} - 4t^{3/2}$
 131. (a) $-\sqrt{x} + C$ (b) $x + C$ (c) $\sqrt{x} + C$
 (d) $-x + C$
 (e) $x - \sqrt{x} + C$
 (f) $-x - \sqrt{x} + C$

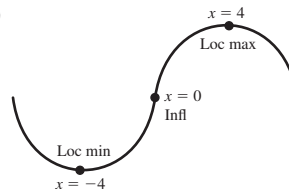
PRACTICE EXERCISES, pp. 305–308

1. Minimum value is 1 at $x = 2$.
 3. Local maximum at $(-2, 17)$; local minimum at $(\frac{4}{3}, -\frac{41}{27})$
 5. Minimum value is 0 at $x = -1$ and $x = 1$.
 7. There is a local minimum at $(0, 1)$.
 9. Maximum value is $\frac{1}{2}$ at $x = 1$; minimum value is $-\frac{1}{2}$ at $x = -1$.
 11. The minimum value is 2 at $x = 0$.
 13. The minimum value is $-\frac{1}{e}$ at $x = \frac{1}{e}$.
 15. The maximum value is $\frac{\pi}{2}$ at $x = 0$; an absolute minimum value is 0 at $x = 1$ and $x = -1$.
 17. No 19. No minimum; absolute maximum: $f(1) = 16$; critical points: $x = 1$ and $11/13$
 21. Absolute minimum: $g(0) = 1$; no absolute maximum; critical point: $x = 0$
 23. Absolute minimum: $2 - 2\ln 2$ at $x = 2$; absolute maximum: 1 at $x = 1$
 25. Yes, except at $x = 0$ 27. No 31. (b) one
 33. (b) 0.8555 99677 2 35. NA
 39. Global minimum value of $\frac{1}{2}$ at $x = 2$

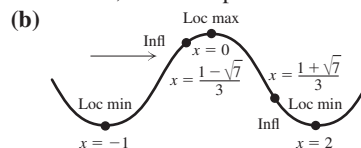
41. (a) $t = 0, 6, 12$ (b) $t = 3, 9$ (c) $6 < t < 12$
 (d) $0 < t < 6, 12 < t < 14$



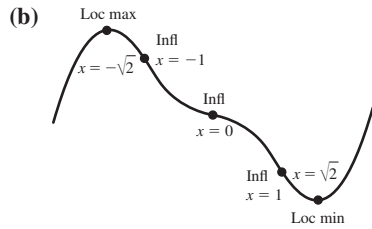
59. (a) Local maximum at $x = 4$, local minimum at $x = -4$, inflection point at $x = 0$
 (b)



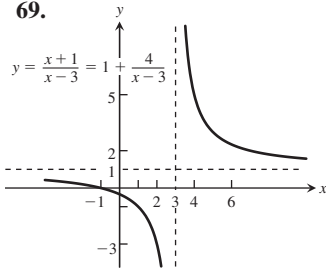
61. (a) Local maximum at $x = 0$, local minima at $x = -1$ and $x = 2$, inflection points at $x = (1 \pm \sqrt{7})/3$



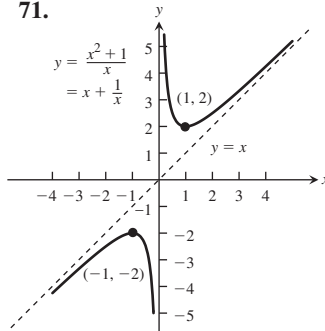
63. (a) Local maximum at $x = -\sqrt{2}$, local minimum at $x = \sqrt{2}$, inflection points at $x = \pm 1$ and 0



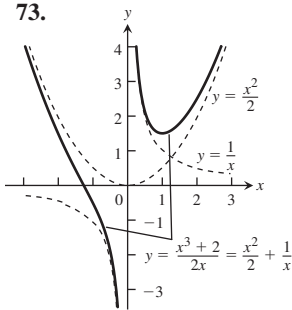
69.



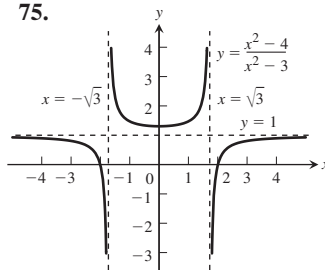
71.



73.



75.



77. 5 79. 0 81. 1 83. $3/7$ 85. 0 87. 1
 89. $\ln 10$ 91. $\ln 2$ 93. 5 95. $-\infty$ 97. 1 99. e^{bk}
 101. $-\infty$ 103. (a) 0, 36 (b) 18, 18 105. 54 square units

107. height = 2, radius = $\sqrt{2}$

109. $x = 5 - \sqrt{5}$ hundred ≈ 276 tires,
 $y = 2(5 - \sqrt{5})$ hundred ≈ 553 tires

111. Dimensions: base is 15 cm by 30 cm,
 height = 5 cm; maximum volume = 2250 cm^3

113. $x_5 = 2.1958\ 23345$ 115. $\frac{x^4}{4} + \frac{5}{2}x^2 - 7x + C$

117. $2t^{3/2} - \frac{4}{t} + C$ 119. $-\frac{1}{r+5} + C$

121. $(\theta^2 + 1)^{3/2} + C$ 123. $\frac{1}{3}(1 + x^4)^{3/4} + C$

125. $10 \tan \frac{s}{10} + C$ 127. $-\frac{1}{\sqrt{2}} \csc \sqrt{2} \theta + C$

129. $\frac{1}{2}x - \sin \frac{x}{2} + C$ 131. $3 \ln x - \frac{x^2}{2} + C$

133. $\frac{1}{2}e^t + e^{-t} + C$ 135. $\frac{\theta^{2-\pi}}{2-\pi} + C$

137. $\frac{3}{2} \operatorname{arcsec} |x| + C$ 139. $y = x - \frac{1}{x} - 1$

141. $r = 4t^{5/2} + 4t^{3/2} - 8t$

143. Yes, $\arcsin x$ and $-\arccos x$ differ by the constant $\pi/2$.

145. $1/\sqrt{2}$ units long by $1/\sqrt{e}$ units high,
 $A = 1/\sqrt{2e} \approx 0.43$ units²

147. Absolute maximum = 0 at $x = e/2$,
 absolute minimum = -0.5 at $x = 0.5$

149. $x = \pm 1$ are the critical points; $y = 1$ is a horizontal asymptote in both directions; absolute minimum value of the function is $e^{-\sqrt{2}/2}$ at $x = -1$, and absolute maximum value is $e^{\sqrt{2}/2}$ at $x = 1$.

151. (a) Absolute maximum of $2/e$ at $x = e^2$, inflection point $(e^{8/3}, (8/3)e^{-4/3})$, concave up on $(e^{8/3}, \infty)$, concave down on $(0, e^{8/3})$

(b) Absolute maximum of 1 at $x = 0$, inflection points $(\pm 1/\sqrt{2}, 1/\sqrt{e})$, concave up on $(-\infty, -1/\sqrt{2}) \cup (1/\sqrt{2}, \infty)$, concave down on $(-1/\sqrt{2}, 1/\sqrt{2})$

(c) Absolute maximum of 1 at $x = 0$, inflection point $(1, 2/e)$, concave up on $(1, \infty)$, concave down on $(-\infty, 1)$

ADDITIONAL AND ADVANCED EXERCISES, pp. 308–311

1. The function is constant on the interval.

3. The extreme points will not be at the end of an open interval.

5. NA

9. No 11. $a = 1, b = 0, c = 1$ 13. Yes

15. Drill the hole at $y = h/2$.

17. $r = \frac{RH}{2(H-R)}$ for $H > 2R$, $r = R$ if $H \leq 2R$

19. $\frac{12}{5}$ and 5

21. (a) $\frac{10}{3}$ (b) $\frac{5}{3}$ (c) $\frac{1}{2}$ (d) 0 (e) $-\frac{1}{2}$ (f) 1

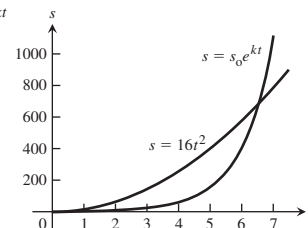
(g) $\frac{1}{2}$ (h) 3

23. (a) $\frac{c-b}{2e}$ (b) $\frac{c+b}{2}$ (c) $\frac{b^2 - 2bc + c^2 + 4ae}{4e}$

(d) $\frac{c+b+t}{2}$

25. $m_0 = 1 - \frac{1}{q}, m_1 = \frac{1}{q}$

27. $s = ce^{kt}$



29. (a) $k = 15$ (b) 7.5 m

31. Yes, $y = x + C$ 33. $v_0 = \frac{2\sqrt{2}}{3} b^{3/4}$ 39. 3

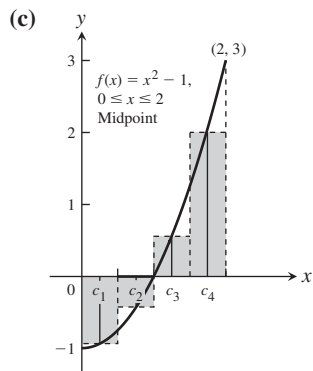
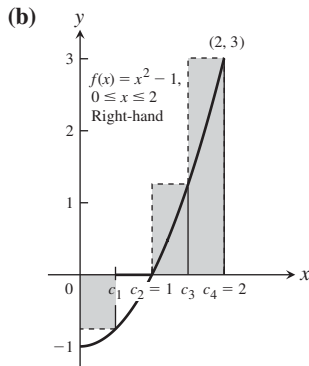
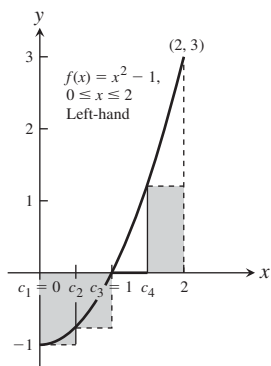
Chapter 5

SECTION 5.1, pp. 320–322

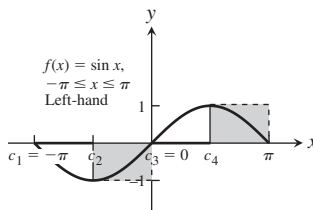
1. (a) 0.125 (b) 0.21875 (c) 0.625 (d) 0.46875
 3. (a) 1.066667 (b) 1.283333 (c) 2.666667 (d) 2.083333
 5. 0.3125, 0.328125 7. 1.5, 1.574603
 9. (a) 87 cm (b) 87 cm 11. (a) 1180 m (b) 1300 m
 13. (a) 22.862 m/s (b) 13.867 m/s (c) 44.893 m
 15. $\frac{31}{16}$ 17. 1
 19. (a) Upper = 758 L, lower = 543 L
 (b) Upper = 2363 L, lower = 1693 L
 (c) ≈ 31.4 h, ≈ 32.4 h
 21. (a) 2 (b) $2\sqrt{2} \approx 2.828$ (c) $8 \sin(\frac{\pi}{8}) \approx 3.061$
 (d) Each area is less than the area inside the circle, π . As n increases, the polygon area approaches π .

SECTION 5.2, pp. 328–329

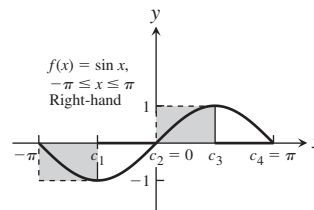
1. $\frac{6(1)}{1+1} + \frac{6(2)}{2+1} = 7$
 3. $\cos(1)\pi + \cos(2)\pi + \cos(3)\pi + \cos(4)\pi = 0$
 5. $\sin \pi - \sin \frac{\pi}{2} + \sin \frac{\pi}{3} = \frac{\sqrt{3}-2}{2}$
 7. All of them 9. b
 11. $\sum_{k=1}^6 k$ 13. $\sum_{k=1}^4 \frac{1}{2^k}$ 15. $\sum_{k=1}^5 (-1)^{k+1} \frac{1}{k}$
 17. (a) -15 (b) 1 (c) 1 (d) -11 (e) 16
 19. (a) 55 (b) 385 (c) 3025 21. -56 23. -73
 25. 240 27. 3376 29. (a) 21 (b) 3500 (c) 2620
 31. (a) $4n$ (b) cn (c) $(n^2 - n)/2$ 33. 2600 35. $-2\sqrt{3}$
 37. (a)



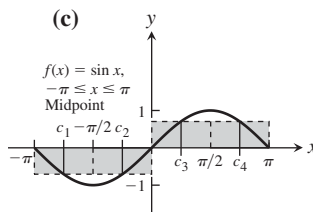
39. (a)



(b)



(c)



$$49. \frac{1}{2} + \frac{1}{n} + \frac{1}{2n^2}, \frac{1}{2}$$

SECTION 5.3, pp. 338–341

1. $\int_0^2 x^2 dx$ 3. $\int_{-7}^5 (x^2 - 3x) dx$ 5. $\int_2^3 \frac{1}{1-x} dx$
 7. $\int_{-\pi/4}^0 \sec x dx$
 9. (a) 0 (b) -8 (c) -12 (d) 10 (e) -2 (f) 16
 11. (a) 5 (b) $5\sqrt{3}$ (c) -5 (d) -5
 13. (a) 4 (b) -4 15. Area = 21 square units
 17. Area = $9\pi/2$ square units 19. Area = 2.5 square units
 21. Area = 3 square units 23. $b^2/4$ 25. $b^2 - a^2$
 27. (a) 2π (b) π 29. $1/2$ 31. $3\pi^2/2$ 33. $7/3$
 35. $1/24$ 37. $3a^2/2$ 39. $b/3$ 41. -14
 43. -2 45. $-7/4$ 47. 7 49. 0
 51. Using n subintervals of length $\Delta x = b/n$ and right-endpoint values:

$$\text{Area} = \int_0^b 3x^2 dx = b^3$$

53. Using n subintervals of length $\Delta x = b/n$ and right-endpoint values:

$$\text{Area} = \int_0^b 2x dx = b^2$$

55. $\text{av}(f) = 0$ 57. $\text{av}(f) = -2$ 59. $\text{av}(f) = 1$
 61. (a) $\text{av}(g) = -1/2$ (b) $\text{av}(g) = 1$ (c) $\text{av}(g) = 1/4$
 63. $c(b-a)$ 65. $b^3/3 - a^3/3$ 67. 9
 69. $b^4/4 - a^4/4$ 71. $a = 0$ and $b = 1$ maximize the integral.
 73. Upper bound = 1, lower bound = $1/2$

$$75. \text{For example, } \int_0^1 \sin(x^2) dx \leq \int_0^1 dx = 1$$

$$77. \int_a^b f(x) dx \geq \int_a^b 0 dx = 0 \quad 79. \text{Upper bound} = 1/2$$

89. Since f is continuous, the Extreme Value Theorem implies that the continuous function f has a maximum value M and a minimum value m (Chapter 4, Theorem 1). The values of f and g are equal except at a single point c where $g(c)$ differs from $f(c)$. The magnitude of the difference is some number between the two numbers $|g(c) - m|$ and $|g(c) - M|$. Call the larger of these two numbers D . In the formula for a Riemann sum given in Equation (1), for a given n , the difference between the Riemann sum for f and the

Riemann sum for g is at most $D(b-a)/n$. Taking the limit as $n \rightarrow \infty$, we see that the difference between the two Riemann sums approaches 0, so that the two limits are equal. A similar bound holds for a general partition, where $D\|P\| \rightarrow 0$.

SECTION 5.4, pp. 351–353

1. $-10/3$ 3. $124/3$ 5. $753/16$ 7. 1 9. $2\sqrt{3}$
11. 0 13. $-\pi/4$ 15. $1 - \frac{\pi}{4}$ 17. $\frac{2 - \sqrt{2}}{4}$ 19. $-8/3$
21. $-3/4$ 23. $\sqrt{2} - \sqrt[4]{8} + 1$ 25. $1/2$ 27. 16
29. $7/3$ 31. $2\pi/3$ 33. $\frac{1}{\pi}(4\pi - 2\pi)$ 35. $\frac{1}{2}(e - 1)$
37. $\sqrt{26} - \sqrt{5}$ 39. $(\cos \sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)$ 41. $4t^5$
43. $3x^2e^{-x^3}$ 45. $\sqrt{1+x^2}$ 47. $-\frac{1}{2}x^{-1/2} \sin x$ 49. 0
51. 1 53. $2xe^{(1/2)x^2}$ 55. 1 57. $28/3$ 59. $1/2$
61. π 63. $\frac{\sqrt{2}\pi}{2}$
65. d, since $y' = \frac{1}{x}$ and $y(\pi) = \int_{\pi}^{\pi} \frac{1}{t} dt - 3 = -3$
67. b, since $y' = \sec x$ and $y(0) = \int_0^0 \sec t dt + 4 = 4$
69. $y = \int_2^x \sec t dt + 3$ 71. $(1 + \ln t)^2$
73. $\frac{2}{3}bh$ 75. \$9.00
77. (a) $T(0) = 20^\circ\text{C}$, $T(16) = 24^\circ\text{C}$, $T(25) = 30^\circ\text{C}$
(b) $\text{av}T = 23.33^\circ\text{C}$
79. $2x - 2$ 81. $-3x + 5$
83. (a) True. Since f is continuous, g is differentiable by Part 1 of the Fundamental Theorem of Calculus.
(b) True: g is continuous because it is differentiable.
(c) True, since $g'(1) = f(1) = 0$.
(d) False, since $g''(1) = f'(1) > 0$.
(e) True, since $g'(1) = 0$ and $g''(1) = f'(1) > 0$.
(f) False: $g''(x) = f'(x) > 0$, so g'' never changes sign.
(g) True, since $g'(1) = f(1) = 0$ and $g'(x) = f(x)$ is an increasing function of x (because $f'(x) > 0$).
85. (a) $v = \frac{ds}{dt} = \frac{d}{dt} \int_0^t f(x) dx = f(t) \Rightarrow v(5) = f(5) = 2 \text{ m/s}$
(b) $a = df/dt$ is negative, since the slope of the tangent line at $t = 5$ is negative.
(c) $s = \int_0^3 f(x) dx = \frac{1}{2}(3)(3) = \frac{9}{2} \text{ m}$, since the integral is the area of the triangle formed by $y = f(x)$, the x -axis, and $x = 3$.
(d) $t = 6$, since after $t = 6$ to $t = 9$, the region lies below the x -axis.
(e) At $t = 4$ and $t = 7$, since there are horizontal tangents there.
(f) Toward the origin between $t = 6$ and $t = 9$, since the velocity is negative on this interval. Away from the origin between $t = 0$ and $t = 6$, since the velocity is positive there.
(g) Right or positive side, because the integral of f from 0 to 9 is positive, there being more area above the x -axis than below.

SECTION 5.5, pp. 360–361

1. $\frac{1}{6}(2x+4)^6 + C$ 3. $-\frac{1}{3}(x^2+5)^{-3} + C$
5. $\frac{1}{10}(3x^2+4x)^5 + C$ 7. $-\frac{1}{3}\cos 3x + C$
9. $\frac{1}{2}\sec 2t + C$ 11. $-6(1-r^3)^{1/2} + C$
13. $\frac{1}{3}(x^{3/2}-1) - \frac{1}{6}\sin(2x^{3/2}-2) + C$
15. (a) $-\frac{1}{4}(\cot^2 2\theta) + C$ (b) $-\frac{1}{4}(\csc^2 2\theta) + C$
17. $-\frac{1}{3}(3-2s)^{3/2} + C$ 19. $-\frac{2}{5}(1-\theta^2)^{5/4} + C$
21. $(-2/(1+\sqrt{x})) + C$ 23. $\frac{1}{3}\tan(3x+2) + C$
25. $\frac{1}{2}\sin^6\left(\frac{x}{3}\right) + C$ 27. $\left(\frac{r^3}{18}-1\right)^6 + C$
29. $-\frac{2}{3}\cos(x^{3/2}+1) + C$ 31. $\frac{1}{2\cos(2t+1)} + C$
33. $-\sin\left(\frac{1}{t}-1\right) + C$ 35. $-\frac{\sin^2(1/\theta)}{2} + C$
37. $\frac{2}{3}(1+x)^{3/2} - 2(1+x)^{1/2} + C$ 39. $\frac{2}{3}\left(2-\frac{1}{x}\right)^{3/2} + C$
41. $\frac{2}{27}\left(1-\frac{3}{x^3}\right)^{3/2} + C$ 43. $\frac{1}{12}(x-1)^{12} + \frac{1}{11}(x-1)^{11} + C$
45. $-\frac{1}{8}(1-x)^8 + \frac{4}{7}(1-x)^7 - \frac{2}{3}(1-x)^6 + C$
47. $\frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C$ 49. $\frac{-1}{4(x^2-4)^2} + C$
51. $e^{\sin x} + C$ 53. $2\tan(e^{\sqrt{x}}+1) + C$ 55. $\ln|\ln x| + C$
57. $z - \ln(1+e^z) + C$ 59. $\frac{5}{6}\tan^{-1}\left(\frac{2r}{3}\right) + C$
61. $e^{\arcsin x} + C$ 63. $\frac{1}{3}(\arcsin x)^3 + C$
65. $\ln|\arctan y| + C$
67. (a) $-\frac{6}{2+\tan^3 x} + C$ (b) $-\frac{6}{2+\tan^3 x} + C$
(c) $-\frac{6}{2+\tan^3 x} + C$
69. $\frac{1}{6}\sin\sqrt{3(2r-1)^2+6} + C$ 73. $s = \frac{1}{2}(3t^2-1)^4 - 5$
75. $s = 4t - 2\sin\left(2t + \frac{\pi}{6}\right) + 9$
77. $s = \sin\left(2t - \frac{\pi}{2}\right) + 100t + 1$ 79. 6 m

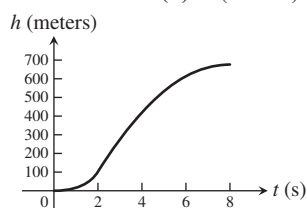
SECTION 5.6, pp. 367–371

1. (a) $14/3$ (b) $2/3$ 3. (a) $1/2$ (b) $-1/2$
5. (a) $15/16$ (b) 0 7. (a) 0 (b) $1/8$ 9. (a) 4 (b) 0
11. (a) $506/375$ (b) $86,744/375$ 13. (a) 0 (b) 0
15. $2\sqrt{3}$ 17. $3/4$ 19. $3^{5/2} - 1$ 21. 3 23. $\pi/3$
25. e 27. $\ln 3$ 29. $(\ln 2)^2$ 31. $\frac{1}{\ln 4}$ 33. $\ln 2$
35. $\ln(2+\sqrt{3}) - \frac{\sqrt{3}}{2}$ 37. π 39. $\pi/12$ 41. $2\pi/3$
43. $\sqrt{3} - 1$ 45. $-\pi/12$ 47. $\pi^2/32$ 49. $16/3$

51. $2^{3/2}$ 53. $\pi/2$ 55. $128/15$ 57. $4/3$ 59. $5/6$
 61. $38/3$ 63. $49/6$ 65. $32/3$ 67. $48/5$ 69. $8/3$
 71. 8 73. $5/3$ (There are three intersection points.) 75. 18
 77. $243/8$ 79. $8/3$ 81. 2 83. $104/15$ 85. $56/15$
 87. 4 89. $\frac{4}{3} - \frac{4}{\pi}$ 91. $\pi/2$ 93. 2 95. $1/2$
 97. 1 99. $\ln 16$ 101. 2 103. $2\ln 5$
 105. (a) $(\pm\sqrt{c}, c)$ (b) $c = 4^{2/3}$ (c) $c = 4^{2/3}$
 107. $11/3$ 109. $3/4$ 111. Neither
 113. The dark blue region has area 2, which is larger than the light blue region, which has an area $(0.6)\pi \approx 1.88$.
 115. The dark blue region has area $e - 2$, or approximately 0.72, which is larger than $1/2$, which is the area of the light blue region.
 117. $F(6) - F(2)$ 119. (a) -3 (b) 3 121. $I = a/2$

PRACTICE EXERCISES, pp. 372–375

1. (a) About 680 m (b)
- h
- (meters)



3. (a) $-1/2$ (b) 31 (c) 13 (d) 0
 5. $\int_1^5 (2x-1)^{-1/2} dx = 2$ 7. $\int_{-\pi}^0 \cos \frac{x}{2} dx = 2$
 9. (a) 4 (b) 2 (c) -2 (d) -2π (e) $8/5$
 11. $8/3$ 13. 62 15. 1 17. $1/6$ 19. 18
 21. $9/8$ 23. $\frac{\pi^2}{32} + \frac{\sqrt{2}}{2} - 1$ 25. 4 27. $\frac{8\sqrt{2}-7}{6}$
 29. Min: -4; max: 0; area: $27/4$ 31. $6/5$ 33. 1
 37. $y = \int_5^x \left(\frac{\sin t}{t}\right) dt - 3$ 39. $y = \arcsin x$
 41. $y = \sec^{-1} x + \frac{2\pi}{3}, x > 1$ 43. $f(x) = e^{x^2/2-1/2}$
 45. $-4(\cos x)^{1/2} + C$ 47. $\theta^2 + \theta + \sin(2\theta + 1) + C$
 49. $\frac{t^3}{3} + \frac{4}{t} + C$ 51. $-\frac{1}{3}\cos(2t^{3/2}) + C$
 53. $\tan(e^x - 7) + C$ 55. $e^{\tan x} + C$ 57. $\frac{-\ln 7}{3}$
 59. $\ln(9/25)$ 61. $-\frac{1}{2}(\ln x)^{-2} + C$ 63. $\frac{1}{2\ln 3}(3^{x^2}) + C$
 65. $\frac{3}{2}\arcsin 2(r-1) + C$ 67. $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C$
 69. $\frac{1}{4}\sec^{-1}\left|\frac{2x-1}{2}\right| + C$ 71. $e^{\arcsin \sqrt{x}} + C$
 73. $2\sqrt{\arctan y} + C$ 75. $\frac{1}{4(\sin 2\theta + \cos 2\theta)^2} + C$ 77. 16
 79. 2 81. 1 83. 8 85. $27\sqrt{3}/160$ 87. $\pi/2$
 89. $\sqrt{3}$ 91. $6\sqrt{3} - 2\pi$ 93. -1 95. 2 97. 1

- 99.
- $15/16 + \ln 2$
- 101.
- $e - 1$
- 103.
- $1/6$
- 105.
- $9/14$

- 107.
- $\frac{9\ln 2}{4}$
- 109.
- π
- 111.
- $\pi/\sqrt{3}$
- 113.
- $\pi/6$

- 115.
- $\pi/12$
117. (a)
- b
- (b)
- b

121. (a)
- $\frac{d}{dx}(x \ln x - x + C) = x \cdot \frac{1}{x} + \ln x - 1 + 0 = \ln x$

(b) $\frac{1}{e-1}$

- 123.
- -4°C
- 125.
- $\sqrt{2 + \cos^3 x}$
- 127.
- $\frac{-6}{3+x^4}$

- 129.
- $\frac{dy}{dx} = \frac{-2}{x}e^{\cos(2\ln x)}$
- 131.
- $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}\sqrt{1-2(\arcsin x)^2}}$

133. Yes 135.
- $-\sqrt{1+x^2}$

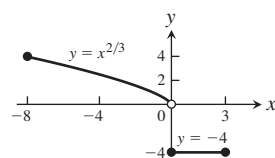
137. Cost
- $\approx \$13,897.50$
- using a lower sum estimate

ADDITIONAL AND ADVANCED EXERCISES, pp. 375–379

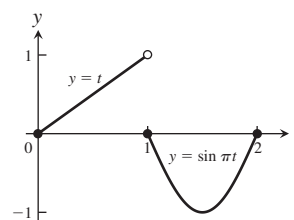
1. (a) Yes (b) No 5. (a)
- $1/4$
- (b)
- $\sqrt[3]{12}$

- 7.
- $f(x) = \frac{x}{\sqrt{x^2+1}}$
- 9.
- $y = x^3 + 2x - 4$

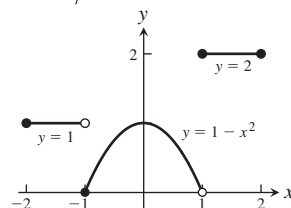
- 11.
- $36/5$



- 13.
- $\frac{1}{2} - \frac{2}{\pi}$



- 15.
- $13/3$



- 17.
- $1/2$
- 19.
- $\pi/2$

21. $\ln 2$
 23. (a) 0 (b) -1
 (c) $-\pi$ (d) $x = 1$
 (e) $y = 2x + 2 - \pi$
 (f) $x = -1, x = 2$
 (g) $[-2\pi, 0]$

- 25.
- $2/x$
- 27.
- $\frac{\sin 4y}{\sqrt{y}} - \frac{\sin y}{2\sqrt{y}}$
- 29.
- $2x \ln |x| - x \ln \frac{|x|}{\sqrt{2}}$

- 31.
- $(\sin x)/x$
- 33.
- $x = 1$

35. (a)
- $\frac{1}{\ln 2}$
- (b)
- $\frac{1}{2\ln 2}$
- (c) 2:1 37.
- $2/17$

- 41.
- $1/6$
- 43.
- $\int_0^1 f(x) dx$
45. (b)
- πr^2

Chapter 6

SECTION 6.1, pp. 387–391

1. 16 3. $16/3$ 5. (a) $2\sqrt{3}$ (b) 8 7. (a) 60 (b) 36
 9. 8π 11. 10 13. (a) s^2h (b) s^2h 15. $8/3$
 17. $\frac{2\pi}{3}$ 19. $4 - \pi$ 21. $\frac{32\pi}{5}$ 23. 36π 25. π

27. $\frac{\pi}{2}\left(1 - \frac{1}{e^2}\right)$ 29. $\frac{\pi}{2} \ln 4$ 31. $\pi\left(\frac{\pi}{2} + 2\sqrt{2} - \frac{11}{3}\right)$
 33. 2π 35. 2π 37. $4\pi \ln 4$ 39. $\pi^2 - 2\pi$ 41. $\frac{2\pi}{3}$
 43. $\frac{117\pi}{5}$ 45. $\pi(\pi - 2)$ 47. $\frac{4\pi}{3}$ 49. 8π 51. $\frac{7\pi}{6}$
 53. (a) 8π (b) $\frac{32\pi}{5}$ (c) $\frac{8\pi}{3}$ (d) $\frac{224\pi}{15}$
 55. (a) $\frac{16\pi}{15}$ (b) $\frac{56\pi}{15}$ (c) $\frac{64\pi}{15}$ 57. $V = 2a^2b\pi^2$
 59. (a) $V = \frac{\pi h^2(3a - h)}{3}$ (b) $\frac{1}{120\pi} \text{ m/s}$
 63. $V = 3308 \text{ cm}^3$ 65. $(4 - b + a)/2$

SECTION 6.2, pp. 396–399

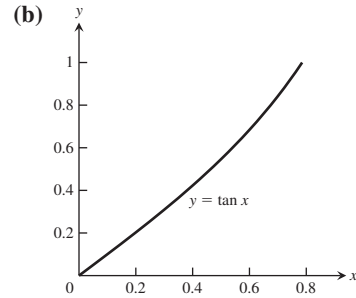
1. 6π 3. 2π 5. $14\pi/3$ 7. 8π 9. $5\pi/6$
 11. $\frac{7\pi}{15}$ 13. (b) 4π 15. $\frac{16\pi}{15}(3\sqrt{2} + 5)$
 17. $\frac{8\pi}{3}$ 19. $\frac{4\pi}{3}$ 21. $\frac{16\pi}{3}$
 23. (a) 16π (b) 32π (c) 28π
 (d) 24π (e) 60π (f) 48π
 25. (a) $\frac{27\pi}{2}$ (b) $\frac{27\pi}{2}$ (c) $\frac{72\pi}{5}$ (d) $\frac{108\pi}{5}$
 27. (a) $\frac{6\pi}{5}$ (b) $\frac{4\pi}{5}$ (c) 2π (d) 2π
 29. (a) About the x-axis: $V = \frac{2\pi}{15}$; about the y-axis: $V = \frac{\pi}{6}$
 (b) About the x-axis: $V = \frac{2\pi}{15}$; about the y-axis: $V = \frac{\pi}{6}$
 31. (a) $\frac{5\pi}{3}$ (b) $\frac{4\pi}{3}$ (c) 2π (d) $\frac{2\pi}{3}$
 33. (a) $\frac{4\pi}{15}$ (b) $\frac{7\pi}{30}$ 35. (a) $\frac{24\pi}{5}$ (b) $\frac{48\pi}{5}$
 37. (a) $\frac{9\pi}{16}$ (b) $\frac{9\pi}{16}$
 39. Disk: 2 integrals; washer: 2 integrals; shell: 1 integral
 41. (a) $\frac{256\pi}{3}$ (b) $\frac{244\pi}{3}$
 47. $\pi\left(1 - \frac{1}{e}\right)$ 49. 2

SECTION 6.3, pp. 403–405

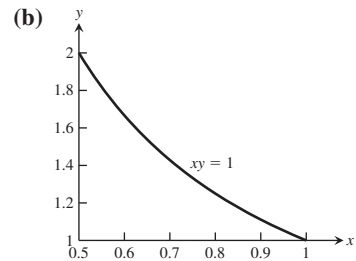
1. 12 3. $\frac{53}{6}$ 5. $\frac{123}{32}$ 7. $\frac{99}{8}$ 9. $\ln 2 + \frac{3}{8}$
 11. $\frac{53}{6}$ 13. $2^{3/2} - (5/4)^{3/2}$ 15. 2
 17. (a) $\int_{-1}^2 \sqrt{1 + 4x^2} dx$ (c) ≈ 6.13
 19. (a) $\int_0^\pi \sqrt{1 + \cos^2 y} dy$ (c) ≈ 3.82
 21. (a) $\int_{-1}^3 \sqrt{1 + (y + 1)^2} dy$ (c) ≈ 9.29
 23. (a) $\int_0^{\pi/6} \sec x dx$ (c) ≈ 0.55
 25. (a) $y = \sqrt{x}$ from $(1, 1)$ to $(4, 2)$
 (b) Only one. We know the derivative of the function and the value of the function at one value of x .
 27. 1 29. $2\sqrt{5}$ 37. $\int_0^x \sqrt{1 + 9t} dt, \frac{2}{27}(10^{3/2} - 1)$

SECTION 6.4, pp. 408–410

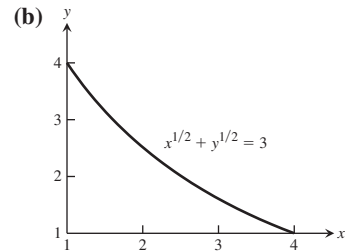
1. (a) $2\pi \int_0^{\pi/4} (\tan x) \sqrt{1 + \sec^4 x} dx$ (c) $S \approx 3.84$



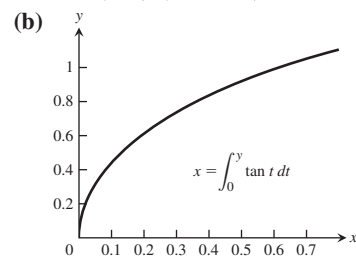
3. (a) $2\pi \int_1^2 \frac{1}{y} \sqrt{1 + y^{-4}} dy$ (c) $S \approx 5.02$



5. (a) $2\pi \int_1^4 (3 - x^{1/2})^2 \sqrt{1 + (1 - 3x^{-1/2})^2} dx$ (c) $S \approx 63.37$



7. (a) $2\pi \int_0^{\pi/3} \left(\int_0^y \tan t dt \right) \sec y dy$ (c) $s \approx 2.08$



9. $4\pi\sqrt{5}$ 11. $3\pi\sqrt{5}$ 13. $98\pi/81$ 15. 2π
 17. $\pi(\sqrt{8} - 1)/9$ 19. $35\pi\sqrt{5}/3$ 21. $\pi\left(\frac{15}{16} + \ln 2\right)$
 23. $253\pi/20$ 27. Order 226.2 liters of each color.

SECTION 6.5, pp. 415–419

1. 116 J 3. 400 N/m 5. 4 cm, 0.08 J
 7. (a) 12,000 N/cm (b) 6000 N-cm, 18,000 N-cm
 9. 780 J 11. 108,000 J 13. 234 J
 15. (a) 235,200,000 J (b) 17 h, 46 min (c) 266 min
 (d) At 9780 N/m³: a) 234,720,000 J b) 17 h, 44 min
 At 9820 N/m³: a) 235,680,000 J b) 17 h, 48 min
 17. 5,772,676.5 J 19. 8,977,666 J 21. 385,369 J
 23. 15,073,099.75 J 27. 120 J 29. 151.3 J

31. 2.175 J 33. 5.144×10^{10} J 35. 9146.7 N

37. (a) 26,989.2 N (b) 25,401.6 N

39. (a) 182,933 N (b) 187,600 N 41. 5808 N

43. (a) 19,600 N (b) 14,700 N (c) 16,135 N

45. (a) 14,933 N (b) 2.94 m 47. $\frac{wb}{2}$

49. No. The tank will overflow because the movable end will have moved only 2.18 m by the time the tank is full.

SECTION 6.6, pp. 429–431

1. $M = 14/3$, $\bar{x} = 93/35$ 3. $M = \ln 4$, $\bar{x} = (3 - \ln 4)/(\ln 4)$

5. $M = 13$, $\bar{x} = 41/26$ 7. $\bar{x} = 0$, $\bar{y} = 12/5$

9. $\bar{x} = 1$, $\bar{y} = -3/5$ 11. $\bar{x} = 16/105$, $\bar{y} = 8/15$

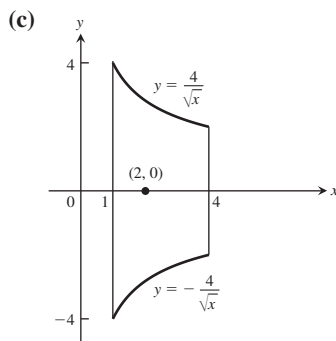
13. $\bar{x} = 0$, $\bar{y} = \pi/8$ 15. $\bar{x} \approx 1.44$, $\bar{y} \approx 0.36$

17. $\bar{x} = \frac{\ln 4}{\pi}$, $\bar{y} = 0$ 19. $\bar{x} = \frac{15}{4 \ln 2}$, $\bar{y} = \frac{15}{128 \ln 2}$

21. $\bar{x} = 5/7$, $\bar{y} = 10/33$. $(\bar{x})^4 < \bar{y}$, so the center of mass is outside the region.

23. $\bar{x} = 3/2$, $\bar{y} = 1/2$

25. (a) $\frac{224\pi}{3}$ (b) $\bar{x} = 2$, $\bar{y} = 0$



29. $\bar{x} = \bar{y} = 1/3$ 31. $\bar{x} = a/3$, $\bar{y} = b/3$ 33. $13\delta/6$

35. $\bar{x} = 0$, $\bar{y} = \frac{a\pi}{4}$ 37. $\bar{x} = 1/2$, $\bar{y} = 4$

39. $\bar{x} = 6/5$, $\bar{y} = 8/7$ 43. $V = 32\pi$, $S = 32\sqrt{2}\pi$

45. $4\pi^2$ 47. $\bar{x} = 0$, $\bar{y} = \frac{2a}{\pi}$ 49. $\bar{x} = 0$, $\bar{y} = \frac{4b}{3\pi}$

51. $\sqrt{2}\pi a^3(4 + 3\pi)/6$ 53. $\bar{x} = \frac{a}{3}$, $\bar{y} = \frac{b}{3}$

PRACTICE EXERCISES, pp. 432–434

1. $\frac{9\pi}{280}$ 3. π^2 5. $\frac{72\pi}{35}$

7. (a) 2π (b) π (c) $12\pi/5$ (d) $26\pi/5$

9. (a) 8π (b) $1088\pi/15$ (c) $512\pi/15$

11. $\pi(3\sqrt{3} - \pi)/3$ 13. $\pi(e - 1)$ 15. $\frac{28\pi}{3} \text{ m}^3$

17. $(\pi/3)(a^2 + ab + b^2)h$ 19. $\frac{10}{3}$ 21. $3 + \frac{1}{8} \ln 2$

23. $\sqrt{2}$ 25. $28\pi\sqrt{2}/3$ 27. 4π 29. 4640 J

31. $\frac{w}{2}(2ar - a^2)$ 33. 65,680,230 J

35. 3,375,000 π J, 257 s 37. (a) 11.31 J (b) 29.18 J

39. $\bar{x} = 0$, $\bar{y} = 8/5$ 41. $\bar{x} = 3/2$, $\bar{y} = 12/5$

43. $\bar{x} = 9/5$, $\bar{y} = 11/10$ 45. 52,267 J 47. 344,960 N

ADDITIONAL AND ADVANCED EXERCISES, pp. 434–435

1. $f(x) = \sqrt{\frac{2x-a}{\pi}}$ 3. $f(x) = \sqrt{C^2 - 1}x + a$, where $C \geq 1$

5. $\frac{\pi}{30\sqrt{2}}$ 7. $28/3$ 9. $\frac{4h\sqrt{3mh}}{3}$

11. $\bar{x} = 0$, $\bar{y} = \frac{n}{2n+1}$, $(0, 1/2)$

15. (a) $\bar{x} = \bar{y} = 4(a^2 + ab + b^2)/(3\pi(a + b))$

(b) $(2a/\pi, 2a/\pi)$ 17. ≈ 365.867 N

Chapter 7

SECTION 7.1, pp. 445–447

1. $\ln\left(\frac{2}{3}\right)$ 3. $\ln|y^2 - 25| + C$ 5. $\ln|6 + 3 \tan t| + C$

7. $\ln(1 + \sqrt{x}) + C$ 9. 1 11. $2(\ln 2)^4$ 13. 2

15. $2e^{\sqrt{x}} + C$ 17. $-e^{-t^2} + C$ 19. $-e^{1/x} + C$

21. $\frac{1}{\pi}e^{\sec \pi t} + C$ 23. 1 25. $\ln(1 + e^r) + C$ 27. $\frac{1}{2 \ln 2}$

29. $\frac{1}{\ln 2}$ 31. $\frac{6}{\ln 7}$ 33. 32760 35. $3^{\sqrt{2}+1}$

37. $\frac{1}{\ln 10} \left(\frac{(\ln x)^2}{2} \right) + C$ 39. $2(\ln 2)^2$ 41. $\frac{3 \ln 2}{2}$ 43. $\ln 10$

45. $(\ln 10) \ln|\ln x| + C$ 47. $y = 1 - \cos(e^t - 2)$

49. $y = 2(e^{-x} + x) - 1$ 51. $y = x + \ln|x| + 2$

53. $\pi \ln 16$ 55. $6 + \ln 2$ 57. (b) 0.00469

77. (a) 1.89279 (b) -0.35621 (c) 0.94575 (d) -2.80735
(e) 5.29595 (f) 0.97041 (g) -1.03972 (h) -1.61181

SECTION 7.2, pp. 454–456

9. $\frac{2}{3}y^{3/2} - x^{1/2} = C$ 11. $e^y - e^x = C$

13. $-x + 2 \tan \sqrt{y} = C$ 15. $e^{-y} + 2e^{\sqrt{x}} = C$

17. $y = \sin(x^2 + C)$ 19. $\frac{1}{3} \ln|y^3 - 2| = x^3 + C$

21. $4 \ln(\sqrt{y} + 2) = e^{x^2} + C$

23. (a) -0.00001 (b) 10,536 years (c) 82%

25. 54.88 g 27. 19.9 m 29. 2.8147498×10^{14}

31. (a) 8 years (b) 32.02 years 33. Yes, $y(20) < 1$

35. 15.28 years 37. 56,562 years

41. (a) 17.5 min (b) 13.26 min 43. -3°C

45. About 6693 years 47. 54.62% 49. $\approx 15,683$ years

SECTION 7.3, pp. 462–465

1. $\cosh x = 5/4$, $\tanh x = -3/5$, $\coth x = -5/3$,
 $\operatorname{sech} x = 4/5$, $\operatorname{csch} x = -4/3$

3. $\sinh x = 8/15$, $\tanh x = 8/17$, $\coth x = 17/8$,
 $\operatorname{sech} x = 15/17$, $\operatorname{csch} x = 15/8$

5. $x + \frac{1}{x}$ 7. e^{5x} 9. e^{4x} 13. $2 \cosh \frac{x}{3}$

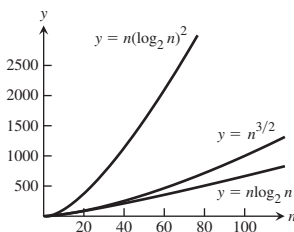
15. $\operatorname{sech}^2 \sqrt{t} + \frac{\tanh \sqrt{t}}{\sqrt{t}}$ 17. $\coth z$

19. $(\ln \operatorname{sech} \theta)(\operatorname{sech} \theta \tanh \theta)$ 21. $\tanh^3 v$ 23. 2

25. $\frac{1}{2\sqrt{x(1+x)}}$ 27. $\frac{1}{1+\theta} - \tanh^{-1}\theta$
29. $\frac{1}{2\sqrt{t}} - \coth^{-1}\sqrt{t}$ 31. $-\operatorname{sech}^{-1}x$ 33. $\frac{\ln 2}{\sqrt{1 + \left(\frac{1}{2}\right)^{2\theta}}}$
35. $|\sec x|$ 41. $\frac{\cosh 2x}{2} + C$
43. $12 \sinh\left(\frac{x}{2} - \ln 3\right) + C$ 45. $7 \ln|e^{x/7} + e^{-x/7}| + C$
47. $\tanh\left(x - \frac{1}{2}\right) + C$ 49. $-2 \operatorname{sech} \sqrt{t} + C$ 51. $\ln \frac{5}{2}$
53. $\frac{3}{32} + \ln 2$ 55. $e - e^{-1}$ 57. $3/4$ 59. $\frac{3}{8} + \ln \sqrt{2}$
61. $\ln(2/3)$ 63. $\frac{-\ln 3}{2}$ 65. $\ln 3$
67. (a) $\sinh^{-1}(\sqrt{3})$ (b) $\ln(\sqrt{3} + 2)$
69. (a) $\coth^{-1}(2) - \coth^{-1}(5/4)$ (b) $\left(\frac{1}{2}\right) \ln\left(\frac{1}{3}\right)$
71. (a) $-\operatorname{sech}^{-1}\left(\frac{12}{13}\right) + \operatorname{sech}^{-1}\left(\frac{4}{5}\right)$
 (b) $-\ln\left(\frac{1 + \sqrt{1 - (12/13)^2}}{(12/13)}\right) + \ln\left(\frac{1 + \sqrt{1 - (4/5)^2}}{(4/5)}\right)$
 $= -\ln\left(\frac{3}{2}\right) + \ln(2) = \ln(4/3)$
73. (a) 0 (b) 0
77. (b) $\sqrt{\frac{mg}{k}}$ (c) $70\sqrt{30/47} \approx 55.93$ m/s
79. 2π 81. $\frac{6}{5}$

SECTION 7.4, pp. 469–470

1. (a) Slower (b) Slower (c) Slower (d) Faster
 (e) Slower (f) Slower (g) Same (h) Slower
3. (a) Same (b) Faster (c) Same (d) Same
 (e) Slower (f) Faster (g) Slower (h) Same
5. (a) Same (b) Same (c) Same (d) Faster
 (e) Faster (f) Same (g) Slower (h) Faster
7. d, a, c, b
9. (a) False (b) False (c) True (d) True (e) True
 (f) True (g) False (h) True
13. When the degree of f is less than or equal to the degree of g .
15. 1, 1
21. (b) $\ln(e^{17000000}) = 17,000,000 < (e^{17 \times 10^6})^{1/10^6}$
 $= e^{17} \approx 24,154,952.75$
- (c) $x \approx 3.4306311 \times 10^{15}$
- (d) They cross at $x \approx 3.4306311 \times 10^{15}$.
23. (a) The algorithm that takes $O(n \log_2 n)$ steps



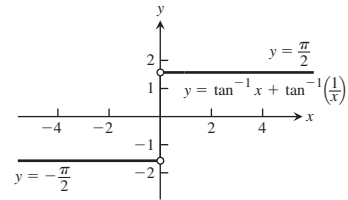
25. It could take one million steps for a sequential search; at most 20 steps for a binary search.

PRACTICE EXERCISES, pp. 471–472

1. $-\cos e^x + C$ 3. $\ln 8$ 5. $2 \ln 2$
7. $\frac{1}{2}(\ln(x-5))^2 + C$ 9. $3 \ln 7$ 11. $2(\sqrt{2} - 1)$
13. $y = \frac{\ln 2}{\ln(3/2)}$ 15. $y = \ln x - \ln 3$ 17. $y = \frac{1}{1 - e^x}$
19. (a) Same rate (b) Same rate (c) Faster (d) Faster
 (e) Same rate (f) Same rate
21. (a) True (b) False (c) False (d) True
 (e) True (f) True
23. $1/3$ 25. $1/e$ m/s 27. $\ln 5x - \ln 3x = \ln(5/3)$
29. $1/2$ 31. $y = \left(\tan^{-1}\left(\frac{x+C}{2}\right)\right)^2$
33. $y^2 = \sin^{-1}(2 \tan x + C)$
35. $y = -2 + \ln(2 - e^{-x})$ 37. $y = 4x - 4\sqrt{x} + 1$
39. 19,035 years 41. $\ln(16/9)$

ADDITIONAL AND ADVANCED EXERCISES, p. 472

1. (a) 1 (b) $\pi/2$ (c) π
3. $\tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right)$ is a constant and the constant is $\frac{\pi}{2}$ for $x > 0$; it is $-\frac{\pi}{2}$ for $x < 0$.



7. $\bar{x} = \frac{\ln 4}{\pi}, \bar{y} = 0$

Chapter 8**SECTION 8.1, pp. 477–478**

1. $\ln 5$ 3. $2 \tan x - 2 \sec x - x + C$
5. $\arcsin x + \sqrt{1 - x^2} + C$ 7. $e^{-\cot z} + C$
9. $\arctan(e^z) + C$ 11. π 13. $t + \cot t + \csc t + C$
15. $\sqrt{2}$ 17. $\frac{1}{8} \ln(1 + 4 \ln^2 y) + C$
19. $\ln|1 + \sin \theta| + C$ 21. $2t^2 - t + 2 \arctan\left(\frac{t}{2}\right) + C$
23. $2(\sqrt{2} - 1) \approx 0.82843$ 25. $\operatorname{arcsec}(e^y) + C$
27. $\arcsin(2 \ln x) + C$ 29. $\ln|\sin x| + \ln|\cos x| + C$
31. $7 + \ln 8$ 33. $[\arcsin y - \sqrt{1 - y^2}]_{-1}^0 = \frac{\pi}{2} - 1$
35. $\operatorname{arcsec}\left|\frac{x-1}{7}\right| + C$ 37. $\frac{\theta^3}{3} - \frac{\theta^2}{2} + \theta + \frac{5}{2} \ln|2\theta - 5| + C$
39. $x - \ln(1 + e^x) + C$ 41. $(1/2)e^{2x} - e^x + \ln(1 + e^x) + C$
43. $2 \arctan(\sqrt{x}) + C$ 45. $2\sqrt{2} - \ln(3 + 2\sqrt{2})$
47. $\ln(2 + \sqrt{3})$ 49. $\bar{x} = 0, \bar{y} = \frac{1}{\ln(3 + 2\sqrt{2})}$
51. $xe^{x^3} + C$ 53. $\frac{1}{30}(x^4 + 1)^{3/2}(3x^4 - 2) + C$

SECTION 8.2, pp. 483–485

1. $-2x \cos(x/2) + 4 \sin(x/2) + C$
3. $t^2 \sin t + 2t \cos t - 2 \sin t + C$
5. $\ln 4 - \frac{3}{4}$ 7. $xe^x - e^x + C$
9. $-(x^2 + 2x + 2)e^{-x} + C$
11. $y \tan^{-1}(y) - \ln \sqrt{1 + y^2} + C$
13. $x \tan x + \ln |\cos x| + C$
15. $(x^3 - 3x^2 + 6x - 6)e^x + C$
17. $(x^2 - 7x + 7)e^x + C$
19. $(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120)e^x + C$
21. $\frac{1}{2}(-e^\theta \cos \theta + e^\theta \sin \theta) + C$
23. $\frac{e^{2x}}{13}(3 \sin 3x + 2 \cos 3x) + C$
25. $\frac{2}{3}(\sqrt{3s+9}e^{\sqrt{3s+9}} - e^{\sqrt{3s+9}}) + C$
27. $\frac{\pi\sqrt{3}}{3} - \ln(2) - \frac{\pi^2}{18}$
29. $\frac{1}{2}[-x \cos(\ln x) + x \sin(\ln x)] + C$
31. $\frac{1}{2} \ln |\sec x^2 + \tan x^2| + C$
33. $\frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C$
35. $-\frac{1}{x} \ln x - \frac{1}{x} + C$ 37. $\frac{1}{4}e^{x^4} + C$
39. $\frac{1}{3}x^2(x^2 + 1)^{3/2} - \frac{2}{15}(x^2 + 1)^{5/2} + C$
41. $-\frac{2}{5} \sin 3x \sin 2x - \frac{3}{5} \cos 3x \cos 2x + C$
43. $\frac{2}{9}x^{3/2}(3 \ln x - 2) + C$
45. $2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C$
47. $\frac{\pi^2 - 4}{8}$ 49. $\frac{5\pi - 3\sqrt{3}}{9}$
51. $\frac{1}{2}(x^2 + 1) \arctan x - \frac{x}{2} + C$ 53. $xe^{x^2} + C$
55. $(2/3)x^{3/2} \arcsin(\sqrt{x}) + (2/9)x\sqrt{1-x} + (4/9)\sqrt{1-x} + C$
57. (a) π (b) 3π (c) 5π (d) $(2n+1)\pi$
59. $2\pi(1 - \ln 2)$ 61. (a) $\pi(\pi - 2)$ (b) 2π
63. (a) 1 (b) $(e - 2)\pi$ (c) $\frac{\pi}{2}(e^2 + 9)$
(d) $\bar{x} = \frac{1}{4}(e^2 + 1)$, $\bar{y} = \frac{1}{2}(e - 2)$
65. $\frac{1}{2\pi}(1 - e^{-2\pi})$ 67. $u = x^n$, $dv = \cos x \, dx$
69. $u = x^n$, $dv = e^{ax} \, dx$ 73. $u = x^n$, $dv = (x + 1)^{-(1/2)} \, dx$
77. $x \arcsin x + \cos(\arcsin x) + C$
79. $x \sec^{-1} x - \ln|x + \sqrt{x^2 - 1}| + C$ 81. Yes
83. (a) $x \sinh^{-1} x - \cosh(\sinh^{-1} x) + C$
(b) $x \sinh^{-1} x - (1 + x^2)^{1/2} + C$
9. $\sin x - \frac{1}{3} \sin^3 x + C$ 11. $\frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + C$
13. $\frac{1}{2}x + \frac{1}{4} \sin 2x + C$ 15. $16/35$ 17. 3π
19. $-4 \sin x \cos^3 x + 2 \cos x \sin x + 2x + C$
21. $-\cos^4 2\theta + C$ 23. 4 25. 2
27. $\sqrt{\frac{3}{2}} - \frac{2}{3}$ 29. $\frac{4}{5}(\frac{3}{2})^{5/2} - \frac{18}{35} - \frac{2}{7}(\frac{3}{2})^{7/2}$ 31. $\sqrt{2}$
33. $\frac{1}{2} \tan^2 x + C$ 35. $\frac{1}{3} \sec^3 x + C$ 37. $\frac{1}{3} \tan^3 x + C$
39. $2\sqrt{3} + \ln(2 + \sqrt{3})$ 41. $\frac{2}{3} \tan \theta + \frac{1}{3} \sec^2 \theta \tan \theta + C$
43. $4/3$ 45. $2 \tan^2 x - 2 \ln(1 + \tan^2 x) + C$
47. $\frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + \ln |\sec x| + C$ 49. $\frac{4}{3} - \ln \sqrt{3}$
51. $\frac{43}{3}$ 53. $-\frac{1}{10} \cos 5x - \frac{1}{2} \cos x + C$ 55. π
57. $\frac{1}{2} \sin x + \frac{1}{14} \sin 7x + C$
59. $\frac{1}{6} \sin 3\theta - \frac{1}{4} \sin \theta - \frac{1}{20} \sin 5\theta + C$
61. $-\frac{2}{5} \cos^5 \theta + C$ 63. $\frac{1}{4} \cos \theta - \frac{1}{20} \cos 5\theta + C$
65. $\sec x - \ln |\csc x + \cot x| + C$ 67. $\cos x + \sec x + C$
69. $\frac{1}{4}x^2 - \frac{1}{4}x \sin 2x - \frac{1}{8} \cos 2x + C$ 71. $\ln(2 + \sqrt{3})$
73. $\pi^2/2$ 75. $\bar{x} = \frac{4\pi}{3}$, $\bar{y} = \frac{8\pi^2 + 3}{12\pi}$ 77. $(\pi/4)(4 - \pi)$

SECTION 8.4, pp. 495–496

1. $\ln|\sqrt{9 + x^2} + x| + C$ 3. $\pi/4$ 5. $\pi/6$
7. $\frac{25}{2} \sin^{-1}\left(\frac{t}{5}\right) + \frac{t\sqrt{25 - t^2}}{2} + C$
9. $\frac{1}{2} \ln \left| \frac{2x}{7} + \frac{\sqrt{4x^2 - 49}}{7} \right| + C$
11. $7 \left[\frac{\sqrt{y^2 - 49}}{7} - \sec^{-1}\left(\frac{y}{7}\right) \right] + C$ 13. $\frac{\sqrt{x^2 - 1}}{x} + C$
15. $\operatorname{arcsec}|x| + C$ 17. $\sqrt{x^2 - 1} + C$
19. $-\sqrt{9 - x^2} + C$ 21. $\frac{1}{3}(x^2 + 4)^{3/2} - 4\sqrt{x^2 + 4} + C$
23. $\frac{-2\sqrt{4 - w^2}}{w} + C$ 25. $\arcsin x - \sqrt{1 - x^2} + C$
27. $4\sqrt{3} - \frac{4\pi}{3}$ 29. $-\frac{x}{\sqrt{x^2 - 1}} + C$
31. $-\frac{1}{5} \left(\frac{\sqrt{1 - x^2}}{x} \right)^5 + C$ 33. $2 \tan^{-1} 2x + \frac{4x}{(4x^2 + 1)} + C$
35. $\frac{1}{2}x^2 + \frac{1}{2} \ln|x^2 - 1| + C$ 37. $\frac{1}{3} \left(\frac{v}{\sqrt{1 - v^2}} \right)^3 + C$
39. $\ln 9 - \ln(1 + \sqrt{10})$ 41. $\pi/6$
43. $\frac{1}{2} \ln |\sqrt{1 + x^4} + x^2| + C$
45. $4 \arcsin \frac{\sqrt{x}}{2} + \sqrt{x} \sqrt{4 - x} + C$
47. $\frac{1}{4} \arcsin \sqrt{x} - \frac{1}{4} \sqrt{x} \sqrt{1 - x} (1 - 2x) + C$

SECTION 8.3, pp. 490–491

1. $\frac{1}{2} \sin 2x + C$ 3. $-\frac{1}{4} \cos^4 x + C$
5. $\frac{1}{3} \cos^3 x - \cos x + C$ 7. $-\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C$

$$49. (9/2)\arcsin\left(\frac{x+1}{3}\right) + (1/2)(x+1)\sqrt{8-2x-x^2} + C$$

$$51. \sqrt{x^2+4x+3} - \operatorname{arcsec}(x+2) + C$$

$$53. y = 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\left(\frac{x}{2}\right)\right]$$

$$55. y = \frac{3}{2}\arctan\left(\frac{x}{2}\right) - \frac{3\pi}{8} \quad 57. 3\pi/4$$

$$59. (a) \frac{1}{12}(\pi + 6\sqrt{3} - 12)$$

$$(b) \bar{x} = \frac{3\sqrt{3} - \pi}{4(\pi + 6\sqrt{3} - 12)}, \bar{y} = \frac{\pi^2 + 12\sqrt{3}\pi - 72}{12(\pi + 6\sqrt{3} - 12)}$$

$$61. (a) -\frac{1}{3}x^2(1-x^2)^{3/2} - \frac{2}{15}(1-x^2)^{5/2} + C$$

$$(b) -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C$$

$$(c) \frac{1}{5}(1-x^2)^{5/2} - \frac{1}{3}(1-x^2)^{3/2} + C$$

$$63. \sqrt{3} - \frac{\sqrt{2}}{2} + \frac{1}{2}\ln\left(\frac{2+\sqrt{3}}{\sqrt{2}+1}\right)$$

SECTION 8.5, pp. 502–504

$$1. \frac{2}{x-3} + \frac{3}{x-2} \quad 3. \frac{1}{x+1} + \frac{3}{(x+1)^2}$$

$$5. \frac{-2}{z} + \frac{-1}{z^2} + \frac{2}{z-1} \quad 7. 1 + \frac{17}{t-3} + \frac{-12}{t-2}$$

$$9. \frac{1}{2}[\ln|1+x| - \ln|1-x|] + C$$

$$11. \frac{1}{7}\ln|(x+6)^2(x-1)^5| + C \quad 13. (\ln 15)/2$$

$$15. -\frac{1}{2}\ln|t| + \frac{1}{6}\ln|t+2| + \frac{1}{3}\ln|t-1| + C \quad 17. 3\ln 2 - 2$$

$$19. \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2(x^2-1)} + C \quad 21. (\pi + 2\ln 2)/8$$

$$23. \tan^{-1}y - \frac{1}{y^2+1} + C$$

$$25. -(s-1)^{-2} + (s-1)^{-1} + \arctan s + C$$

$$27. \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln|x^2+x+1| - \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$29. \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\arctan x + C$$

$$31. \frac{-1}{\theta^2+2\theta+2} + \ln(\theta^2+2\theta+2) - \arctan(\theta+1) + C$$

$$33. x^2 + \ln\left|\frac{x-1}{x}\right| + C$$

$$35. 9x + 2\ln|x| + \frac{1}{x} + 7\ln|x-1| + C$$

$$37. \frac{y^2}{2} - \ln|y| + \frac{1}{2}\ln(1+y^2) + C \quad 39. \ln\left(\frac{e^t+1}{e^t+2}\right) + C$$

$$41. \frac{1}{5}\ln\left|\frac{\sin y - 2}{\sin y + 3}\right| + C$$

$$43. \frac{(\tan^{-1} 2x)^2}{4} - 3\ln|x-2| + \frac{6}{x-2} + C$$

$$45. \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C$$

$$47. 2\sqrt{1+x} + \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$$

$$49. \frac{1}{4}\ln\left|\frac{x^4}{x^4+1}\right| + C$$

$$51. \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}\cos\theta+1}{\sqrt{2}\cos\theta-1}\right| + \frac{1}{2}\ln\left|\frac{1-\cos\theta}{1+\cos\theta}\right| + C$$

$$53. 4\sqrt{1+\sqrt{x}} + 2\ln\left|\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}\right| + C$$

$$55. \frac{1}{3}x^3 - 2x^2 + 5x - 10\ln|x+2| + C$$

$$57. \frac{1}{\ln 2}\ln(2^x + 2^{-x}) + C \quad 59. \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\arctan x + C$$

$$61. \frac{1}{2}\ln|(\ln x + 1)(\ln x + 3)| + C$$

$$63. \ln|x + \sqrt{x^2-1}| + C$$

$$65. \frac{2}{9}x^3(x^3+1)^{3/2} - \frac{4}{45}(x^3+1)^{5/2} + C$$

$$67. x = \ln|t-2| - \ln|t-1| + \ln 2$$

$$69. x = \frac{6t}{t+2} - 1 \quad 71. 3\pi\ln 25$$

$$73. \ln(3) - \frac{1}{2} \quad 75. 1.10$$

$$77. (a) x = \frac{1000e^{4t}}{499 + e^{4t}} \quad (b) 1.55 \text{ days}$$

SECTION 8.6, pp. 508–510

$$1. \frac{2}{\sqrt{3}}\left(\arctan\sqrt{\frac{x-3}{3}}\right) + C$$

$$3. \sqrt{x-2}\left(\frac{2(x-2)}{3} + 4\right) + C \quad 5. \frac{(2x-3)^{3/2}(x+1)}{5} + C$$

$$7. \frac{-\sqrt{9-4x}}{x} - \frac{2}{3}\ln\left|\frac{\sqrt{9-4x}-3}{\sqrt{9-4x}+3}\right| + C$$

$$9. \frac{(x+2)(2x-6)\sqrt{4x-x^2}}{6} + 4\arcsin\left(\frac{x-2}{2}\right) + C$$

$$11. -\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7} + \sqrt{7+x^2}}{x}\right| + C$$

$$13. \sqrt{4-x^2} - 2\ln\left|\frac{2+\sqrt{4-x^2}}{x}\right| + C$$

$$15. \frac{e^{2t}}{13}(2\cos 3t + 3\sin 3t) + C$$

$$17. \frac{x^2}{2}\arccos x + \frac{1}{4}\arcsin x - \frac{1}{4}x\sqrt{1-x^2} + C$$

$$19. \frac{x^3}{3}\arctan x - \frac{x^2}{6} + \frac{1}{6}\ln(1+x^2) + C$$

$$21. -\frac{\cos 5x}{10} - \frac{\cos x}{2} + C$$

$$23. 8\left[\frac{\sin(7t/2)}{7} - \frac{\sin(9t/2)}{9}\right] + C$$

$$25. 6\sin(\theta/12) + \frac{6}{7}\sin(7\theta/12) + C$$

$$27. \frac{1}{2}\ln(x^2+1) + \frac{x}{2(1+x^2)} + \frac{1}{2}\tan^{-1}x + C$$

$$29. \left(x - \frac{1}{2}\right)\arcsin \sqrt{x} + \frac{1}{2}\sqrt{x-x^2} + C$$

$$31. \arcsin \sqrt{x} - \sqrt{x-x^2} + C$$

33. $\sqrt{1 - \sin^2 t} - \ln \left| \frac{1 + \sqrt{1 - \sin^2 t}}{\sin t} \right| + C$
35. $\ln \left| \ln y + \sqrt{3 + (\ln y)^2} \right| + C$
37. $\ln \left| x + 1 + \sqrt{x^2 + 2x + 5} \right| + C$
39. $\frac{x+2}{2} \sqrt{5-4x-x^2} + \frac{9}{2} \arcsin \left(\frac{x+2}{3} \right) + C$
41. $-\frac{\sin^4 2x \cos 2x}{10} - \frac{2 \sin^2 2x \cos 2x}{15} - \frac{4 \cos 2x}{15} + C$
43. $\frac{\sin^3 2\theta \cos^2 2\theta}{10} + \frac{\sin^3 2\theta}{15} + C$
45. $\tan^2 2x - 2 \ln |\sec 2x| + C$
47. $\frac{(\sec \pi x)(\tan \pi x)}{\pi} + \frac{1}{\pi} \ln |\sec \pi x + \tan \pi x| + C$
49. $-\frac{\csc^3 x \cot x}{4} - \frac{3 \csc x \cot x}{8} - \frac{3}{8} \ln |\csc x + \cot x| + C$
51. $\frac{1}{2} [\sec(e^t - 1) \tan(e^t - 1) + \ln |\sec(e^t - 1) + \tan(e^t - 1)|] + C$
53. $\sqrt{2} + \ln(\sqrt{2} + 1)$ 55. $\pi/3$
57. $2\pi\sqrt{3} + \pi\sqrt{2} \ln(\sqrt{2} + \sqrt{3})$ 59. $\bar{x} = 4/3, \bar{y} = \ln\sqrt{2}$
61. 7.62 63. $\pi/8$ 67. (c) $\pi/4$

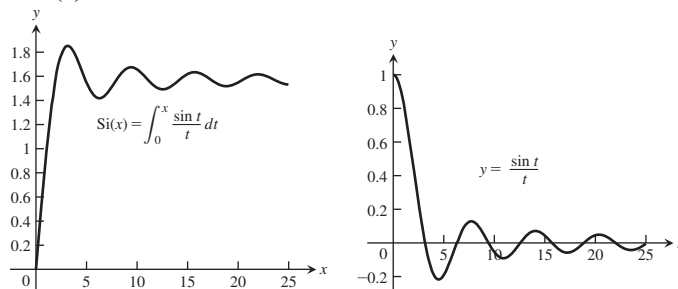
SECTION 8.7, pp. 518–520

1. I: (a) 1.5, 0 (b) 1.5, 0 (c) 0%
 II: (a) 1.5, 0 (b) 1.5, 0 (c) 0%
 III: (a) 1.5, 0 (b) 1.5, 0 (c) 0%
3. I: (a) 2.63, 0.04 (b) 2.67, 0.04 (c) 1.6%
 II: (a) 2.75, 0.08 (b) 2.67, 0.08 (c) 3.1%
 III: (a) 2.67, 0 (b) 2.67, 0 (c) 0%
5. I: (a) 5.88, 0.25 (b) 6, 0.13 (c) 2.1%
 II: (a) 6.25, 0.5 (b) 6, 0.25 (c) 4.2%
 III: (a) 6, 0 (b) 6, 0 (c) 0%
7. I: (a) 0.496, 0.016 (b) 0.5, 0.004 (c) 0.9%
 II: (a) 0.509, 0.031 (b) 0.5, 0.009 (c) 1.8%
 III: (a) 0.5004, 0.0026 (b) 0.5, 0.0004 (c) 0.1%
9. I: (a) 2.052, 0.081 (b) 2, 0.052 (c) 2.6%
 II: (a) 1.896, 0.161 (b) 2, 0.104 (c) 5.2%
 III: (a) 2.005, 0.0066 (b) 2, 0.0046 (c) 0.2%
11. (a) 1 (b) 2 13. (a) 116 (b) 2
15. (a) 283 (b) 2 17. (a) 71 (b) 10
19. (a) 76 (b) 12 21. (a) 82 (b) 8
23. 106.6 m³ 25. ≈ 2.55 m
27. (a) ≈ 0.00021 (b) ≈ 1.37079 (c) $\approx 0.015\%$
31. (a) ≈ 5.870 (b) $|E_T| \leq 0.0032$
33. 21.07 cm 35. 14.42 39. ≈ 28.7 mg

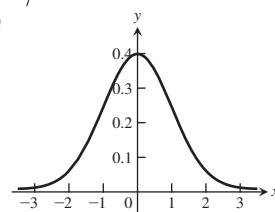
SECTION 8.8, pp. 528–531

1. $\pi/2$ 3. 2 5. 6 7. $\pi/2$ 9. $\ln 3$ 11. $\ln 4$
13. 0 15. $\sqrt{3}$ 17. π 19. $\ln \left(1 + \frac{\pi}{2} \right)$
21. -1 23. 1 25. -1/4 27. $\pi/2$ 29. $\pi/3$
31. 6 33. $\ln 2$ 35. Diverges 37. Diverges
39. Diverges 41. Diverges 43. Converges
45. Converges 47. Diverges 49. Converges
51. Converges 53. Diverges 55. Converges
57. Converges 59. Diverges 61. Converges

63. Diverges 65. Converges 67. Converges
69. Diverges 71. Converges; 20 73. Converges; $\frac{1}{81}$
75. Diverges 77. Diverges 79. Converges; $-\frac{1}{375}$
81. (a) Converges when $p < 1$ (b) Converges when $p > 1$
83. 1 85. 2π 87. $\ln 2$
89. (a) 1 (b) $\pi/3$ (c) Diverges
91. (a) $\pi/2$ (b) π 93. (b) ≈ 0.88621
95. (a)

(b) $\pi/2$

97. (a)

(b) $\approx 0.683, \approx 0.954, \approx 0.997$ 103. ≈ 0.16462

PRACTICE EXERCISES, pp. 532–534

1. $(x+1)(\ln(x+1)) - (x+1) + C$
3. $x \arctan(3x) - \frac{1}{6} \ln(1+9x^2) + C$
5. $(x+1)^2 e^x - 2(x+1)e^x + 2e^x + C$
7. $\frac{2e^x \sin 2x}{5} + \frac{e^x \cos 2x}{5} + C$
9. $2 \ln|x-2| - \ln|x-1| + C$
11. $\ln|x| - \ln|x+1| + \frac{1}{x+1} + C$
13. $-\frac{1}{3} \ln \left| \frac{\cos \theta - 1}{\cos \theta + 2} \right| + C$
15. $4 \ln|x| - \frac{1}{2} \ln(x^2 + 1) + 4 \arctan x + C$
17. $\frac{1}{16} \ln \left| \frac{(v-2)^5(v+2)}{v^6} \right| + C$
19. $\frac{1}{2} \arctan t - \frac{\sqrt{3}}{6} \tan^{-1} \frac{t}{\sqrt{3}} + C$
21. $\frac{x^2}{2} + \frac{4}{3} \ln|x+2| + \frac{2}{3} \ln|x-1| + C$
23. $\frac{x^2}{2} - \frac{9}{2} \ln|x+3| + \frac{3}{2} \ln|x+1| + C$
25. $\frac{1}{3} \ln \left| \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} \right| + C$ 27. $\ln|1 - e^{-x}| + C$
29. $-\sqrt{16-y^2} + C$ 31. $-\frac{1}{2} \ln|4-x^2| + C$
33. $\ln \frac{1}{\sqrt{9-x^2}} + C$ 35. $\frac{1}{6} \ln \left| \frac{x+3}{x-3} \right| + C$

37. $-\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$ 39. $\frac{\tan^5 x}{5} + C$
 41. $\frac{\cos \theta}{2} - \frac{\cos 11\theta}{22} + C$ 43. $4\sqrt{1 - \cos(t/2)} + C$
 45. At least 16 47. All three rules yield π 49. -4°C
 51. (a) ≈ 2.42 L (c) ≈ 24.83 km/L
 53. $\pi/2$ 55. 6 57. $\ln 3$ 59. 2 61. $\pi/6$
 63. Diverges 65. Diverges 67. Converges
 69. $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$ 71. $2 \tan x - x + C$
 73. $x \tan x - \ln|\sec x| + C$ 75. $-\frac{1}{3}(\cos x)^3 + C$
 77. $1 + \frac{1}{2}\ln\left(\frac{2}{1+e^2}\right)$ 79. $2\ln\left|1 - \frac{1}{x}\right| + \frac{4x+1}{2x^2} + C$
 81. $\frac{e^{2x}-1}{e^x} + C$ 83. $9/4$ 85. $256/15$
 87. $-\frac{1}{3}\csc^3 x + C$
 89. $\frac{2x^{3/2}}{3} - x + 2\sqrt{x} - 2\ln(\sqrt{x} + 1) + C$
 91. $\frac{1}{2}\sin^{-1}(x-1) + \frac{1}{2}(x-1)\sqrt{2x-x^2} + C$
 93. $-2\cot x - \ln|\csc x + \cot x| + \csc x + C$
 95. $\frac{1}{12}\ln\left|\frac{3+v}{3-v}\right| + \frac{1}{6}\tan^{-1}\frac{v}{3} + C$
 97. $\frac{\theta \sin(2\theta+1)}{2} + \frac{\cos(2\theta+1)}{4} + C$
 99. $\frac{1}{4}\sec^2 \theta + C$ 101. $2\left(\frac{(\sqrt{2-x})^3}{3} - 2\sqrt{2-x}\right) + C$
 103. $\arctan(y-1) + C$
 105. $\frac{1}{4}\ln|z| - \frac{1}{4z} - \frac{1}{4}\left[\frac{1}{2}\ln(z^2+4) + \frac{1}{2}\arctan\left(\frac{z}{2}\right)\right] + C$
 107. $-\frac{1}{4}\sqrt{9-4t^2} + C$ 109. $\ln\left(\frac{e^t+1}{e^t+2}\right) + C$
 111. $1/4$ 113. $\frac{2}{3}x^{3/2} + C$ 115. $-\frac{1}{5}\arctan(\cos 5t) + C$
 117. $2\sqrt{r} - 2\ln(1+\sqrt{r}) + C$ 119. $\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + C$
 121. $\frac{2}{3}\ln|x+1| + \frac{1}{6}\ln|x^2-x+1| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$
 123. $\frac{4}{7}(1+\sqrt{x})^{7/2} - \frac{8}{5}(1+\sqrt{x})^{5/2} + \frac{4}{3}(1+\sqrt{x})^{3/2} + C$
 125. $2\ln|\sqrt{x}+\sqrt{1+x}| + C$ 127. $\ln x - \ln|1+\ln x| + C$
 129. $\frac{1}{2}x^{\ln x} + C$ 131. $\frac{1}{2}\ln\left|\frac{1-\sqrt{1-x^4}}{x^2}\right| + C$
 133. $x - \frac{1}{\sqrt{2}}\arctan(\sqrt{2}\tan x) + C$ 135. $\frac{\pi}{4}$

ADDITIONAL AND ADVANCED EXERCISES, pp. 534–537

1. $x(\arcsin x)^2 + 2(\arcsin x)\sqrt{1-x^2} - 2x + C$
 3. $\frac{x^2 \arcsin x}{2} + \frac{x\sqrt{1-x^2} - \arcsin x}{4} + C$
 5. $\frac{1}{2}(\ln(t - \sqrt{1-t^2}) - \arcsin t) + C$ 7. 0

9. $\ln(4) - 1$ 11. 1 13. $32\pi/35$ 15. 2π
 17. (a) π (b) $\pi(2e-5)$
 19. (b) $\pi\left(\frac{8(\ln 2)^2}{3} - \frac{16(\ln 2)}{9} + \frac{16}{27}\right)$ 21. $\left(\frac{e^2+1}{4}, \frac{e-2}{2}\right)$
 23. $\sqrt{1+e^2} - \ln\left(\frac{\sqrt{1+e^2}}{e} + \frac{1}{e}\right) - \sqrt{2} + \ln(1+\sqrt{2})$
 25. $\frac{12\pi}{5}$ 27. $a = \frac{1}{2}, -\frac{\ln 2}{4}$ 29. $\frac{1}{2} < p \leq 1$
 33. $\frac{2}{1 - \tan(x/2)} + C$ 35. 1 37. $\frac{\sqrt{3}\pi}{9}$
 39. $\frac{1}{\sqrt{2}}\ln\left|\frac{\tan(t/2)+1-\sqrt{2}}{\tan(t/2)+1+\sqrt{2}}\right| + C$
 41. $\ln\left|\frac{1+\tan(\theta/2)}{1-\tan(\theta/2)}\right| + C$

Chapter 9**SECTION 9.1, pp. 547–551**

1. $a_1 = 0, a_2 = -1/4, a_3 = -2/9, a_4 = -3/16$
 3. $a_1 = 1, a_2 = -1/3, a_3 = 1/5, a_4 = -1/7$
 5. $a_1 = 1/2, a_2 = 1/2, a_3 = 1/2, a_4 = 1/2$
 7. $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \frac{63}{32}, \frac{127}{64}, \frac{255}{128}, \frac{511}{256}, \frac{1023}{512}$
 9. $2, 1, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, -\frac{1}{64}, \frac{1}{128}, \frac{1}{256}$
 11. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55
 13. $a_n = (-1)^{n+1}, n \geq 1$
 15. $a_n = (-1)^{n+1}(n)^2, n \geq 1$ 17. $a_n = \frac{2^{n-1}}{3(n+2)}, n \geq 1$
 19. $a_n = n^2 - 1, n \geq 1$ 21. $a_n = 4n - 3, n \geq 1$
 23. $a_n = \frac{3n+2}{n!}, n \geq 1$ 25. $a_n = \frac{1+(-1)^{n+1}}{2}, n \geq 1$
 27. $a_n = \frac{1}{(n+1)(n+2)}$ 29. $a_n = \sin\left(\frac{\sqrt{n+1}}{1+(n+1)^2}\right)$
 31. Converges, 2 33. Converges, -1 35. Converges, -5
 37. Diverges 39. Diverges 41. Converges, $1/2$
 43. Converges, 0 45. Converges, $\sqrt{2}$ 47. Converges, 1
 49. Converges, 0 51. Converges, 0 53. Converges, 0
 55. Converges, 1 57. Converges, e^7 59. Converges, 1
 61. Converges, 1 63. Diverges 65. Converges, 4
 67. Converges, 0 69. Diverges 71. Converges, e^{-1}
 73. Diverges 75. Converges, 0 77. Diverges
 79. Converges, $e^{2/3}$ 81. Converges, $x(x > 0)$
 83. Converges, 0 85. Converges, 1 87. Converges, $1/2$
 89. Converges, 1 91. Converges, $\pi/2$ 93. Converges, 0
 95. Converges, 0 97. Converges, $1/2$ 99. Converges, 0
 101. 8 103. 4 105. 5 107. $1 + \sqrt{2}$ 109. $x_n = 2^{n-2}$
 111. (a) $f(x) = x^2 - 2, 1.414213562 \approx \sqrt{2}$
 (b) $f(x) = \tan(x) - 1, 0.7853981635 \approx \pi/4$
 (c) $f(x) = e^x$, diverges
 113. (b) 1

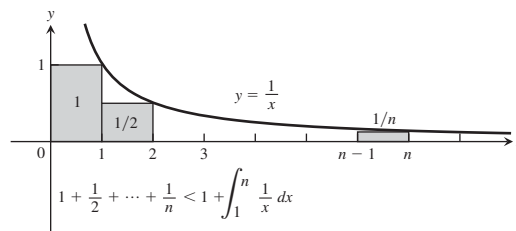
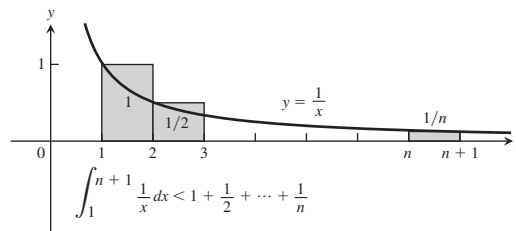
121. Monotonic, bounded
 123. Not monotonic, bounded
 125. Monotonic, bounded, converges
 127. Monotonic, bounded, converges
 129. Not monotonic, bounded, diverges
 131. Monotonic, bounded, converges
 133. Monotonic, bounded, converges
 145. (b) $\sqrt{3}$

SECTION 9.2, pp. 558–560

1. $s_n = \frac{2(1 - (1/3)^n)}{1 - (1/3)}, 3$ 3. $s_n = \frac{1 - (-1/2)^n}{1 - (-1/2)}, 2/3$
 5. $s_n = \frac{1}{2} - \frac{1}{n+2}, \frac{1}{2}$ 7. $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \dots, \frac{4}{5}$
 9. $-\frac{3}{4} + \frac{9}{16} + \frac{57}{64} + \frac{249}{256} + \dots$, diverges.
 11. $(5+1) + (\frac{5}{2} + \frac{1}{3}) + (\frac{5}{4} + \frac{1}{9}) + (\frac{5}{8} + \frac{1}{27}) + \dots, \frac{23}{2}$
 13. $(1+1) + (\frac{1}{2} - \frac{1}{5}) + (\frac{1}{4} + \frac{1}{25}) + (\frac{1}{8} - \frac{1}{125}) + \dots, \frac{17}{6}$
 15. Converges, $5/3$ 17. Converges, $1/7$
 19. Converges, $\frac{e}{e+2}$ 21. Diverges 23. $23/99$
 25. $7/9$ 27. $1/15$ 29. $41333/33300$ 31. Diverges
 33. Inconclusive 35. Diverges 37. Diverges
 39. $s_n = 1 - \frac{1}{n+1}$; converges, 1
 41. $s_n = \ln \sqrt{n+1}$; diverges
 43. $s_n = \frac{\pi}{3} - \arccos(\frac{1}{n+2})$; converges, $-\frac{\pi}{6}$ 45. 1 47. 5
 49. 1 51. $-\frac{1}{\ln 2}$ 53. Converges, $2 + \sqrt{2}$
 55. Converges, 1 57. Diverges
 59. Converges, $\frac{e^2}{e^2 - 1}$
 61. Converges, $2/9$ 63. Converges, $3/2$ 65. Diverges
 67. Converges, 4 69. Diverges 71. Converges, $\frac{\pi}{\pi - e}$
 73. Converges, $-5/6$ 75. Diverges
 77. $a = 1, r = -x$; converges to $1/(1+x)$ for $|x| < 1$
 79. $a = 3, r = (x-1)/2$; converges to $6/(3-x)$ for x in $(-1, 3)$
 81. $|x| < \frac{1}{2}, \frac{1}{1-2x}$ 83. $-2 < x < 0, \frac{1}{2+x}$
 85. $x \neq (2k+1)\frac{\pi}{2}, k$ an integer; $\frac{1}{1 - \sin x}$
 87. (a) $\sum_{n=-2}^{\infty} \frac{1}{(n+4)(n+5)}$ (b) $\sum_{n=0}^{\infty} \frac{1}{(n+2)(n+3)}$
 (c) $\sum_{n=5}^{\infty} \frac{1}{(n-3)(n-2)}$
 97. (a) $r = 3/5$ (b) $r = -3/10$ 99. $|r| < 1, \frac{1+2r}{1-r^2}$
 101. (a) 16.84 mg, 17.79 mg (b) 17.84 mg
 103. (a) $0, \frac{1}{27}, \frac{2}{27}, \frac{1}{9}, \frac{2}{9}, \frac{7}{27}, \frac{8}{27}, \frac{1}{3}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}, 1$
 (b) $\sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{2}{3}\right)^{n-1} = 1$ 105. $(4/3)\pi$

SECTION 9.3, pp. 565–567

1. Converges 3. Converges 5. Converges 7. Diverges
 9. Converges 11. Diverges
 13. Converges; geometric series, $r = \frac{1}{10} < 1$
 15. Diverges; $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$
 17. Diverges; p -series, $p < 1$
 19. Converges; geometric series, $r = \frac{1}{8} < 1$
 21. Diverges; Integral Test
 23. Converges; geometric series, $r = 2/3 < 1$
 25. Diverges; Integral Test
 27. Diverges; $\lim_{n \rightarrow \infty} \frac{2^n}{n+1} \neq 0$
 29. Diverges; $\lim_{n \rightarrow \infty} (\sqrt{n}/\ln n) \neq 0$
 31. Diverges; geometric series, $r = \frac{1}{\ln 2} > 1$
 33. Converges; Integral Test
 35. Diverges; n th-Term Test
 37. Converges; Integral Test
 39. Diverges; n th-Term Test
 41. Converges; by taking limit of partial sums
 43. Converges; Integral Test
 45. Converges; Integral Test 47. $a = 1$
 49. (a)



- (b) ≈ 41.55
 51. True 53. $n \geq 251,415$
 55. $s_8 = \sum_{n=1}^8 \frac{1}{n^3} \approx 1.195$ 57. 10^{60}
 65. (a) $1.20166 \leq S \leq 1.20253$
 (b) $S \approx 1.2021$, error < 0.0005
 67. $\left(\frac{\pi^2}{6} - 1\right) \approx 0.64493$

SECTION 9.4, pp. 571–572

1. Converges; compare with $\sum (1/n^2)$
 3. Diverges; compare with $\sum (1/\sqrt{n})$
 5. Converges; compare with $\sum (1/n^{3/2})$
 7. Converges; compare with $\sum \sqrt{\frac{n+4n}{n^4+0}} = \sqrt{5} \sum \frac{1}{n^{3/2}}$
 9. Converges

11. Diverges; limit comparison with $\sum (1/n)$
 13. Diverges; limit comparison with $\sum (1/\sqrt{n})$
 15. Diverges
 17. Diverges; limit comparison with $\sum (1/\sqrt{n})$
 19. Converges; compare with $\sum (1/2^n)$
 21. Diverges; n th-Term Test
 23. Converges; compare with $\sum (1/n^2)$
 25. Converges; $\left(\frac{n}{3n+1}\right)^n < \left(\frac{n}{3n}\right)^n = \left(\frac{1}{3}\right)^n$
 27. Diverges; direct comparison with $\sum (1/n)$
 29. Diverges; limit comparison with $\sum (1/n)$
 31. Diverges; limit comparison with $\sum (1/n)$
 33. Converges; compare with $\sum (1/n^{3/2})$
 35. Converges; $\frac{1}{n2^n} \leq \frac{1}{2^n}$ 37. Converges; $\frac{1}{3^{n-1}+1} < \frac{1}{3^{n-1}}$
 39. Converges; comparison with $\sum (1/5n^2)$
 41. Diverges; comparison with $\sum (1/n)$
 43. Converges; comparison with $\sum \frac{1}{n(n-1)}$ or limit comparison with $\sum (1/n^2)$
 45. Diverges; limit comparison with $\sum (1/n)$
 47. Converges; $\frac{\tan^{-1} n}{n^{1.1}} < \frac{\pi/2}{n^{1.1}}$
 49. Converges; compare with $\sum (1/n^2)$
 51. Diverges; limit comparison with $\sum (1/n)$
 53. Converges; limit comparison with $\sum (1/n^2)$
 55. Diverges n th-Term Test
 67. Converges 69. Converges 71. Converges

SECTION 9.5, pp. 577–578

1. Converges 3. Diverges 5. Converges
 7. Converges 9. Converges 11. Diverges
 13. Converges 15. Converges 17. Converges; Ratio Test
 19. Diverges; Ratio Test 21. Converges; Ratio Test
 23. Converges; compare with $\sum (3/(1.25)^n)$
 25. Diverges; $\lim_{n \rightarrow \infty} \left(1 - \frac{3}{n}\right)^n = e^{-3} \neq 0$
 27. Converges; compare with $\sum (1/n^2)$
 29. Diverges; compare with $\sum (1/(2n))$ 31. Diverges; $a_n \not\rightarrow 0$
 33. Converges; Ratio Test 35. Converges; Ratio Test
 37. Converges; Ratio Test 39. Converges; Root Test
 41. Converges; compare with $\sum (1/n^2)$
 43. Converges; Ratio Test 45. Diverges; Ratio Test
 47. Converges; Ratio Test 49. Diverges; Ratio Test
 51. Converges; Ratio Test 53. Converges; Ratio Test
 55. Diverges; $a_n = \left(\frac{1}{3}\right)^{(1/n!)} \rightarrow 1$ 57. Converges; Ratio Test
 59. Diverges; Root Test 61. Converges; Root Test
 63. Converges; Ratio Test
 65. (a) Diverges; n th-Term Test (b) Diverges; Root Test
 (c) Converges; Root Test (d) Converges; Ratio Test
 69. Yes 71. Converges absolutely

SECTION 9.6, pp. 583–585

1. Converges by Alternating Series Test
 3. Converges; Alternating Series Test
 5. Converges; Alternating Series Test
 7. Diverges; $a_n \not\rightarrow 0$
 9. Diverges; $a_n \not\rightarrow 0$
 11. Converges; Alternating Series Test
 13. Converges by Alternating Series Test
 15. Converges absolutely. Series of absolute values is a convergent geometric series.
 17. Converges conditionally; $1/\sqrt{n} \rightarrow 0$ but $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.
 19. Converges absolutely; compare with $\sum_{n=1}^{\infty} (1/n^2)$.
 21. Converges conditionally; $1/(n+3) \rightarrow 0$ but $\sum_{n=1}^{\infty} \frac{1}{n+3}$ diverges (compare with $\sum_{n=1}^{\infty} (1/n)$).
 23. Diverges; $\frac{3+n}{5+n} \rightarrow 1$
 25. Converges conditionally; $\left(\frac{1}{n^2} + \frac{1}{n}\right) \rightarrow 0$ but $(1+n)/n^2 > 1/n$
 27. Converges absolutely; Ratio Test
 29. Converges absolutely by Integral Test
 31. Diverges; $a_n \not\rightarrow 0$
 33. Converges absolutely by Ratio Test
 35. Converges absolutely, since $\left|\frac{\cos n\pi}{n\sqrt{n}}\right| = \left|\frac{(-1)^{n+1}}{n^{3/2}}\right| = \frac{1}{n^{3/2}}$ (convergent p -series)
 37. Converges absolutely by Root Test
 39. Diverges; $a_n \rightarrow \infty$
 41. Converges conditionally;
 $\sqrt{n+1} - \sqrt{n} = 1/(\sqrt{n} + \sqrt{n+1}) \rightarrow 0$, but series of absolute values diverges (compare with $\sum (1/\sqrt{n})$).
 43. Diverges, $a_n \rightarrow 1/2 \neq 0$
 45. Converges absolutely;
 $\operatorname{sech} n = \frac{2}{e^n + e^{-n}} = \frac{2e^n}{e^{2n} + 1} < \frac{2e^n}{e^{2n}} = \frac{2}{e^n}$, a term from a convergent geometric series.
 47. Converges conditionally; $\sum (-1)^{n+1} \frac{1}{2(n+1)}$ converges by Alternating Series Test; $\sum \frac{1}{2(n+1)}$ diverges by limit comparison with $\sum (1/n)$.
 49. |Error| < 0.2 51. |Error| < 2×10^{-11}
 53. $n \geq 31$ 55. $n \geq 4$ 57. Converges; Root Test
 59. Converges; Limit of Partial Sums
 61. Converges; Ratio Test 63. Diverges; p -series Test
 65. Converges; Root Test 67. Converges; Limit Comparison Test
 69. Diverges; Limit of Partial Sums
 71. Diverges; Limit Comparison Test
 73. Diverges; n th-Term Test
 75. Diverges; Limit of Partial Sums
 77. Converges; Limit Comparison Test
 79. Converges; Limit Comparison Test
 81. Converges; Ratio Test
 83. 0.54030 85. (a) $a_n \geq a_{n+1}$ (b) $-1/2$

SECTION 9.7, pp. 594–597

1. (a) $1, -1 < x < 1$ (b) $-1 < x < 1$ (c) none
 3. (a) $1/4, -1/2 < x < 0$ (b) $-1/2 < x < 0$ (c) none
 5. (a) $10, -8 < x < 12$ (b) $-8 < x < 12$ (c) none

7. (a) $1, -1 < x < 1$ (b) $-1 < x < 1$ (c) none
 9. (a) $3, -3 \leq x \leq 3$ (b) $-3 \leq x \leq 3$ (c) none
 11. (a) ∞ , for all x (b) for all x (c) none
 13. (a) $1/2, -1/2 < x < 1/2$ (b) $-1/2 < x < 1/2$ (c) none
 15. (a) $1, -1 \leq x < 1$ (b) $-1 < x < 1$ (c) $x = -1$
 17. (a) $5, -8 < x < 2$ (b) $-8 < x < 2$ (c) none
 19. (a) $3, -3 < x < 3$ (b) $-3 < x < 3$ (c) none
 21. (a) $1, -2 < x < 0$ (b) $-2 < x < 0$ (c) none
 23. (a) $1, -1 < x < 1$ (b) $-1 < x < 1$ (c) none
 25. (a) $0, x = 0$ (b) $x = 0$ (c) none
 27. (a) $2, -4 < x \leq 0$ (b) $-4 < x < 0$ (c) $x = 0$
 29. (a) $1, -1 \leq x \leq 1$ (b) $-1 \leq x \leq 1$ (c) none
 31. (a) $1/4, 1 \leq x \leq 3/2$ (b) $1 \leq x \leq 3/2$ (c) none
 33. (a) ∞ , for all x (b) for all x (c) none
 35. (a) $1, -1 \leq x < 1$ (b) $-1 < x < 1$ (c) -1
 37. 3 39. 8 41. ∞
 43. $-1/3 < x < 1/3, 1/(1-3x)$
 45. $-1 < x < 3, 4/(3+2x-x^2)$
 47. $0 < x < 16, 2/(4-\sqrt{x})$
 49. $-\sqrt{2} < x < \sqrt{2}, 3/(2-x^2)$
 51. $\frac{2}{x} = \sum_{n=0}^{\infty} 2(-1)^n (x-1)^n, 0 < x < 2$
 53. $\sum_{n=0}^{\infty} (-\frac{1}{3})^n (x-5)^n, 2 < x < 8$
 55. $1 < x < 5, 2/(x-1), \sum_{n=1}^{\infty} (-\frac{1}{2})^n n(x-3)^{n-1},$
 $1 < x < 5, -2/(x-1)^2$
 57. (a) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots;$
 converges for all x
 (b) Same answer as part (c)
 (c) $2x - \frac{2^3 x^3}{3!} + \frac{2^5 x^5}{5!} - \frac{2^7 x^7}{7!} + \frac{2^9 x^9}{9!} - \frac{2^{11} x^{11}}{11!} + \dots$
 59. (a) $\frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31x^{10}}{14175}, -\frac{\pi}{2} < x < \frac{\pi}{2}$
 (b) $1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} + \dots, -\frac{\pi}{2} < x < \frac{\pi}{2}$
 65. (a) T (b) T (c) F (d) T (e) N (f) F (g) N (h) T

SECTION 9.8, pp. 601–602

1. $P_0(x) = 1, P_1(x) = 1 + 2x, P_2(x) = 1 + 2x + 2x^2,$
 $P_3(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3$
 3. $P_0(x) = 0, P_1(x) = x - 1, P_2(x) = (x - 1) - \frac{1}{2}(x - 1)^2,$
 $P_3(x) = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3$
 5. $P_0(x) = \frac{1}{2}, P_1(x) = \frac{1}{2} - \frac{1}{4}(x - 2),$
 $P_2(x) = \frac{1}{2} - \frac{1}{4}(x - 2) + \frac{1}{8}(x - 2)^2,$
 $P_3(x) = \frac{1}{2} - \frac{1}{4}(x - 2) + \frac{1}{8}(x - 2)^2 - \frac{1}{16}(x - 2)^3$

7. $P_0(x) = \frac{\sqrt{2}}{2}, P_1(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right),$
 $P_2(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2,$
 $P_3(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2$
 $- \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3$
 9. $P_0(x) = 2, P_1(x) = 2 + \frac{1}{4}(x - 4),$
 $P_2(x) = 2 + \frac{1}{4}(x - 4) - \frac{1}{64}(x - 4)^2,$
 $P_3(x) = 2 + \frac{1}{4}(x - 4) - \frac{1}{64}(x - 4)^2 + \frac{1}{512}(x - 4)^3$
 11. $\sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$
 13. $\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \dots$
 15. $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!}$ 17. $7 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ 19. $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$
 21. $x^4 - 2x^3 - 5x + 4$ 23. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{(2n-1)!}$
 25. $8 + 10(x - 2) + 6(x - 2)^2 + (x - 2)^3$
 27. $21 - 36(x + 2) + 25(x + 2)^2 - 8(x + 2)^3 + (x + 2)^4$
 29. $\sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$ 31. $\sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n$
 33. $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^{2n}}{(2n)!} \left(x - \frac{\pi}{4}\right)^{2n}$
 35. $-1 - 2x - \frac{5}{2}x^2 - \dots, -1 < x < 1$
 37. $x^2 - \frac{1}{2}x^3 + \frac{1}{6}x^4 + \dots, -1 < x < 1$
 39. $x^4 + x^6 + \frac{x^8}{2} + \dots, (-\infty, \infty)$
 41. $L(x) = 0, Q(x) = -x^2/2$ 43. $L(x) = 1, Q(x) = 1 + x^2/2$
 45. $L(x) = x, Q(x) = x$

SECTION 9.9, pp. 608–609

1. $\sum_{n=0}^{\infty} \frac{(-5x)^n}{n!} = 1 - 5x + \frac{5^2 x^2}{2!} - \frac{5^3 x^3}{3!} + \dots$
 3. $\sum_{n=0}^{\infty} \frac{5(-1)^n (-x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{5(-1)^{n+1} x^{2n+1}}{(2n+1)!}$
 $= -5x + \frac{5x^3}{3!} - \frac{5x^5}{5!} + \frac{5x^7}{7!} + \dots$
 5. $\sum_{n=0}^{\infty} \frac{(-1)^n (5x^2)^{2n}}{(2n)!} = 1 - \frac{25x^4}{2!} + \frac{625x^8}{4!} - \dots$
 7. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n} = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \dots$
 9. $\sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{4}\right)^n x^{3n} = 1 - \frac{3}{4}x^3 + \frac{3^2}{4^2}x^6 - \frac{3^3}{4^3}x^9 + \dots$
 11. $\ln 3 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n x^n}{n} = \ln 3 + 2x - 2x^2 + \frac{8}{3}x^3 - \dots$
 13. $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \dots$

15. $\sum_{n=2}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots$
17. $x - \frac{\pi^2 x^3}{2!} + \frac{\pi^4 x^5}{4!} - \frac{\pi^6 x^7}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} x^{2n+1}}{(2n)!}$
19. $1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{2 \cdot (2n)!} =$
 $1 - \frac{(2x)^2}{2 \cdot 2!} + \frac{(2x)^4}{2 \cdot 4!} - \frac{(2x)^6}{2 \cdot 6!} + \frac{(2x)^8}{2 \cdot 8!} - \dots$
21. $x^2 \sum_{n=0}^{\infty} (2x)^n = x^2 + 2x^3 + 4x^4 + \dots$
23. $\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \dots$
25. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n-1}}{2n-1} = x^3 - \frac{x^7}{3} + \frac{x^{11}}{5} - \frac{x^{15}}{7} + \dots$
27. $\sum_{n=0}^{\infty} \left(\frac{1}{n!} + (-1)^n \right) x^n = 2 + \frac{3}{2}x^2 - \frac{5}{6}x^3 + \frac{25}{24}x^4 - \dots$
29. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n+1}}{3n} = \frac{x^3}{3} - \frac{x^5}{6} + \frac{x^7}{9} - \dots$
31. $x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \dots$
33. $x^2 - \frac{2}{3}x^4 + \frac{23}{45}x^6 - \frac{44}{105}x^8 + \dots$
35. $1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \dots$ 37. $1 - \frac{x^2}{2} - \frac{x^3}{2} - \frac{x^4}{4} - \dots$
39. $|\text{Error}| \leq \frac{1}{10^4 \cdot 4!} < 4.2 \times 10^{-6}$
41. $|x| < (0.06)^{1/5} < 0.56968$
43. $|\text{Error}| < (10^{-3})^3/6 < 1.67 \times 10^{-10}, \quad -10^{-3} < x < 0$
45. $|\text{Error}| < (3^{0.1})(0.1)^3/6 < 1.87 \times 10^{-4}$
53. (a) $Q(x) = 1 + kx + \frac{k(k-1)}{2}x^2$ (b) $0 \leq x < 100^{-1/3}$

SECTION 9.10, pp. 616–618

1. $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$ 3. $1 + 3x + 6x^2 + 10x^3$
5. $1 - x + \frac{3x^2}{4} - \frac{x^3}{2}$ 7. $1 - \frac{x^3}{2} + \frac{3x^6}{8} - \frac{5x^9}{16}$
9. $1 + \frac{3}{4}x^2 + \frac{3}{32}x^4 - \frac{1}{128}x^6$
11. $(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$
13. $(1-2x)^3 = 1 - 6x + 12x^2 - 8x^3$ 15. 0.0713362
17. 0.4969536 19. 0.0999445 21. 0.10000
23. $\frac{1}{13 \cdot 6!} \approx 0.00011$ 25. $\frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!}$
27. (a) $\frac{x^2}{2} - \frac{x^4}{12}$
 (b) $\frac{x^2}{2} - \frac{x^4}{3 \cdot 4} + \frac{x^6}{5 \cdot 6} - \frac{x^8}{7 \cdot 8} + \dots + (-1)^{15} \frac{x^{32}}{31 \cdot 32}$
29. $1/2$ 31. $-1/24$ 33. $1/3$ 35. -1 37. 2
39. $3/2$ 41. e 43. $\cos \frac{3}{4}$ 45. $\frac{\sqrt{3}}{2}$ 47. $\frac{x^3}{1-x}$
49. $\frac{x^3}{1+x^2}$ 51. $\frac{-1}{(1+x)^2}$ 55. 500 terms 57. 4 terms
59. (a) $x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112}$, radius of convergence = 1

(b) $\frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \frac{5x^7}{112}$

61. $1 - 2x + 3x^2 - 4x^3 + \dots$
67. (a) -1 (b) $(1/\sqrt{2})(1+i)$ (c) $-i$
71. $x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \dots$, for all x

PRACTICE EXERCISES, pp. 619–621

1. Converges to 1 3. Converges to -1 5. Diverges
7. Converges to 0 9. Converges to 1 11. Converges to e^{-5}
13. Converges to 3 15. Converges to $\ln 2$ 17. Diverges
19. $1/6$ 21. $3/2$ 23. $e/(e-1)$ 25. Diverges
27. Converges conditionally 29. Converges conditionally
31. Converges absolutely 33. Converges absolutely
35. Converges absolutely 37. Converges absolutely
39. Converges absolutely 41. Converges absolutely
43. Diverges
47. (a) $3, -7 \leq x < -1$ (b) $-7 < x < -1$ (c) $x = -7$
49. (a) $1/3, 0 \leq x \leq 2/3$ (b) $0 \leq x \leq 2/3$ (c) None
51. (a) ∞ , for all x (b) For all x (c) None
53. (a) $\sqrt{3}, -\sqrt{3} < x < \sqrt{3}$ (b) $-\sqrt{3} < x < \sqrt{3}$ (c) None
55. (a) $e, -e < x < e$ (b) $-e < x < e$ (c) Empty set
57. $\frac{1}{1+x}, \frac{1}{4}, \frac{4}{5}$ 59. $\sin x, \pi, 0$ 61. $e^x, \ln 2, 2$
63. $\sum_{n=0}^{\infty} 2^n x^n$ 65. $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1} x^{2n+1}}{(2n+1)!}$
67. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{10n/3}}{(2n)!}$ 69. $\sum_{n=0}^{\infty} \frac{((\pi x)/2)^n}{n!}$
71. $2 - \frac{(x+1)}{2 \cdot 1!} + \frac{3(x+1)^2}{2^3 \cdot 2!} + \frac{9(x+1)^3}{2^5 \cdot 3!} + \dots$
73. $\frac{1}{4} - \frac{1}{4^2}(x-3) + \frac{1}{4^3}(x-3)^2 - \frac{1}{4^4}(x-3)^3$
75. 0.4849171431 77. 0.4872223583 79. $7/2$ 81. $1/12$
83. -2 85. $r = -3, s = 9/2$ 87. $2/3$
89. $\ln\left(\frac{n+1}{2n}\right)$; the series converges to $\ln\left(\frac{1}{2}\right)$.
91. (a) ∞ (b) $a = 1, b = 0$ 93. It converges.
101. (a) Converges; Limit Comparison Test
 (b) Converges; Direct Comparison Test
 (c) Diverges; n th-Term Test
103. 2

ADDITIONAL AND ADVANCED EXERCISES, pp. 621–623

1. Converges; Comparison Test
3. Diverges; n th-Term Test
5. Converges; Comparison Test
7. Diverges; n th-Term Test
9. With $a = \pi/3$, $\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \pi/3) - \frac{1}{4}(x - \pi/3)^2 + \frac{\sqrt{3}}{12}(x - \pi/3)^3 + \dots$
11. With $a = 0$, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
13. With $a = 22\pi$, $\cos x = 1 - \frac{1}{2}(x - 22\pi)^2 + \frac{1}{4!}(x - 22\pi)^4 - \frac{1}{6!}(x - 22\pi)^6 + \dots$
15. Converges, limit = b 17. $\pi/2$ 21. $b = \pm \frac{1}{5}$
23. $a = 2, L = -7/6$ 27. (b) Yes

33. (a) $\sum_{n=1}^{\infty} nx^{n-1}$ (b) 6 (c) $\frac{1}{q}$

35. (a) $R_n = C_0 e^{-kt_0} (1 - e^{-nkt_0}) / (1 - e^{-kt_0})$,
 $R = C_0 (e^{-kt_0}) / (1 - e^{-kt_0}) = C_0 / (e^{kt_0} - 1)$

(b) $R_1 = 1/e \approx 0.368$,

$R_{10} = R(1 - e^{-10}) \approx R(0.9999546) \approx 0.58195$;

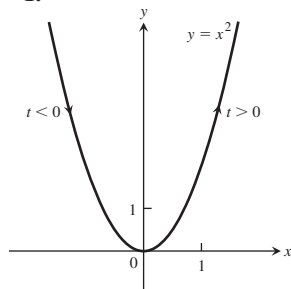
$R \approx 0.58198$; $0 < (R - R_{10})/R < 0.0001$

(c) 7

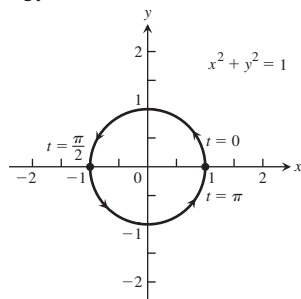
Chapter 10

SECTION 10.1, pp. 630–633

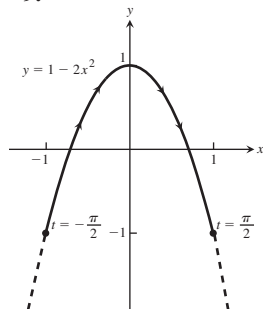
1.



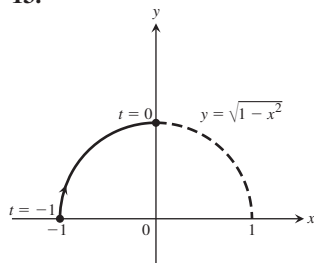
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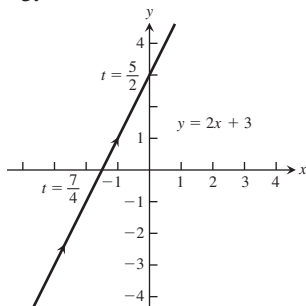
9.



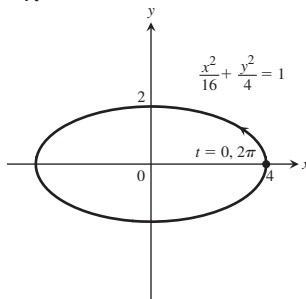
13.



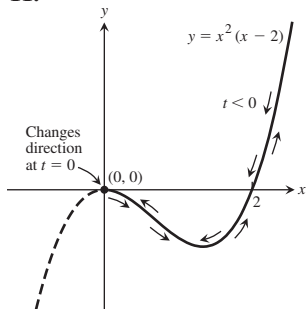
3.



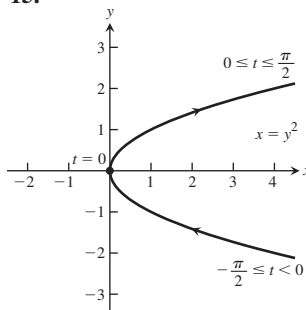
7.



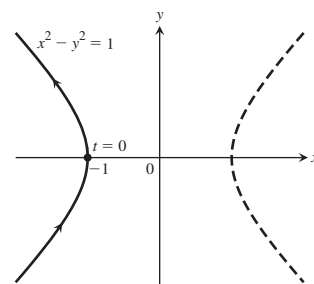
11.



15.



17.

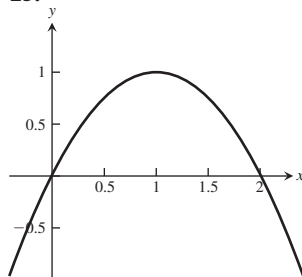


19. D

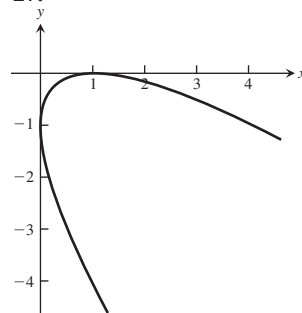
21. E

23. C

25.



27.



29. (a) $x = a \cos t$, $y = -a \sin t$, $0 \leq t \leq 2\pi$

(b) $x = a \cos t$, $y = a \sin t$, $0 \leq t \leq 2\pi$

(c) $x = a \cos t$, $y = -a \sin t$, $0 \leq t \leq 4\pi$

(d) $x = a \cos t$, $y = a \sin t$, $0 \leq t \leq 4\pi$

31. Possible answer: $x = -1 + 5t$, $y = -3 + 4t$, $0 \leq t \leq 1$

33. Possible answer: $x = t^2 + 1$, $y = t$, $t \leq 0$

35. Possible answer: $x = 2 - 3t$, $y = 3 - 4t$, $t \geq 0$

37. Possible answer: $x = 2 \cos t$, $y = 2|\sin t|$, $0 \leq t \leq 4\pi$

39. Possible answer: $x = \frac{-at}{\sqrt{1+t^2}}$, $y = \frac{a}{\sqrt{1+t^2}}$, $-\infty < t < \infty$

41. Possible answer: $x = \frac{4}{1+2\tan\theta}$, $y = \frac{4\tan\theta}{1+2\tan\theta}$,

$0 \leq \theta < \pi/2$ and $x = 0$, $y = 2$ if $\theta = \pi/2$

43. Possible answer: $x = 2 - \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$

45. $x = 2 \cot t$, $y = 2 \sin^2 t$, $0 < t < \pi$

47. $x = a \sin^2 t \tan t$, $y = a \sin^2 t$, $0 \leq t < \pi/2$ 49. (1, 1)

SECTION 10.2, pp. 640–642

1. $y = -x + 2\sqrt{2}$, $\frac{d^2y}{dx^2} = -\sqrt{2}$

3. $y = -\frac{1}{2}x + 2\sqrt{2}$, $\frac{d^2y}{dx^2} = -\frac{\sqrt{2}}{4}$

5. $y = x + \frac{1}{4}$, $\frac{d^2y}{dx^2} = -2$ 7. $y = 2x - \sqrt{3}$, $\frac{d^2y}{dx^2} = -3\sqrt{3}$

9. $y = x - 4$, $\frac{d^2y}{dx^2} = \frac{1}{2}$

11. $y = \sqrt{3}x - \frac{\pi\sqrt{3}}{3} + 2$, $\frac{d^2y}{dx^2} = -4$

13. $y = 9x - 1$, $\frac{d^2y}{dx^2} = 108$ 15. $-\frac{3}{16}$ 17. -6

19. 1 21. $3a^2\pi$ 23. $|ab|\pi$ 25. 4 27. 12

29. π^2 31. $8\pi^2$ 33. $\frac{52\pi}{3}$ 35. $3\pi\sqrt{5}$

37. $(\bar{x}, \bar{y}) = \left(\frac{12}{\pi} - \frac{24}{\pi^2}, \frac{24}{\pi^2} - 2\right)$

39. $(\bar{x}, \bar{y}) = \left(\frac{1}{3}, \pi - \frac{4}{3}\right)$ 41. (a) π (b) π

43. (a) $x = 1, y = 0, \frac{dy}{dx} = \frac{1}{2}$ (b) $x = 0, y = 3, \frac{dy}{dx} = 0$

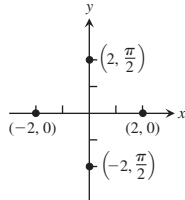
(c) $x = \frac{\sqrt{3}-1}{2}, y = \frac{3-\sqrt{3}}{2}, \frac{dy}{dx} = \frac{2\sqrt{3}-1}{\sqrt{3}-2}$

45. $\left(\frac{\sqrt{2}}{2}, 1\right), y = 2x$ at $t = 0, y = -2x$ at $t = \pi$

47. (a) $8a$ (b) $\frac{64\pi}{3}$ 49. $32\pi/15$

SECTION 10.3, pp. 645–646

1. a, e; b, g; c, h; d, f 3.



(a) $\left(2, \frac{\pi}{2} + 2n\pi\right)$ and $\left(-2, \frac{\pi}{2} + (2n+1)\pi\right), n$ an integer

(b) $(2, 2n\pi)$ and $(-2, (2n+1)\pi), n$ an integer

(c) $\left(2, \frac{3\pi}{2} + 2n\pi\right)$ and $\left(-2, \frac{3\pi}{2} + (2n+1)\pi\right), n$ an integer

(d) $(2, (2n+1)\pi)$ and $(-2, 2n\pi), n$ an integer

5. (a) $(3, 0)$ (b) $(-3, 0)$ (c) $(-1, \sqrt{3})$ (d) $(1, \sqrt{3})$

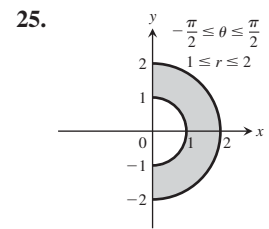
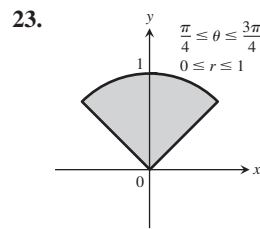
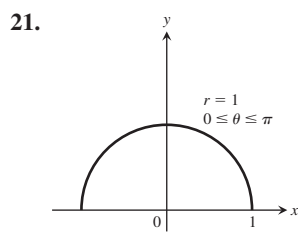
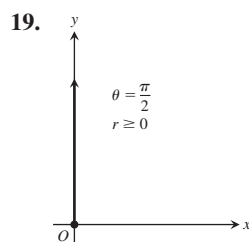
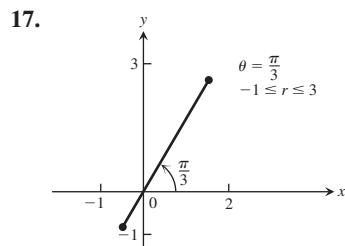
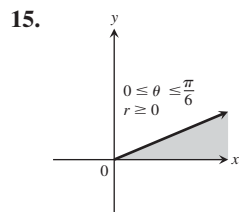
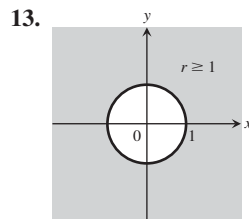
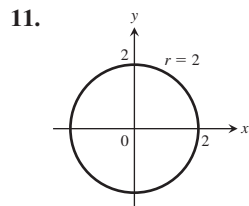
(e) $(3, 0)$ (f) $(1, \sqrt{3})$ (g) $(-3, 0)$ (h) $(-1, \sqrt{3})$

7. (a) $\left(\sqrt{2}, \frac{\pi}{4}\right)$ (b) $(3, \pi)$ (c) $\left(2, \frac{11\pi}{6}\right)$

(d) $\left(5, \pi - \tan^{-1} \frac{4}{3}\right)$

9. (a) $(-3\sqrt{2}, \frac{5\pi}{4})$ (b) $(-1, 0)$ (c) $(-2, \frac{5\pi}{3})$

(d) $(-5, \pi - \tan^{-1} \frac{3}{4})$



27. $x = 2$, vertical line through $(2, 0)$ 29. $y = 0$, the x -axis

31. $y = 4$, horizontal line through $(0, 4)$

33. $x + y = 1$, line, $m = -1, b = 1$

35. $x^2 + y^2 = 1$, circle, $C(0, 0)$, radius 1

37. $y - 2x = 5$, line, $m = 2, b = 5$

39. $y^2 = x$, parabola, vertex $(0, 0)$, opens right

41. $y = e^x$, graph of natural exponential function

43. $x + y = \pm 1$, two straight lines of slope -1 , y -intercepts $b = \pm 1$

45. $(x + 2)^2 + y^2 = 4$, circle, $C(-2, 0)$, radius 2

47. $x^2 + (y - 4)^2 = 16$, circle, $C(0, 4)$, radius 4

49. $(x - 1)^2 + (y - 1)^2 = 2$, circle, $C(1, 1)$, radius $\sqrt{2}$

51. $\sqrt{3}y + x = 4$ 53. $r \cos \theta = 7$ 55. $\theta = \pi/4$

57. $r = 2$ or $r = -2$ 59. $4r^2 \cos^2 \theta + 9r^2 \sin^2 \theta = 36$

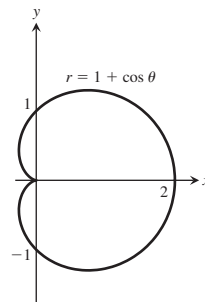
61. $r \sin^2 \theta = 4 \cos \theta$ 63. $r = 4 \sin \theta$

65. $r^2 = 6r \cos \theta - 2r \sin \theta - 6$

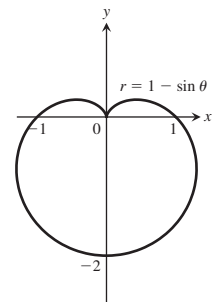
67. $(0, \theta)$, where θ is any angle

SECTION 10.4, pp. 649–650

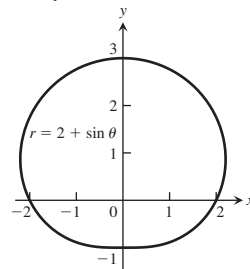
1. x -axis



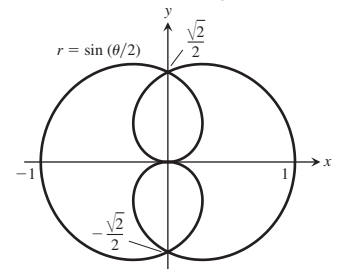
3. y -axis



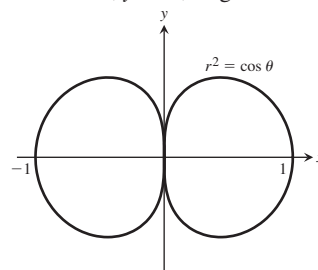
5. y -axis



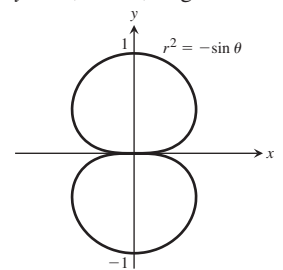
7. x -axis, y -axis, origin



9. x -axis, y -axis, origin

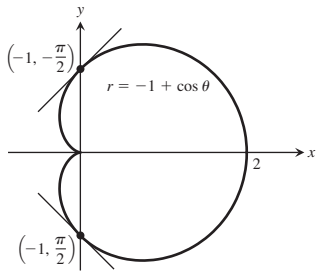


11. y -axis, x -axis, origin

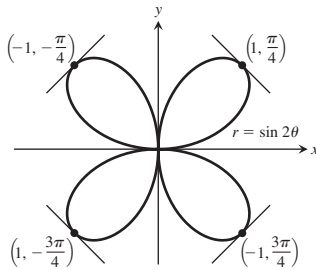


13. x-axis, y-axis, origin 15. Origin

17. The slope at $(-1, \pi/2)$ is -1 , at $(-1, -\pi/2)$ is 1 .



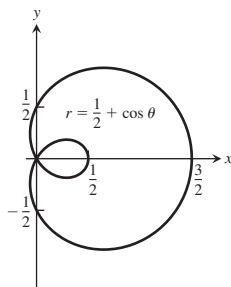
19. The slope at $(1, \pi/4)$ is -1 , at $(-1, -\pi/4)$ is 1 , at $(-1, 3\pi/4)$ is 1 , at $(1, -3\pi/4)$ is -1 .



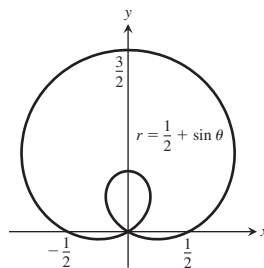
21. At $\pi/6$: slope $\sqrt{3}$, concavity 16 (concave up); at $\pi/3$: slope $-\sqrt{3}$, concavity -16 (concave down).

23. At 0 : slope 0 , concavity 2 (concave up); at $\pi/2$: slope $-2/\pi$, concavity $-2(8 + \pi^2)/\pi^3$ (concave down).

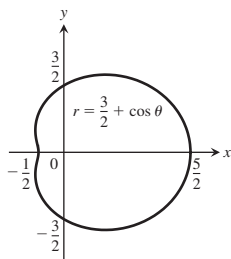
25. (a)



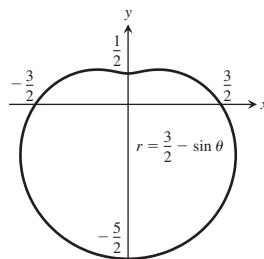
(b)



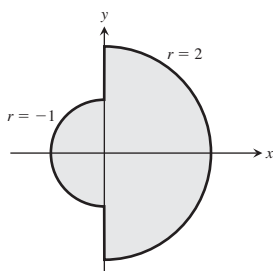
27. (a)



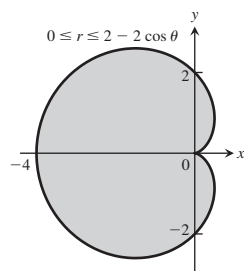
(b)



29.



31.



33. Equation (a)

SECTION 10.5, pp. 654-655

1. $\frac{1}{6}\pi^3$ 3. 18π 5. $\frac{\pi}{8}$ 7. 2 9. $\frac{\pi}{2} - 1$

11. $5\pi - 8$ 13. $3\sqrt{3} - \pi$ 15. $\frac{\pi}{3} + \frac{\sqrt{3}}{2}$

17. $\frac{8\pi}{3} + \sqrt{3}$ 19. (a) $\frac{3}{2} - \frac{\pi}{4}$ 21. $19/3$ 23. 8

25. $3(\sqrt{2} + \ln(1 + \sqrt{2}))$ 27. $\frac{\pi}{8} + \frac{3}{8}$

31. (a) a (b) a (c) $2a/\pi$

SECTION 10.6, pp. 661-663

1. $y^2 = 8x$, $F(2, 0)$, directrix: $x = -2$

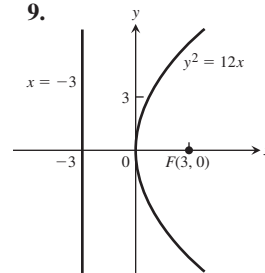
3. $x^2 = -6y$, $F(0, -3/2)$, directrix: $y = 3/2$

5. $\frac{x^2}{4} - \frac{y^2}{9} = 1$, $F(\pm\sqrt{13}, 0)$, $V(\pm 2, 0)$,

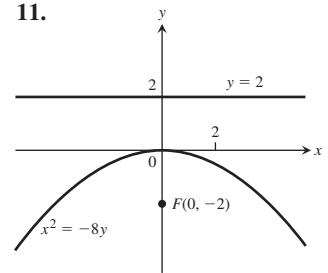
asymptotes: $y = \pm \frac{3}{2}x$

7. $\frac{x^2}{2} + y^2 = 1$, $F(\pm 1, 0)$, $V(\pm\sqrt{2}, 0)$

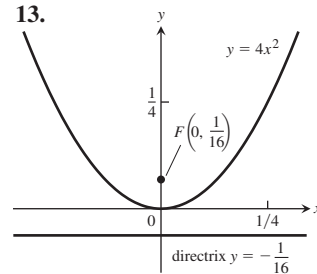
9.



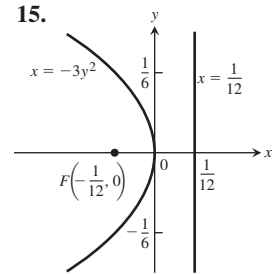
11.



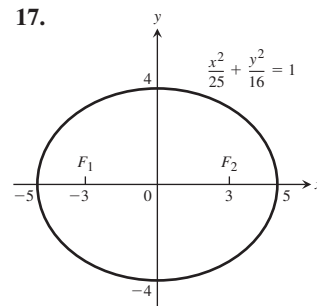
13.



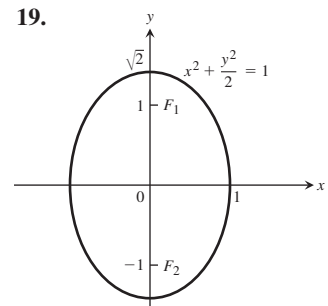
15.



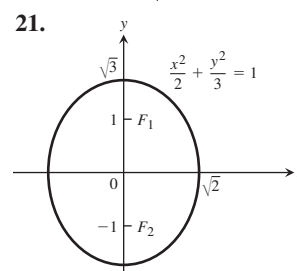
17.



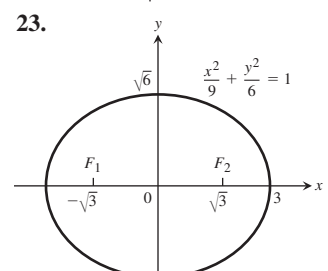
19.



21.

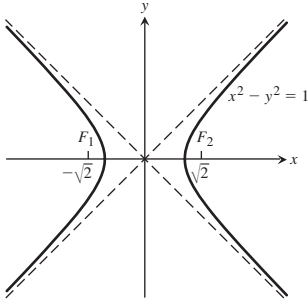


23.

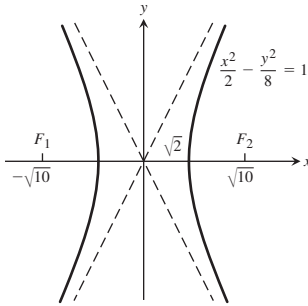


25. $\frac{x^2}{4} + \frac{y^2}{2} = 1$

27. Asymptotes: $y = \pm x$

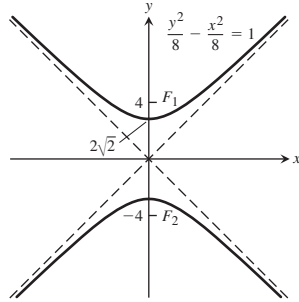


31. Asymptotes: $y = \pm 2x$

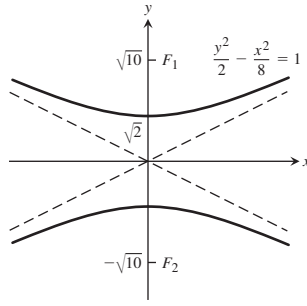


35. $y^2 - x^2 = 1$

29. Asymptotes: $y = \pm x$



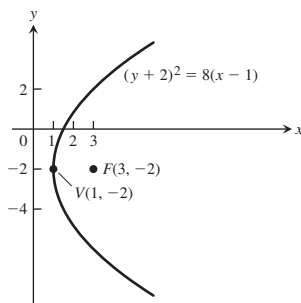
33. Asymptotes: $y = \pm x/2$



37. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

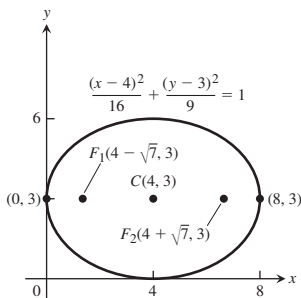
39. (a) Vertex: $(1, -2)$; focus: $(3, -2)$; directrix: $x = -1$

(b)



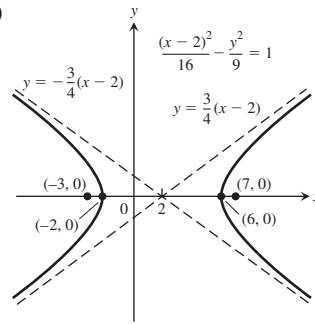
41. (a) Foci: $(4 \pm \sqrt{7}, 3)$; vertices: $(8, 3)$ and $(0, 3)$; center: $(4, 3)$

(b)



43. (a) Center: $(2, 0)$; foci: $(7, 0)$ and $(-3, 0)$; vertices: $(6, 0)$ and $(-2, 0)$; asymptotes: $y = \pm \frac{3}{4}(x - 2)$

(b)



45. $(y + 3)^2 = 4(x + 2)$, $V(-2, -3)$, $F(-1, -3)$, directrix: $x = -3$

47. $(x - 1)^2 = 8(y + 7)$, $V(1, -7)$, $F(1, -5)$, directrix: $y = -9$

49. $\frac{(x + 2)^2}{6} + \frac{(y + 1)^2}{9} = 1$, $F(-2, \pm\sqrt{3} - 1)$, $V(-2, \pm 3 - 1)$, $C(-2, -1)$

51. $\frac{(x - 2)^2}{3} + \frac{(y - 3)^2}{2} = 1$, $F(3, 3)$ and $F(1, 3)$, $V(\pm\sqrt{3} + 2, 3)$, $C(2, 3)$

53. $\frac{(x - 2)^2}{4} - \frac{(y - 2)^2}{5} = 1$, $C(2, 2)$, $F(5, 2)$ and $F(-1, 2)$, $V(4, 2)$ and $V(0, 2)$; asymptotes: $(y - 2) = \pm \frac{\sqrt{5}}{2}(x - 2)$

55. $(y + 1)^2 - (x + 1)^2 = 1$, $C(-1, -1)$, $F(-1, \sqrt{2} - 1)$ and $F(-1, -\sqrt{2} - 1)$, $V(-1, 0)$ and $V(-1, -2)$; asymptotes $(y + 1) = \pm(x + 1)$

57. $C(-2, 0)$, $a = 4$ 59. $V(-1, 1)$, $F(-1, 0)$

61. Ellipse: $\frac{(x + 2)^2}{5} + y^2 = 1$, $C(-2, 0)$, $F(0, 0)$ and $F(-4, 0)$, $V(\sqrt{5} - 2, 0)$ and $V(-\sqrt{5} - 2, 0)$

63. Ellipse: $\frac{(x - 1)^2}{2} + (y - 1)^2 = 1$, $C(1, 1)$, $F(2, 1)$ and $F(0, 1)$, $V(\sqrt{2} + 1, 1)$ and $V(-\sqrt{2} + 1, 1)$

65. Hyperbola: $(x - 1)^2 - (y - 2)^2 = 1$, $C(1, 2)$, $F(1 + \sqrt{2}, 2)$ and $F(1 - \sqrt{2}, 2)$, $V(2, 2)$ and $V(0, 2)$; asymptotes: $(y - 2) = \pm(x - 1)$

67. Hyperbola: $\frac{(y - 3)^2}{6} - \frac{x^2}{3} = 1$, $C(0, 3)$, $F(0, 6)$ and $F(0, 0)$, $V(0, \sqrt{6} + 3)$ and $V(0, -\sqrt{6} + 3)$; asymptotes: $y = \sqrt{2}x + 3$ or $y = -\sqrt{2}x + 3$

69. (b) 1:1 73. Length = $2\sqrt{2}$, width = $\sqrt{2}$, area = 4

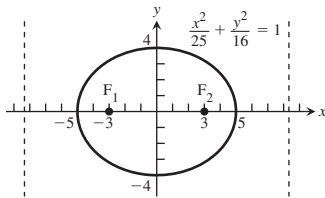
75. 24π

77. $x = 0$, $y = 0$: $y = -2x$; $x = 0$, $y = 2$: $y = 2x + 2$; $x = 4$, $y = 0$: $y = 2x - 8$

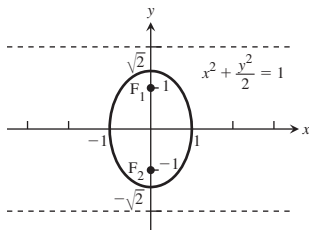
79. $\bar{x} = 0$, $\bar{y} = \frac{16}{3\pi}$

SECTION 10.7, pp. 668–669

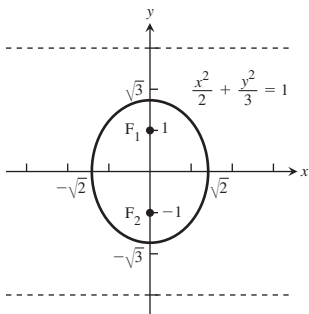
1. $e = \frac{3}{5}$, $F(\pm 3, 0)$;
directrices are $x = \pm \frac{25}{3}$.



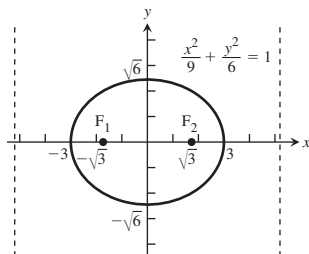
3. $e = \frac{1}{\sqrt{2}}$; $F(0, \pm 1)$;
directrices are $y = \pm 2$.



5. $e = \frac{1}{\sqrt{3}}$; $F(0, \pm 1)$;
directrices are $y = \pm 3$.



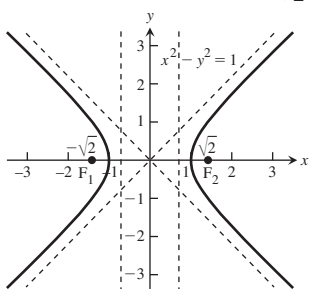
7. $e = \frac{\sqrt{3}}{3}$; $F(\pm \sqrt{3}, 0)$;
directrices are $x = \pm 3\sqrt{3}$.



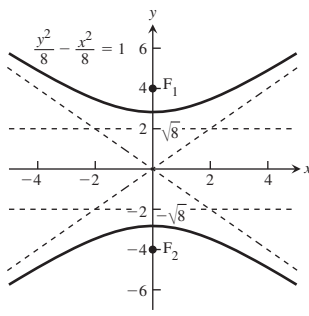
9. $\frac{x^2}{27} + \frac{y^2}{36} = 1$ 11. $\frac{x^2}{4851} + \frac{y^2}{4900} = 1$

13. $\frac{x^2}{9} + \frac{y^2}{4} = 1$ 15. $\frac{x^2}{64} + \frac{y^2}{48} = 1$

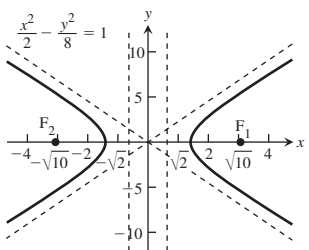
17. $e = \sqrt{2}$; $F(\pm \sqrt{2}, 0)$;
directrices are $x = \pm \frac{1}{\sqrt{2}}$.



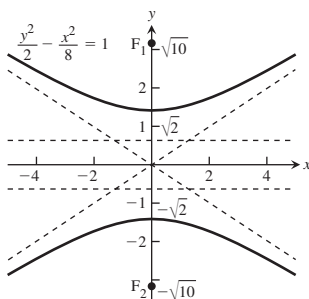
19. $e = \sqrt{2}$; $F(0, \pm 4)$;
directrices are $y = \pm 2$.



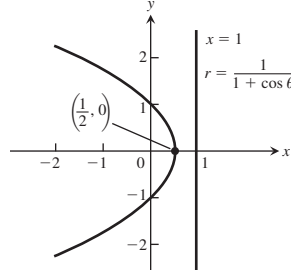
21. $e = \sqrt{5}$; $F(\pm \sqrt{10}, 0)$;
directrices are $x = \pm \frac{2}{\sqrt{10}}$.



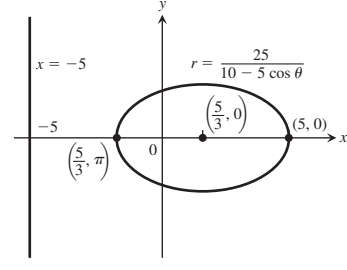
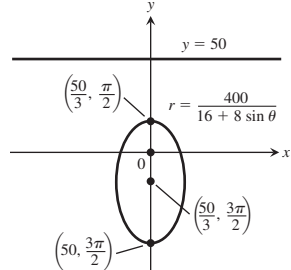
23. $e = \sqrt{5}$; $F(0, \pm \sqrt{10})$;
directrices are $y = \pm \frac{2}{\sqrt{10}}$.



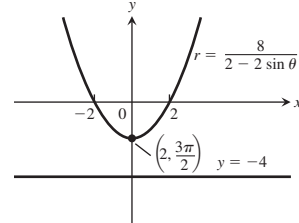
25. $y^2 - \frac{x^2}{8} = 1$ 27. $x^2 - \frac{y^2}{8} = 1$ 29. $r = \frac{2}{1 + \cos \theta}$
31. $r = \frac{30}{1 - 5 \sin \theta}$ 33. $r = \frac{1}{2 + \cos \theta}$ 35. $r = \frac{10}{5 - \sin \theta}$
37.



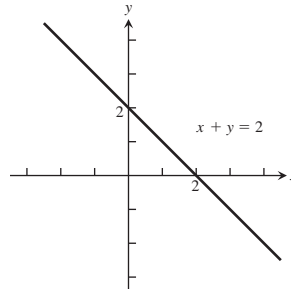
41.



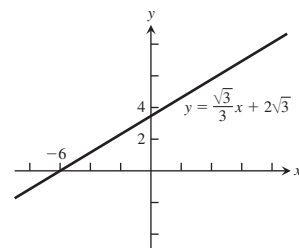
43.



45. $y = 2 - x$

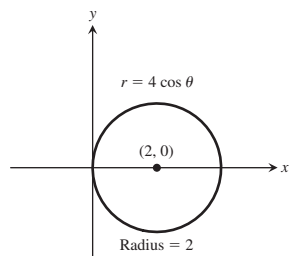


47. $y = \frac{\sqrt{3}}{3}x + 2\sqrt{3}$

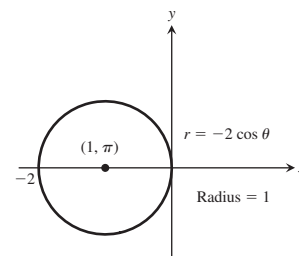


49. $r \cos(\theta - \frac{\pi}{4}) = 3$ 51. $r \cos(\theta + \frac{\pi}{2}) = 5$

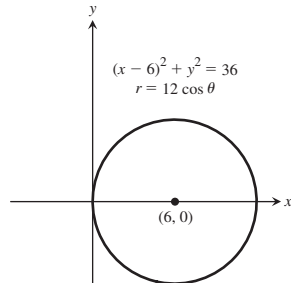
53.



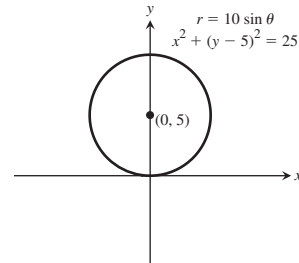
55.



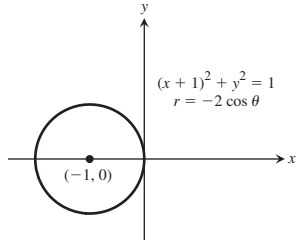
57. $r = 12 \cos \theta$



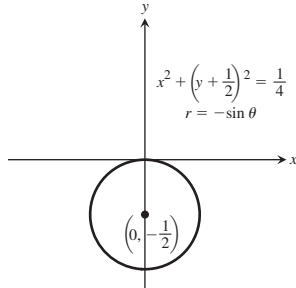
59. $r = 10 \sin \theta$



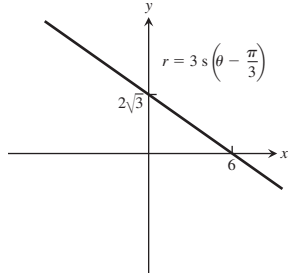
61. $r = -2 \cos \theta$



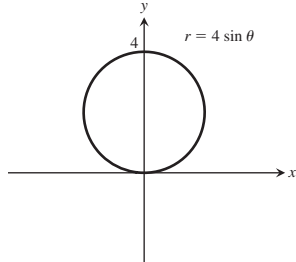
63. $r = -\sin \theta$



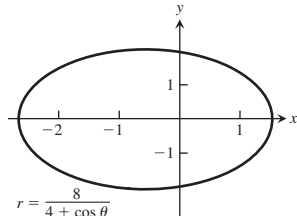
65.



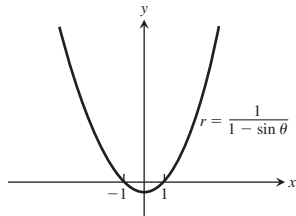
67.



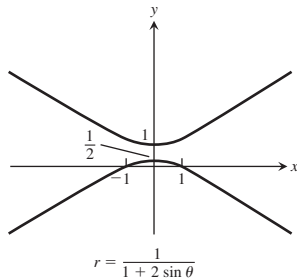
69.



71.



73.

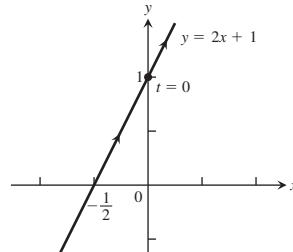


75. (b)

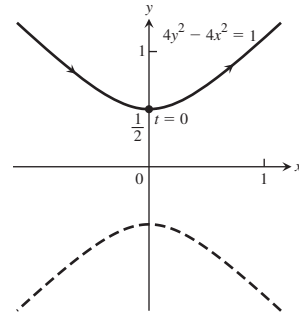
Planet	Perihelion	Aphelion
Mercury	0.3075 AU	0.4667 AU
Venus	0.7184 AU	0.7282 AU
Earth	0.9833 AU	1.0167 AU
Mars	1.3817 AU	1.6663 AU
Jupiter	4.9512 AU	5.4548 AU
Saturn	9.0210 AU	10.0570 AU
Uranus	18.2977 AU	20.0623 AU
Neptune	29.8135 AU	30.3065 AU

PRACTICE EXERCISES, pp. 670–672

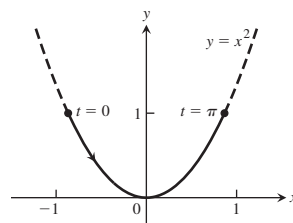
1.



3.



5.



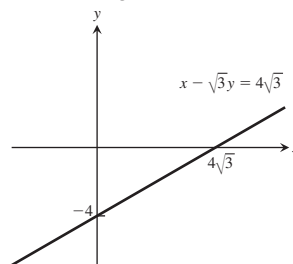
7. $x = 3 \cos t, y = 4 \sin t, 0 \leq t \leq 2\pi$

9. $y = \frac{\sqrt{3}}{2}x + \frac{1}{4}, \frac{1}{4}$

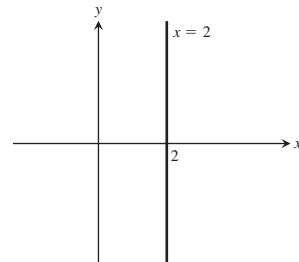
11. (a) $y = \frac{\pm|x|^{3/2}}{8} - 1$ (b) $y = \frac{\pm\sqrt{1-x^2}}{x}$

13. $\frac{10}{3}$ 15. $\frac{285}{8}$ 17. 10 19. $\frac{9\pi}{2}$ 21. $\frac{76\pi}{3}$

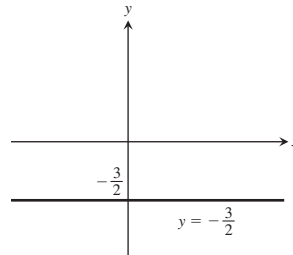
23. $y = \frac{\sqrt{3}}{3}x - 4$



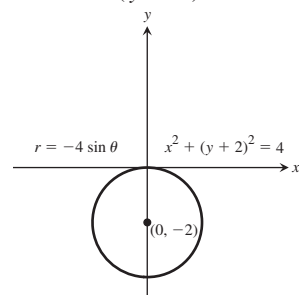
25. $x = 2$



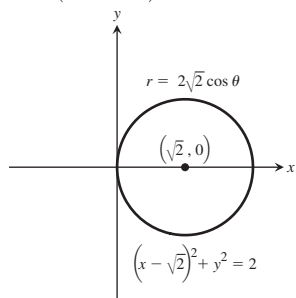
27. $y = -\frac{3}{2}$



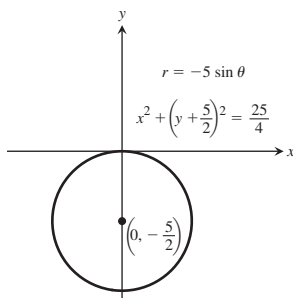
29. $x^2 + (y + 2)^2 = 4$



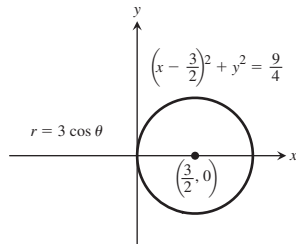
31. $(x - \sqrt{2})^2 + y^2 = 2$



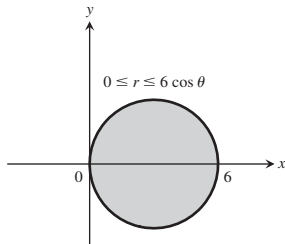
33. $r = -5 \sin \theta$



35. $r = 3 \cos \theta$



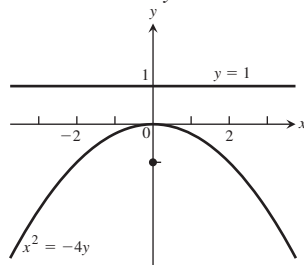
37.



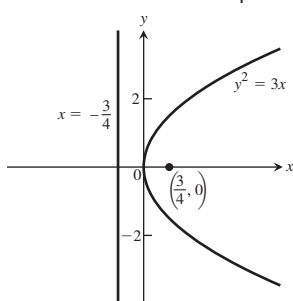
39. d 41. l 43. k 45. i 47. $\frac{9}{2}\pi$

49. $2 + \frac{\pi}{4}$ 51. 8 53. $\pi - 3$

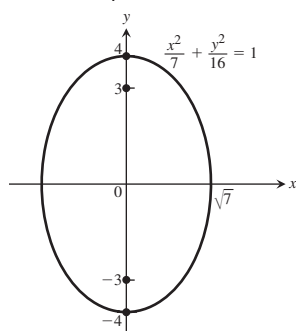
55. Focus is $(0, -1)$;
directrix is $y = 1$.



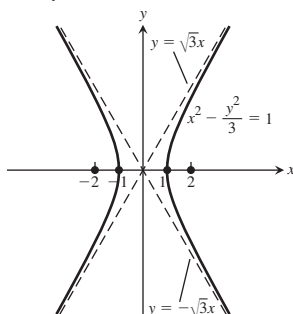
57. Focus is $(\frac{3}{4}, 0)$;
directrix is $x = -\frac{3}{4}$.



59. $e = \frac{3}{4}$



61. $e = 2$; the asymptotes are
 $y = \pm\sqrt{3}x$.



63. $(x - 2)^2 = -12(y - 3)$, $V(2, 3)$, $F(2, 0)$, directrix is $y = 6$.

65. $\frac{(x + 3)^2}{9} + \frac{(y + 5)^2}{25} = 1$, $C(-3, -5)$, $F(-3, -1)$ and
 $F(-3, -9)$, $V(-3, -10)$ and $V(-3, 0)$.

67. $\frac{(y - 2\sqrt{2})^2}{8} - \frac{(x - 2)^2}{2} = 1$, $C(2, 2\sqrt{2})$,

$F(2, 2\sqrt{2} \pm \sqrt{10})$, $V(2, 4\sqrt{2})$ and $V(2, 0)$, the asymptotes
are $y = 2x - 4 + 2\sqrt{2}$ and $y = -2x + 4 + 2\sqrt{2}$.

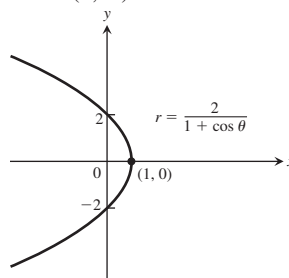
69. Hyperbola: $C(2, 0)$, $V(0, 0)$ and $V(4, 0)$, the foci are
 $F(2 \pm \sqrt{5}, 0)$, and the asymptotes are $y = \pm \frac{x - 2}{2}$.

71. Parabola: $V(-3, 1)$, $F(-7, 1)$, and the directrix is $x = 1$.

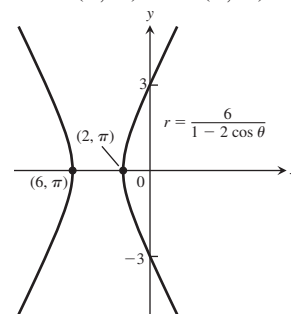
73. Ellipse: $C(-3, 2)$, $F(-3 \pm \sqrt{7}, 2)$, $V(1, 2)$ and $V(-7, 2)$

75. Circle: $C(1, 1)$ and radius = $\sqrt{2}$

77. $V(1, 0)$



79. $V(2, \pi)$ and $V(6, \pi)$



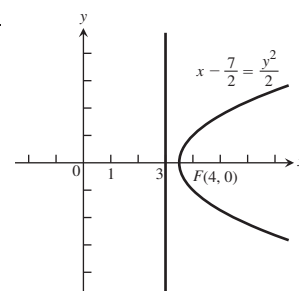
81. $r = \frac{4}{1 + 2 \cos \theta}$

83. $r = \frac{2}{2 + \sin \theta}$

85. (a) 24π (b) 16π

ADDITIONAL AND ADVANCED EXERCISES, pp. 672-674

1. $x - \frac{7}{2} = \frac{y^2}{2}$



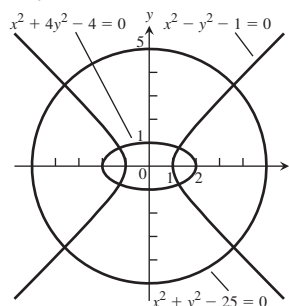
3. $3x^2 + 3y^2 - 8y + 4 = 0$

5. $F(0, \pm 1)$

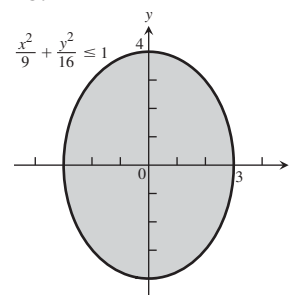
7. (a) $\frac{(y - 1)^2}{16} - \frac{x^2}{48} = 1$

(b) $\frac{(y + \frac{3}{4})^2}{(\frac{25}{16})} - \frac{x^2}{(\frac{75}{2})} = 1$

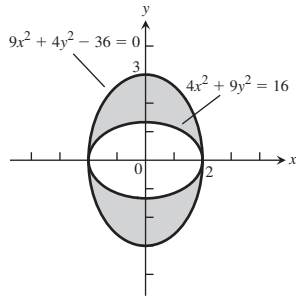
11.



13.



15.



17. (a) $r = e^{2\theta}$ (b) $\frac{\sqrt{5}}{2}(e^{4\pi} - 1)$

19. $r = \frac{4}{1 + 2\cos\theta}$ 21. $r = \frac{2}{2 + \sin\theta}$

23. $x = (a + b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right),$
 $y = (a + b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right)$

27. $\frac{\pi}{2}$

Chapter 11**SECTION 11.1, pp. 678–680**

1. The line through the point $(2, 3, 0)$ parallel to the z -axis
3. The x -axis
5. The circle $x^2 + y^2 = 4$ in the xy -plane
7. The circle $x^2 + z^2 = 4$ in the xz -plane
9. The circle $y^2 + z^2 = 1$ in the yz -plane
11. The circle $x^2 + y^2 = 16$ in the xy -plane
13. The ellipse formed by the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $z = y$
15. The parabola $y = x^2$ in the xy -plane
17. (a) The first quadrant of the xy -plane
(b) The fourth quadrant of the xy -plane
19. (a) The ball of radius 1 centered at the origin
(b) All points more than 1 unit from the origin
21. (a) The ball of radius 2 centered at the origin with the interior of the ball of radius 1 centered at the origin removed
(b) The solid upper hemisphere of radius 1 centered at the origin
23. (a) The region on or inside the parabola $y = x^2$ in the xy -plane and all points above this region
(b) The region on or to the left of the parabola $x = y^2$ in the xy -plane and all points above it that are 2 units or less away from the xy -plane
25. 3 27. 7 29. $2\sqrt{3}$ 31. (a) 2 (b) 3 (c) 4
33. (a) 3 (b) 4 (c) 5
35. (a) $x = 3$ (b) $y = -1$ (c) $z = -2$
37. (a) $z = 1$ (b) $x = 3$ (c) $y = -1$
39. (a) $x^2 + (y - 2)^2 = 4, z = 0$
(b) $(y - 2)^2 + z^2 = 4, x = 0$ (c) $x^2 + z^2 = 4, y = 2$
41. (a) $y = 3, z = -1$ (b) $x = 1, z = -1$ (c) $x = 1, y = 3$
43. $x^2 + y^2 + z^2 = 25, z = 3$ 45. $0 \leq z \leq 1$ 47. $z \leq 0$
49. (a) $(x - 1)^2 + (y - 1)^2 + (z - 1)^2 < 1$
(b) $(x - 1)^2 + (y - 1)^2 + (z - 1)^2 > 1$
51. $C(-2, 0, 2), a = 2\sqrt{2}$ 53. $C(\sqrt{2}, \sqrt{2}, -\sqrt{2}), a = \sqrt{2}$
55. $C(-2, 0, 2), a = \sqrt{8}$ 57. $C\left(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right), a = \frac{5\sqrt{3}}{4}$

59. $C(2, -3, 5), a = 7$

61. $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 14$

63. $(x + 1)^2 + \left(y - \frac{1}{2}\right)^2 + \left(z + \frac{2}{3}\right)^2 = \frac{16}{81}$

65. (a) $\sqrt{y^2 + z^2}$ (b) $\sqrt{x^2 + z^2}$ (c) $\sqrt{x^2 + y^2}$

67. $\sqrt{17} + \sqrt{33} + 6$ 69. $y = 1$

71. (a) $(0, 3, -3)$ (b) $(0, 5, -5)$

73. $z = x^2/4 + 1$ 75. (a) $z^2 = x^2$ (b) $y^2 = x^2$

SECTION 11.2, pp. 688–690

1. (a) $\langle 9, -6 \rangle$ (b) $3\sqrt{13}$ 3. (a) $\langle 1, 3 \rangle$ (b) $\sqrt{10}$

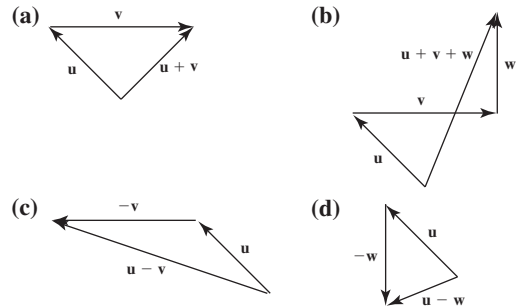
5. (a) $\langle 12, -19 \rangle$ (b) $\sqrt{505}$

7. (a) $\langle \frac{1}{5}, \frac{14}{5} \rangle$ (b) $\frac{\sqrt{197}}{5}$ 9. $\langle 1, -4 \rangle$

11. $\langle -2, -3 \rangle$ 13. $\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$ 15. $\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2} \rangle$

17. $-3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ 19. $-3\mathbf{i} + 16\mathbf{j}$ 21. $3\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}$

23. The vector $\mathbf{v}$ is horizontal and 25 mm long. The vectors $\mathbf{u}$ and $\mathbf{w}$ are 17 mm long. $\mathbf{w}$ is vertical and $\mathbf{u}$ makes a 45° angle with the horizontal. All vectors must be drawn to scale.



25. $3\left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right)$ 27. $5(\mathbf{k})$

29. $\sqrt{\frac{1}{2}}\left(\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right)$

31. (a) $2\mathbf{i}$ (b) $-\sqrt{3}\mathbf{k}$ (c) $\frac{3}{10}\mathbf{j} + \frac{2}{5}\mathbf{k}$ (d) $6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

33. $\frac{7}{13}(12\mathbf{i} - 5\mathbf{k})$

35. (a) $\frac{3}{5\sqrt{2}}\mathbf{i} + \frac{4}{5\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}$ (b) $(1/2, 3, 5/2)$

37. (a) $-\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$ (b) $\left(\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\right)$

39. $A(4, -3, 5)$ 41. $a = \frac{3}{2}, b = \frac{1}{2}$

43. $a = -1, b = 2, c = 1$ 45. $\approx \langle -338.095, 725.046 \rangle$

47. $|\mathbf{F}_1| \approx 73.205 \text{ N},$

$|\mathbf{F}_2| \approx 89.658 \text{ N},$

$\mathbf{F}_1 = \langle -|\mathbf{F}_1|\cos 30^\circ, |\mathbf{F}_1|\sin 30^\circ \rangle \approx \langle -63.397, 36.603 \rangle,$

$\mathbf{F}_2 = \langle |\mathbf{F}_2|\cos 45^\circ, |\mathbf{F}_2|\sin 45^\circ \rangle \approx \langle 63.397, 63.397 \rangle$

49. $w \approx 126.093 \text{ N},$

$|\mathbf{F}_1| \approx 106.933 \text{ N}$

51. (a) $(5 \cos 60^\circ, 5 \sin 60^\circ) = \left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$

(b) $(5 \cos 60^\circ + 10 \cos 315^\circ, 5 \sin 60^\circ + 10 \sin 315^\circ) =$
 $\left(\frac{5 + 10\sqrt{2}}{2}, \frac{5\sqrt{3} - 10\sqrt{2}}{2}\right)$

53. (a) $\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}$ (b) $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ (c) $(2, 2, 1)$

59. (a) $\langle 0, 0, 0 \rangle$ (b) $\langle 0, 0, 0 \rangle$

61. (a) $\langle 0, 16, 4, -12 \rangle$ (b) $4\sqrt{26}$

63. (a) $\langle -2, -1, -1, 2 \rangle$ (b) $\sqrt{10}$

65. (a) $\langle 8, -24, -3, 13 \rangle$ (b) $\sqrt{818}$

SECTION 11.3, pp. 697–699

1. (a) $-25, 5, 5$ (b) -1 (c) -5 (d) $-2\mathbf{i} + 4\mathbf{j} - \sqrt{5}\mathbf{k}$

3. (a) $25, 15, 5$ (b) $\frac{1}{3}$ (c) $\frac{5}{3}$ (d) $\frac{1}{9}(10\mathbf{i} + 11\mathbf{j} - 2\mathbf{k})$

5. (a) $2, \sqrt{34}, \sqrt{3}$ (b) $\frac{2}{\sqrt{3}\sqrt{34}}$ (c) $\frac{2}{\sqrt{34}}$

(d) $\frac{1}{17}(5\mathbf{j} - 3\mathbf{k})$

7. (a) $10 + \sqrt{17}, \sqrt{26}, \sqrt{21}$ (b) $\frac{10 + \sqrt{17}}{\sqrt{546}}$ (c) $\frac{10 + \sqrt{17}}{\sqrt{26}}$

(d) $\frac{10 + \sqrt{17}}{26}(5\mathbf{i} + \mathbf{j})$

9. 0.75 rad 11. 1.77 rad

13. Angle at $A = \arccos\left(\frac{1}{\sqrt{5}}\right) \approx 63.435$ degrees, angle at

$B = \arccos\left(\frac{3}{5}\right) \approx 53.130$ degrees, angle at

$C = \arccos\left(\frac{1}{\sqrt{5}}\right) \approx 63.435$ degrees.

17. $\arccos\left(\frac{3}{\sqrt{10}}\right) \approx 0.322$ radian or 18.43 degrees

25. Horizontal component: ≈ 396 m/s, vertical component: ≈ 55.7 m/s

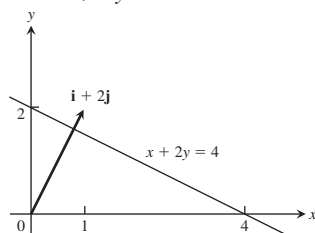
27. (a) Since $|\cos \theta| \leq 1$, we have

$|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\cos \theta| \leq |\mathbf{u}||\mathbf{v}|(1) = |\mathbf{u}||\mathbf{v}|.$

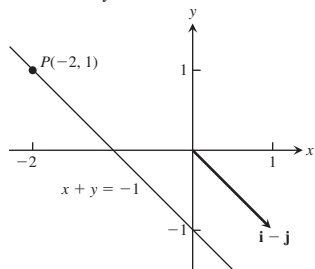
(b) We have equality precisely when $|\cos \theta| = 1$ or when one or both of $\mathbf{u}$ and $\mathbf{v}$ are $\mathbf{0}$. In the case of nonzero vectors, we have equality when $\theta = 0$ or π , that is, when the vectors are parallel.

29. a

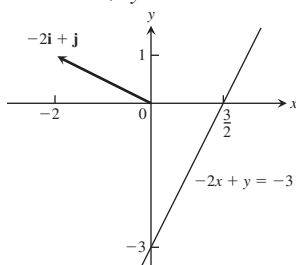
35. $x + 2y = 4$



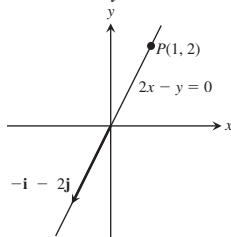
39. $x + y = -1$



37. $-2x + y = -3$



41. $2x - y = 0$



43. 5 J 45. 3464 J 47. $\frac{\pi}{4}$ 49. $\frac{\pi}{6}$ 51. 0.14

53. (a) 3 (b) not orthogonal

55. (a) 0 (b) orthogonal

SECTION 11.4, pp. 704–705

1. $|\mathbf{u} \times \mathbf{v}| = 3$, direction is $\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$; $|\mathbf{v} \times \mathbf{u}| = 3$,

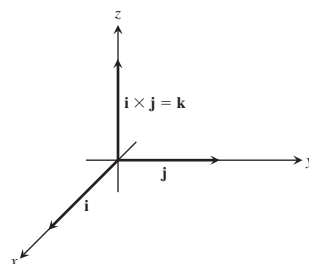
direction is $-\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

3. $|\mathbf{u} \times \mathbf{v}| = 0$, no direction; $|\mathbf{v} \times \mathbf{u}| = 0$, no direction

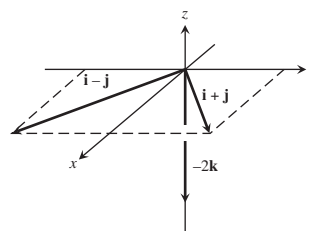
5. $|\mathbf{u} \times \mathbf{v}| = 6$, direction is $-\mathbf{k}$; $|\mathbf{v} \times \mathbf{u}| = 6$, direction is $\mathbf{k}$

7. $|\mathbf{u} \times \mathbf{v}| = 6\sqrt{5}$, direction is $\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{k}$; $|\mathbf{v} \times \mathbf{u}| = 6\sqrt{5}$, direction is $-\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}$

9.



13.



15. (a) $2\sqrt{6}$ (b) $\pm \frac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} + \mathbf{k})$

17. (a) $\frac{\sqrt{2}}{2}$ (b) $\pm \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$

19. 8 21. 7 23. (a) None (b) $\mathbf{u}$ and $\mathbf{w}$

25. $1.5\sqrt{3}$ N-m

27. (a) True (b) Not always true (c) True (d) True (e) Not always true (f) True (g) True (h) True

29. (a) $\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}$ (b) $k\mathbf{u} \times \mathbf{v}$ (k any constant)

(c) $k(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ (k any constant)

(d) $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ (e) $k(\mathbf{u} \times \mathbf{v}) \times (\mathbf{u} \times \mathbf{w})$ (k any constant)

(f) $|\mathbf{u}| \frac{\mathbf{v}}{|\mathbf{v}|}$

31. (a) Yes (b) No (c) Yes (d) No

33. No, $\mathbf{v}$ need not equal $\mathbf{w}$. For example, $\mathbf{i} + \mathbf{j} \neq -\mathbf{i} + \mathbf{j}$, but $\mathbf{i} \times (\mathbf{i} + \mathbf{j}) = \mathbf{i} \times \mathbf{i} + \mathbf{i} \times \mathbf{j} = \mathbf{0} + \mathbf{k} = \mathbf{k}$ and $\mathbf{i} \times (-\mathbf{i} + \mathbf{j}) = -\mathbf{i} \times \mathbf{i} + \mathbf{i} \times \mathbf{j} = \mathbf{0} + \mathbf{k} = \mathbf{k}$.

35. 2 37. 13 39. $\sqrt{129}$ 41. $\frac{11}{2}$

43. $\frac{25}{2}$ 45. $\frac{3}{2}$ 47. $\frac{\sqrt{21}}{2}$

49. If $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$, then

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k},$$

and the triangle's area is

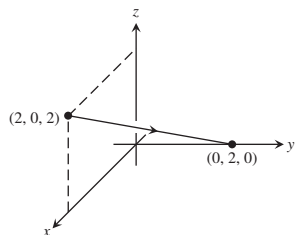
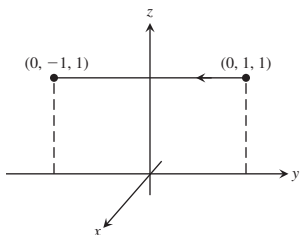
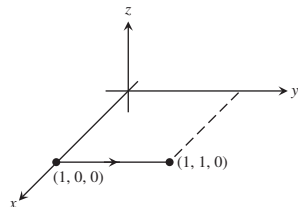
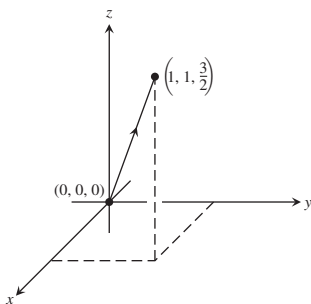
$$\frac{1}{2}|\mathbf{A} \times \mathbf{B}| = \pm \frac{1}{2} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

The applicable sign is (+) if the acute angle from $\mathbf{A}$ to $\mathbf{B}$ runs counterclockwise in the xy -plane, and (−) if it runs clockwise.

51. 4 53. $44/3$ 55. Coplanar 57. Not coplanar

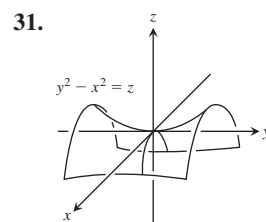
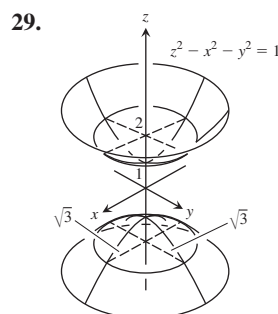
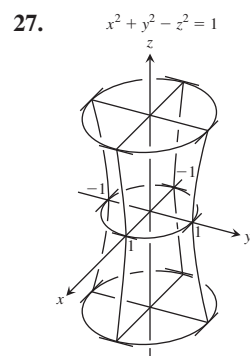
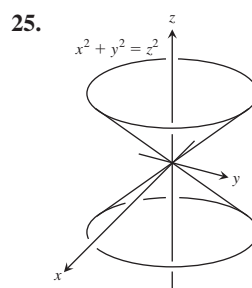
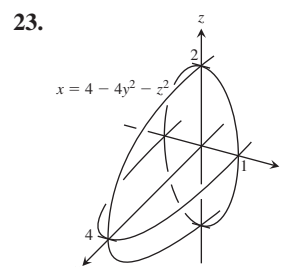
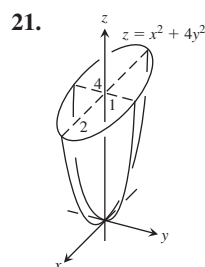
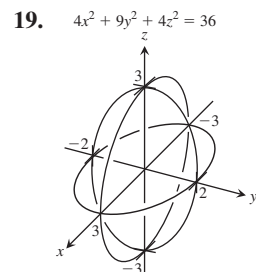
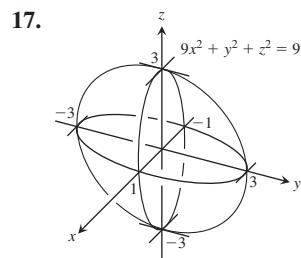
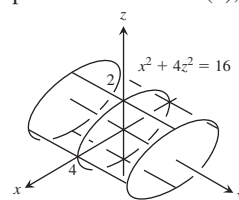
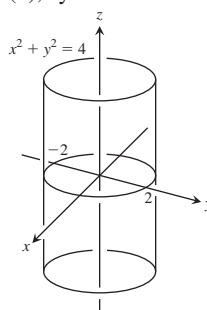
SECTION 11.5, pp. 712–714

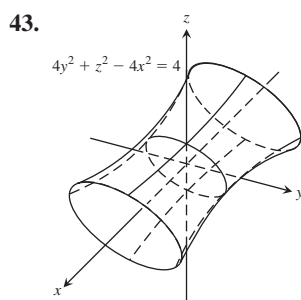
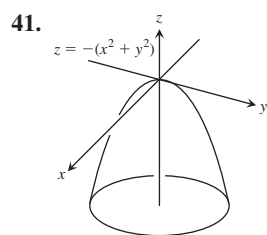
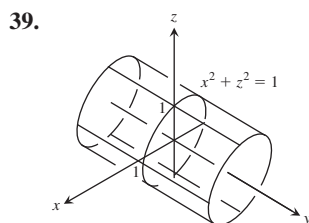
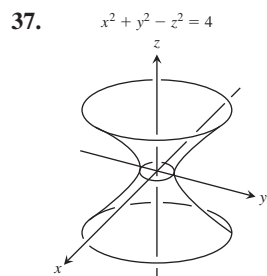
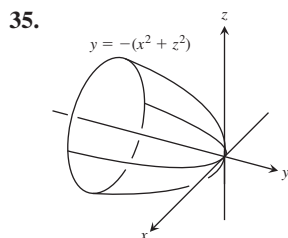
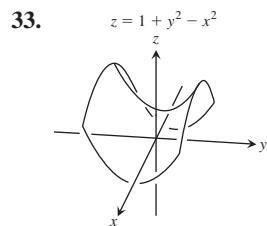
1. $x = 3 + t$, $y = -4 + t$, $z = -1 + t$
 3. $x = -2 + 5t$, $y = 5t$, $z = 3 - 5t$
 5. $x = 0$, $y = 2t$, $z = t$
 7. $x = 1$, $y = 1$, $z = 1 + t$
 9. $x = t$, $y = -7 + 2t$, $z = 2t$
 11. $x = t$, $y = 0$, $z = 0$
 13. $x = t$, $y = t$, $z = \frac{3}{2}t$, $0 \leq t \leq 1$
 15. $x = 1$, $y = 1 + t$, $z = 0$, $-1 \leq t \leq 0$
 17. $x = 0$, $y = 1 - 2t$, $z = 1$, $0 \leq t \leq 1$
 19. $x = 2 - 2t$, $y = 2t$, $z = 2 - 2t$, $0 \leq t \leq 1$
 21. $3x - 2y - z = -3$ 23. $7x - 5y - 4z = 6$
 25. $x + 3y + 4z = 34$ 27. $(1, 2, 3), -20x + 12y + z = 7$
 29. $y + z = 3$ 31. $x - y + z = 0$ 33. $2\sqrt{30}$ 35. 0
 37. $\frac{9\sqrt{42}}{7}$ 39. 3 41. $19/5$ 43. $5/3$ 45. $9/\sqrt{41}$
 47. $\pi/4$ 49. $\pi - \arccos(-1/6) \approx \pi - 1.738 \approx 1.403$ radians
 51. $\arcsin(2/\sqrt{154}) \approx 0.161$ radian 53. 1.38 rad
 55. 0.82 rad 57. $(\frac{3}{2}, -\frac{3}{2}, \frac{1}{2})$ 59. $(1, 1, 0)$
 61. $x = 1 - t$, $y = 1 + t$, $z = -1$
 63. $x = 4$, $y = 3 + 6t$, $z = 1 + 3t$
 65. L_1 intersects L_2 ; L_2 is parallel to L_3 , $\sqrt{5}/3$; L_1 and L_3 are skew, $10\sqrt{2}/3$
 67. $x = 2 + 2t$, $y = -4 - t$, $z = 7 + 3t$; $x = -2 - t$, $y = -2 + (1/2)t$, $z = 1 - (3/2)t$
 69. $(0, -\frac{1}{2}, -\frac{3}{2}), (-1, 0, -3), (1, -1, 0)$
 73. Many possible answers. One possibility: $x + y = 3$ and $2y + z = 7$.
 75. $(x/a) + (y/b) + (z/c) = 1$ describes all planes *except* those through the origin or parallel to a coordinate axis.



SECTION 11.6, pp. 718–719

1. (d), ellipsoid 3. (a), cylinder 5. (l), hyperbolic paraboloid
 7. (b), cylinder 9. (k), hyperbolic paraboloid 11. (h), cone
 13. 15.

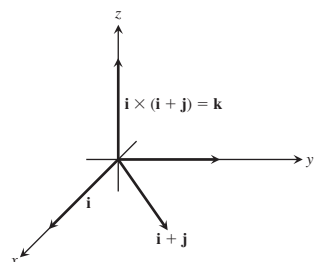




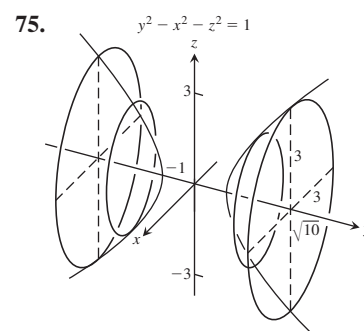
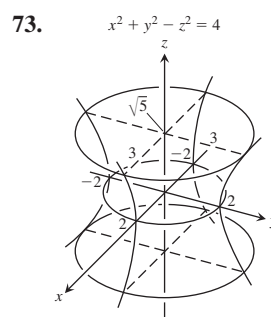
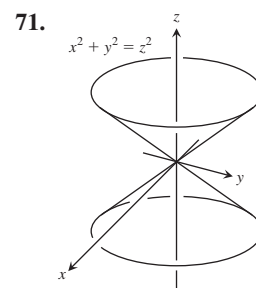
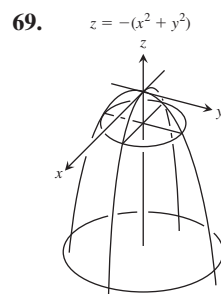
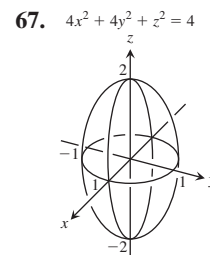
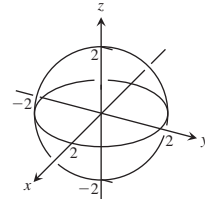
45. (a) $\frac{2\pi(9 - c^2)}{9}$ (b) 8π (c) $\frac{4\pi abc}{3}$

PRACTICE EXERCISES, pp. 720-722

1. (a) $\langle -17, 32 \rangle$ (b) $\sqrt{1313}$ 3. (a) $\langle 6, -8 \rangle$ (b) 10
5. $\left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$ [assuming counterclockwise] 7. $\left\langle \frac{8}{\sqrt{17}}, -\frac{2}{\sqrt{17}} \right\rangle$
9. Length = 2; direction is $\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$.
11. Length = 2; direction is $-\mathbf{i}$.
13. Length = 7; direction is $\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$.
15. $\frac{8}{\sqrt{33}}\mathbf{i} - \frac{2}{\sqrt{33}}\mathbf{j} + \frac{8}{\sqrt{33}}\mathbf{k}$
17. $|\mathbf{v}| = \sqrt{2}$, $|\mathbf{u}| = 3$, $\mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v} = 3$, $\mathbf{v} \times \mathbf{u} = -2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$,
 $\mathbf{u} \times \mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $|\mathbf{v} \times \mathbf{u}| = 3$, $\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$,
 $|\mathbf{u}|\cos\theta = \frac{3}{\sqrt{2}}$, $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{3}{2}(\mathbf{i} + \mathbf{j})$
19. $\frac{4}{3}(2\mathbf{i} + \mathbf{j} - \mathbf{k})$
21. $\mathbf{u} \times \mathbf{v} = \mathbf{k}$



23. $2\sqrt{7}$ 25. (a) $\sqrt{14}$ (b) 1 29. $\sqrt{78}/3$
31. $x = 1 - 3t$, $y = 2$, $z = 3 + 7t$ 33. $\sqrt{2}$
35. $2x + y + z = 5$ 37. $-9x + y + 7z = 4$
39. $\left(0, -\frac{1}{2}, -\frac{3}{2}\right), (-1, 0, -3), (1, -1, 0)$
41. $\pi/3$ 43. $x = -5 + 5t$, $y = 3 - t$, $z = -3t$
45. (b) $x = -12t$, $y = 19/12 + 15t$, $z = 1/6 + 6t$
47. Yes; $\mathbf{v}$ is parallel to the plane. 49. 3 51. $-3\mathbf{j} + 3\mathbf{k}$
53. $\frac{2}{\sqrt{35}}(5\mathbf{i} - \mathbf{j} - 3\mathbf{k})$ 55. $\left(\frac{11}{9}, \frac{26}{9}, -\frac{7}{9}\right)$
57. $(1, -2, -1)$; $x = 1 - 5t$, $y = -2 + 3t$, $z = -1 + 4t$
59. $2x + 7y + 2z + 10 = 0$
61. (a) No (b) No (c) No (d) No (e) Yes
63. $11/\sqrt{107}$
65. $x^2 + y^2 + z^2 = 4$



ADDITIONAL AND ADVANCED EXERCISES, pp. 722-724

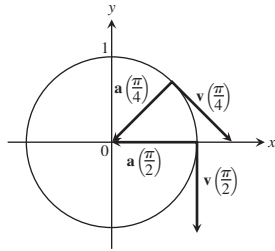
1. $(26, 23, -1/3)$ 3. $|\mathbf{F}| = 88.7 \text{ N}$
5. (a) $|\mathbf{F}_1| = 80 \text{ N}$, $|\mathbf{F}_2| = 60 \text{ N}$, $\mathbf{F}_1 = \langle -48, 64 \rangle$,
 $\mathbf{F}_2 = \langle 48, 36 \rangle$, $\alpha = \arctan \frac{4}{3}$, $\beta = \arctan \frac{3}{4}$
- (b) $|\mathbf{F}_1| = \frac{2400}{13} \approx 184.615 \text{ N}$, $|\mathbf{F}_2| = \frac{1000}{13} \approx 76.923 \text{ N}$,
 $\mathbf{F}_1 = \left\langle \frac{-12,000}{169}, \frac{28,800}{169} \right\rangle \approx \langle -71.006, 170.414 \rangle$,
 $\mathbf{F}_2 = \left\langle \frac{12,000}{169}, \frac{5000}{169} \right\rangle \approx \langle 71.006, 29.586 \rangle$,
 $\alpha = \arctan \frac{12}{5}$, $\beta = \arctan \frac{5}{12}$

9. (a) $\theta = \arctan \sqrt{2} \approx 54.74^\circ$
 (b) $\theta = \arctan 2\sqrt{2} \approx 70.53^\circ$
 13. (b) $\frac{6}{\sqrt{14}}$ (c) $2x - y + 2z = 8$
 (d) $x - 2y + z = 3 + 5\sqrt{6}$ and $x - 2y + z = 3 - 5\sqrt{6}$
 15. $\frac{32}{41}\mathbf{i} + \frac{23}{41}\mathbf{j} - \frac{13}{41}\mathbf{k}$
 17. (a) $0, 0$ (b) $-10\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}, -9\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$
 (c) $-4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}, \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$
 (d) $-10\mathbf{i} - 10\mathbf{k}, -12\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$
 19. The formula is always true.

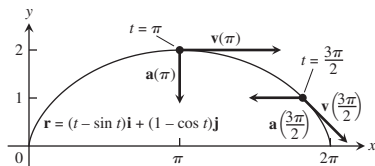
Chapter 12

SECTION 12.1, pp. 732–734

1. $\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}$
 3. $2\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\pi}{4}\mathbf{k}$
 5. $y = x^2 - 2x$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{a} = 2\mathbf{j}$
 7. $y = \frac{2}{9}x^2$, $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{a} = 3\mathbf{i} + 8\mathbf{j}$
 9. $t = \frac{\pi}{4}$: $\mathbf{v} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$, $\mathbf{a} = \frac{-\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$;
 $t = \pi/2$: $\mathbf{v} = -\mathbf{j}$, $\mathbf{a} = -\mathbf{i}$



11. $t = \pi$: $\mathbf{v} = 2\mathbf{i}$, $\mathbf{a} = -\mathbf{j}$; $t = \frac{3\pi}{2}$: $\mathbf{v} = \mathbf{i} - \mathbf{j}$, $\mathbf{a} = -\mathbf{i}$



13. $\mathbf{v} = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}$; $\mathbf{a} = 2\mathbf{j}$; speed: 3; direction: $\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$;
 $\mathbf{v}(1) = 3\left(\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right)$
 15. $\mathbf{v} = (-2 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 4\mathbf{k}$;
 $\mathbf{a} = (-2 \cos t)\mathbf{i} - (3 \sin t)\mathbf{j}$; speed: $2\sqrt{5}$;
 direction: $(-1/\sqrt{5})\mathbf{i} + (2/\sqrt{5})\mathbf{j}$;
 $\mathbf{v}(\pi/2) = 2\sqrt{5}\left[(-1/\sqrt{5})\mathbf{i} + (2/\sqrt{5})\mathbf{j}\right]$
 17. $\mathbf{v} = \left(\frac{2}{t+1}\right)\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$; $\mathbf{a} = \left(\frac{-2}{(t+1)^2}\right)\mathbf{i} + 2\mathbf{j} + \mathbf{k}$;
 speed: $\sqrt{6}$; direction: $\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}$;
 $\mathbf{v}(1) = \sqrt{6}\left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}\right)$

19. $\pi/2$ 21. $\pi/2$ 23. $x = t$, $y = -1$, $z = 1 + t$

25. $x = t$, $y = \frac{1}{3}t$, $z = t$ 27. 4, -2 29. 2, -2

31. E 33. D 35. C

37. (a) (i): It has constant speed 1. (ii): Yes
 (iii): Counterclockwise (iv): Yes
 (b) (i): It has constant speed 2. (ii): Yes
 (iii): Counterclockwise (iv): Yes
 (c) (i): It has constant speed 1. (ii): Yes
 (iii): Counterclockwise
 (iv): It starts at $(0, -1)$ instead of $(1, 0)$.
 (d) (i): It has constant speed 1. (ii): Yes
 (iii): Clockwise (iv): Yes
 (e) (i): It has variable speed. (ii): No
 (iii): Counterclockwise (iv): Yes

39. $\mathbf{v} = 2\sqrt{5}\mathbf{i} + \sqrt{5}\mathbf{j}$

SECTION 12.2, pp. 739–742

1. $(1/4)\mathbf{i} + 7\mathbf{j} + (3/2)\mathbf{k}$ 3. $\left(\frac{\pi + 2\sqrt{2}}{2}\right)\mathbf{j} + \mathbf{k}$
 5. $(\ln 4)\mathbf{i} + (\ln 4)\mathbf{j} + (\ln 2)\mathbf{k}$
 7. $\frac{e-1}{2}\mathbf{i} + \frac{e-1}{e}\mathbf{j} + \mathbf{k}$
 9. $\mathbf{i} - \mathbf{j} + \frac{\pi}{4}\mathbf{k}$
 11. $\mathbf{r}(t) = \left(\frac{-t^2}{2} + 1\right)\mathbf{i} + \left(\frac{-t^2}{2} + 2\right)\mathbf{j} + \left(\frac{-t^2}{2} + 3\right)\mathbf{k}$
 13. $\mathbf{r}(t) = ((t+1)^{3/2} - 1)\mathbf{i} + (-e^{-t} + 1)\mathbf{j} + (\ln(t+1) + 1)\mathbf{k}$
 15. $\mathbf{r}(t) = (3 + \ln|\sec t|)\mathbf{i} + (-2 + 2\sin(t/2))\mathbf{j}$
 $+ (1 - (1/2)\ln|\sec 2t + \tan 2t|)\mathbf{k}$
 17. $\mathbf{r}(t) = 8t\mathbf{i} + 8t\mathbf{j} + (-16t^2 + 100)\mathbf{k}$
 19. $\mathbf{r}(t) = (e^t - 2t + 2)\mathbf{i} + (-e^{-t} + 3t + 2)\mathbf{j} + (e^{2t} - 2t + 1)\mathbf{k}$
 21. $\mathbf{r}(t) = \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t + 1\right)\mathbf{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t - 2\right)\mathbf{j}$
 $+ \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t + 3\right)\mathbf{k} = \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)(3\mathbf{i} - \mathbf{j} + \mathbf{k})$
 $+ (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

23. 50 s

25. (a) 72.2 s; 25,510 m (b) 4020 m (c) 6378 m

27. (a) $v_0 \approx 9.9$ m/s (b) $\alpha \approx 18.4^\circ$ or 71.6°

29. 39.3° or 50.7° 35. (b) $\mathbf{v}_0$ would bisect $\angle AOR$.

37. (a) (Assuming that “x” is zero at the point of impact)
 $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$, where $x(t) = (12 \cos 27^\circ)t$ and
 $y(t) = 1.3 + (12 \sin 27^\circ)t - 4.9t^2$.
 (b) At $t \approx 0.556$ s, it reaches its maximum height of about
 2.814 m.
 (c) Range ≈ 14 m; flight time ≈ 1.31 s
 (d) At $t \approx 0.232$ and $t \approx 0.880$ s, when it is ≈ 11.52 and
 ≈ 4.59 m from where it will land
 (e) Yes. It changes things because the ball won’t clear the net.

39. Height is about 10.8 m.

47. (a) $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$; where

$$x(t) = \left(\frac{1}{0.08}\right)(1 - e^{-0.08t})(50 \cos 20^\circ - 5) \text{ and}$$

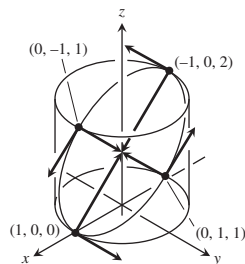
$$y(t) = 1 + \left(\frac{50}{0.08}\right)(1 - e^{-0.08t})(\sin 20^\circ)$$

$$+ \left(\frac{9.8}{0.08^2}\right)(1 - 0.08t - e^{-0.08t})$$

- (b) At $t \approx 1.633$ s, it reaches a maximum height of about 14.66 m.
 (c) Range ≈ 125.11 m; flight time ≈ 3.404 s
 (d) At $t \approx 0.670$ and 2.622 s, when it is about 27.39 and 99.30 m from home plate
 (e) No

SECTION 12.3, p. 746

1. $\mathbf{T} = \left(-\frac{2}{3}\sin t\right)\mathbf{i} + \left(\frac{2}{3}\cos t\right)\mathbf{j} + \frac{\sqrt{5}}{3}\mathbf{k}$, 3π
 3. $\mathbf{T} = \frac{1}{\sqrt{1+t}}\mathbf{i} + \frac{\sqrt{t}}{\sqrt{1+t}}\mathbf{k}$, $\frac{52}{3}$
 5. $\mathbf{T} = -\cos t\mathbf{j} + \sin t\mathbf{k}$, $\frac{3}{2}$
 7. $\mathbf{T} = \left(\frac{\cos t - t \sin t}{t+1}\right)\mathbf{i} + \left(\frac{\sin t + t \cos t}{t+1}\right)\mathbf{j}$
 $+ \left(\frac{\sqrt{2}t^{1/2}}{t+1}\right)\mathbf{k}$, $\frac{\pi^2}{2} + \pi$
 9. $(0, 5, 24\pi)$
 11. $s(t) = 5t$, $L = \frac{5\pi}{2}$
 13. $s(t) = \sqrt{3}e^t - \sqrt{3}$, $L = \frac{3\sqrt{3}}{4}$
 15. $\sqrt{2} + \ln(1 + \sqrt{2})$
 17. (a) Cylinder is $x^2 + y^2 = 1$; plane is $x + z = 1$.
 (b) and (c)



(d) $L = \int_0^{2\pi} \sqrt{1 + \sin^2 t} dt$ (e) $L \approx 7.64$

SECTION 12.4, pp. 751–752

1. $\mathbf{T} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$, $\mathbf{N} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}$, $\kappa = \cos t$
 3. $\mathbf{T} = \frac{1}{\sqrt{1+t^2}}\mathbf{i} - \frac{t}{\sqrt{1+t^2}}\mathbf{j}$, $\mathbf{N} = \frac{-t}{\sqrt{1+t^2}}\mathbf{i} - \frac{1}{\sqrt{1+t^2}}\mathbf{j}$, $\kappa = \frac{1}{2(\sqrt{1+t^2})^3}$
 5. (b) $\cos x$
 7. (b) $\mathbf{N} = \frac{-2e^{2t}}{\sqrt{1+4e^{4t}}}\mathbf{i} + \frac{1}{\sqrt{1+4e^{4t}}}\mathbf{j}$
 (c) $\mathbf{N} = -\frac{1}{2}(\sqrt{4-t^2}\mathbf{i} + t\mathbf{j})$
 9. $\mathbf{T} = \frac{3\cos t}{5}\mathbf{i} - \frac{3\sin t}{5}\mathbf{j} + \frac{4}{5}\mathbf{k}$,
 $\mathbf{N} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}$, $\kappa = \frac{3}{25}$
 11. $\mathbf{T} = \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\cos t + \sin t}{\sqrt{2}}\right)\mathbf{j}$,
 $\mathbf{N} = \left(\frac{-\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{-\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j}$, $\kappa = \frac{1}{e^t\sqrt{2}}$

13. $\mathbf{T} = \frac{t}{\sqrt{t^2+1}}\mathbf{i} + \frac{1}{\sqrt{t^2+1}}\mathbf{j}$,
 $\mathbf{N} = \frac{\mathbf{i}}{\sqrt{t^2+1}} - \frac{t\mathbf{j}}{\sqrt{t^2+1}}$, $\kappa = \frac{1}{t(t^2+1)^{3/2}}$
 15. $\mathbf{T} = \left(\operatorname{sech} \frac{t}{a}\right)\mathbf{i} + \left(\tanh \frac{t}{a}\right)\mathbf{k}$,
 $\mathbf{N} = \left(-\tanh \frac{t}{a}\right)\mathbf{i} + \left(\operatorname{sech} \frac{t}{a}\right)\mathbf{k}$,
 $\kappa = \frac{1}{a}\operatorname{sech}^2 \frac{t}{a}$
 19. $1/(2b)$
 21. $\left(x - \frac{\pi}{2}\right)^2 + y^2 = 1$
 23. $\kappa(x) = 2/(1+4x^2)^{3/2}$
 25. $\kappa(x) = |\sin x|/(1+\cos^2 x)^{3/2}$
 27. maximum curvature $2/(3\sqrt{3})$ at $x = 1/\sqrt{2}$

SECTION 12.5, p. 758

1. $\mathbf{a} = \frac{2t}{\sqrt{1+t^2}}\mathbf{T} + \frac{2}{\sqrt{1+t^2}}\mathbf{N}$ 3. $\mathbf{a} = |\mathbf{a}|\mathbf{N}$
 5. $\mathbf{a}(0) = \sqrt{2}\mathbf{N}$ 7. $\mathbf{a}(1) = \frac{4}{3}\mathbf{T} + \frac{2\sqrt{5}}{3}\mathbf{N}$ 9. $\mathbf{a}(0) = 2\mathbf{N}$
 11. $\mathbf{r}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} - \mathbf{k}$, $\mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$,
 $\mathbf{N}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$, $\mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{k}$; osculating plane:
 $z = -1$; normal plane: $-x + y = 0$; rectifying plane:
 $x + y = \sqrt{2}$
 13. $\mathbf{B} = \left(\frac{4}{5}\cos t\right)\mathbf{i} - \left(\frac{4}{5}\sin t\right)\mathbf{j} - \frac{3}{5}\mathbf{k}$, $\tau = -\frac{4}{25}$
 15. $\mathbf{B} = \mathbf{k}$, $\tau = 0$ 17. $\mathbf{B} = -\mathbf{k}$, $\tau = 0$ 19. $\mathbf{B} = -\mathbf{j}$, $\tau = 0$
 21. Yes. If the car is moving on a curved path ($\kappa \neq 0$), then
 $a_N = \kappa|\mathbf{v}|^2 \neq 0$ and $\mathbf{a} \neq \mathbf{0}$.
 27. $\kappa = \frac{1}{t}$, $\rho = t$
 31. Components of $\mathbf{v}$: -1.8701, 0.7089, 1.0000
 Components of $\mathbf{a}$: -1.6960, -2.0307, 0
 Speed: 2.2361; Components of $\mathbf{T}$: -0.8364, 0.3170, 0.4472
 Components of $\mathbf{N}$: -0.4143, -0.8998, -0.1369
 Components of $\mathbf{B}$: 0.3590, -0.2998, 0.8839; Curvature: 0.5060
 Torsion: 0.2813; Tangential component of acceleration: 0.7746
 Normal component of acceleration: 2.5298
 33. Components of $\mathbf{v}$: 2.0000, 0, -0.1629
 Components of $\mathbf{a}$: 0, -1.0000, -0.0086; Speed: 2.0066
 Components of $\mathbf{T}$: 0.9967, 0, -0.0812
 Components of $\mathbf{N}$: -0.0007, -1.0000, -0.0086
 Components of $\mathbf{B}$: -0.0812, 0.0086, 0.9967;
 Curvature: 0.2484
 Torsion: 0.0411; Tangential component of acceleration: 0.0007
 Normal component of acceleration: 1.0000

SECTION 12.6, p. 762

1. $\mathbf{v} = 2\mathbf{u}_r + 2\theta\mathbf{u}_\theta$
 $\mathbf{a} = -4\theta\mathbf{u}_r + 8\mathbf{u}_\theta$
 3. $\mathbf{v} = (3a \sin \theta)\mathbf{u}_r + 3a(1 - \cos \theta)\mathbf{u}_\theta$
 $\mathbf{a} = 9a(2 \cos \theta - 1)\mathbf{u}_r + (18a \sin \theta)\mathbf{u}_\theta$

$$5. \mathbf{v} = 2ae^{a\theta}\mathbf{u}_r + 2e^{a\theta}\mathbf{u}_\theta$$

$$\mathbf{a} = 4e^{a\theta}(a^2 - 1)\mathbf{u}_r + 8ae^{a\theta}\mathbf{u}_\theta$$

$$7. \mathbf{v} = (-8 \sin 4t)\mathbf{u}_r + (4 \cos 4t)\mathbf{u}_\theta$$

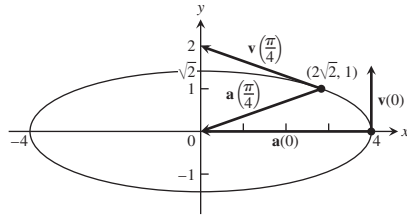
$$\mathbf{a} = (-40 \cos 4t)\mathbf{u}_r - (32 \sin 4t)\mathbf{u}_\theta$$

$$13. \approx 29.93 \times 10^{10} \text{ m} \quad 15. \approx 2.25 \times 10^9 \text{ km}^2/\text{s}$$

$$17. \approx 1.876 \times 10^{27} \text{ kg}$$

PRACTICE EXERCISES, pp. 763–764

$$1. \frac{x^2}{16} + \frac{y^2}{2} = 1$$



$$\text{At } t = 0: a_T = 0, a_N = 4, \kappa = 2;$$

$$\text{At } t = \frac{\pi}{4}: a_T = \frac{7}{3}, a_N = \frac{4\sqrt{2}}{3}, \kappa = \frac{4\sqrt{2}}{27}$$

$$3. |\mathbf{v}|_{\max} = 1 \quad 5. \kappa = 1/5 \quad 7. dy/dt = -x; \text{ clockwise}$$

$$11. \text{Shot put is on the ground, about 21.88 m from the stopboard.}$$

$$15. \text{Length} = \frac{\pi}{4} \sqrt{1 + \frac{\pi^2}{16}} + \ln \left(\frac{\pi}{4} + \sqrt{1 + \frac{\pi^2}{16}} \right)$$

$$17. \mathbf{T}(0) = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}; \quad \mathbf{N}(0) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j};$$

$$\mathbf{B}(0) = -\frac{1}{3\sqrt{2}}\mathbf{i} + \frac{1}{3\sqrt{2}}\mathbf{j} + \frac{4}{3\sqrt{2}}\mathbf{k}; \quad \kappa = \frac{\sqrt{2}}{3}; \tau = \frac{1}{6}$$

$$19. \mathbf{T}(\ln 2) = \frac{1}{\sqrt{17}}\mathbf{i} + \frac{4}{\sqrt{17}}\mathbf{j}; \quad \mathbf{N}(\ln 2) = -\frac{4}{\sqrt{17}}\mathbf{i} + \frac{1}{\sqrt{17}}\mathbf{j};$$

$$\mathbf{B}(\ln 2) = \mathbf{k}; \kappa = \frac{8}{17\sqrt{17}}; \tau = 0$$

$$21. \mathbf{a}(0) = 10\mathbf{T} + 6\mathbf{N}$$

$$23. \mathbf{T} = \left(\frac{1}{\sqrt{2}} \cos t \right) \mathbf{i} - (\sin t) \mathbf{j} + \left(\frac{1}{\sqrt{2}} \cos t \right) \mathbf{k};$$

$$\mathbf{N} = \left(-\frac{1}{\sqrt{2}} \sin t \right) \mathbf{i} - (\cos t) \mathbf{j} - \left(\frac{1}{\sqrt{2}} \sin t \right) \mathbf{k};$$

$$\mathbf{B} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{k}; \kappa = \frac{1}{\sqrt{2}}; \tau = 0$$

$$25. \frac{\pi}{3} \quad 27. x = 1 + t, \quad y = t, \quad z = -t \quad 31. \kappa = \frac{1}{a}$$

ADDITIONAL AND ADVANCED EXERCISES, pp. 765–766

$$1. (a) \left. \frac{d\theta}{dt} \right|_{\theta=2\pi} = 2\sqrt{\frac{\pi gb}{a^2 + b^2}}$$

$$(b) \theta = \frac{gbt^2}{2(a^2 + b^2)}, \quad z = \frac{gb^2t^2}{2(a^2 + b^2)}$$

$$(c) \mathbf{v}(t) = \frac{gbt}{\sqrt{a^2 + b^2}} \mathbf{T};$$

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{bg}{\sqrt{a^2 + b^2}} \mathbf{T} + a \left(\frac{bgt}{a^2 + b^2} \right)^2 \mathbf{N}$$

There is no component in the direction of $\mathbf{B}$.

$$5. (a) \frac{dx}{dt} = \dot{r} \cos \theta - r\dot{\theta} \sin \theta, \quad \frac{dy}{dt} = \dot{r} \sin \theta + r\dot{\theta} \cos \theta$$

$$(b) \frac{dr}{dt} = \dot{x} \cos \theta + \dot{y} \sin \theta, \quad r \frac{d\theta}{dt} = -\dot{x} \sin \theta + \dot{y} \cos \theta$$

$$7. (a) \mathbf{a}(1) = -9\mathbf{u}_r - 6\mathbf{u}_\theta, \quad \mathbf{v}(1) = -\mathbf{u}_r + 3\mathbf{u}_\theta$$

$$(b) 6.5 \text{ cm}$$

$$9. (c) \mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{k}, \quad \mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta + \ddot{z}\mathbf{k}$$

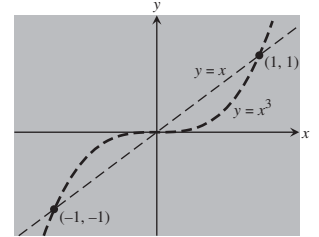
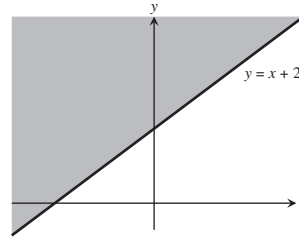
Chapter 13

SECTION 13.1, pp. 773–775

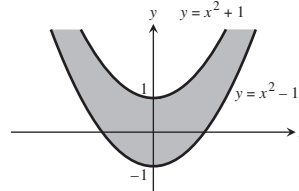
$$1. (a) 0 \quad (b) 0 \quad (c) 58 \quad (d) 33$$

$$3. (a) 4/5 \quad (b) 8/5 \quad (c) 3 \quad (d) 0$$

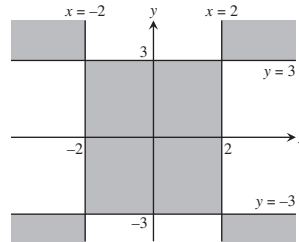
$$5. \text{Domain: all points } (x, y) \text{ on or above line } y = x + 2 \text{ or } y = x^3$$



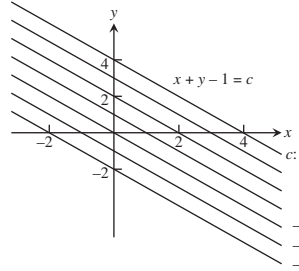
$$9. \text{Domain: all points } (x, y) \text{ satisfying } x^2 - 1 \leq y \leq x^2 + 1$$



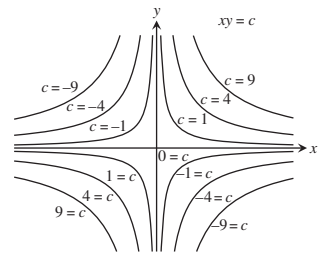
$$11. \text{Domain: all points } (x, y) \text{ for which } (x - 2)(x + 2)(y - 3)(y + 3) \geq 0$$



13.



15.



$$17. (a) \text{All points in the } xy\text{-plane} \quad (b) \text{All reals}$$

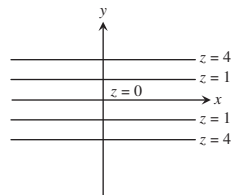
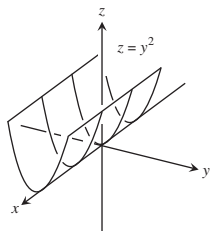
$$(c) \text{The lines } y - x = c \quad (d) \text{No boundary points}$$

$$(e) \text{Both open and closed} \quad (f) \text{Unbounded}$$

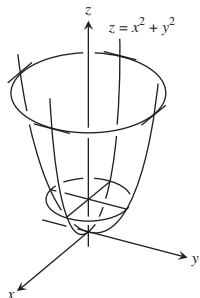
$$19. (a) \text{All points in the } xy\text{-plane} \quad (b) z \geq 0$$

$$(c) \text{For } f(x, y) = 0, \text{ the origin; for } f(x, y) = c, c > 0, \text{ ellipses with the center } (0, 0), \text{ and major and minor axes along the } x\text{- and } y\text{-axes, respectively}$$

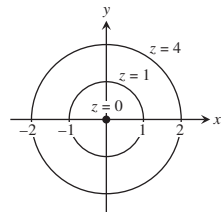
- (d) No boundary points (e) Both open and closed
 (f) Unbounded
21. (a) All points in the xy -plane (b) All reals
 (c) For $f(x, y) = 0$, the x - and y -axes; for $f(x, y) = c, c \neq 0$, hyperbolas with the x - and y -axes as asymptotes
 (d) No boundary points (e) Both open and closed
 (f) Unbounded
23. (a) All (x, y) satisfying $x^2 + y^2 < 16$ (b) $z \geq 1/4$
 (c) Circles centered at the origin with radii $r < 4$
 (d) Boundary is the circle $x^2 + y^2 = 16$
 (e) Open (f) Bounded
25. (a) $(x, y) \neq (0, 0)$ (b) All reals
 (c) The circles with center $(0, 0)$ and radii $r > 0$
 (d) Boundary is the single point $(0, 0)$
 (e) Open (f) Unbounded
27. (a) All (x, y) satisfying $-1 \leq y - x \leq 1$
 (b) $-\pi/2 \leq z \leq \pi/2$
 (c) Straight lines of the form $y - x = c$, where $-1 \leq c \leq 1$
 (d) Boundary is two straight lines $y = 1 + x$ and $y = -1 + x$
 (e) Closed (f) Unbounded
29. (a) Domain: all points (x, y) outside the circle $x^2 + y^2 = 1$
 (b) Range: all reals
 (c) Circles centered at the origin with radii $r > 1$
 (d) Boundary: $x^2 + y^2 = 1$
 (e) Open (f) Unbounded
31. (f), (h) 33. (a), (i) 35. (d), (j)
37. (a)



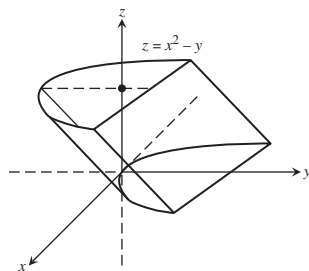
39. (a)



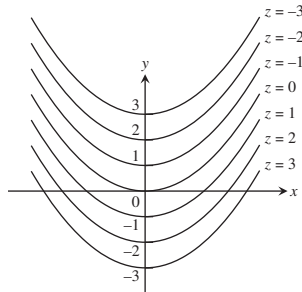
(b)



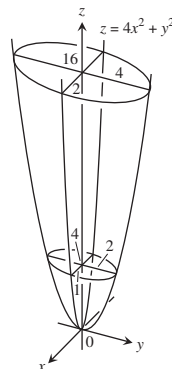
41. (a)



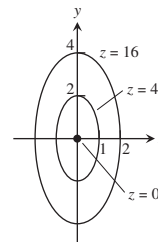
(b)



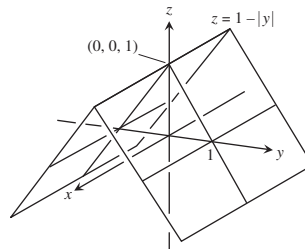
43. (a)



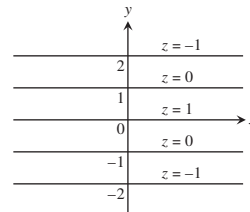
(b)



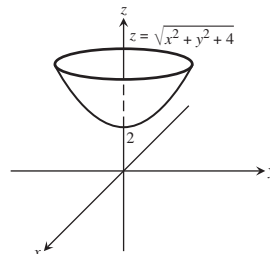
45. (a)



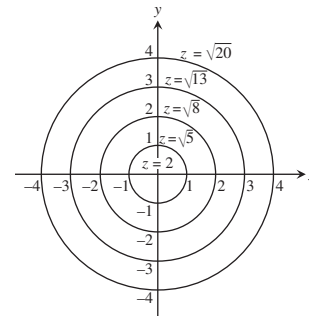
(b)



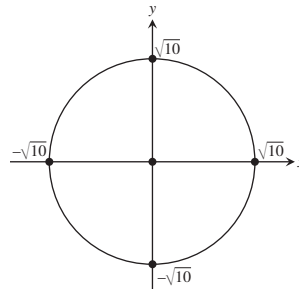
47. (a)



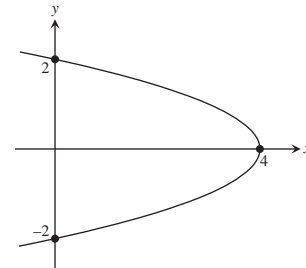
(b)



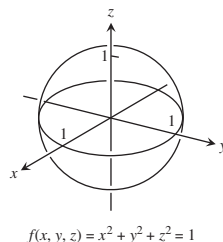
49. $x^2 + y^2 = 10$



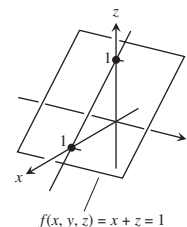
51. $x + y^2 = 4$



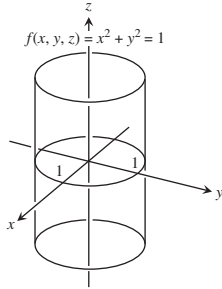
53.



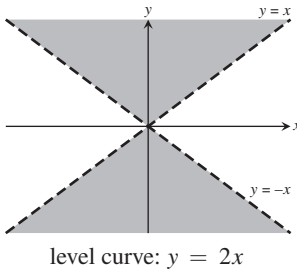
55.



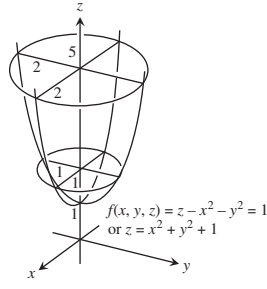
57.



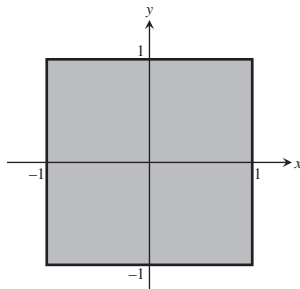
61. $\sqrt{x-y} - \ln z = 2$

65. Domain: all points (x, y) satisfying $|x| < |y|$ level curve: $y = 2x$

59.



63. $x^2 + y^2 + z^2 = 4$

67. Domain: all points (x, y) satisfying $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$ level curve:
 $\sin^{-1} y - \sin^{-1} x = \frac{\pi}{2}$ **SECTION 13.2, pp. 781-784**

1. $5/2$ 3. $2\sqrt{6}$ 5. 1 7. $1/2$ 9. 1
 11. $1/4$ 13. 0 15. -1 17. 2 19. $1/4$
 21. 1 23. 3 25. $19/12$ 27. 2 29. 3
 31. (a) All (x, y) (b) All (x, y) except $(0, 0)$
 33. (a) All (x, y) except where $x = 0$ or $y = 0$ (b) All (x, y)
 35. (a) All (x, y, z)
 (b) All (x, y, z) except the interior of the cylinder $x^2 + y^2 = 1$
 37. (a) All (x, y, z) with $z \neq 0$ (b) All (x, y, z) with $x^2 + z^2 \neq 1$
 39. (a) All points (x, y, z) satisfying $z > x^2 + y^2 + 1$
 (b) All points (x, y, z) satisfying $z \neq \sqrt{x^2 + y^2}$
 41. Consider paths along $y = x, x > 0$, and along $y = x, x < 0$.
 43. Consider the paths $y = kx^2, k$ a constant.
 45. Consider the paths $y = mx, m$ a constant, $m \neq -1$.
 47. Consider the paths $y = kx^2, k$ a constant, $k \neq 0$.
 49. Consider the paths $x = 1$ and $y = x$.
 51. Along $y = 1$ the limit is 0; along $y = e^x$ the limit is $1/2$.
 53. Along $y = 0$ the limit is 1; along $y = -\sin x$ the limit is 0.
 55. (a) 1 (b) 0 (c) Does not exist
 59. The limit is 1. 61. The limit is 0.
 63. (a) $f(x, y)|_{y=mx} = \sin 2\theta$, where $\tan \theta = m$ 65. 0
 67. Does not exist 69. $\pi/2$ 71. $f(0, 0) = \ln 3$
 73. $\delta = 0.1$ 75. $\delta = 0.005$ 77. $\delta = 0.04$
 79. $\delta = \sqrt{0.015}$ 81. $\delta = 0.005$

SECTION 13.3, pp. 793-796

1. $\frac{\partial f}{\partial x} = 4x, \frac{\partial f}{\partial y} = -3$ 3. $\frac{\partial f}{\partial x} = 2x(y+2), \frac{\partial f}{\partial y} = x^2 - 1$
 5. $\frac{\partial f}{\partial x} = 2y(xy-1), \frac{\partial f}{\partial y} = 2x(xy-1)$

7. $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$
 9. $\frac{\partial f}{\partial x} = \frac{-1}{(x+y)^2}, \frac{\partial f}{\partial y} = \frac{-1}{(x+y)^2}$
 11. $\frac{\partial f}{\partial x} = \frac{-y^2-1}{(xy-1)^2}, \frac{\partial f}{\partial y} = \frac{-x^2-1}{(xy-1)^2}$
 13. $\frac{\partial f}{\partial x} = e^{x+y+1}, \frac{\partial f}{\partial y} = e^{x+y+1}$ 15. $\frac{\partial f}{\partial x} = \frac{1}{x+y}, \frac{\partial f}{\partial y} = \frac{1}{x+y}$
 17. $\frac{\partial f}{\partial x} = 2 \sin(x-3y) \cos(x-3y),$
 $\frac{\partial f}{\partial y} = -6 \sin(x-3y) \cos(x-3y)$
 19. $\frac{\partial f}{\partial x} = yx^{y-1}, \frac{\partial f}{\partial y} = x^y \ln x$ 21. $\frac{\partial f}{\partial x} = -g(x), \frac{\partial f}{\partial y} = g(y)$
 23. $f_x = y^2, f_y = 2xy, f_z = -4z$
 25. $f_x = 1, f_y = -y(y^2 + z^2)^{-1/2}, f_z = -z(y^2 + z^2)^{-1/2}$
 27. $f_x = \frac{yz}{\sqrt{1-x^2y^2z^2}}, f_y = \frac{xz}{\sqrt{1-x^2y^2z^2}}, f_z = \frac{xy}{\sqrt{1-x^2y^2z^2}}$
 29. $f_x = \frac{1}{x+2y+3z}, f_y = \frac{2}{x+2y+3z}, f_z = \frac{3}{x+2y+3z}$
 31. $f_x = -2xe^{-(x^2+y^2+z^2)}, f_y = -2ye^{-(x^2+y^2+z^2)}, f_z = -2ze^{-(x^2+y^2+z^2)}$
 33. $f_x = \operatorname{sech}^2(x+2y+3z), f_y = 2 \operatorname{sech}^2(x+2y+3z),$
 $f_z = 3 \operatorname{sech}^2(x+2y+3z)$
 35. $\frac{\partial f}{\partial t} = -2\pi \sin(2\pi t - \alpha), \frac{\partial f}{\partial \alpha} = \sin(2\pi t - \alpha)$
 37. $\frac{\partial h}{\partial \rho} = \sin \phi \cos \theta, \frac{\partial h}{\partial \theta} = \rho \cos \phi \cos \theta, \frac{\partial h}{\partial \theta} = -\rho \sin \phi \sin \theta$
 39. $W_P(P, V, \delta, v, g) = V, W_V(P, V, \delta, v, g) = P + \frac{\delta v^2}{2g},$
 $W_\delta(P, V, \delta, v, g) = \frac{Vv^2}{2g}, W_v(P, V, \delta, v, g) = \frac{V\delta v}{g},$
 $W_g(P, V, \delta, v, g) = -\frac{V\delta v^2}{2g^2}$
 41. $\frac{\partial f}{\partial x} = 1+y, \frac{\partial f}{\partial y} = 1+x, \frac{\partial^2 f}{\partial x^2} = 0, \frac{\partial^2 f}{\partial y^2} = 0, \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 1$
 43. $\frac{\partial g}{\partial x} = 2xy + y \cos x, \frac{\partial g}{\partial y} = x^2 - \sin y + \sin x,$
 $\frac{\partial^2 g}{\partial x^2} = 2y - y \sin x, \frac{\partial^2 g}{\partial y^2} = -\cos y,$
 $\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial^2 g}{\partial x \partial y} = 2x + \cos x$
 45. $\frac{\partial r}{\partial x} = \frac{1}{x+y}, \frac{\partial r}{\partial y} = \frac{1}{x+y}, \frac{\partial^2 r}{\partial x^2} = \frac{-1}{(x+y)^2}, \frac{\partial^2 r}{\partial y^2} = \frac{-1}{(x+y)^2},$
 $\frac{\partial^2 r}{\partial y \partial x} = \frac{\partial^2 r}{\partial x \partial y} = \frac{-1}{(x+y)^2}$
 47. $\frac{\partial w}{\partial x} = x^2y \sec^2(xy) + 2x \tan(xy), \frac{\partial w}{\partial y} = x^3 \sec^2(xy),$
 $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = 2x^3y \sec^2(xy) \tan(xy) + 3x^2 \sec^2(xy)$
 $\frac{\partial^2 w}{\partial x^2} = 4xy \sec^2(xy) + 2x^2y^2 \sec^2(xy) \tan(xy) + 2 \tan(xy)$
 $\frac{\partial^2 w}{\partial y^2} = 2x^4 \sec^2(xy) \tan(xy)$

$$49. \frac{\partial w}{\partial x} = \sin(x^2 y) + 2x^2 y \cos(x^2 y), \frac{\partial w}{\partial y} = x^3 \cos(x^2 y),$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = 3x^2 \cos(x^2 y) - 2x^4 y \sin(x^2 y)$$

$$\frac{\partial^2 w}{\partial x^2} = 6xy \cos(x^2 y) - 4x^3 y^2 \sin(x^2 y)$$

$$\frac{\partial^2 w}{\partial y^2} = -x^5 \sin(x^2 y)$$

$$51. \frac{\partial f}{\partial x} = 2xy^3 - 4x^3, \frac{\partial f}{\partial y} = 3x^2 y^2 + 5y^4,$$

$$\frac{\partial^2 f}{\partial x^2} = 2y^3 - 12x^2, \frac{\partial^2 f}{\partial y^2} = 6x^2 y + 20y^3,$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 6xy^2$$

$$53. \frac{\partial z}{\partial x} = 2x \cos(2x - y^2) + \sin(2x - y^2),$$

$$\frac{\partial z}{\partial y} = -2xy \cos(2x - y^2),$$

$$\frac{\partial^2 z}{\partial x^2} = 4 \cos(2x - y^2) - 4x \sin(2x - y^2),$$

$$\frac{\partial^2 z}{\partial y^2} = -4xy^2 \sin(2x - y^2) - 2x \cos(2x - y^2),$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 4xy \sin(2x - y^2) - 2y \cos(2x - y^2)$$

$$55. \frac{\partial w}{\partial x} = \frac{2}{2x+3y}, \frac{\partial w}{\partial y} = \frac{3}{2x+3y}, \frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = \frac{-6}{(2x+3y)^2}$$

$$57. \frac{\partial w}{\partial x} = y^2 + 2xy^3 + 3x^2 y^4, \frac{\partial w}{\partial y} = 2xy + 3x^2 y^2 + 4x^3 y^3,$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = 2y + 6xy^2 + 12x^2 y^3$$

$$59. \frac{\partial w}{\partial x} = \frac{2x}{y^3}, \frac{\partial w}{\partial y} = \frac{-3x^2}{y^4}$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{-6x}{y^4}, \frac{\partial^2 w}{\partial x \partial y} = \frac{-6x}{y^4}$$

$$61. \text{(a) } x \text{ first} \quad \text{(b) } y \text{ first} \quad \text{(c) } x \text{ first}$$

$$\text{(d) } x \text{ first} \quad \text{(e) } y \text{ first} \quad \text{(f) } y \text{ first}$$

$$63. f_x(1, 2) = -13, f_y(1, 2) = -2$$

$$65. f_x(-2, 3) = 1/2, f_y(-2, 3) = 3/4$$

$$67. 12 \quad 69. -2$$

$$71. \frac{\partial A}{\partial a} = \frac{a}{bc \sin A}, \frac{\partial A}{\partial b} = \frac{c \cos A - b}{bc \sin A}$$

$$73. v_x = \frac{\ln v}{(\ln u)(\ln v) - 1}$$

$$75. \text{(a) } 3 \quad \text{(b) } 2$$

$$77. \frac{\partial f}{\partial x} = 3x^2 y^2 - 2x \Rightarrow$$

$$f(x, y) = x^3 y^2 - x^2 + g(y) \Rightarrow$$

$$\frac{\partial f}{\partial y} = 2x^3 y + g'(y) = 2x^3 y + 64 \Rightarrow$$

$$g'(y) = 64 \Rightarrow g(y) = 64y \text{ works} \Rightarrow$$

$$f(x, y) = x^3 y^2 - x^2 + 64y \text{ works}$$

$$79. \frac{\partial^2 f}{\partial y \partial x} = \frac{2x - 2y}{(x + y)^3} \neq \frac{\partial^2 f}{\partial x \partial y} = \frac{2y - 2x}{(x + y)^3} \text{ so impossible}$$

$$81. f_x(x, y) = 0 \text{ for all points } (x, y),$$

$$f_y(x, y) = \begin{cases} 3y^2, & y \geq 0 \\ -2y, & y < 0 \end{cases}$$

$$f_{xy}(x, y) = f_{yx}(x, y) = 0 \text{ for all points } (x, y)$$

99. Yes

SECTION 13.4, pp. 803–805

$$1. \text{(a) } \frac{dw}{dt} = 0, \quad \text{(b) } \frac{dw}{dt}(\pi) = 0$$

$$3. \text{(a) } \frac{dw}{dt} = 1, \quad \text{(b) } \frac{dw}{dt}(3) = 1$$

$$5. \text{(a) } \frac{dw}{dt} = 4t \tan^{-1} t + 1, \quad \text{(b) } \frac{dw}{dt}(1) = \pi + 1$$

$$7. \text{(a) } \frac{\partial z}{\partial u} = 4 \cos v \ln(u \sin v) + 4 \cos v,$$

$$\frac{\partial z}{\partial v} = -4u \sin v \ln(u \sin v) + \frac{4u \cos^2 v}{\sin v}$$

$$\text{(b) } \frac{\partial z}{\partial u} = \sqrt{2}(\ln 2 + 2), \quad \frac{\partial z}{\partial v} = -2\sqrt{2}(\ln 2 - 2)$$

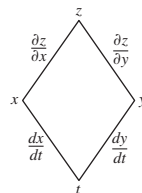
$$9. \text{(a) } \frac{\partial w}{\partial u} = 2u + 4uv, \quad \frac{\partial w}{\partial v} = -2v + 2u^2$$

$$\text{(b) } \frac{\partial w}{\partial u} = 3, \quad \frac{\partial w}{\partial v} = -\frac{3}{2}$$

$$11. \text{(a) } \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = \frac{z}{(z-y)^2}, \quad \frac{\partial u}{\partial z} = \frac{-y}{(z-y)^2}$$

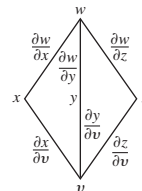
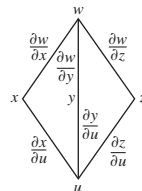
$$\text{(b) } \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial u}{\partial z} = -2$$

$$13. \frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

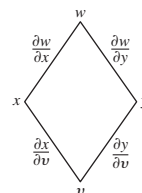
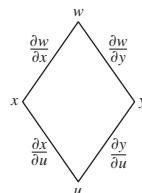


$$15. \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u},$$

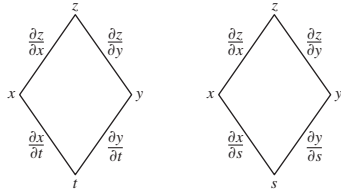
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}$$



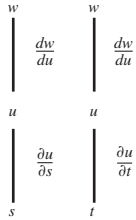
$$17. \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v}.$$



$$19. \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}, \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

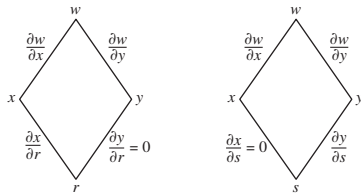


$$21. \frac{\partial w}{\partial s} = \frac{dw}{du} \frac{\partial u}{\partial s}, \frac{\partial w}{\partial t} = \frac{dw}{du} \frac{\partial u}{\partial t}$$



$$23. \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} \text{ since } \frac{\partial y}{\partial r} = 0,$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} = \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} \text{ since } \frac{\partial x}{\partial s} = 0$$



$$25. 4/3 \quad 27. -4/5 \quad 29. 20 \quad 31. \frac{\partial z}{\partial x} = \frac{1}{4}, \frac{\partial z}{\partial y} = -\frac{3}{4}$$

$$33. \frac{\partial z}{\partial x} = -1, \frac{\partial z}{\partial y} = -1 \quad 35. 12 \quad 37. -7$$

$$39. \frac{\partial z}{\partial u} = 2, \frac{\partial z}{\partial v} = 1 \quad 41. \frac{\partial w}{\partial t} = 2te^{s^3+t^2}, \frac{\partial w}{\partial s} = 3s^2e^{s^3+t^2}$$

$$43. 23 \quad 45. -16, 2 \quad 47. -0.00005 \text{ amp/s}$$

$$53. (\cos 1, \sin 1, 1) \text{ and } (\cos(-2), \sin(-2), -2)$$

$$55. \text{(a) Maximum at } \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \text{ and } \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right); \text{ minimum at } \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \text{ and } \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

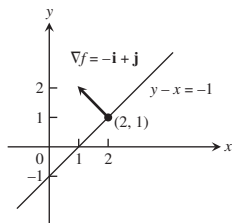
$$\text{(b) Max} = 6; \text{ min} = 2$$

$$57. 5^\circ\text{C/s} \quad 59. 2x\sqrt{x^8+x^3} + \int_0^{x^2} \frac{3x^2}{2\sqrt{t^4+x^3}} dt$$

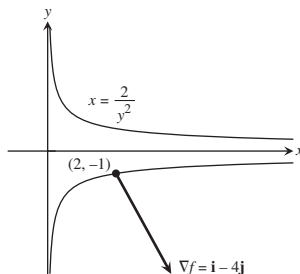
$$61. 0.19 \text{ m/min} \quad 63. -8/61 \text{ rad/min}$$

SECTION 13.5, pp. 814–815

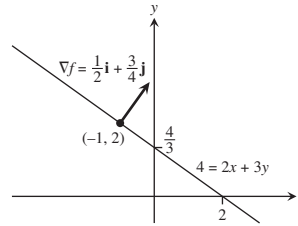
1.



3.



5.



$$7. \nabla f = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} \quad 9. \nabla f = -\frac{26}{27}\mathbf{i} + \frac{23}{54}\mathbf{j} - \frac{23}{54}\mathbf{k}$$

$$11. -4 \quad 13. 21/13 \quad 15. 3 \quad 17. 2$$

$$19. \mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}, (D_{\mathbf{u}}f)_{P_0} = \sqrt{2}; -\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j},$$

$$(D_{-\mathbf{u}}f)_{P_0} = -\sqrt{2}$$

$$21. \mathbf{u} = \frac{1}{3\sqrt{3}}\mathbf{i} - \frac{5}{3\sqrt{3}}\mathbf{j} - \frac{1}{3\sqrt{3}}\mathbf{k}, (D_{\mathbf{u}}f)_{P_0} = 3\sqrt{3};$$

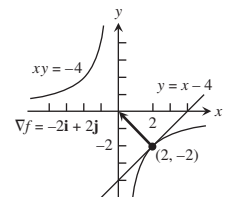
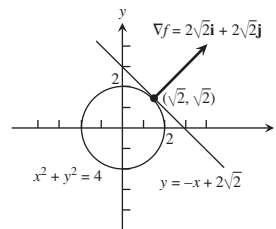
$$-\mathbf{u} = -\frac{1}{3\sqrt{3}}\mathbf{i} + \frac{5}{3\sqrt{3}}\mathbf{j} + \frac{1}{3\sqrt{3}}\mathbf{k}, (D_{-\mathbf{u}}f)_{P_0} = -3\sqrt{3}$$

$$23. \mathbf{u} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}), (D_{\mathbf{u}}f)_{P_0} = 2\sqrt{3};$$

$$-\mathbf{u} = -\frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}), (D_{-\mathbf{u}}f)_{P_0} = -2\sqrt{3}$$

25.

27.



$$29. \text{(a) } \mathbf{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}, D_{\mathbf{u}}f(1, -1) = 5$$

$$\text{(b) } \mathbf{u} = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}, D_{\mathbf{u}}f(1, -1) = -5$$

$$\text{(c) } \mathbf{u} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}, \mathbf{u} = -\frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$$

$$\text{(d) } \mathbf{u} = -\mathbf{j}, \mathbf{u} = \frac{24}{25}\mathbf{i} - \frac{7}{25}\mathbf{j}$$

$$\text{(e) } \mathbf{u} = -\mathbf{i}, \mathbf{u} = \frac{7}{25}\mathbf{i} + \frac{24}{25}\mathbf{j}$$

$$31. \mathbf{u} = \frac{7}{\sqrt{53}}\mathbf{i} - \frac{2}{\sqrt{53}}\mathbf{j}, -\mathbf{u} = -\frac{7}{\sqrt{53}}\mathbf{i} + \frac{2}{\sqrt{53}}\mathbf{j}$$

$$33. \text{No, the maximum rate of change is } \sqrt{185} < 14. \quad 35. -7/\sqrt{5}$$

$$41. \mathbf{r}(t) = (-3 - 6t)\mathbf{i} + (4 + 8t)\mathbf{j}, -\infty < t < \infty$$

$$43. \mathbf{r}(t) = (3 + 6t)\mathbf{i} + (-2 - 4t)\mathbf{j} + (1 + 2t)\mathbf{k}, -\infty < t < \infty$$

$$45. \left\langle \frac{14}{3}, \frac{1}{6}, -12, -\frac{4}{9} \right\rangle \quad 47. \left\langle 30, 1, 90, \frac{1}{e}, 9 \right\rangle$$

$$49. \frac{3}{20e^2} - 2 \quad 51. \frac{\pi}{2} + \frac{1}{12}$$

SECTION 13.6, pp. 822–825

$$1. \text{(a) } x + y + z = 3$$

$$\text{(b) } x = 1 + 2t, y = 1 + 2t, z = 1 + 2t$$

$$3. \text{(a) } 2x - z - 2 = 0$$

$$\text{(b) } x = 2 - 4t, y = 0, z = 2 + 2t$$

$$5. \text{(a) } 2x + 2y + z - 4 = 0$$

$$\text{(b) } x = 2t, y = 1 + 2t, z = 2 + t$$

$$7. \text{(a) } x + y + z - 1 = 0$$

$$\text{(b) } x = t, y = 1 + t, z = t$$

9. (a) $-x + 3y + z/e = 2$
 (b) $x = 2 - t, y = 1 + 3t, z = e + (1/e)t$
11. $2x - z - 2 = 0$ 13. $x - y + 2z - 1 = 0$
15. $x = 1, y = 1 + 2t, z = 1 - 2t$
17. $x = 1 - 2t, y = 1, z = \frac{1}{2} + 2t$
19. $x = 1 + 90t, y = 1 - 90t, z = 3$
21. $df = \frac{9}{11,830} \approx 0.0008$ 23. $dg = 0$
25. (a) $\frac{\sqrt{3}}{2} \sin \sqrt{3} - \frac{1}{2} \cos \sqrt{3} \approx 0.935$ °C/m
 (b) $\sqrt{3} \sin \sqrt{3} - \cos \sqrt{3} \approx 1.87$ °C/s
27. (a) $L(x, y) = 1$ (b) $L(x, y) = 2x + 2y - 1$
29. (a) $L(x, y) = 3x - 4y + 5$
 (b) $L(x, y) = 3x - 4y + 5$
31. (a) $L(x, y) = 1 + x$
 (b) $L(x, y) = -y + \frac{\pi}{2}$
33. (a) $W(30, -5) = -13$ °C, $W(50, -25) = -42.2$ °C,
 $W(30, -10) = -19.5$ °C
 (b) $W(15, -40) \approx -53.7$ °C, $W(80, -40) \approx -66.6$ °C,
 $W(90, 0) \approx -10.2$ °C
 (c) $L(v, T) \approx 1.336T - 0.11v - 2.952$
 (d) i) $L(39, 9) \approx -19.3$ °C
 ii) $L(42, -12) \approx -23.6$ °C
 iii) $L(10, -25) \approx -37.5$ °C
35. $L(x, y) = 7 + x - 6y$; 0.06
37. $L(x, y) = x + y + 1$; 0.08 39. $L(x, y) = 1 + x$; 0.0222
41. (a) $L(x, y, z) = 2x + 2y + 2z - 3$
 (b) $L(x, y, z) = y + z$ (c) $L(x, y, z) = 0$
43. (a) $L(x, y, z) = x$
 (b) $L(x, y, z) = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$
 (c) $L(x, y, z) = \frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z$
45. (a) $L(x, y, z) = 2 + x$
 (b) $L(x, y, z) = x - y - z + \frac{\pi}{2} + 1$
 (c) $L(x, y, z) = x - y - z + \frac{\pi}{2} + 1$
47. $L(x, y, z) = 2x - 6y - 2z + 6$, 0.0024
49. $L(x, y, z) = x + y - z - 1$, 0.00135
51. Maximum error (estimate) ≤ 0.31 in magnitude
53. Pay more attention to the smaller of the two dimensions. It will generate the larger partial derivative.
55. f is most sensitive to a change in d .
61. (a) 1.75% (b) 1.75%

SECTION 13.7, pp. 832-834

1. $f(-3, 3) = -5$, local minimum 3. $f(-2, 1)$, saddle point
5. $f\left(3, \frac{3}{2}\right) = \frac{17}{2}$, local maximum
7. $f(2, -1) = -6$, local minimum 9. $f(1, 2)$, saddle point
11. $f\left(\frac{16}{7}, 0\right) = -\frac{16}{7}$, local maximum
13. $f(0, 0)$, saddle point; $f\left(-\frac{2}{3}, \frac{2}{3}\right) = \frac{170}{27}$, local maximum
15. $f(0, 0) = 0$, local minimum; $f(1, -1)$, saddle point
17. $f(0, \pm\sqrt{5})$, saddle points; $f(-2, -1) = 30$, local maximum;
 $f(2, 1) = -30$, local minimum
19. $f(0, 0)$, saddle point; $f(1, 1) = 2, f(-1, -1) = 2$, local maxima

21. $f(0, 0) = -1$, local maximum
23. $f(n\pi, 0)$, saddle points, for every integer n
25. $f(2, 0) = e^{-4}$, local minimum
27. $f(0, 0) = 0$, local minimum; $f(0, 2)$, saddle point
29. $f\left(\frac{1}{2}, 1\right) = \ln\left(\frac{1}{4}\right) - 3$, local maximum
31. Absolute maximum: 1 at $(0, 0)$; absolute minimum: -5 at $(1, 2)$
33. Absolute maximum: 4 at $(0, 2)$; absolute minimum: 0 at $(0, 0)$
35. Absolute maximum: 11 at $(0, -3)$; absolute minimum: -10 at $(4, -2)$
37. Absolute maximum: 4 at $(2, 0)$; absolute minimum: $\frac{3\sqrt{2}}{2}$ at
 $\left(3, -\frac{\pi}{4}\right), \left(3, \frac{\pi}{4}\right), \left(1, -\frac{\pi}{4}\right)$, and $\left(1, \frac{\pi}{4}\right)$
39. $a = -3, b = 2$
41. Hottest is $2\frac{1}{4}^\circ$ at $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$; coldest is
 $-\frac{1}{4}^\circ$ at $\left(\frac{1}{2}, 0\right)$.
43. (a) $f(0, 0)$, saddle point (b) $f(1, 2)$, local minimum
 (c) $f(1, -2)$, local minimum; $f(-1, -2)$, saddle point
49. $\left(\frac{1}{6}, \frac{1}{3}, \frac{355}{36}\right)$ 51. $\left(\frac{9}{7}, \frac{6}{7}, \frac{3}{7}\right)$ 53. 3, 3, 3 55. 12
57. $\frac{4}{\sqrt{3}} \times \frac{4}{\sqrt{3}} \times \frac{4}{\sqrt{3}}$ 59. $2 \text{ m} \times 2 \text{ m} \times 1 \text{ m}$
61. Points $(0, 2, 0)$ and $(0, -2, 0)$ have distance 2 from the origin.
63. (a) On the semicircle, max $f = 2\sqrt{2}$ at $t = \pi/4$, min $f = -2$
 at $t = \pi$. On the quarter circle, max $f = 2\sqrt{2}$ at
 $t = \pi/4$, min $f = 2$ at $t = 0, \pi/2$.
 (b) On the semicircle, max $g = 2$ at $t = \pi/4$, min $g = -2$
 at $t = 3\pi/4$. On the quarter circle, max $g = 2$ at
 $t = \pi/4$, min $g = 0$ at $t = 0, \pi/2$.
 (c) On the semicircle, max $h = 8$ at $t = 0, \pi$; min $h = 4$
 at $t = \pi/2$. On the quarter circle, max $h = 8$ at
 $t = 0$, min $h = 4$ at $t = \pi/2$.
65. i) min $f = -1/2$ at $t = -1/2$; no max
 ii) max $f = 0$ at $t = -1, 0$; min $f = -1/2$ at $t = -1/2$
 iii) max $f = 4$ at $t = 1$; min $f = 0$ at $t = 0$
69. $y = -\frac{20}{13}x + \frac{9}{13}, y\Big|_{x=4} = -\frac{71}{13}$

SECTION 13.8, pp. 841-844

1. $\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right), \left(\pm\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$ 3. 39 5. $(3, \pm 3\sqrt{2})$
7. (a) 8 (b) 64
9. $r = 2 \text{ cm}, h = 4 \text{ cm}$ 11. Length $= 4\sqrt{2}$, width $= 3\sqrt{2}$
13. $f(0, 0) = 0$ is minimum; $f(2, 4) = 20$ is maximum.
15. Lowest $= 0^\circ$, highest $= 125^\circ$
17. $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$ 19. 1 21. $(0, 0, 2), (0, 0, -2)$
23. $f(1, -2, 5) = 30$ is maximum; $f(-1, 2, -5) = -30$ is minimum.
25. 3, 3, 3 27. $\frac{2}{\sqrt{3}}$ by $\frac{2}{\sqrt{3}}$ by $\frac{2}{\sqrt{3}}$ units
29. $(\pm 4/3, -4/3, -4/3)$ 31. (b) $\approx 24,322$ units
33. $U(8, 14) = \$128$ 37. $f(2/3, 4/3, -4/3) = \frac{4}{3}$
39. $(2, 4, 4)$ 41. Maximum is $1 + 6\sqrt{3}$ at $(\pm\sqrt{6}, \sqrt{3}, 1)$;
 minimum is $1 - 6\sqrt{3}$ at $(\pm\sqrt{6}, -\sqrt{3}, 1)$.
43. Maximum is 4 at $(0, 0, \pm 2)$; minimum is 2 at $(\pm\sqrt{2}, \pm\sqrt{2}, 0)$.

SECTION 13.9, p. 848

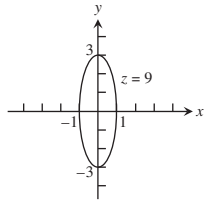
1. Quadratic: $x + xy$; cubic: $x + xy + \frac{1}{2}xy^2$
3. Quadratic: xy ; cubic: xy
5. Quadratic: $y + \frac{1}{2}(2xy - y^2)$;
cubic: $y + \frac{1}{2}(2xy - y^2) + \frac{1}{6}(3x^2y - 3xy^2 + 2y^3)$
7. Quadratic: $\frac{1}{2}(2x^2 + 2y^2) = x^2 + y^2$; cubic: $x^2 + y^2$
9. Quadratic: $1 + (x + y) + (x + y)^2$;
cubic: $1 + (x + y) + (x + y)^2 + (x + y)^3$
11. Quadratic: $1 - \frac{1}{2}x^2 - \frac{1}{2}y^2$; $E(x, y) \leq 0.00134$

SECTION 13.10, p. 852

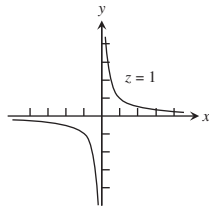
1. (a) 0 (b) $1 + 2z$ (c) $1 + 2z$
3. (a) $\frac{\partial U}{\partial P} + \frac{\partial U}{\partial T} \left(\frac{V}{nR} \right)$ (b) $\frac{\partial U}{\partial P} \left(\frac{nR}{V} \right) + \frac{\partial U}{\partial T}$
5. (a) 5 (b) 5
7. $\left(\frac{\partial x}{\partial r} \right)_\theta = \cos \theta \left(\frac{\partial r}{\partial x} \right)_y = \frac{x}{\sqrt{x^2 + y^2}}$

PRACTICE EXERCISES, pp. 853–856

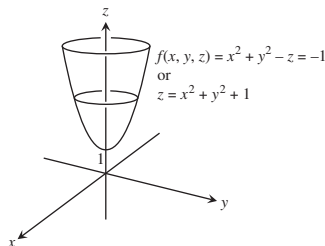
1. Domain: all points in the xy -plane; range: $z \geq 0$. Level curves are ellipses with major axis along the y -axis and minor axis along the x -axis.



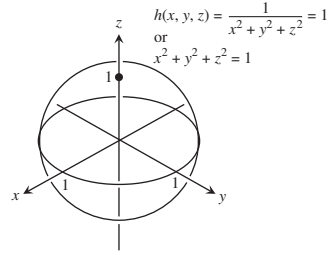
3. Domain: all (x, y) such that $x \neq 0$ and $y \neq 0$; range: $z \neq 0$. Level curves are hyperbolas with the x - and y -axes as asymptotes.



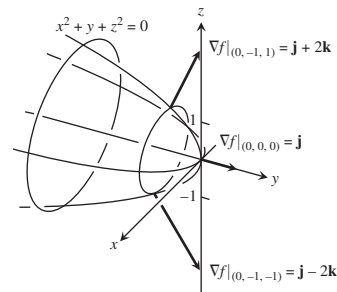
5. Domain: all points in xyz -space; range: all real numbers. Level surfaces are paraboloids of revolution with the z -axis as axis.



7. Domain: all (x, y, z) such that $(x, y, z) \neq (0, 0, 0)$; range: positive real numbers. Level surfaces are spheres with center $(0, 0, 0)$ and radius $r > 0$.

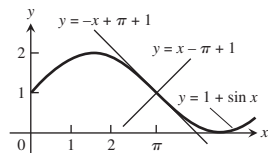


9. -2 11. $1/2$ 13. 1 15. Let $y = kx^2$, $k \neq 1$
17. No; $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ does not exist.
19. $\frac{\partial g}{\partial r} = \cos \theta + \sin \theta$, $\frac{\partial g}{\partial \theta} = -r \sin \theta + r \cos \theta$
21. $\frac{\partial f}{\partial R_1} = -\frac{1}{R_1^2}$, $\frac{\partial f}{\partial R_2} = -\frac{1}{R_2^2}$, $\frac{\partial f}{\partial R_3} = -\frac{1}{R_3^2}$
23. $\frac{\partial P}{\partial n} = \frac{RT}{V}$, $\frac{\partial P}{\partial R} = \frac{nT}{V}$, $\frac{\partial P}{\partial T} = \frac{nR}{V}$, $\frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$
25. $\frac{\partial^2 g}{\partial x^2} = 0$, $\frac{\partial^2 g}{\partial y^2} = \frac{2x}{y^3}$, $\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial^2 g}{\partial x \partial y} = -\frac{1}{y^2}$
27. $\frac{\partial^2 f}{\partial x^2} = -30x + \frac{2 - 2x^2}{(x^2 + 1)^2}$, $\frac{\partial^2 f}{\partial y^2} = 0$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 1$
29. $\left. \frac{dw}{dt} \right|_{t=0} = -1$
31. $\left. \frac{\partial w}{\partial r} \right|_{(r,s)=(\pi,0)} = 2$, $\left. \frac{\partial w}{\partial s} \right|_{(r,s)=(\pi,0)} = 2 - \pi$
33. $\left. \frac{df}{dt} \right|_{t=1} = -(\sin 1 + \cos 2)(\sin 1) + (\cos 1 + \cos 2)(\cos 1) - 2(\sin 1 + \cos 1)(\sin 2)$
35. $\left. \frac{dy}{dx} \right|_{(x,y)=(0,1)} = -1$
37. Increases most rapidly in the direction $\mathbf{u} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$;
decreases most rapidly in the direction $-\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$;
 $D_{\mathbf{u}}f = \frac{\sqrt{2}}{2}$; $D_{-\mathbf{u}}f = -\frac{\sqrt{2}}{2}$; $D_{\mathbf{u}_1}f = -\frac{7}{10}$ where $\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|}$
39. Increases most rapidly in the direction $\mathbf{u} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$;
decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} - \frac{6}{7}\mathbf{k}$;
 $D_{\mathbf{u}}f = 7$; $D_{-\mathbf{u}}f = -7$; $D_{\mathbf{u}_1}f = 7$ where $\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|}$
41. $\pi/\sqrt{2}$ 43. (a) $f_x(1,2) = f_y(1,2) = 2$ (b) $14/5$
- 45.



47. Tangent: $4x - y - 5z = 4$; normal line:

$$x = 2 + 4t, y = -1 - t, z = 1 - 5t$$

49. $2y - z - 2 = 0$ 51. Tangent: $x + y = \pi + 1$; normal line: $y = x - \pi + 1$ 53. $x = 1 - 2t, y = 1, z = 1/2 + 2t$

55. Answers will depend on the upper bound used for

$$|f_{xx}|, |f_{xy}|, |f_{yy}|. \text{ With } M = \sqrt{2}/2, |E| \leq 0.0142. \text{ With } M = 1, |E| \leq 0.02.$$

57. $L(x, y, z) = y - 3z, L(x, y, z) = x + y - z - 1$

59. Be more careful with the diameter.

61. $dI = 0.038\%$, % change in $I = 15.83\%$, more sensitive to voltage change63. (a) 5% 65. Local minimum of -8 at $(-2, -2)$ 67. Saddle point at $(0, 0), f(0, 0) = 0$; local maximum of $1/4$ at $(-1/2, -1/2)$ 69. Saddle point at $(0, 0), f(0, 0) = 0$; local minimum of -4 at $(0, 2)$; local maximum of 4 at $(-2, 0)$; saddle point at $(-2, 2), f(-2, 2) = 0$ 71. Absolute maximum: 28 at $(0, 4)$; absolute minimum: $-9/4$ at $(3/2, 0)$ 73. Absolute maximum: 18 at $(2, -2)$; absolute minimum: $-17/4$ at $(-2, 1/2)$ 75. Absolute maximum: 8 at $(-2, 0)$; absolute minimum: -1 at $(1, 0)$ 77. Absolute maximum: 4 at $(1, 0)$; absolute minimum: -4 at $(0, -1)$ 79. Absolute maximum: 1 at $(0, \pm 1)$ and $(1, 0)$; absolute minimum: -1 at $(-1, 0)$ 81. Maximum: 5 at $(0, 1)$; minimum: $-1/3$ at $(0, -1/3)$ 83. Maximum: $\sqrt{3}$ at $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$; minimum: $-\sqrt{3}$ at $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ 85. Width $= (\frac{c^2 V}{ab})^{1/3}$, depth $= (\frac{b^2 V}{ac})^{1/3}$, height $= (\frac{a^2 V}{bc})^{1/3}$ 87. Maximum: $\frac{3}{2}$ at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2})$; minimum: $\frac{1}{2}$ at $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2})$ and $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2})$ 89. $\frac{\partial w}{\partial x} = \cos \theta \frac{\partial w}{\partial r} - \frac{\sin \theta}{r} \frac{\partial w}{\partial \theta}, \frac{\partial w}{\partial y} = \sin \theta \frac{\partial w}{\partial r} + \frac{\cos \theta}{r} \frac{\partial w}{\partial \theta}$ 95. $(t, -t \pm 4, t)$, t a real number101. (a) $(2y + x^2 z)e^{yz}$ (b) $x^2 e^{yz} \left(y - \frac{z}{2y} \right)$ (c) $(1 + x^2 y)e^{yz}$

ADDITIONAL AND ADVANCED EXERCISES, pp. 856–858

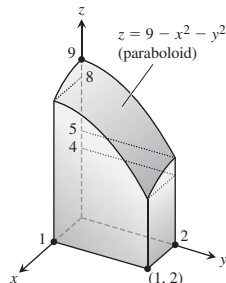
1. $f_{xy}(0, 0) = -1, f_{yx}(0, 0) = 1$ 7. (c) $\frac{r^2}{2} = \frac{1}{2}(x^2 + y^2 + z^2)$ 13. $V = \frac{\sqrt{3}abc}{2}$ 17. $f(x, y) = \frac{y}{2} + 4, g(x, y) = \frac{x}{2} + \frac{9}{2}$ 19. $y = 2 \ln |\sin x| + \ln 2$ 21. (a) $\frac{1}{\sqrt{53}}(2\mathbf{i} + 7\mathbf{j})$ (b) $\frac{-1}{\sqrt{29,097}}(98\mathbf{i} - 127\mathbf{j} + 58\mathbf{k})$ 23. $w = e^{-c^2 \pi^2 t} \sin \pi x$

Chapter 14

SECTION 14.1, pp. 863–864

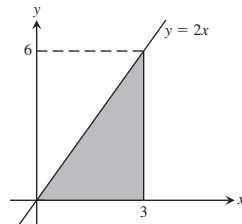
1. 24 3. 1 5. 16 7. $2 \ln 2 - 1$ 9. $(3/2)(5 - e)$ 11. $3/2$ 13. $\ln 2$ 15. $3/2, -2$ 17. 14 19. 021. $1/2$ 23. $2 \ln 2$ 25. $(\ln 2)^2$

27.

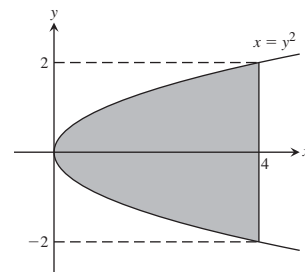
29. $8/3$ 31. 1 33. $\sqrt{2}$ 35. $2/27$ 37. $\frac{3}{2} \ln 3 - 1$ 39. (a) $1/3$ (b) $2/3$

SECTION 14.2, pp. 870–873

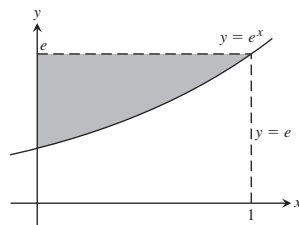
1.



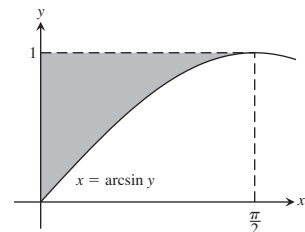
3.



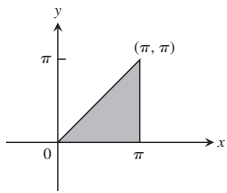
5.



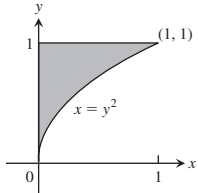
7.

9. (a) $\int_0^2 \int_{x^3}^8 dy dx$ (b) $\int_0^8 \int_0^{y^{1/3}} dx dy$ 11. (a) $\int_0^3 \int_{x^2}^{3x} dy dx$ (b) $\int_0^9 \int_{y/3}^{\sqrt{y}} dx dy$ 13. (a) $\int_0^9 \int_0^{\sqrt{x}} dy dx$ (b) $\int_0^3 \int_{y^2}^9 dx dy$ 15. (a) $\int_0^{\ln 3} \int_{e^{-x}}^1 dy dx$ (b) $\int_{1/3}^1 \int_{-\ln y}^{\ln 3} dx dy$ 17. (a) $\int_0^1 \int_x^{3-2x} dy dx$ (b) $\int_0^1 \int_0^y dx dy + \int_1^3 \int_0^{(3-y)/2} dx dy$ 19. 42 21. $21/40$ 23. $\pi/2$ 25. $49/5$

27. $\frac{\pi^2}{2} + 2$



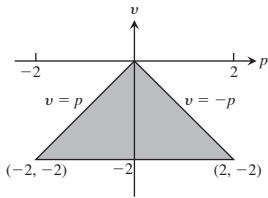
31. $e - 2$



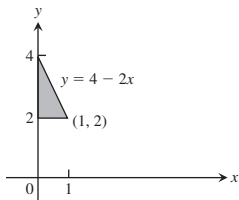
33. $\frac{3}{2} \ln 2$

35. $-1/10$

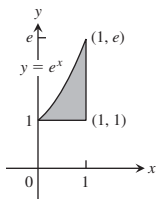
37. 8



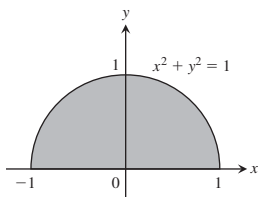
41. $\int_2^4 \int_0^{(4-y)/2} dx dy$



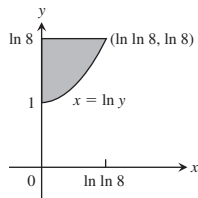
45. $\int_1^e \int_{\ln y}^1 dx dy$



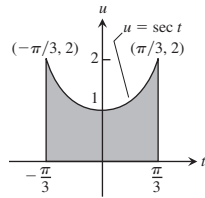
49. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3y dy dx$



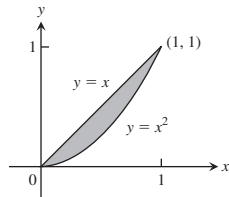
29. $8 \ln 8 - 16 + e$



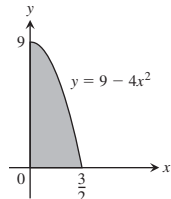
39. 2π



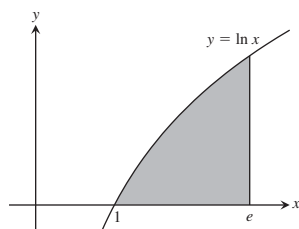
43. $\int_0^1 \int_{x^2}^x dy dx$



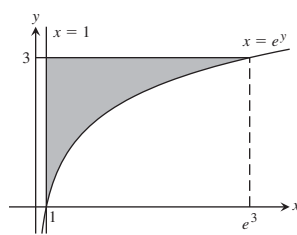
47. $\int_0^9 \int_0^{(\sqrt{9-y})/2} 16x dx dy$



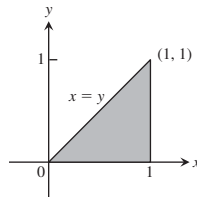
51. $\int_0^1 \int_{e^y}^e xy dx dy$



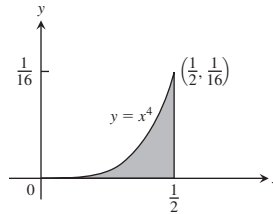
53. $\int_1^{e^3} \int_{\ln x}^3 (x + y) dy dx$



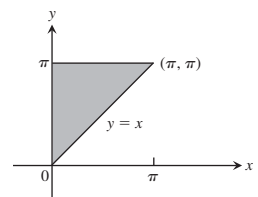
57. $\frac{e-2}{2}$



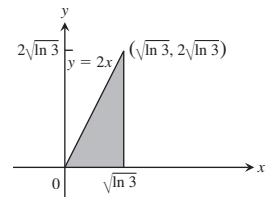
61. $1/(80\pi)$



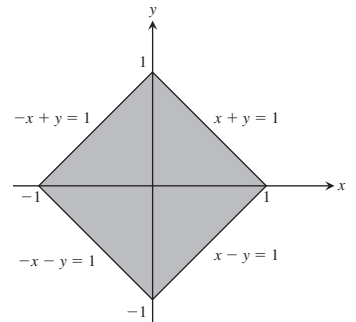
55. 2



59. 2



63. $-2/3$



65. $4/3$

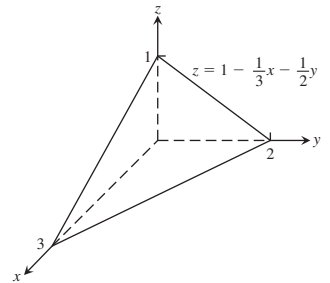
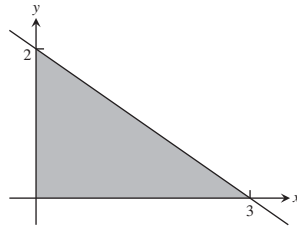
67. $625/12$

69. 16

71. 20

73. $2(1 + \ln 2)$

75.



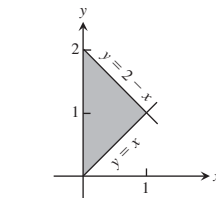
77. 1

79. π^2

81. $-\frac{3}{32}$

83. $\frac{20\sqrt{3}}{9}$

85. $\int_0^1 \int_x^{2-x} (x^2 + y^2) dy dx = \frac{4}{3}$



87. R is the set of points (x, y) such that $x^2 + 2y^2 \leq 4$.

89. No, by Fubini's Theorem, the two orders of integration must give the same result.

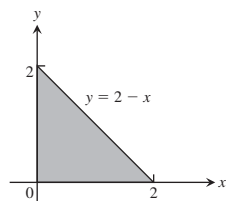
93. 0.603

95. 0.233

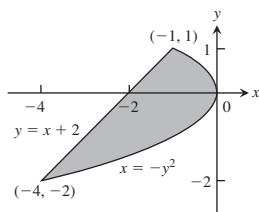
SECTION 14.3, pp. 875–876

1. $\int_0^2 \int_0^{2-x} dy dx = 2$ or

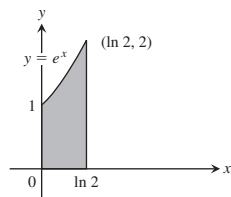
$$\int_0^2 \int_0^{2-y} dx dy = 2$$



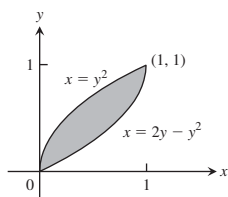
3. $\int_{-2}^1 \int_{y-2}^{-y^2} dx dy = \frac{9}{2}$



5. $\int_0^{\ln 2} \int_0^{e^x} dy dx = 1$

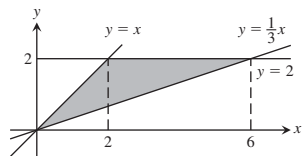


7. $\int_0^1 \int_{y^2}^{2y-y^2} dx dy = \frac{1}{3}$



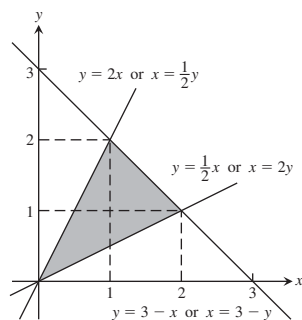
9. $\int_0^2 \int_y^{3y} 1 dx dy = 4$ or

$$\int_0^2 \int_{x/3}^x 1 dy dx + \int_2^6 \int_{x/3}^2 1 dy dx = 4$$

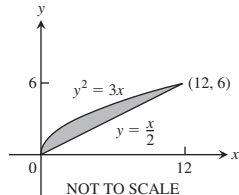
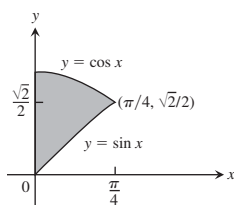
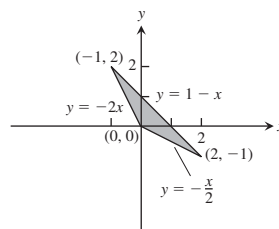


11. $\int_0^1 \int_{x/2}^{2x} 1 dy dx + \int_1^2 \int_{x/2}^{3-x} 1 dy dx = \frac{3}{2}$ or

$$\int_0^1 \int_{y/2}^{2y} 1 dx dy + \int_1^2 \int_{y/2}^2 1 dx dy = \frac{3}{2}$$



13. 12

15. $\sqrt{2} - 1$ 17. $\frac{3}{2}$ 19. (a) 0 (b) $4/\pi^2$ 21. $8/3$ 23. $\pi - 2$ 25. $40,000(1 - e^{-2}) \ln(7/2) \approx 43,329$

SECTION 14.4, pp. 881–883

1. $\frac{\pi}{2} \leq \theta \leq 2\pi, 0 \leq r \leq 9$ 3. $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}, 0 \leq r \leq \csc \theta$

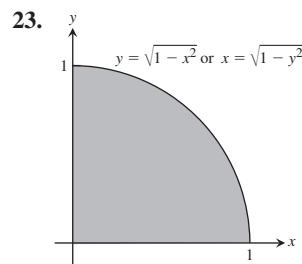
5. $0 \leq \theta \leq \frac{\pi}{6}, 1 \leq r \leq 2\sqrt{3} \sec \theta$

$$\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}, 1 \leq r \leq 2 \csc \theta$$

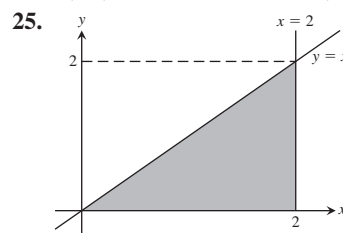
7. $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq 2 \cos \theta$ 9. $\frac{\pi}{2}$

11. 2π 13. 36 15. $2 - \sqrt{3}$ 17. $(1 - \ln 2)\pi$

19. $(2 \ln 2 - 1)(\pi/2)$ 21. $\frac{2(1 + \sqrt{2})}{3}$



$$\int_0^1 \int_0^{\sqrt{1-x^2}} xy dy dx \quad \text{or} \quad \int_0^1 \int_0^{\sqrt{1-y^2}} xy dx dy$$



$$\int_0^2 \int_0^x y^2(x^2 + y^2) dy dx \quad \text{or} \quad \int_0^2 \int_y^2 y^2(x^2 + y^2) dx dy$$

27. $2(\pi - 2)$ 29. 12π 31. $(3\pi/8) + 1$ 33. $\frac{2a}{3}$ 35. $\frac{2a}{3}$ 37. $2\pi(2 - \sqrt{e})$ 39. $\frac{4}{3} + \frac{5\pi}{8}$ 41. (a) $\frac{\sqrt{\pi}}{2}$ (b) 1 43. $\pi \ln 4$, no 45. $\frac{1}{2}(a^2 + 2h^2)$ 47. $\frac{8}{9}(3\pi - 4)$

SECTION 14.5, pp. 891–8931. $1/6$

$$3. \int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} dz \, dy \, dx, \int_0^2 \int_0^{1-y/2} \int_0^{3-3x-3y/2} dz \, dx \, dy, \\ \int_0^1 \int_0^{3-3x} \int_0^{2-2x-2z/3} dy \, dz \, dx, \int_0^3 \int_0^{1-z/3} \int_0^{2-2x-2z/3} dy \, dx \, dz, \\ \int_0^2 \int_0^{3-3y/2} \int_0^{1-y/2-z/3} dx \, dz \, dy, \int_0^3 \int_0^{2-2z/3} \int_0^{1-y/2-z/3} dx \, dy \, dz.$$

The value of all six integrals is 1.

$$5. \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-x^2-y^2} 1 \, dz \, dx \, dy, \int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{x^2+y^2}^{8-x^2-y^2} 1 \, dz \, dx \, dy, \\ \int_{-2}^2 \int_4^{8-y^2} \int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} 1 \, dx \, dz \, dy + \int_{-2}^2 \int_{y^2}^4 \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} 1 \, dx \, dz \, dy, \\ \int_4^8 \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} 1 \, dx \, dy \, dz + \int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} 1 \, dx \, dy \, dz, \\ \int_{-2}^2 \int_4^{8-x^2} \int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} 1 \, dy \, dz \, dx + \int_{-2}^2 \int_{x^2}^4 \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} 1 \, dy \, dz \, dx, \\ \int_4^8 \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} 1 \, dy \, dx \, dz + \int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} 1 \, dy \, dx \, dz.$$

The value of all six integrals is 16π .

$$7. 1 \quad 9. 6 \quad 11. \frac{5(2-\sqrt{3})}{4} \quad 13. 18$$

$$15. 7/6 \quad 17. 0 \quad 19. \frac{1}{2} - \frac{\pi}{8}$$

$$21. (a) \int_{-1}^1 \int_0^{1-x^2} \int_{x^2}^{1-z} dy \, dz \, dx \quad (b) \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1-z} dy \, dx \, dz$$

$$(c) \int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} dx \, dy \, dz \quad (d) \int_0^1 \int_0^{1-y} \int_{-\sqrt{y}}^{\sqrt{y}} dx \, dz \, dy$$

$$(e) \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{1-y} dz \, dx \, dy$$

$$23. 2/3 \quad 25. 20/3 \quad 27. 1 \quad 29. 16/3 \quad 31. 8\pi - \frac{32}{3}$$

$$33. 2 \quad 35. 4\pi \quad 37. 31/3 \quad 39. 1 \quad 41. 2 \sin 4$$

$$43. 4 \quad 45. a = 3 \text{ or } a = 13/3$$

$$47. \text{The domain is the set of all points } (x, y, z) \text{ such that } 4x^2 + 4y^2 + z^2 \leq 4.$$

SECTION 14.6, pp. 900–903

$$1. \bar{x} = 5/14, \bar{y} = 38/35 \quad 3. \bar{x} = 64/35, \bar{y} = 5/7$$

$$5. \bar{x} = \bar{y} = 4a/(3\pi)$$

$$7. I_x = I_y = 4\pi \text{ gm/cm}^2, I_0 = 8\pi \text{ gm/cm}^2$$

$$9. \bar{x} = -1, \bar{y} = 1/4 \quad 11. I_x = 64/105$$

$$13. \bar{x} = 3/8, \bar{y} = 17/16 \quad 15. \bar{x} = 11/3, \bar{y} = 14/27, I_y = 432$$

$$17. \bar{x} = 0, \bar{y} = 13/31, I_y = 7/5$$

$$19. \bar{x} = 0, \bar{y} = 7/10; I_x = 9/10 \text{ kg/m}^2, I_y = 3/10 \text{ kg/m}^2, \\ I_0 = 6/5 \text{ kg/m}^2$$

$$21. I_x = \frac{M}{3}(b^2 + c^2), I_y = \frac{M}{3}(a^2 + c^2), I_z = \frac{M}{3}(a^2 + b^2)$$

$$23. \bar{x} = \bar{y} = 0, \bar{z} = 12/5, I_x = 7904/105 \approx 75.28, \\ I_y = 4832/63 \approx 76.70, I_z = 256/45 \approx 5.69$$

$$25. (a) \bar{x} = \bar{y} = 0, \bar{z} = 8/3 \quad (b) c = 2\sqrt{2}$$

$$27. I_L = 1386$$

$$29. (a) 4/3 \text{ gm} \quad (b) \bar{x} = 4/5 \text{ cm}, \bar{y} = \bar{z} = 2/5 \text{ cm}$$

$$31. (a) 5/2 \quad (b) \bar{x} = \bar{y} = \bar{z} = 8/15 \quad (c) I_x = I_y = I_z = 11/6$$

$$33. 3 \text{ kg}$$

$$37. (a) I_{\text{c.m.}} = \frac{abc(a^2 + b^2)}{12}$$

$$(b) I_L = \frac{abc(a^2 + 7b^2)}{3}$$

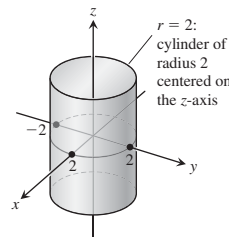
$$39. \mu_x = \mu_y = 7/12$$

$$41. \mu_x = 3/4, \mu_y = 2/3$$

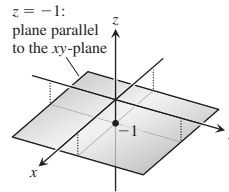
$$43. f(x, y) = 1/6, P(X < Y) = 2/3$$

SECTION 14.7, pp. 911–915

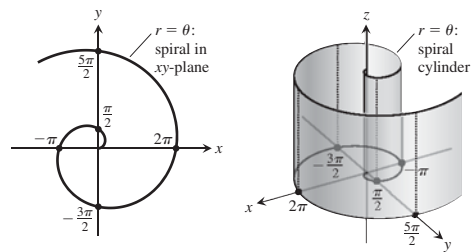
1.



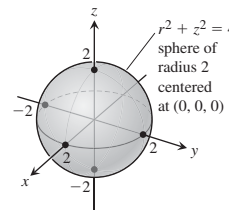
3.



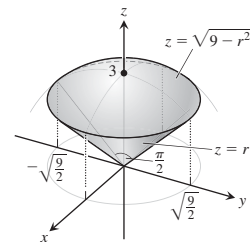
5.



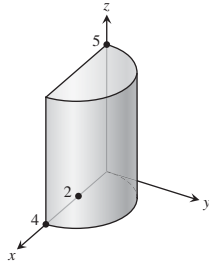
7.



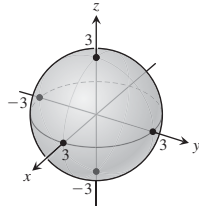
$$9. r \leq z \leq \sqrt{9 - r^2} : \text{cone with vertex angle } \frac{\pi}{2} \text{ below a sphere of radius 3 centered at } (0, 0, 0), \text{ and its interior}$$



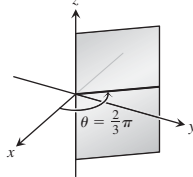
11. $0 \leq r \leq 4 \cos \theta$, $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq z \leq 5$: half-cylinder of height 5, radius 2, and tangent to the z -axis, and its interior



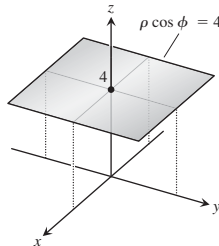
13. $\rho = 3$: sphere of radius 3 centered at $(0, 0, 0)$



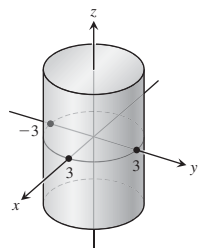
15. $\theta = \frac{2}{3}\pi$: closed half-plane along the z -axis



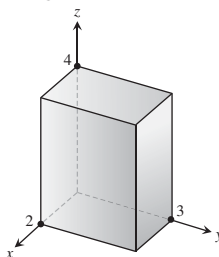
17. $\rho \cos \phi = 4$: plane with z -intercept 4 and parallel to the xy -plane



19. $0 \leq \rho \leq 3 \csc \phi \Rightarrow 0 \leq \rho \sin \phi \leq 3$: cylinder of radius 3 centered on the z -axis, and its interior



21. $0 \leq \rho \cos \theta \sin \phi \leq 2$, $0 \leq \rho \sin \theta \sin \phi \leq 3$, $0 \leq \rho \cos \phi \leq 4$: rectangular box $2 \times 3 \times 4$, and its interior



23. $\frac{4\pi(\sqrt{2}-1)}{3}$ 25. $\frac{17\pi}{5}$ 27. $\pi(6\sqrt{2}-8)$

29. $\frac{3\pi}{10}$ 31. $\pi/3$

33. (a) $\int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$

(b) $\int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^1 r \, dr \, dz \, d\theta + \int_0^{2\pi} \int_{\sqrt{3}}^2 \int_0^{\sqrt{4-z^2}} r \, dr \, dz \, d\theta$

(c) $\int_0^1 \int_0^{\sqrt{4-r^2}} \int_0^{2\pi} r \, d\theta \, dz \, dr$

35. $\int_{-\pi/2}^{\pi/2} \int_0^{\cos \theta} \int_0^{3r^2} f(r, \theta, z) \, r \, dz \, dr \, d\theta$

37. $\int_0^\pi \int_0^{2\sin \theta} \int_0^{4-r\sin \theta} f(r, \theta, z) \, r \, dz \, dr \, d\theta$

39. $\int_{-\pi/2}^{\pi/2} \int_1^{1+\cos \theta} \int_0^4 f(r, \theta, z) \, r \, dz \, dr \, d\theta$

41. $\int_0^{\pi/4} \int_0^{\sec \theta} \int_0^{2-r\sin \theta} f(r, \theta, z) \, r \, dz \, dr \, d\theta$

43. π^2 45. $\pi/3$ 47. 5π 49. 2π 51. $\left(\frac{8-5\sqrt{2}}{2}\right)\pi$

53. (a) $\int_0^{2\pi} \int_0^{\pi/6} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta + \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^{\csc \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

(b) $\int_0^{2\pi} \int_1^2 \int_{\pi/6}^{\sin^{-1}(1/\rho)} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta +$

$\int_0^{2\pi} \int_0^2 \int_0^{\pi/6} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta +$

$\int_0^{2\pi} \int_0^1 \int_{\pi/6}^{\pi/2} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta$

55. $\int_0^{2\pi} \int_0^{\pi/2} \int_{\cos \phi}^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{31\pi}{6}$

57. $\int_0^{2\pi} \int_0^\pi \int_0^{1-\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{8\pi}{3}$

59. $\int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{2\cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{\pi}{3}$

61. (a) $8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

(b) $8 \int_0^{\pi/2} \int_0^2 \int_0^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$

(c) $8 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} dz \, dy \, dx$

63. (a) $\int_0^{2\pi} \int_0^{\pi/3} \int_{\sec \phi}^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

(b) $\int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$

(c) $\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} dz \, dy \, dx$ (d) $5\pi/3$

65. $8\pi/3$ 67. $9/4$ 69. $\frac{3\pi-4}{18}$ 71. $\frac{2\pi a^3}{3}$

73. $\frac{(4\sqrt{2}-5)\pi}{3}$ 75. $\pi/2$ 77. $\frac{4(2\sqrt{2}-1)\pi}{3}$ 79. 16π

81. $5\pi/2$ 83. $\frac{4\pi(8-3\sqrt{3})}{3}$ 85. $2/3$ 87. $3/4$

89. $\bar{x} = \bar{y} = 0$, $\bar{z} = 3/8$ 91. $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 3/8)$

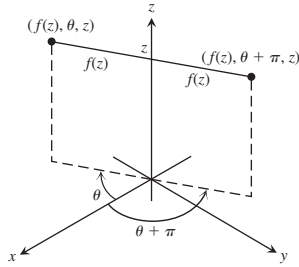
93. $\bar{x} = \bar{y} = 0, \bar{z} = 5/6$ 95. $I_x = \pi/4$ 97. $\frac{a^4 h \pi}{10}$

99. (a) $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, \frac{4}{5})$, $I_z = \frac{\pi}{12}$

(b) $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, \frac{5}{6})$, $I_z = \frac{\pi}{14}$

101. $\frac{3M}{\pi R^3}$

103. The surface's equation $r = f(z)$ tells us that the point $(r, \theta, z) = (f(z), \theta, z)$ will lie on the surface for all θ . In particular, $(f(z), \theta + \pi, z)$ lies on the surface whenever $(f(z), \theta, z)$ lies on the surface, so the surface is symmetric with respect to the z -axis.



SECTION 14.8, pp. 923–924

1. (a) $x = \frac{u+v}{3}, y = \frac{v-2u}{3}; \frac{1}{3}$

(b) Triangular region with boundaries $u = 0, v = 0$, and $u + v = 3$

3. (a) $x = \frac{1}{5}(2u - v), y = \frac{1}{10}(3v - u); \frac{1}{10}$

(b) Triangular region with boundaries $3v = u, v = 2u$, and $3u + v = 10$

7. $64/5$ 9. $\int_1^2 \int_1^3 (u+v) \frac{2u}{v} du dv = 8 + \frac{52}{3} \ln 2$

11. $\frac{\pi ab(a^2 + b^2)}{4}$ 13. $\frac{1}{3} \left(1 + \frac{3}{e^2}\right) \approx 0.4687$

15. $\frac{225}{16}$ 17. 12 19. $\frac{a^2 b^2 c^2}{6}$

21. (a) $\begin{vmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \end{vmatrix} = u \cos^2 v + u \sin^2 v = u$

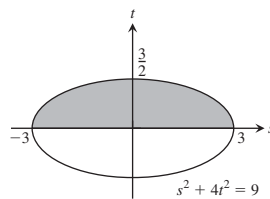
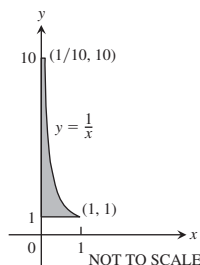
(b) $\begin{vmatrix} \sin v & u \cos v \\ \cos v & -u \sin v \end{vmatrix} = -u \sin^2 v - u \cos^2 v = -u$

27. $\frac{3}{2} \ln 2$

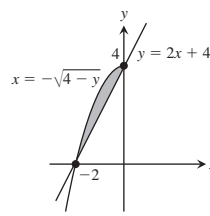
PRACTICE EXERCISES, pp. 925–927

1. $9e - 9$

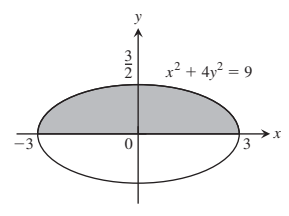
3. $9/2$



5. $\int_{-2}^0 \int_{2x+4}^{4-x^2} dy dx = \frac{4}{3}$



7. $\int_{-3}^3 \int_0^{(1/2)\sqrt{9-x^2}} y dy dx = \frac{9}{2}$



9. $\sin 4$ 11. $\frac{\ln 17}{4}$ 13. $4/3$ 15. $4/3$ 17. $1/4$

19. π 21. $\frac{\pi - 2}{4}$ 23. 0 25. $8/35$ 27. $\pi/2$

29. $\frac{2(31 - 3^{5/2})}{3}$

31. (a) $\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 3 dz dx dy$

(b) $\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec \phi} 3 \rho^2 \sin \phi d\rho d\phi d\theta$ (c) $2\pi(8 - 4\sqrt{2})$

33. $\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec \phi} \rho^2 \sin \phi d\rho d\phi d\theta = \frac{\pi}{3}$

35. $\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} z^2 xy dz dy dx$
 $+ \int_1^{\sqrt{3}} \int_0^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} z^2 xy dz dy dx$

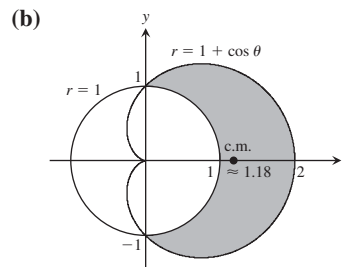
37. (a) $\frac{8\pi(4\sqrt{2} - 5)}{3}$ (b) $\frac{8\pi(4\sqrt{2} - 5)}{3}$

39. $I_z = \frac{8\pi\delta(b^5 - a^5)}{15}$ 41. $\bar{x} = \bar{y} = \frac{1}{2 - \ln 4}$

43. $I_0 = 104$ 45. $I_x = 2\delta$

47. $M = 4, M_x = 0, M_y = 0$ 49. $\bar{x} = \frac{3\sqrt{3}}{\pi}, \bar{y} = 0$

51. (a) $\bar{x} = \frac{15\pi + 32}{6\pi + 48}, \bar{y} = 0$



ADDITIONAL AND ADVANCED EXERCISES, pp. 927–929

1. (a) $\int_{-3}^2 \int_x^{6-x^2} x^2 dy dx$ (b) $\int_{-3}^2 \int_x^{6-x^2} \int_0^{x^2} dz dy dx$
 (c) $125/4$

3. 2π 5. $3\pi/2$

7. (a) Hole radius = 1, sphere radius = 2 (b) $4\sqrt{3}\pi$

9. $\pi/4$ 11. $\ln\left(\frac{b}{a}\right)$ 15. $1/\sqrt[3]{3}$

17. Mass = $a^2 \cos^{-1}\left(\frac{b}{a}\right) - b\sqrt{a^2 - b^2}$,

$I_0 = \frac{a^4}{2} \cos^{-1}\left(\frac{b}{a}\right) - \frac{b^3}{2} \sqrt{a^2 - b^2} - \frac{b^3}{6} (a^2 - b^2)^{3/2}$

19. $\frac{1}{ab}(e^{a^2b^2} - 1)$ 21. (b) 1 (c) 0

25. $h = \sqrt{180}$ cm, $h = \sqrt{540}$ cm

27. $2\pi\left[\frac{1}{3} - \left(\frac{1}{3}\right)\frac{\sqrt{2}}{2}\right]$

Chapter 15

SECTION 15.1, pp. 935–937

1. Graph (c) 3. Graph (g) 5. Graph (d) 7. Graph (f)

9. $\sqrt{2}$ 11. $\frac{13}{2}$ 13. $3\sqrt{14}$ 15. $\frac{1}{6}(5\sqrt{5} + 9)$

17. $\sqrt{3}\ln\left(\frac{b}{a}\right)$ 19. (a) $4\sqrt{5}$ (b) $\frac{1}{12}(17^{3/2} - 1)$

21. $\frac{15}{32}(e^{16} - e^{64})$ 23. $\frac{1}{27}(40^{3/2} - 13^{3/2})$

25. $\frac{1}{6}(5^{3/2} + 7\sqrt{2} - 1)$ 27. $\frac{10\sqrt{5} - 2}{3}$ 29. 8

31. $\frac{1}{6}(17^{3/2} - 1)$ 33. $2\sqrt{2} - 1$

35. (a) $4\sqrt{2} - 2$ (b) $\sqrt{2} + \ln(1 + \sqrt{2})$ 37. $I_z = 2\pi\delta a^3$

39. (a) $I_z = 2\pi\sqrt{2}\delta$ (b) $I_z = 4\pi\sqrt{2}\delta$ 41. $I_x = 2\pi - 2$

SECTION 15.2, pp. 947–950

1. $\nabla f = -(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})(x^2 + y^2 + z^2)^{-3/2}$

3. $\nabla g = -\left(\frac{2x}{x^2 + y^2}\right)\mathbf{i} - \left(\frac{2y}{x^2 + y^2}\right)\mathbf{j} + e^z\mathbf{k}$

5. $\mathbf{F} = -\frac{kx}{(x^2 + y^2)^{3/2}}\mathbf{i} - \frac{ky}{(x^2 + y^2)^{3/2}}\mathbf{j}$, any $k > 0$

7. (a) $9/2$ (b) $13/3$ (c) $9/2$

9. (a) $1/3$ (b) $-1/5$ (c) 0

11. (a) 2 (b) $3/2$ (c) $1/2$

13. $-15/2$ 15. 36 17. (a) $-5/6$ (b) 0 (c) $-7/12$

19. $1/2$ 21. $-\pi$ 23. $69/4$ 25. $-39/2$ 27. $25/6$

29. (a) $\text{Circ}_1 = 0$, $\text{circ}_2 = 2\pi$, $\text{flux}_1 = 2\pi$, $\text{flux}_2 = 0$

(b) $\text{Circ}_1 = 0$, $\text{circ}_2 = 8\pi$, $\text{flux}_1 = 8\pi$, $\text{flux}_2 = 0$

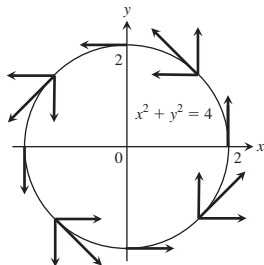
31. $\text{Circ} = 0$, $\text{flux} = a^2\pi$ 33. $\text{Circ} = a^2\pi$, $\text{flux} = 0$

35. (a) $-\frac{\pi}{2}$ (b) 0 (c) 1 37. $(.0001)\pi$ kg/s

39. (a) 32 (b) 32 (c) 32 41. 115.2 J

43. $5/3 - (3/2)\ln 2$ m²/s 45. $5/3$ g/s

47.



49. (a) $\mathbf{G} = -y\mathbf{i} + x\mathbf{j}$ (b) $\mathbf{G} = \sqrt{x^2 + y^2}\mathbf{F}$

51. $\mathbf{F} = -\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}$ 55. 48 57. π 59. 0 61. $\frac{1}{2}$

SECTION 15.3, pp. 959–961

1. Conservative 3. Not conservative 5. Not conservative

7. $f(x, y, z) = x^2 + \frac{3y^2}{2} + 2z^2 + C$

9. $f(x, y, z) = xe^{y+2z} + C$

11. $f(x, y, z) = x\ln x - x + \tan(x + y) + \frac{1}{2}\ln(y^2 + z^2) + C$

13. 49 15. -16 17. 1 19. $9\ln 2$ 21. 0 23. -3

27. $\mathbf{F} = \nabla\left(\frac{x^2 - 1}{y}\right)$ 29. (a) 1 (b) 1 (c) 1

31. (a) 2 (b) 2 33. (a) $c = b = 2a$ (b) $c = b = 2$

35. It does not matter what path you use. The work will be the same on any path because the field is conservative.

37. The force $\mathbf{F}$ is conservative because all partial derivatives of M , N , and P are zero. $f(x, y, z) = ax + by + cz + C$; $A = (xa, ya, za)$ and $B = (xb, yb, zb)$. Therefore, $\int \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A) = a(xb - xa) + b(yb - ya) + c(zb - za) = \mathbf{F} \cdot \overline{AB}$.

SECTION 15.4, pp. 971–973

1. $2y - 1$ 3. $ye^x - xe^y$ 5. $\sin y - \sin x$

7. $\text{Flux} = 0$, $\text{circ} = 2\pi a^2$ 9. $\text{Flux} = -\pi a^2$, $\text{circ} = 0$

11. $\text{Flux} = 2$, $\text{circ} = 0$ 13. $\text{Flux} = -9$, $\text{circ} = 9$

15. $\text{Flux} = -11/60$, $\text{circ} = -7/60$

17. $\text{Flux} = 64/9$, $\text{circ} = 0$ 19. $\text{Flux} = 1/2$, $\text{circ} = 1/2$

21. $\text{Flux} = 1/5$, $\text{circ} = -1/12$ 23. 0 25. $2/33$

27. 0 29. -16π 31. πa^2 33. $3\pi/8$

35. (a) 0 if C is traversed counterclockwise(b) $(h - k)(\text{area of the region})$ 37. 045. (a) 4π (b) 4π if $(0, 0)$ lies inside K , 0 otherwise

SECTION 15.5, pp. 981–983

1. $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r^2\mathbf{k}$, $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$

3. $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (r/2)\mathbf{k}$, $0 \leq r \leq 6$, $0 \leq \theta \leq \pi/2$

5. $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + \sqrt{9 - r^2}\mathbf{k}$, $0 \leq r \leq 3\sqrt{2}/2$, $0 \leq \theta \leq 2\pi$; Also:

$\mathbf{r}(\phi, \theta) = (3 \sin \phi \cos \theta)\mathbf{i} + (3 \sin \phi \sin \theta)\mathbf{j} + (3 \cos \phi)\mathbf{k}$, $0 \leq \phi \leq \pi/4$, $0 \leq \theta \leq 2\pi$

7. $\mathbf{r}(\phi, \theta) = (\sqrt{3} \sin \phi \cos \theta)\mathbf{i} + (\sqrt{3} \sin \phi \sin \theta)\mathbf{j} + (\sqrt{3} \cos \phi)\mathbf{k}$, $\pi/3 \leq \phi \leq 2\pi/3$, $0 \leq \theta \leq 2\pi$

9. $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (4 - y^2)\mathbf{k}$, $0 \leq x \leq 2$, $-2 \leq y \leq 2$

11. $\mathbf{r}(u, v) = u\mathbf{i} + (3 \cos v)\mathbf{j} + (3 \sin v)\mathbf{k}$, $0 \leq u \leq 3$, $0 \leq v \leq 2\pi$

13. (a) $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (1 - r \cos \theta - r \sin \theta)\mathbf{k}$, $0 \leq r \leq 3$, $0 \leq \theta \leq 2\pi$

(b) $\mathbf{r}(u, v) = (1 - u \cos v - u \sin v)\mathbf{i} + (u \cos v)\mathbf{j} + (u \sin v)\mathbf{k}$, $0 \leq u \leq 3$, $0 \leq v \leq 2\pi$

15. $\mathbf{r}(u, v) = (4 \cos^2 v)\mathbf{i} + u\mathbf{j} + (4 \cos v \sin v)\mathbf{k}$, $0 \leq u \leq 3$, $-(\pi/2) \leq v \leq (\pi/2)$; Another way: $\mathbf{r}(u, v) = (2 + 2 \cos v)\mathbf{i} + u\mathbf{j} + (2 \sin v)\mathbf{k}$, $0 \leq u \leq 3$, $0 \leq v \leq 2\pi$

17. $\int_0^{2\pi} \int_0^1 \frac{\sqrt{5}}{2} r dr d\theta = \frac{\pi\sqrt{5}}{2}$

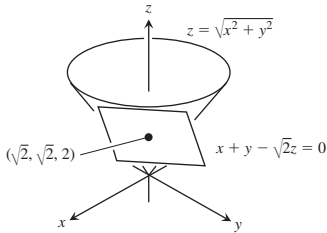
19. $\int_0^{2\pi} \int_1^3 r\sqrt{5} dr d\theta = 8\pi\sqrt{5}$

$$21. \int_0^{2\pi} \int_1^4 1 \, du \, dv = 6\pi$$

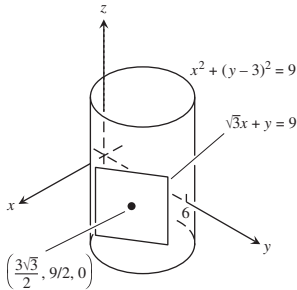
$$23. \int_0^{2\pi} \int_0^1 u\sqrt{4u^2+1} \, du \, dv = \frac{(5\sqrt{5}-1)\pi}{6}$$

$$25. \int_0^{2\pi} \int_{\pi/4}^{\pi} 2 \sin \phi \, d\phi \, d\theta = (4 + 2\sqrt{2})\pi$$

27.



29.



$$33. \text{(b)} A = \int_0^{2\pi} \int_0^{\pi} [a^2 b^2 \sin^2 \phi \cos^2 \phi + b^2 c^2 \sin^4 \phi \cos^2 \theta + a^2 c^2 \sin^4 \phi \sin^2 \theta]^{1/2} d\phi \, d\theta$$

$$35. x_0 x + y_0 y = 25 \quad 37. 13\pi/3 \quad 39. 4$$

$$41. 6\sqrt{6} - 2\sqrt{2} \quad 43. \pi\sqrt{c^2+1}$$

$$45. \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5}) \quad 47. 3 + 2\ln 2$$

$$49. \frac{\pi}{6}(13\sqrt{13} - 1) \quad 51. 5\pi\sqrt{2} \quad 53. \frac{2}{3}(5\sqrt{5} - 1)$$

SECTION 15.6, pp. 991–993

$$1. \iint_S x \, d\sigma = \int_0^3 \int_0^2 u\sqrt{4u^2+1} \, du \, dv = \frac{17\sqrt{17}-1}{4}$$

$$3. \iint_S x^2 \, d\sigma = \int_0^{2\pi} \int_0^{\pi} \sin^3 \phi \cos^2 \theta \, d\phi \, d\theta = \frac{4\pi}{3}$$

$$5. \iint_S z \, d\sigma = \int_0^1 \int_0^1 (4-u-v)\sqrt{3} \, dv \, du = 3\sqrt{3}$$

(for $x = u, y = v$)

$$7. \iint_S x^2 \sqrt{5-4z} \, d\sigma = \int_0^1 \int_0^{2\pi} u^2 \cos^2 v \cdot \sqrt{4u^2+1} \cdot u \sqrt{4u^2+1} \, dv \, du = \int_0^1 \int_0^{2\pi} u^3 (4u^2+1) \cos^2 v \, dv \, du = \frac{11\pi}{12}$$

$$9. 9a^3 \quad 11. \frac{abc}{4}(ab+ac+bc) \quad 13. 2$$

$$15. \frac{1}{30}(\sqrt{2}+6\sqrt{6}) \quad 17. \sqrt{6}/30 \quad 19. -32 \quad 21. \frac{\pi a^3}{6}$$

$$23. 13a^4/6 \quad 25. 2\pi/3 \quad 27. -73\pi/6 \quad 29. 18$$

$$31. \frac{\pi a^3}{6} \quad 33. \frac{\pi a^2}{4} \quad 35. \frac{\pi a^3}{2} \quad 37. -32$$

$$39. -4 \quad 41. 3a^4 \quad 43. \left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$$

$$45. (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{14}{9}\right), I_z = \frac{15\pi\sqrt{2}}{2}\delta$$

$$47. \frac{8\pi}{3}a^4\delta \quad 49. 70/3 \text{ mg}$$

SECTION 15.7, pp. 1004–1006

$$1. -\mathbf{i} - 4\mathbf{j} + \mathbf{k} \quad 3. (1-y)\mathbf{i} + (1-z)\mathbf{j} + (1-x)\mathbf{k}$$

$$5. x(z^2 - y^2)\mathbf{i} + y(x^2 - z^2)\mathbf{j} + z(y^2 - x^2)\mathbf{k} \quad 7. 4\pi$$

$$9. -5/6 \quad 11. 0 \quad 13. -6\pi \quad 15. 2\pi a^2$$

$$17. -\pi \quad 19. 12\pi \quad 21. -\pi/4 \quad 23. -15\pi$$

$$25. -8\pi \quad 33. 16I_y + 16I_x$$

SECTION 15.8, pp. 1016–1018

$$1. 0 \quad 3. (y^2 z + x z^2 + x^2 y)e^{xyz} \quad 5. 0 \quad 7. 0$$

$$9. -16 \quad 11. -8\pi \quad 13. 3\pi \quad 15. -40/3$$

$$17. 12\pi \quad 19. 12\pi(4\sqrt{2}-1) \quad 23. \text{No}$$

25. The integral's value never exceeds the surface area of S .

$$27. 184/35$$

PRACTICE EXERCISES, pp. 1019–1021

$$1. \text{Path 1: } 2\sqrt{3}; \text{ path 2: } 1 + 3\sqrt{2} \quad 3. 4a^2 \quad 5. 0$$

$$7. 8\pi \sin(1) \quad 9. 0 \quad 11. \pi\sqrt{3}$$

$$13. 2\pi\left(1 - \frac{1}{\sqrt{2}}\right) \quad 15. \frac{abc}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \quad 17. 50$$

$$19. \mathbf{r}(\phi, \theta) = (6 \sin \phi \cos \theta)\mathbf{i} + (6 \sin \phi \sin \theta)\mathbf{j} + (6 \cos \phi)\mathbf{k},$$

$$\frac{\pi}{6} \leq \phi \leq \frac{2\pi}{3}, 0 \leq \theta \leq 2\pi$$

$$21. \mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (1+r)\mathbf{k}, 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi$$

$$23. \mathbf{r}(u, v) = (u \cos v)\mathbf{i} + 2u^2\mathbf{j} + (u \sin v)\mathbf{k}, 0 \leq u \leq 1, 0 \leq v \leq \pi$$

$$25. \sqrt{6} \quad 27. \pi[\sqrt{2} + \ln(1 + \sqrt{2})] \quad 29. \text{Conservative}$$

$$31. \text{Not conservative} \quad 33. f(x, y, z) = y^2 + yz + 2x + z$$

$$35. \text{Path 1: } 2; \text{ path 2: } 8/3 \quad 37. \text{(a)} 1 - e^{-2\pi} \quad \text{(b)} 1 - e^{-2\pi}$$

$$39. 0 \quad 41. \text{(a)} 4\sqrt{2} - 2 \quad \text{(b)} \sqrt{2} + \ln(1 + \sqrt{2})$$

$$43. (\bar{x}, \bar{y}, \bar{z}) = \left(1, \frac{16}{15}, \frac{2}{3}\right); I_x = \frac{232}{45}, I_y = \frac{64}{15}, I_z = \frac{56}{9}$$

$$45. \bar{z} = \frac{3}{2}, I_z = \frac{7\sqrt{3}}{3} \quad 47. (\bar{x}, \bar{y}, \bar{z}) = (0, 0, 49/12), I_z = 640\pi$$

$$49. \text{Flux: } 3/2; \text{ circ: } -1/2 \quad 53. 3$$

$$55. \frac{2\pi}{3}(7 - 8\sqrt{2}) \quad 57. 0 \quad 59. \pi$$

ADDITIONAL AND ADVANCED EXERCISES, pp. 1021–1023

$$1. 6\pi \quad 3. 2/3$$

$$5. \text{(a)} \mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$$

$$\text{(b)} \mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{k}$$

$$\text{(c)} \mathbf{F}(x, y, z) = z\mathbf{i}$$

$$7. \frac{16\pi R^3}{3} \quad 9. a = 2, b = 1. \text{ The minimum flux is } -4.$$

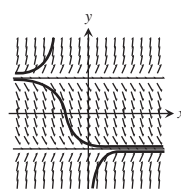
$$11. \text{(b)} \frac{16}{3}g \quad \text{(c)} \text{Work} = \left(\int_C gxy \, ds\right)\bar{y} = g \int_C xy^2 \, ds = \frac{16}{3}g$$

$$13. \text{(c)} \frac{4}{3}\pi w \quad 19. \text{False if } \mathbf{F} = y\mathbf{i} + x\mathbf{j}$$

Chapter 16**SECTION 16.1, pp. 1030–1032**

$$1. \text{(d)} \quad 3. \text{(a)}$$

5.



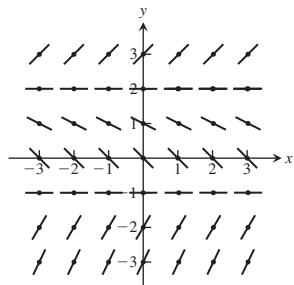
$$7. y' = x - y; y(1) = -1$$

$$9. y' = -(1 + y) \sin x; y(0) = 2$$

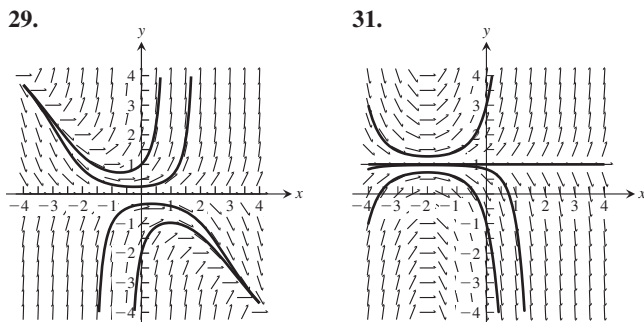
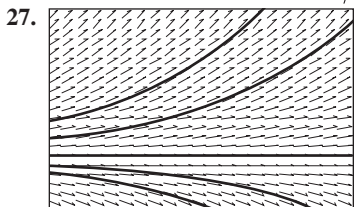
$$11. y' = 1 + xe^y; y(-2) = 2$$

13.

y	$f(y) = \frac{dy}{dx}$
-3	2
-2	1.5
-1	0
0	-1
1	-0.75
2	0
3	1



15. $y(\text{exact}) = -x^2$, $y_1 = -2$, $y_2 = -3.3333$, $y_3 = -5$
 17. $y(\text{exact}) = 3e^{x(x+2)}$, $y_1 = 4.2$, $y_2 = 6.216$, $y_3 = 9.697$
 19. $y(\text{exact}) = e^{x^2} + 1$, $y_1 = 2.0$, $y_2 = 2.0202$, $y_3 = 2.0618$
 21. $y \approx 2.48832$; exact value is e .
 23. $y \approx -0.2272$; exact value is $1/(1 - 2\sqrt{5}) \approx -0.2880$.



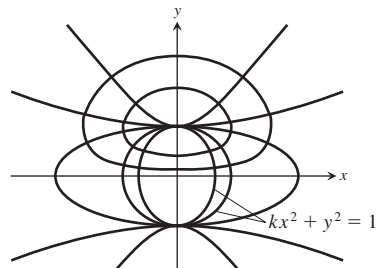
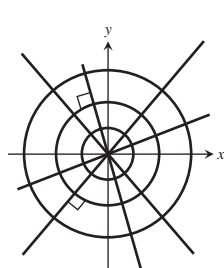
39. Euler's method gives $y \approx 3.45835$; the exact solution is $y = 1 + e \approx 3.71828$.
 41. $y \approx 1.5000$; exact value is 1.5275.

SECTION 16.2, pp. 1036–1038

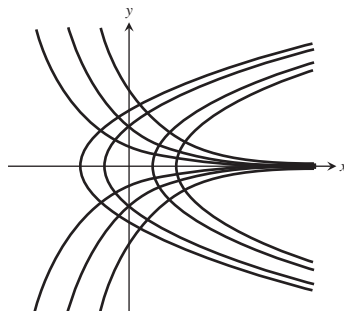
1. $y = \frac{e^x + C}{x}$, $x > 0$ 3. $y = \frac{C - \cos x}{x^3}$, $x > 0$
 5. $y = \frac{1}{2} - \frac{1}{x} + \frac{C}{x^2}$, $x > 0$ 7. $y = \frac{1}{2}xe^{x/2} + Ce^{x/2}$
 9. $y = x(\ln x)^2 + Cx$
 11. $s = \frac{t^3}{3(t-1)^4} - \frac{t}{(t-1)^4} + \frac{C}{(t-1)^4}$
 13. $r = (\csc \theta)(\ln |\sec \theta| + C)$, $0 < \theta < \pi/2$
 15. $y = \frac{3}{2} - \frac{1}{2}e^{-2t}$ 17. $y = -\frac{1}{\theta} \cos \theta + \frac{\pi}{2\theta}$
 19. $y = 6e^{x^2} - \frac{e^{x^2}}{x+1}$ 21. $y = y_0 e^{kt}$
 23. (b) is correct, but (a) is not. 25. $t = \frac{L}{R} \ln 2$ sec
 27. (a) $i = \frac{V}{R} - \frac{V}{R}e^{-3} = \frac{V}{R}(1 - e^{-3}) \approx 0.95 \frac{V}{R}$ amp (b) 86%
 29. $y = \frac{1}{1 + Ce^{-x}}$ 31. $y^3 = 1 + Cx^{-3}$

SECTION 16.3, pp. 1043–1044

1. (a) 168.5 m (b) 41.13 s
 3. $s(t) = 4.91(1 - e^{-(22.36/39.92)t})$
 5. $x^2 + y^2 = C$ 7. $\ln |y| - \frac{1}{2}y^2 = \frac{1}{2}x^2 + C$



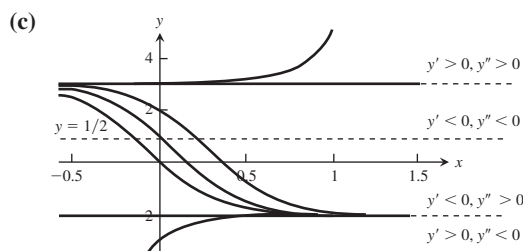
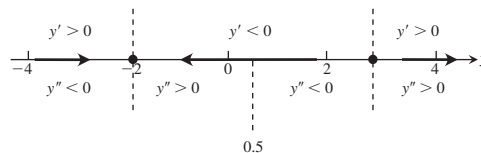
9. $y = \pm\sqrt{2x + C}$



13. (a) 4 kg/min (b) $(400 + 4t)$ L (c) $4\left(\frac{y}{100 + t}\right)$ kg/min
 (d) $\frac{dy}{dt} = 4 - \frac{4y}{100 + t}$, $y(0) = 20$,
 $y = 0.8(100 + t) - \frac{60}{\left(1 + \frac{t}{100}\right)^4}$
 (e) Concentration = $\frac{y(25)}{\text{amt. brine in tank}}$
 $= \frac{75.424}{500} \approx 0.151$ kg/L
 15. $y(27.8) \approx 5.93$ kg, $t \approx 27.8$ min

SECTION 16.4, pp. 1052–1053

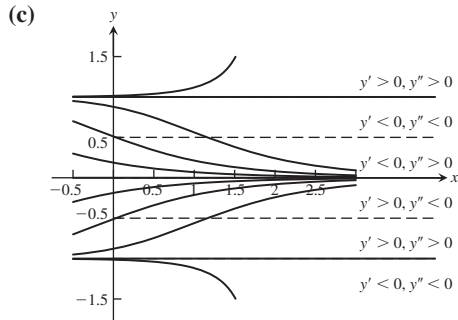
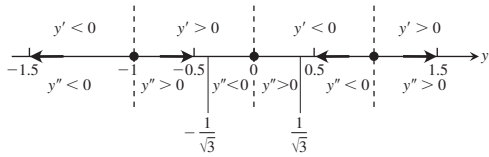
1. $y' = (y + 2)(y - 3)$
 (a) $y = -2$ is a stable equilibrium value, and $y = 3$ is an unstable equilibrium value.
 (b) $y'' = 2(y + 2)\left(y - \frac{1}{2}\right)(y - 3)$



3. $y' = y^3 - y = (y + 1)y(y - 1)$

(a) $y = -1$ and $y = 1$ are unstable equilibria, and $y = 0$ is a stable equilibrium.

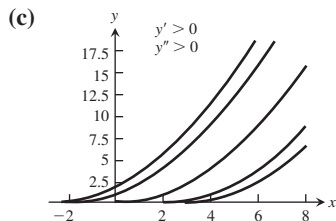
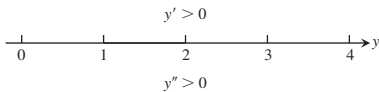
(b) $y'' = (3y^2 - 1)y'$
 $= 3(y + 1)(y + 1/\sqrt{3})y(y - 1/\sqrt{3})(y - 1)$



5. $y' = \sqrt{y}$, $y > 0$

(a) There are no equilibrium values.

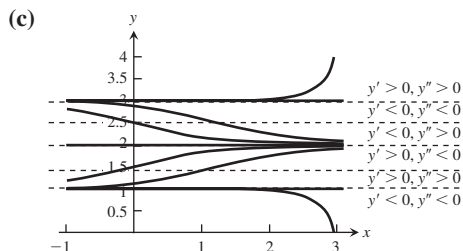
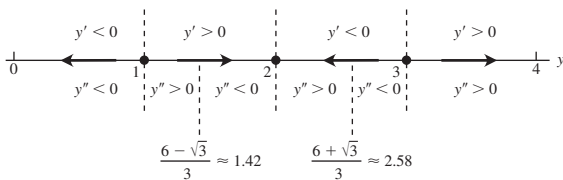
(b) $y'' = \frac{1}{2}$



7. $y' = (y - 1)(y - 2)(y - 3)$

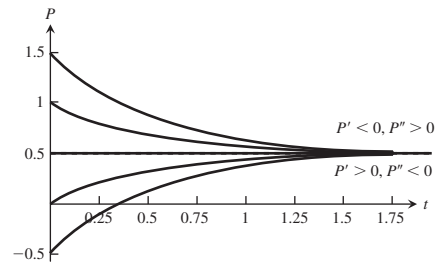
(a) $y = 1$ and $y = 3$ are unstable equilibria, and $y = 2$ is a stable equilibrium.

(b) $y'' = (3y^2 - 12y + 11)(y - 1)(y - 2)(y - 3) =$
 $3(y - 1)\left(y - \frac{6 - \sqrt{3}}{3}\right)(y - 2)\left(y - \frac{6 + \sqrt{3}}{3}\right)(y - 3)$

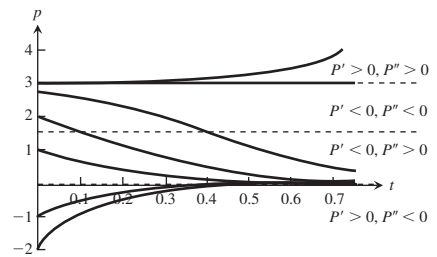
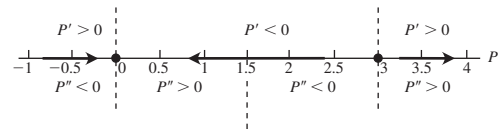


9. $\frac{dP}{dt} = 1 - 2P$ has a stable equilibrium at $P = \frac{1}{2}$;

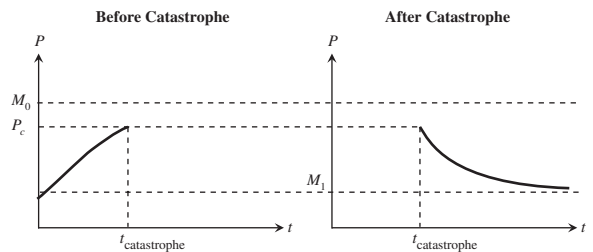
$\frac{d^2P}{dt^2} = -2\frac{dP}{dt} = -2(1 - 2P).$



11. $\frac{dP}{dt} = 2P(P - 3)$ has a stable equilibrium at $P = 0$ and an unstable equilibrium at $P = 3$; $\frac{d^2P}{dt^2} = 2(2P - 3)\frac{dP}{dt} = 4P(2P - 3)(P - 3).$



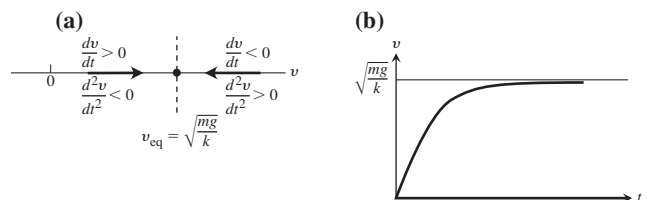
13. Before the catastrophe, the population exhibits logistic growth and $P(t)$ increases toward M_0 , the stable equilibrium. After the catastrophe, the population declines logistically and $P(t)$ decreases toward M_1 , the new stable equilibrium.



15. $\frac{dv}{dt} = g - \frac{k}{m}v^2$, $g, k, m > 0$ and $v(t) \geq 0$

Equilibrium: $\frac{dv}{dt} = g - \frac{k}{m}v^2 = 0 \Rightarrow v = \sqrt{\frac{mg}{k}}$

Concavity: $\frac{d^2v}{dt^2} = -2\left(\frac{k}{m}v\right)\frac{dv}{dt} = -2\left(\frac{k}{m}v\right)\left(g - \frac{k}{m}v^2\right)$

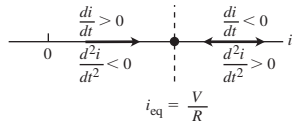


(c) $v_{\text{terminal}} = \sqrt{\frac{441}{0.2}} = 46.96 \text{ m/s} = 169 \text{ km/h}$ for the 45-kg skydiver
 $v_{\text{terminal}} = \sqrt{\frac{7.84}{0.2}} = 62.6 \text{ m/s} = 225 \text{ km/h}$ for the 80-kg skydiver

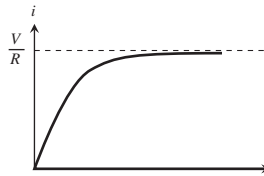
17. $F = F_p - F_r$; $ma = 50 - 5|v|$; $\frac{dv}{dt} = \frac{1}{m}(50 - 5|v|)$.

The maximum velocity occurs when $\frac{dv}{dt} = 0$, or $v = 4$ m/s.

19. Phase line:



If the switch is closed at $t = 0$, then $i(0) = 0$, and the graph of the solution looks like this:



As $t \rightarrow \infty$, $i(t) \rightarrow i_{\text{steady state}} = \frac{V}{R}$.

21. $y = \frac{1}{1 + e^{-20x}}$.

SECTION 16.5, pp. 1056–1059

1. Seasonal variations, nonconformity of the environments, effects of other interactions, unexpected disasters, etc.
3. This model assumes that the number of interactions is proportional to the product of x and y :

$$\frac{dx}{dt} = (a - by)x, \quad a < 0,$$

$$\frac{dy}{dt} = m\left(1 - \frac{y}{M}\right)y - nxy = y\left(m - \frac{m}{M}y - nx\right).$$

Rest points are $(0, 0)$, unstable, and $(0, M)$, stable.

5. (a) Logistic growth occurs in the absence of the competitor, and involves a simple interaction between the species. Growth dominates the competition when either population is small, so it is difficult to drive either species to extinction.

- (b) a : per capita growth rate for trout
 m : per capita growth rate for bass
 b : intensity of competition to the trout
 n : intensity of competition to the bass
 k_1 : environmental carrying capacity for the trout
 k_2 : environmental carrying capacity for the bass
 $\frac{a}{b}$: growth versus competition or net growth of trout
 $\frac{m}{n}$: relative survival of bass

(c) $\frac{dx}{dt} = 0$ when $x = 0$ or $y = \frac{a}{b} - \frac{a}{bk_1}x$,

$$\frac{dy}{dt} = 0 \text{ when } y = 0 \text{ or } y = k_2 - \frac{k_2n}{m}x.$$

By picking $a/b > k_2$ and $m/n > k_1$, we ensure that an equilibrium point exists inside the first quadrant.

PRACTICE EXERCISES, pp. 1059–1060

1. $y = -\ln\left(C - \frac{2}{5}(x-2)^{5/2} - \frac{4}{3}(x-2)^{3/2}\right)$
3. $\tan y = -x \sin x - \cos x + C$
5. $(y+1)e^{-y} = -\ln|x| + C$

7. $y = C \frac{x-1}{x}$ 9. $y = \frac{x^2}{4}e^{x/2} + Ce^{x/2}$

11. $y = \frac{x^2 - 2x + C}{2x^2}$ 13. $y = \frac{e^{-x} + C}{1 + e^x}$ 15. $xy + y^3 = C$

17. $\tan y = -\cos x + (1/3)\cos^3 x + C$

19. $e^{x^2}y = (x-1)e^x + C$

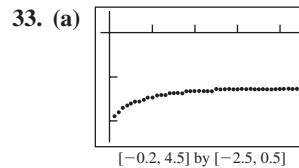
21. $\ln|\ln y| = (1/2)x^2 \ln x - (1/4)x^2 + C$

23. $y = \frac{2x^3 + 3x^2 + 6}{6(x+1)^2}$ 25. $y = \frac{1}{3}(1 - 4e^{-x^3})$

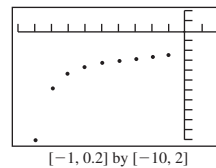
27. $y = e^{-x}(3x^3 - 3x^2)$

29.			
x	y	x	y
0	0	1.1	1.6241
0.1	0.1000	1.2	1.8319
0.2	0.2095	1.3	2.0513
0.3	0.3285	1.4	2.2832
0.4	0.4568	1.5	2.5285
0.5	0.5946	1.6	2.7884
0.6	0.7418	1.7	3.0643
0.7	0.8986	1.8	3.3579
0.8	1.0649	1.9	3.6709
0.9	1.2411	2.0	4.0057
1.0	1.4273		

31. $y(3) \approx 0.9131$



- (b) Note that we choose a small interval of x values because the y values decrease very rapidly and our calculator cannot handle the calculations for $x \leq -1$. (This occurs because the analytic solution is $y = -2 + \ln(2 - e^{-x})$, which has an asymptote at $x = -\ln 2 \approx -0.69$. Obviously, the Euler approximations are misleading for $x \leq -0.7$.)

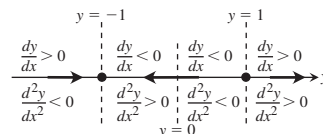


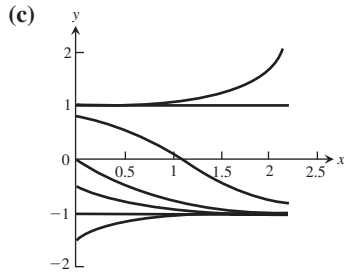
35. $y(\text{exact}) = \frac{1}{2}x^2 - \frac{3}{2}$; $y(2) \approx 0.4$; exact value is $\frac{1}{2}$.

37. $y(\text{exact}) = -e^{(x^2-1)/2}$; $y(2) \approx -3.4192$; exact value is $-e^{3/2} \approx -4.4817$.

39. (a) $y = -1$ is stable and $y = 1$ is unstable.

(b) $\frac{d^2y}{dx^2} = 2y \frac{dy}{dx} = 2y(y^2 - 1)$





43. (a) 1 min: 608 L; 10 min: 680 L; 60 min: 1080 L
 (b) $S(1) \approx 11.76$ kg, $S(10) \approx 57.89$ kg, $S(60) \approx 225.56$ kg

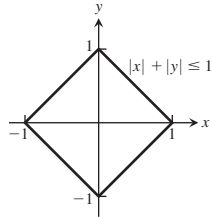
ADDITIONAL AND ADVANCED EXERCISES, p. 1061

1. (a) $y = c + (y_0 - c)e^{-k(A/V)t}$
 (b) Steady-state solution: $y_\infty = c$
 5. $x^2(x^2 + 2y^2) = C$
 7. $\ln|x| + e^{-y/x} = C$
 9. $\ln|x| - \ln|\sec(y/x - 1) + \tan(y/x - 1)| = C$

Appendices

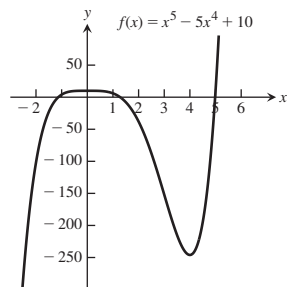
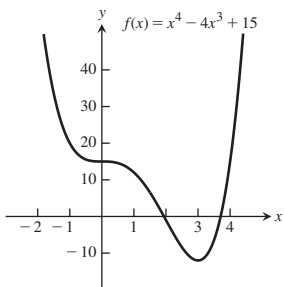
APPENDIX A.1, p. AP-6

1. $0.\overline{1}, 0.\overline{2}, 0.\overline{3}, 0.\overline{8}, 0.\overline{9}$ or 1
 3. $x < -2$
 5. $x \leq -\frac{1}{3}$
 7. $3, -3$ 9. $7/6, 25/6$
 11. $[-2, 4]$
 13. $[0, 10]$
 15. $(-\infty, -2] \cup [2, \infty)$
 17. $(-\infty, -3] \cup [1, \infty)$
 19. $(-3, -2) \cup (2, 3)$
 21. $(0, 1)$
 23. $(-\infty, 1]$
 27. The graph of $|x| + |y| \leq 1$ is the interior and boundary of the “diamond-shaped” region.

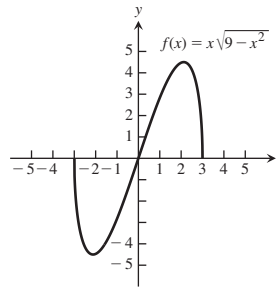


APPENDIX A.2, p. AP-10

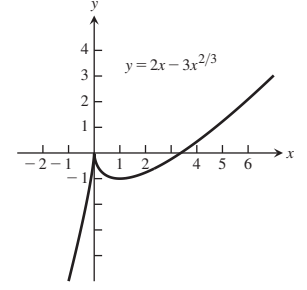
1. d 3. d
 5. $[-3, 5]$ by $[-15, 40]$
 7. $[-3, 6]$ by $[-250, 50]$



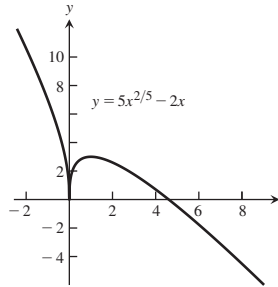
9. $[-5, 5]$ by $[-6, 6]$



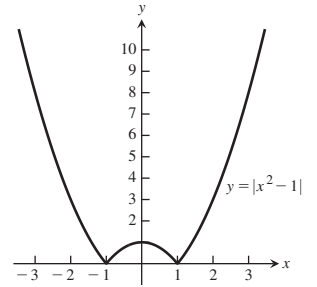
11. $[-2, 6]$ by $[-5, 4]$



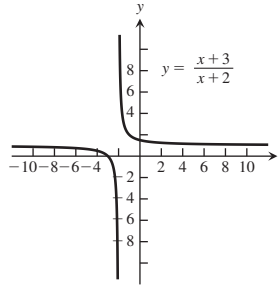
13. $[-2, 8]$ by $[-5, 10]$



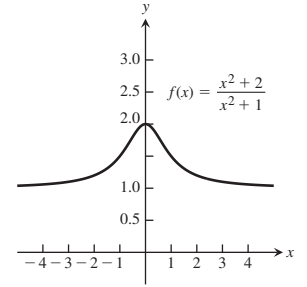
15. $[-3, 3]$ by $[0, 10]$



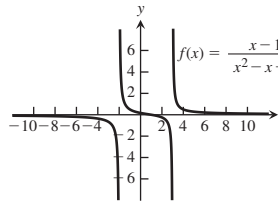
17. $[-10, 10]$ by $[-10, 10]$



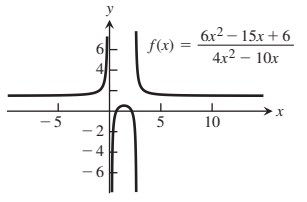
19. $[-4, 4]$ by $[0, 3]$



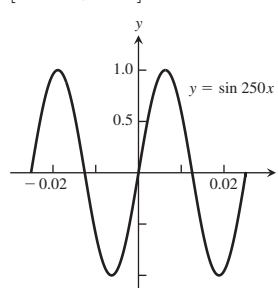
21. $[-10, 10]$ by $[-6, 6]$



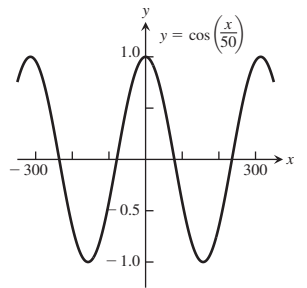
23. $[-6, 10]$ by $[-6, 6]$



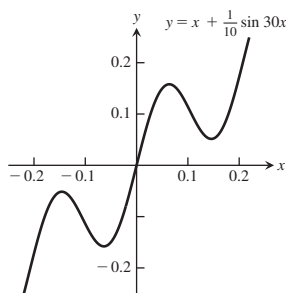
25. $\left[-\frac{\pi}{125}, \frac{\pi}{125}\right]$ by $[-1.25, 1.25]$



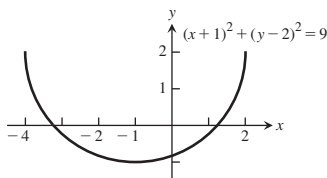
27. $[-100\pi, 100\pi]$ by $[-1.25, 1.25]$



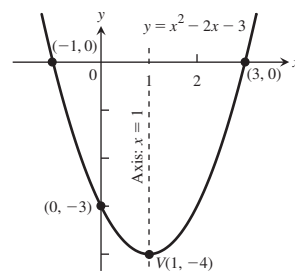
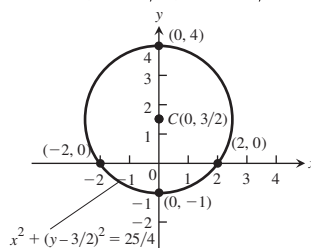
29. $\left[-\frac{\pi}{15}, \frac{\pi}{15}\right]$ by $[-0.25, 0.25]$



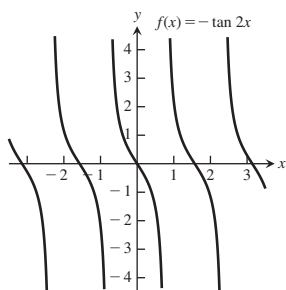
31.



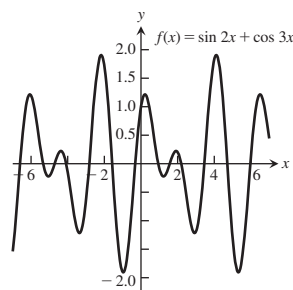
25. $x^2 + (y - 3/2)^2 = 25/4$ 27.



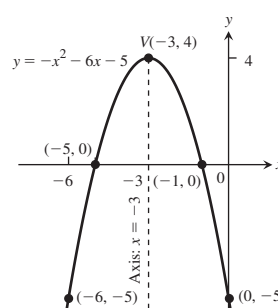
33.



35.



29.



31. Exterior points of a circle of radius $\sqrt{7}$, centered at the origin

33. The washer between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ (points with distance from the origin between 1 and 2)

35. $(x + 2)^2 + (y - 1)^2 < 6$

37. $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$

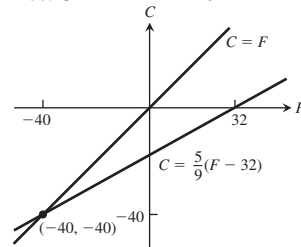
39. $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{3}\right), \left(\frac{1}{\sqrt{3}}, -\frac{1}{3}\right)$

41. (a) $\approx 0.1^\circ\text{C}/\text{mm}$ (b) $\approx -0.31^\circ\text{C}/\text{mm}$

(c) $\approx 0.15^\circ\text{C}/\text{mm}$

42. 604.9 kPa

45. Yes: $C = F - 40^\circ$

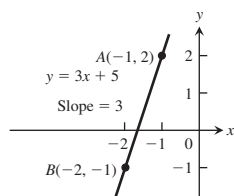


51. $k = -8, k = 1/2$

APPENDIX A.4, pp. AP-21-AP-22

1. 2, -4; $2\sqrt{5}$ 3. Unit circle

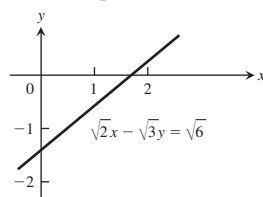
5. $m_\perp = -\frac{1}{3}$



7. (a) $x = -1$ (b) $y = 4/3$ 9. $y = -x$

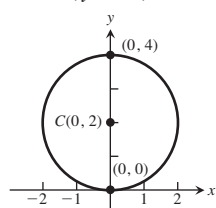
11. $y = -\frac{5}{4}x + 6$ 13. $y = 4x + 4$ 15. $y = -\frac{x}{2} + 12$

17. x-intercept = $\sqrt{3}$, y-intercept = $-\sqrt{2}$

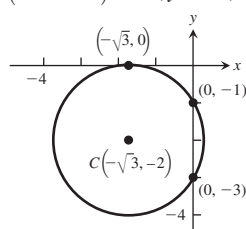


19. (3, -3)

21. $x^2 + (y - 2)^2 = 4$



23. $(x + \sqrt{3})^2 + (y + 2)^2 = 4$



APPENDIX A.8, pp. AP-40-AP-43

1. No 3. Yes 5. Yes 7. Yes 11. ≈ 0.537

13. ≈ 0.688 15. ≈ 0.0502 17. $\sqrt{21}$ 19. $\frac{1}{2} \ln 2$

21. $\frac{1}{\pi}, \frac{1}{\pi} \left(\tan^{-1} 2 - \frac{\pi}{4} \right) \approx 0.10242$

25. mean = $\frac{8}{3} \approx 2.67$, median = $\sqrt{8} \approx 2.83$

27. mean = 2, median = $\sqrt{2} \approx 1.41$

29. $P(X < \frac{1}{2}) \approx 0.3935$

31. (a) ≈ 0.57 , so about 57 in every 100 bulbs will fail.

(b) ≈ 832 hr

- 33.** ≈ 60 hydra **35. (a)** ≈ 0.393 **(b)** ≈ 0.135 **(c)** 0
(d) The probability that any customer waits longer than 3 minutes is $1 - (0.997521)^{200} \approx 0.391 < 1/2$. So the most likely outcome is that all 200 would be served within 3 minutes.
37. \$10,256 **39.** $\approx 323, \approx 262$ **41.** ≈ 0.89435
43. (a) $\approx 16\%$ **(b)** ≈ 0.23832 **45.** ≈ 618 women
47. ≈ 61 adults **49.** ≈ 289 shafts
- 51. (a)** ≈ 0.977 **(b)** ≈ 0.159 **(c)** ≈ 0.838
55. (a) {LLL, LLD, LDL, DLL, LLU, LUL, ULL, LDD, LDU, LUD, LUU, DLD, DLU, ULD, ULU, DDL, DUL, UDL, UUL, DDD, DDU, DUD, UDD, DUU, UDU, UUD, UUU}
(c) $7/27 \approx 0.26$ **(d)** $20/27 \approx 0.74$

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- Page numbers with the prefix “17” are in the online Chapter 17 Second-Order Differential Equations
- Page numbers with the prefix “18” are in the online Chapter 18 Complex Functions
- Page numbers with the prefix “19” are in the online Chapter 19 Fourier Series and Wavelets
- Page numbers with the prefix “AP” are in Appendix A
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A Brief Table of Integrals

Basic Forms

1. $\int k \, dx = kx + C, \quad k \text{ any number}$
2. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x \, dx = e^x + C$
5. $\int a^x \, dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$
6. $\int \sin x \, dx = -\cos x + C$
7. $\int \cos x \, dx = \sin x + C$
8. $\int \sec^2 x \, dx = \tan x + C$
9. $\int \csc^2 x \, dx = -\cot x + C$
10. $\int \sec x \tan x \, dx = \sec x + C$
11. $\int \csc x \cot x \, dx = -\csc x + C$
12. $\int \tan x \, dx = \ln|\sec x| + C$
13. $\int \cot x \, dx = \ln|\sin x| + C$
14. $\int \sinh x \, dx = \cosh x + C$
15. $\int \cosh x \, dx = \sinh x + C$
16. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$
17. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$
18. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \left| \frac{x}{a} \right| + C$
19. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \frac{x}{a} + C \quad (a > 0)$
20. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} + C \quad (x > a > 0)$

Forms Involving $ax + b$

21. $\int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{a(n+1)} + C, \quad n \neq -1$
22. $\int x(ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{a^2} \left[\frac{ax + b}{n+2} - \frac{b}{n+1} \right] + C, \quad n \neq -1, -2$
23. $\int (ax + b)^{-1} \, dx = \frac{1}{a} \ln|ax + b| + C$
24. $\int x(ax + b)^{-1} \, dx = \frac{x}{a} - \frac{b}{a^2} \ln|ax + b| + C$
25. $\int x(ax + b)^{-2} \, dx = \frac{1}{a^2} \left[\ln|ax + b| + \frac{b}{ax + b} \right] + C$
26. $\int \frac{dx}{x(ax + b)} = \frac{1}{b} \ln \left| \frac{x}{ax + b} \right| + C$
27. $\int (\sqrt{ax + b})^n \, dx = \frac{2(\sqrt{ax + b})^{n+2}}{a(n+2)} + C, \quad n \neq -2$
28. $\int \frac{\sqrt{ax + b}}{x} \, dx = 2\sqrt{ax + b} + b \int \frac{dx}{x\sqrt{ax + b}}$

$$29. \text{ (a) } \int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}} \ln \left| \frac{\sqrt{ax+b} - \sqrt{b}}{\sqrt{ax+b} + \sqrt{b}} \right| + C$$

$$\text{ (b) } \int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}} \arctan \sqrt{\frac{ax-b}{b}} + C$$

$$30. \int \frac{\sqrt{ax+b}}{x^2} dx = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2} \int \frac{dx}{x\sqrt{ax+b}} + C$$

$$31. \int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b} \int \frac{dx}{x\sqrt{ax+b}} + C$$

Forms Involving $a^2 + x^2$

$$32. \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$33. \int \frac{dx}{(a^2 + x^2)^2} = \frac{x}{2a^2(a^2 + x^2)} + \frac{1}{2a^3} \arctan \frac{x}{a} + C$$

$$34. \int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \frac{x}{a} + C = \ln(x + \sqrt{a^2 + x^2}) + C$$

$$35. \int \sqrt{a^2 + x^2} dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \ln(x + \sqrt{a^2 + x^2}) + C$$

$$36. \int x^2 \sqrt{a^2 + x^2} dx = \frac{x}{8} (a^2 + 2x^2) \sqrt{a^2 + x^2} - \frac{a^4}{8} \ln(x + \sqrt{a^2 + x^2}) + C$$

$$37. \int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \ln \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right| + C$$

$$38. \int \frac{\sqrt{a^2 + x^2}}{x^2} dx = \ln(x + \sqrt{a^2 + x^2}) - \frac{\sqrt{a^2 + x^2}}{x} + C$$

$$39. \int \frac{x^2}{\sqrt{a^2 + x^2}} dx = -\frac{a^2}{2} \ln(x + \sqrt{a^2 + x^2}) + \frac{x\sqrt{a^2 + x^2}}{2} + C$$

$$40. \int \frac{dx}{x\sqrt{a^2 + x^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right| + C$$

$$41. \int \frac{dx}{x^2\sqrt{a^2 + x^2}} = -\frac{\sqrt{a^2 + x^2}}{a^2 x} + C$$

Forms Involving $a^2 - x^2$

$$42. \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right| + C$$

$$43. \int \frac{dx}{(a^2 - x^2)^2} = \frac{x}{2a^2(a^2 - x^2)} + \frac{1}{4a^3} \ln \left| \frac{x+a}{x-a} \right| + C$$

$$44. \int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$$

$$45. \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C$$

$$46. \int x^2 \sqrt{a^2 - x^2} dx = \frac{a^4}{8} \arcsin \frac{x}{a} - \frac{1}{8} x \sqrt{a^2 - x^2} (a^2 - 2x^2) + C$$

$$47. \int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right| + C$$

$$48. \int \frac{\sqrt{a^2 - x^2}}{x^2} dx = -\arcsin \frac{x}{a} - \frac{\sqrt{a^2 - x^2}}{x} + C$$

$$49. \int \frac{x^2}{\sqrt{a^2 - x^2}} dx = \frac{a^2}{2} \arcsin \frac{x}{a} - \frac{1}{2} x \sqrt{a^2 - x^2} + C$$

$$50. \int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right| + C$$

$$51. \int \frac{dx}{x^2\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x} + C$$

Forms Involving $x^2 - a^2$

$$52. \int \frac{dx}{\sqrt{x^2 - a^2}} = \ln |x + \sqrt{x^2 - a^2}| + C$$

$$53. \int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln |x + \sqrt{x^2 - a^2}| + C$$

$$54. \int (\sqrt{x^2 - a^2})^n dx = \frac{x(\sqrt{x^2 - a^2})^n}{n+1} - \frac{na^2}{n+1} \int (\sqrt{x^2 - a^2})^{n-2} dx, \quad n \neq -1$$

$$55. \int \frac{dx}{(\sqrt{x^2 - a^2})^n} = \frac{x(\sqrt{x^2 - a^2})^{2-n}}{(2-n)a^2} - \frac{n-3}{(n-2)a^2} \int \frac{dx}{(\sqrt{x^2 - a^2})^{n-2}}, \quad n \neq 2$$

$$56. \int x(\sqrt{x^2 - a^2})^n dx = \frac{(\sqrt{x^2 - a^2})^{n+2}}{n+2} + C, \quad n \neq -2$$

$$57. \int x^2 \sqrt{x^2 - a^2} dx = \frac{x}{8}(2x^2 - a^2)\sqrt{x^2 - a^2} - \frac{a^4}{8} \ln|x + \sqrt{x^2 - a^2}| + C$$

$$58. \int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \operatorname{arcsec}\left|\frac{x}{a}\right| + C$$

$$59. \int \frac{\sqrt{x^2 - a^2}}{x^2} dx = \ln|x + \sqrt{x^2 - a^2}| - \frac{\sqrt{x^2 - a^2}}{x} + C$$

$$60. \int \frac{x^2}{\sqrt{x^2 - a^2}} dx = \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + \frac{x}{2} \sqrt{x^2 - a^2} + C$$

$$61. \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec}\left|\frac{x}{a}\right| + C = \frac{1}{a} \arccos\left|\frac{a}{x}\right| + C$$

$$62. \int \frac{dx}{x^2 \sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a^2 x} + C$$

Trigonometric Forms

$$63. \int \sin ax dx = -\frac{1}{a} \cos ax + C$$

$$64. \int \cos ax dx = \frac{1}{a} \sin ax + C$$

$$65. \int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$$

$$66. \int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C$$

$$67. \int \sin^n ax dx = -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n} \int \sin^{n-2} ax dx$$

$$68. \int \cos^n ax dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n} \int \cos^{n-2} ax dx$$

$$69. (a) \int \sin ax \cos bx dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} + C, \quad a^2 \neq b^2$$

$$(b) \int \sin ax \sin bx dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} + C, \quad a^2 \neq b^2$$

$$(c) \int \cos ax \cos bx dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} + C, \quad a^2 \neq b^2$$

$$70. \int \sin ax \cos ax dx = -\frac{\cos 2ax}{4a} + C$$

$$71. \int \sin^n ax \cos ax dx = \frac{\sin^{n+1} ax}{(n+1)a} + C, \quad n \neq -1$$

$$72. \int \frac{\cos ax}{\sin ax} dx = \frac{1}{a} \ln|\sin ax| + C$$

$$73. \int \cos^n ax \sin ax dx = -\frac{\cos^{n+1} ax}{(n+1)a} + C, \quad n \neq -1$$

$$74. \int \frac{\sin ax}{\cos ax} dx = -\frac{1}{a} \ln|\cos ax| + C$$

$$75. \int \sin^n ax \cos^m ax dx = -\frac{\sin^{n-1} ax \cos^{m+1} ax}{a(m+n)} + \frac{n-1}{m+n} \int \sin^{n-2} ax \cos^m ax dx, \quad n \neq -m \quad (\text{reduces } \sin^n ax)$$

$$76. \int \sin^n ax \cos^m ax dx = \frac{\sin^{n+1} ax \cos^{m-1} ax}{a(m+n)} + \frac{m-1}{m+n} \int \sin^n ax \cos^{m-2} ax dx, \quad m \neq -n \quad (\text{reduces } \cos^m ax)$$

77. $\int \frac{dx}{b + c \sin ax} = \frac{-2}{a\sqrt{b^2 - c^2}} \arctan \left[\sqrt{\frac{b-c}{b+c}} \tan \left(\frac{\pi}{4} - \frac{ax}{2} \right) \right] + C, \quad b^2 > c^2$
78. $\int \frac{dx}{b + c \sin ax} = \frac{-1}{a\sqrt{c^2 - b^2}} \ln \left| \frac{c + b \sin ax + \sqrt{c^2 - b^2} \cos ax}{b + c \sin ax} \right| + C, \quad b^2 < c^2$
79. $\int \frac{dx}{1 + \sin ax} = -\frac{1}{a} \tan \left(\frac{\pi}{4} - \frac{ax}{2} \right) + C$
80. $\int \frac{dx}{1 - \sin ax} = \frac{1}{a} \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right) + C$
81. $\int \frac{dx}{b + c \cos ax} = \frac{2}{a\sqrt{b^2 - c^2}} \arctan \left[\sqrt{\frac{b-c}{b+c}} \tan \frac{ax}{2} \right] + C, \quad b^2 > c^2$
82. $\int \frac{dx}{b + c \cos ax} = \frac{1}{a\sqrt{c^2 - b^2}} \ln \left| \frac{c + b \cos ax + \sqrt{c^2 - b^2} \sin ax}{b + c \cos ax} \right| + C, \quad b^2 < c^2$
83. $\int \frac{dx}{1 + \cos ax} = \frac{1}{a} \tan \frac{ax}{2} + C$
84. $\int \frac{dx}{1 - \cos ax} = -\frac{1}{a} \cot \frac{ax}{2} + C$
85. $\int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax + C$
86. $\int x \cos ax \, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax + C$
87. $\int x^n \sin ax \, dx = -\frac{x^n}{a} \cos ax + \frac{n}{a} \int x^{n-1} \cos ax \, dx$
88. $\int x^n \cos ax \, dx = \frac{x^n}{a} \sin ax - \frac{n}{a} \int x^{n-1} \sin ax \, dx$
89. $\int \tan ax \, dx = \frac{1}{a} \ln |\sec ax| + C$
90. $\int \cot ax \, dx = \frac{1}{a} \ln |\sin ax| + C$
91. $\int \tan^2 ax \, dx = \frac{1}{a} \tan ax - x + C$
92. $\int \cot^2 ax \, dx = -\frac{1}{a} \cot ax - x + C$
93. $\int \tan^n ax \, dx = \frac{\tan^{n-1} ax}{a(n-1)} - \int \tan^{n-2} ax \, dx, \quad n \neq 1$
94. $\int \cot^n ax \, dx = -\frac{\cot^{n-1} ax}{a(n-1)} - \int \cot^{n-2} ax \, dx, \quad n \neq 1$
95. $\int \sec ax \, dx = \frac{1}{a} \ln |\sec ax + \tan ax| + C$
96. $\int \csc ax \, dx = -\frac{1}{a} \ln |\csc ax + \cot ax| + C$
97. $\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C$
98. $\int \csc^2 ax \, dx = -\frac{1}{a} \cot ax + C$
99. $\int \sec^n ax \, dx = \frac{\sec^{n-2} ax \tan ax}{a(n-1)} + \frac{n-2}{n-1} \int \sec^{n-2} ax \, dx, \quad n \neq 1$
100. $\int \csc^n ax \, dx = -\frac{\csc^{n-2} ax \cot ax}{a(n-1)} + \frac{n-2}{n-1} \int \csc^{n-2} ax \, dx, \quad n \neq 1$
101. $\int \sec^n ax \tan ax \, dx = \frac{\sec^n ax}{na} + C, \quad n \neq 0$
102. $\int \csc^n ax \cot ax \, dx = -\frac{\csc^n ax}{na} + C, \quad n \neq 0$

Inverse Trigonometric Forms

103. $\int \arcsin ax \, dx = x \arcsin ax + \frac{1}{a} \sqrt{1 - a^2 x^2} + C$
104. $\int \arccos ax \, dx = x \arccos ax - \frac{1}{a} \sqrt{1 - a^2 x^2} + C$
105. $\int \arctan ax \, dx = x \arctan ax - \frac{1}{2a} \ln(1 + a^2 x^2) + C$
106. $\int x^n \arcsin ax \, dx = \frac{x^{n+1}}{n+1} \arcsin ax - \frac{a}{n+1} \int \frac{x^{n+1} dx}{\sqrt{1 - a^2 x^2}}, \quad n \neq -1$

$$107. \int x^n \arccos ax \, dx = \frac{x^{n+1}}{n+1} \arccos ax + \frac{a}{n+1} \int \frac{x^{n+1} dx}{\sqrt{1-a^2x^2}}, \quad n \neq -1$$

$$108. \int x^n \arctan ax \, dx = \frac{x^{n+1}}{n+1} \arctan ax - \frac{a}{n+1} \int \frac{x^{n+1} dx}{1+a^2x^2}, \quad n \neq -1$$

Exponential and Logarithmic Forms

$$109. \int e^{ax} \, dx = \frac{1}{a} e^{ax} + C$$

$$110. \int b^{ax} \, dx = \frac{1}{a \ln b} b^{ax} + C, \quad b > 0, b \neq 1$$

$$111. \int x e^{ax} \, dx = \frac{e^{ax}}{a^2} (ax - 1) + C$$

$$112. \int x^n e^{ax} \, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx$$

$$113. \int x^n b^{ax} \, dx = \frac{x^n b^{ax}}{a \ln b} - \frac{n}{a \ln b} \int x^{n-1} b^{ax} \, dx, \quad b > 0, b \neq 1$$

$$114. \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C$$

$$115. \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C \quad 116. \int \ln ax \, dx = x \ln ax - x + C$$

$$117. \int x^n (\ln ax)^m \, dx = \frac{x^{n+1} (\ln ax)^m}{n+1} - \frac{m}{n+1} \int x^n (\ln ax)^{m-1} \, dx, \quad n \neq -1$$

$$118. \int x^{-1} (\ln ax)^m \, dx = \frac{(\ln ax)^{m+1}}{m+1} + C, \quad m \neq -1$$

$$119. \int \frac{dx}{x \ln ax} = \ln |\ln ax| + C$$

Forms Involving $\sqrt{2ax - x^2}$, $a > 0$

$$120. \int \frac{dx}{\sqrt{2ax - x^2}} = \arcsin\left(\frac{x-a}{a}\right) + C$$

$$121. \int \sqrt{2ax - x^2} \, dx = \frac{x-a}{2} \sqrt{2ax - x^2} + \frac{a^2}{2} \arcsin\left(\frac{x-a}{a}\right) + C$$

$$122. \int (\sqrt{2ax - x^2})^n \, dx = \frac{(x-a)(\sqrt{2ax - x^2})^n}{n+1} + \frac{na^2}{n+1} \int (\sqrt{2ax - x^2})^{n-2} \, dx$$

$$123. \int \frac{dx}{(\sqrt{2ax - x^2})^n} = \frac{(x-a)(\sqrt{2ax - x^2})^{2-n}}{(n-2)a^2} + \frac{n-3}{(n-2)a^2} \int \frac{dx}{(\sqrt{2ax - x^2})^{n-2}}$$

$$124. \int x \sqrt{2ax - x^2} \, dx = \frac{(x+a)(2x-3a)\sqrt{2ax - x^2}}{6} + \frac{a^3}{2} \arcsin\left(\frac{x-a}{a}\right) + C$$

$$125. \int \frac{\sqrt{2ax - x^2}}{x} \, dx = \sqrt{2ax - x^2} + a \arcsin\left(\frac{x-a}{a}\right) + C$$

$$126. \int \frac{\sqrt{2ax - x^2}}{x^2} \, dx = -2 \sqrt{\frac{2a-x}{x}} - \arcsin\left(\frac{x-a}{a}\right) + C$$

$$127. \int \frac{x \, dx}{\sqrt{2ax - x^2}} = a \arcsin\left(\frac{x-a}{a}\right) - \sqrt{2ax - x^2} + C$$

$$128. \int \frac{dx}{x \sqrt{2ax - x^2}} = -\frac{1}{a} \sqrt{\frac{2a-x}{x}} + C$$

Hyperbolic Forms

$$129. \int \sinh ax \, dx = \frac{1}{a} \cosh ax + C$$

$$130. \int \cosh ax \, dx = \frac{1}{a} \sinh ax + C$$

$$131. \int \sinh^2 ax \, dx = \frac{\sinh 2ax}{4a} - \frac{x}{2} + C$$

$$132. \int \cosh^2 ax \, dx = \frac{\sinh 2ax}{4a} + \frac{x}{2} + C$$

$$133. \int \sinh^n ax \, dx = \frac{\sinh^{n-1} ax \cosh ax}{na} - \frac{n-1}{n} \int \sinh^{n-2} ax \, dx, \quad n \neq 0$$

$$134. \int \cosh^n ax \, dx = \frac{\cosh^{n-1} ax \sinh ax}{na} + \frac{n-1}{n} \int \cosh^{n-2} ax \, dx, \quad n \neq 0$$

$$135. \int x \sinh ax \, dx = \frac{x}{a} \cosh ax - \frac{1}{a^2} \sinh ax + C$$

$$136. \int x \cosh ax \, dx = \frac{x}{a} \sinh ax - \frac{1}{a^2} \cosh ax + C$$

$$137. \int x^n \sinh ax \, dx = \frac{x^n}{a} \cosh ax - \frac{n}{a} \int x^{n-1} \cosh ax \, dx$$

$$138. \int x^n \cosh ax \, dx = \frac{x^n}{a} \sinh ax - \frac{n}{a} \int x^{n-1} \sinh ax \, dx$$

$$139. \int \tanh ax \, dx = \frac{1}{a} \ln(\cosh ax) + C$$

$$140. \int \coth ax \, dx = \frac{1}{a} \ln|\sinh ax| + C$$

$$141. \int \tanh^2 ax \, dx = x - \frac{1}{a} \tanh ax + C$$

$$142. \int \coth^2 ax \, dx = x - \frac{1}{a} \coth ax + C$$

$$143. \int \tanh^n ax \, dx = -\frac{\tanh^{n-1} ax}{(n-1)a} + \int \tanh^{n-2} ax \, dx, \quad n \neq 1$$

$$144. \int \coth^n ax \, dx = -\frac{\coth^{n-1} ax}{(n-1)a} + \int \coth^{n-2} ax \, dx, \quad n \neq 1$$

$$145. \int \operatorname{sech} ax \, dx = \frac{1}{a} \arcsin(\tanh ax) + C$$

$$146. \int \operatorname{csch} ax \, dx = \frac{1}{a} \ln \left| \tanh \frac{ax}{2} \right| + C$$

$$147. \int \operatorname{sech}^2 ax \, dx = \frac{1}{a} \tanh ax + C$$

$$148. \int \operatorname{csch}^2 ax \, dx = -\frac{1}{a} \coth ax + C$$

$$149. \int \operatorname{sech}^n ax \, dx = \frac{\operatorname{sech}^{n-2} ax \tanh ax}{(n-1)a} + \frac{n-2}{n-1} \int \operatorname{sech}^{n-2} ax \, dx, \quad n \neq 1$$

$$150. \int \operatorname{csch}^n ax \, dx = -\frac{\operatorname{csch}^{n-2} ax \coth ax}{(n-1)a} - \frac{n-2}{n-1} \int \operatorname{csch}^{n-2} ax \, dx, \quad n \neq 1$$

$$151. \int \operatorname{sech}^n ax \tanh ax \, dx = -\frac{\operatorname{sech}^n ax}{na} + C, \quad n \neq 0$$

$$152. \int \operatorname{csch}^n ax \coth ax \, dx = -\frac{\operatorname{csch}^n ax}{na} + C, \quad n \neq 0$$

$$153. \int e^{ax} \sinh bx \, dx = \frac{e^{ax}}{2} \left[\frac{e^{bx}}{a+b} - \frac{e^{-bx}}{a-b} \right] + C, \quad a^2 \neq b^2$$

$$154. \int e^{ax} \cosh bx \, dx = \frac{e^{ax}}{2} \left[\frac{e^{bx}}{a+b} + \frac{e^{-bx}}{a-b} \right] + C, \quad a^2 \neq b^2$$

Some Definite Integrals

$$155. \int_0^\infty x^{n-1} e^{-x} \, dx = \Gamma(n) = (n-1)!, \quad n > 0$$

$$156. \int_0^\infty e^{-ax^2} \, dx = \frac{1}{2} \sqrt{\frac{\pi}{a}}, \quad a > 0$$

$$157. \int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx = \begin{cases} \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} \cdot \frac{\pi}{2}, & \text{if } n \text{ is an even integer } \geq 2 \\ \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{3 \cdot 5 \cdot 7 \cdots n}, & \text{if } n \text{ is an odd integer } \geq 3 \end{cases}$$

Basic Algebra Formulas

Arithmetic Operations

$$a(b + c) = ab + ac, \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$
$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \quad \frac{a/b}{c/d} = \frac{a}{b} \cdot \frac{d}{c}$$

Laws of Signs

$$-(-a) = a, \quad \frac{-a}{b} = -\frac{a}{b} = \frac{a}{-b}$$

Zero Division by zero is not defined.

$$\text{If } a \neq 0: \frac{0}{a} = 0, \quad a^0 = 1, \quad 0^a = 0$$

$$\text{For any number } a: a \cdot 0 = 0 \cdot a = 0$$

Laws of Exponents

$$a^m a^n = a^{m+n}, \quad (ab)^m = a^m b^m, \quad (a^m)^n = a^{mn}, \quad a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

If $a \neq 0$, then

$$\frac{a^m}{a^n} = a^{m-n}, \quad a^0 = 1, \quad a^{-m} = \frac{1}{a^m}.$$

The Binomial Theorem For any positive integer n ,

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2$$
$$+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \cdots + nab^{n-1} + b^n.$$

For instance,

$$(a + b)^2 = a^2 + 2ab + b^2, \quad (a - b)^2 = a^2 - 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, \quad (a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

Factoring the Difference of Like Integer Powers, $n > 1$

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})$$

For instance,

$$a^2 - b^2 = (a - b)(a + b),$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2),$$
$$a^4 - b^4 = (a - b)(a^3 + a^2b + ab^2 + b^3).$$

Completing the Square If $a \neq 0$, then

$$ax^2 + bx + c = au^2 + C \quad \left(u = x + (b/2a), C = c - \frac{b^2}{4a}\right).$$

The Quadratic Formula

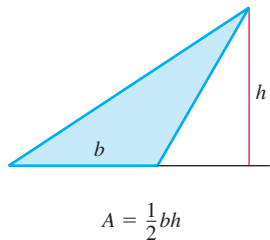
If $a \neq 0$ and $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

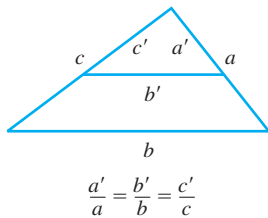
Geometry Formulas

A = area, B = area of base, C = circumference, S = surface area, V = volume

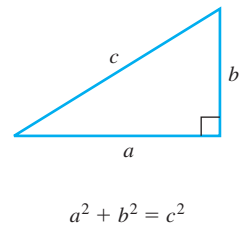
Triangle



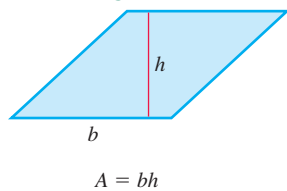
Similar Triangles



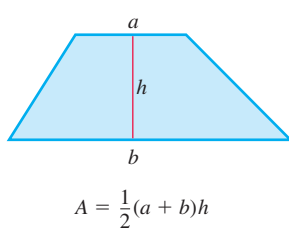
Pythagorean Theorem



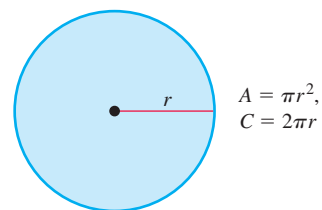
Parallelogram



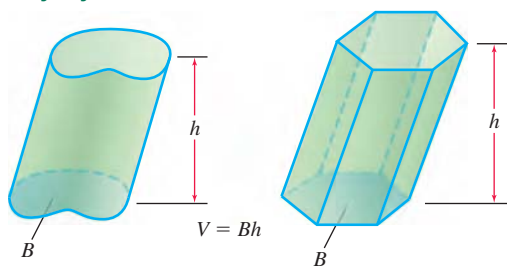
Trapezoid



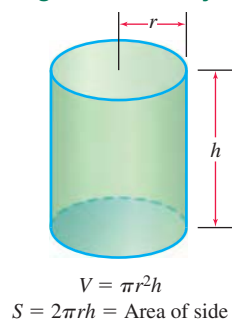
Circle



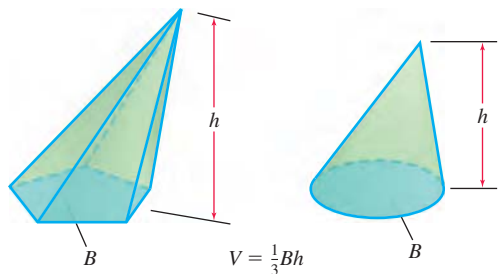
Any Cylinder or Prism with Parallel Bases



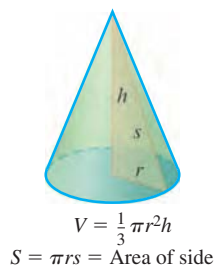
Right Circular Cylinder



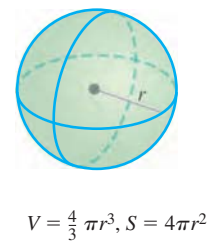
Any Cone or Pyramid



Right Circular Cone



Sphere



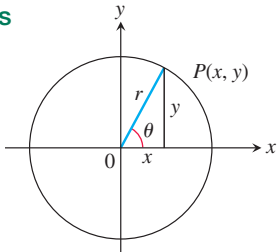
Trigonometry Formulas

Definitions and Fundamental Identities

Sine: $\sin \theta = \frac{y}{r} = \frac{1}{\csc \theta}$

Cosine: $\cos \theta = \frac{x}{r} = \frac{1}{\sec \theta}$

Tangent: $\tan \theta = \frac{y}{x} = \frac{1}{\cot \theta}$



Identities

$$\sin(-\theta) = -\sin \theta, \quad \cos(-\theta) = \cos \theta$$

$$\sin^2 \theta + \cos^2 \theta = 1, \quad \sec^2 \theta = 1 + \tan^2 \theta, \quad \csc^2 \theta = 1 + \cot^2 \theta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta, \quad \cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}, \quad \sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\sin\left(A - \frac{\pi}{2}\right) = -\cos A, \quad \cos\left(A - \frac{\pi}{2}\right) = \sin A$$

$$\sin\left(A + \frac{\pi}{2}\right) = \cos A, \quad \cos\left(A + \frac{\pi}{2}\right) = -\sin A$$

$$\sin A \sin B = \frac{1}{2} \cos(A - B) - \frac{1}{2} \cos(A + B)$$

$$\cos A \cos B = \frac{1}{2} \cos(A - B) + \frac{1}{2} \cos(A + B)$$

$$\sin A \cos B = \frac{1}{2} \sin(A - B) + \frac{1}{2} \sin(A + B)$$

$$\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$$

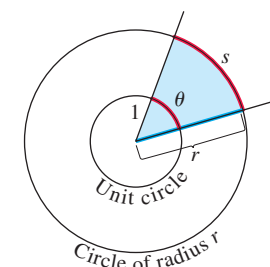
$$\sin A - \sin B = 2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$$

$$\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$$

$$\cos A - \cos B = -2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$$

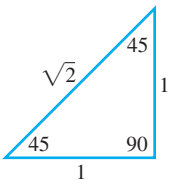
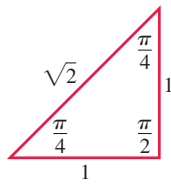
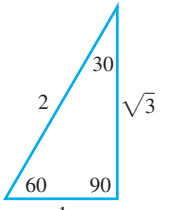
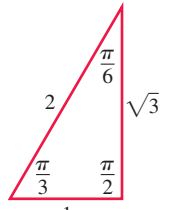
Trigonometric Functions

Radian Measure

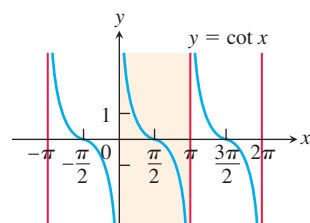
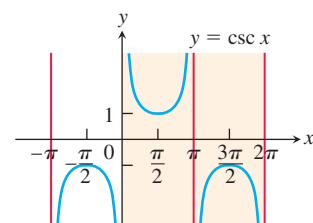
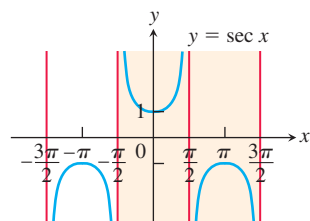
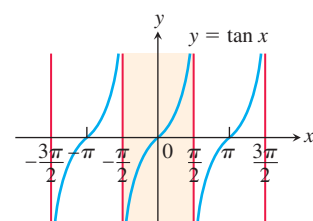
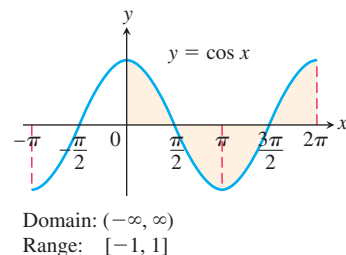
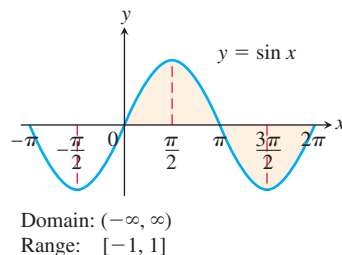


$$\frac{s}{r} = \frac{\theta}{1} = \theta \quad \text{or} \quad \theta = \frac{s}{r},$$

$$180^\circ = \pi \text{ radians.}$$

Degrees	Radians
	
	

The angles of two common triangles, in degrees and radians.



Series

Tests for Convergence of Infinite Series

- 1. The n th-Term Test:** Unless $a_n \rightarrow 0$, the series diverges.
- 2. Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
- 3. p -series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
- 4. Series with nonnegative terms:** Try the Integral Test, Ratio Test, or Root Test. Try comparing to a known series with the Comparison Test or the Limit Comparison Test.
- 5. Series with some negative terms:** Does $\sum |a_n|$ converge? If yes, so does $\sum a_n$ because absolute convergence implies convergence.
- 6. Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

Taylor Series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \leq 1$$

$$\ln \frac{1+x}{1-x} = 2 \tanh^{-1} x = 2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots \right) = 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}, \quad |x| < 1$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \leq 1$$

Binomial Series

$$\begin{aligned} (1+x)^m &= 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + \cdots + \frac{m(m-1)(m-2)\cdots(m-k+1)x^k}{k!} + \cdots \\ &= 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k, \quad |x| < 1, \end{aligned}$$

where

$$\binom{m}{1} = m, \quad \binom{m}{2} = \frac{m(m-1)}{2!}, \quad \binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!} \quad \text{for } k \geq 3.$$

Vector Operator Formulas (Cartesian Form)

Formulas for Grad, Div, Curl, and the Laplacian

Cartesian (x, y, z) **i, j, and k** are unit vectors in the directions of increasing x , y , and z . M , N , and P are the scalar components of $\mathbf{F}(x, y, z)$ in these directions.

Gradient $\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$

Divergence $\nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$

Curl $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$

Laplacian $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

Vector Triple Products

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

Vector Identities

In the identities here, f and g are differentiable scalar functions; $\mathbf{F}$, $\mathbf{F}_1$, and $\mathbf{F}_2$ are differentiable vector fields; and a and b are real constants.

$$\nabla \times (\nabla f) = \mathbf{0}$$

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla \cdot (g\mathbf{F}) = g\nabla \cdot \mathbf{F} + \nabla g \cdot \mathbf{F}$$

$$\nabla \times (g\mathbf{F}) = g\nabla \times \mathbf{F} + \nabla g \times \mathbf{F}$$

$$\nabla \cdot (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \cdot \mathbf{F}_1 + b\nabla \cdot \mathbf{F}_2$$

$$\nabla \times (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \times \mathbf{F}_1 + b\nabla \times \mathbf{F}_2$$

$$\nabla(\mathbf{F}_1 \cdot \mathbf{F}_2) = (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 +$$

$$\mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) + \mathbf{F}_2 \times (\nabla \times \mathbf{F}_1)$$

The Fundamental Theorem of Line Integrals

Part 1 Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then there exists a differentiable function f such that

$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

if and only if, for all points A and B in D , the value of $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining A to B in D .

Part 2 If the integral is independent of the path from A to B , its value is

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Green's Theorem and Its Generalization to Three Dimensions

Tangential form of Green's Theorem: $\oint_C \mathbf{F} \cdot \mathbf{T} ds = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA$

Stokes' Theorem: $\oint_C \mathbf{F} \cdot \mathbf{T} ds = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma$

Normal form of Green's Theorem: $\oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint_R (\nabla \cdot \mathbf{F}) dA$

Divergence Theorem: $\iiint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot \mathbf{F} dV$

$$\nabla \cdot (\mathbf{F}_1 \times \mathbf{F}_2) = \mathbf{F}_2 \cdot (\nabla \times \mathbf{F}_1) - \mathbf{F}_1 \cdot (\nabla \times \mathbf{F}_2)$$

$$\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2) = (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 - (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\nabla \cdot \mathbf{F}_2)\mathbf{F}_1 - (\nabla \cdot \mathbf{F}_1)\mathbf{F}_2$$

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - (\nabla \cdot \nabla)\mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$$

$$(\nabla \times \mathbf{F}) \times \mathbf{F} = (\mathbf{F} \cdot \nabla)\mathbf{F} - \frac{1}{2}\nabla(\mathbf{F} \cdot \mathbf{F})$$

Limits

General Laws

If L , M , c , and k are real numbers and

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = M, \quad \text{then}$$

Sum Rule: $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$

Difference Rule: $\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$

Product Rule: $\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$

Constant Multiple Rule: $\lim_{x \rightarrow c} (k \cdot f(x)) = k \cdot L$

Quotient Rule: $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$

The Sandwich Theorem

If $g(x) \leq f(x) \leq h(x)$ in an open interval containing c , except possibly at $x = c$, and if

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L,$$

then $\lim_{x \rightarrow c} f(x) = L$.

Inequalities

If $f(x) \leq g(x)$ in an open interval containing c , except possibly at $x = c$, and both limits exist, then

$$\lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} g(x).$$

Continuity

If g is continuous at L and $\lim_{x \rightarrow c} f(x) = L$, then

$$\lim_{x \rightarrow c} g(f(x)) = g(L).$$

Specific Formulas

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \rightarrow c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

If $f(x)$ is continuous at $x = c$, then

$$\lim_{x \rightarrow c} f(x) = f(c).$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

L'Hôpital's Rule

If $f(a) = g(a) = 0$, both f' and g' exist in an open interval I containing a , and $g'(x) \neq 0$ on I if $x \neq a$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right side exists.

Differentiation Rules

General Formulas

Assume u and v are differentiable functions of x .

Constant: $\frac{d}{dx}(c) = 0$

Sum: $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$

Difference: $\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}$

Constant Multiple: $\frac{d}{dx}(cu) = c \frac{du}{dx}$

Product: $\frac{d}{dx}(uv) = u \frac{dv}{dx} + \frac{du}{dx} v$

Quotient: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

Power: $\frac{d}{dx} x^n = nx^{n-1}$

Chain Rule: $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$

Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \quad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \quad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Exponential and Logarithmic Functions

$$\frac{d}{dx} e^x = e^x \quad \frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} a^x = a^x \ln a \quad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Inverse Trigonometric Functions

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} \quad \frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{1+x^2} \quad \frac{d}{dx}(\operatorname{arccsc} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \quad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \quad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x \quad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}} \quad \frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \quad \frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2} \quad \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{1+x^2}}$$

Parametric Equations

If $x = f(t)$ and $y = g(t)$ are differentiable, then

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}.$$

Integration Rules

General Formulas

Zero: $\int_a^a f(x) dx = 0$

Order of Integration: $\int_b^a f(x) dx = -\int_a^b f(x) dx$

Constant Multiples: $\int_a^b kf(x) dx = k \int_a^b f(x) dx, \quad k \text{ any number}$

$$\int_a^b -f(x) dx = -\int_a^b f(x) dx, \quad k = -1$$

Sums and Differences: $\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$

Additivity: $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$

Max-Min Inequality: If $\max f$ and $\min f$ are the maximum and minimum values of f on $[a, b]$, then

$$\min f \cdot (b - a) \leq \int_a^b f(x) dx \leq \max f \cdot (b - a).$$

Domination: $f(x) \geq g(x) \text{ on } [a, b] \text{ implies } \int_a^b f(x) dx \geq \int_a^b g(x) dx.$

$$f(x) \geq 0 \text{ on } [a, b] \text{ implies } \int_a^b f(x) dx \geq 0.$$

The Fundamental Theorem of Calculus

Part 1 If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t) dt$ is continuous on $[a, b]$ and differentiable on (a, b) , and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x).$$

Part 2 If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a).$$

Substitution in Definite Integrals

$$\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

Integration by Parts

$$\int_a^b u(x)v'(x) dx = u(x)v(x) \Big|_a^b - \int_a^b v(x)u'(x) dx$$