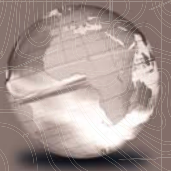


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Early Transcendentals

FIFTEENTH EDITION IN SI UNITS

Hass • Heil • Bogacki • Weir



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THOMAS' CALCULUS

Early Transcendentals

FIFTEENTH EDITION IN SI UNITS

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Contents

Preface 9

1

Functions 21

- 1.1 Functions and Their Graphs 21
- 1.2 Combining Functions; Shifting and Scaling Graphs 34
- 1.3 Trigonometric Functions 41
- 1.4 Exponential Functions 49
- 1.5 Inverse Functions and Logarithms 54
- Questions to Guide Your Review* 67
- Practice Exercises* 67
- Additional and Advanced Exercises* 69
- Technology Application Projects* 71

2

Limits and Continuity 72

- 2.1 Rates of Change and Tangent Lines to Curves 72
- 2.2 Limit of a Function and Limit Laws 79
- 2.3 The Precise Definition of a Limit 90
- 2.4 One-Sided Limits 99
- 2.5 Limits Involving Infinity; Asymptotes of Graphs 106
- 2.6 Continuity 120
- Questions to Guide Your Review* 132
- Practice Exercises* 133
- Additional and Advanced Exercises* 134
- Technology Application Projects* 137

3

Derivatives 138

- 3.1 Tangent Lines and the Derivative at a Point 138
- 3.2 The Derivative as a Function 142
- 3.3 Differentiation Rules 151
- 3.4 The Derivative as a Rate of Change 161
- 3.5 Derivatives of Trigonometric Functions 170
- 3.6 The Chain Rule 176
- 3.7 Implicit Differentiation 184
- 3.8 Derivatives of Inverse Functions and Logarithms 189
-  ~~3.9~~ Inverse Trigonometric Functions 200
- 3.10 Related Rates 206
- 3.11 Linearization and Differentials 214
- Questions to Guide Your Review* 226
- Practice Exercises* 227
- Additional and Advanced Exercises* 231
- Technology Application Projects* 234

4

Applications of Derivatives 235

- 4.1 Extreme Values of Functions on Closed Intervals 235
- 4.2 The Mean Value Theorem 243
- 4.3 Monotonic Functions and the First Derivative Test 250
- 4.4 Concavity and Curve Sketching 255
- ~~4.5 Indeterminate Forms and L'Hôpital's Rule 268~~
- 4.6 Applied Optimization 277
- 4.7 Newton's Method 289
- ~~4.8 Antiderivatives 294~~

Questions to Guide Your Review 304*Practice Exercises* 305*Additional and Advanced Exercises* 308*Technology Application Projects* 311**Start Here**

5

Integrals 312

- 5.1 Area and Estimating with Finite Sums 312
- ~~5.2 Sigma Notation and Limits of Finite Sums 322~~
- ~~5.3 The Definite Integral 329~~
- ~~5.4 The Fundamental Theorem of Calculus 342~~
- ~~5.5 Indefinite Integrals and the Substitution Method 354~~
- ~~5.6 Definite Integral Substitutions and the Area Between Curves 361~~

Questions to Guide Your Review 372*Practice Exercises* 372*Additional and Advanced Exercises* 375*Technology Application Projects* 379

6

Applications of Definite Integrals 380

- ~~6.1 Volumes Using Cross-Sections 380~~
- ~~6.2 Volumes Using Cylindrical Shells 391~~
- ~~6.3 Arc Length 399~~
- ~~6.4 Areas of Surfaces of Revolution 405~~
- 6.5 Work and Fluid Forces 410
- 6.6 Moments and Centers of Mass 420

Questions to Guide Your Review 431*Practice Exercises* 432*Additional and Advanced Exercises* 434*Technology Application Projects* 435

7

Integrals and Transcendental Functions 436

- ~~7.1 The Logarithm Defined as an Integral 436~~
- ~~7.2 Exponential Change and Separable Differential Equations 447~~
- ~~7.3 Hyperbolic Functions 457~~
- 7.4 Relative Rates of Growth 465

Questions to Guide Your Review 470*Practice Exercises* 471*Additional and Advanced Exercises* 472

8

Techniques of Integration 473

- ~~8.1~~ Using Basic Integration Formulas 473
- ~~8.2~~ Integration by Parts 478
- ~~8.3~~ Trigonometric Integrals 486
- ~~8.4~~ Trigonometric Substitutions 492
- ~~8.5~~ Integration of Rational Functions by Partial Fractions 497
- 8.6 Integral Tables and Computer Algebra Systems 504
- 8.7 Numerical Integration 510
- ~~8.8~~ Improper Integrals 520
- Questions to Guide Your Review* 531
- Practice Exercises* 532
- Additional and Advanced Exercises* 534
- Technology Application Projects* 537

9

Infinite Sequences and Series 538

- 9.1 Sequences 538
- 9.2 Infinite Series 551
- 9.3 The Integral Test 561
- 9.4 Comparison Tests 567
- 9.5 Absolute Convergence; The Ratio and Root Tests 572
- 9.6 Alternating Series and Conditional Convergence 579
- 9.7 Power Series 586
- 9.8 Taylor and Maclaurin Series 597
- 9.9 Convergence of Taylor Series 602
- 9.10 Applications of Taylor Series 609
- Questions to Guide Your Review* 618
- Practice Exercises* 619
- Additional and Advanced Exercises* 621
- Technology Application Projects* 623

10

Parametric Equations and Polar Coordinates 624

- 10.1 Parametrizations of Plane Curves 624
- 10.2 Calculus with Parametric Curves 633
- ~~10.3~~ Polar Coordinates 642
- ~~10.4~~ Graphing Polar Coordinate Equations 646
- ~~10.5~~ Areas and Lengths in Polar Coordinates 650
- 10.6 Conic Sections 655
- 10.7 Conics in Polar Coordinates 663
- Questions to Guide Your Review* 669
- Practice Exercises* 670
- Additional and Advanced Exercises* 672
- Technology Application Projects* 674

11

Vectors and the Geometry of Space 675

- 11.1 Three-Dimensional Coordinate Systems 675
- 11.2 Vectors 680
- 11.3 The Dot Product 691

- 11.4 The Cross Product 699
- 11.5 Lines and Planes in Space 705
- 11.6 Cylinders and Quadric Surfaces 714
 - Questions to Guide Your Review* 720
 - Practice Exercises* 720
 - Additional and Advanced Exercises* 722
 - Technology Application Projects* 724

12 Vector-Valued Functions and Motion in Space 725

- 12.1 Curves in Space and Their Tangents 725
- 12.2 Integrals of Vector Functions; Projectile Motion 734
- 12.3 Arc Length in Space 743
- 12.4 Curvature and Normal Vectors of a Curve 747
- 12.5 Tangential and Normal Components of Acceleration 753
- 12.6 Velocity and Acceleration in Polar Coordinates 759
 - Questions to Guide Your Review* 762
 - Practice Exercises* 763
 - Additional and Advanced Exercises* 765
 - Technology Application Projects* 766

13 Partial Derivatives 767

- 13.1 Functions of Several Variables 767
- 13.2 Limits and Continuity in Higher Dimensions 775
- 13.3 Partial Derivatives 784
- 13.4 The Chain Rule 796
- 13.5 Directional Derivatives and Gradient Vectors 806
- 13.6 Tangent Planes and Differentials 815
- 13.7 Extreme Values and Saddle Points 825
- 13.8 Lagrange Multipliers 834
- 13.9 Taylor's Formula for Two Variables 844
- 13.10 Partial Derivatives with Constrained Variables 848
 - Questions to Guide Your Review* 852
 - Practice Exercises* 853
 - Additional and Advanced Exercises* 856
 - Technology Application Projects* 858

14 Multiple Integrals 859

- 14.1 Double and Iterated Integrals over Rectangles 859
- 14.2 Double Integrals over General Regions 864
- 14.3 Area by Double Integration 873
- 14.4 Double Integrals in Polar Form 876
- 14.5 Triple Integrals in Rectangular Coordinates 883
- 14.6 Applications 893
- 14.7 Triple Integrals in Cylindrical and Spherical Coordinates 903
- 14.8 Substitutions in Multiple Integrals 915
 - Questions to Guide Your Review* 924
 - Practice Exercises* 925
 - Additional and Advanced Exercises* 927
 - Technology Application Projects* 929

15

Integrals and Vector Fields 930

- 15.1 Line Integrals of Scalar Functions 930
- 15.2 Vector Fields and Line Integrals: Work, Circulation, and Flux 937
- 15.3 Path Independence, Conservative Fields, and Potential Functions 950
- 15.4 Green's Theorem in the Plane 961
- 15.5 Surfaces and Area 973
- 15.6 Surface Integrals 983
- 15.7 Stokes' Theorem 993
- 15.8 The Divergence Theorem and a Unified Theory 1006
- Questions to Guide Your Review* 1018
- Practice Exercises* 1019
- Additional and Advanced Exercises* 1021
- Technology Application Projects* 1023

16

First-Order Differential Equations 1024

- 16.1 Solutions, Slope Fields, and Euler's Method 1024
- 16.2 First-Order Linear Equations 1032
- 16.3 Applications 1038
- 16.4 Graphical Solutions of Autonomous Equations 1044
- 16.5 Systems of Equations and Phase Planes 1053
- Questions to Guide Your Review* 1059
- Practice Exercises* 1059
- Additional and Advanced Exercises* 1061
- Technology Application Projects* 1061

17

Second-Order Differential Equations (Online)

- 17.1 Second-Order Linear Equations
- 17.2 Nonhomogeneous Linear Equations
- 17.3 Applications
- 17.4 Euler Equations
- 17.5 Power-Series Solutions

18

Complex Functions (Online)

- 18.1 Complex Numbers
- 18.2 Functions of a Complex Variable
- 18.3 Derivatives
- 18.4 The Cauchy-Riemann Equations
- 18.5 Complex Power Series
- 18.6 Some Complex Functions
- 18.7 Conformal Maps
- Questions to Guide Your Review*
- Additional and Advanced Exercises*

19

Fourier Series and Wavelets (Online)

- 19.1** Periodic Functions
- 19.2** Summing Sines and Cosines
- 19.3** Vectors and Approximation in Three and More Dimensions
- 19.4** Approximation of Functions
- 19.5** Advanced Topic: The Haar System and Wavelets

Questions to Guide Your Review
Additional and Advanced Exercises

Appendix A AP-1

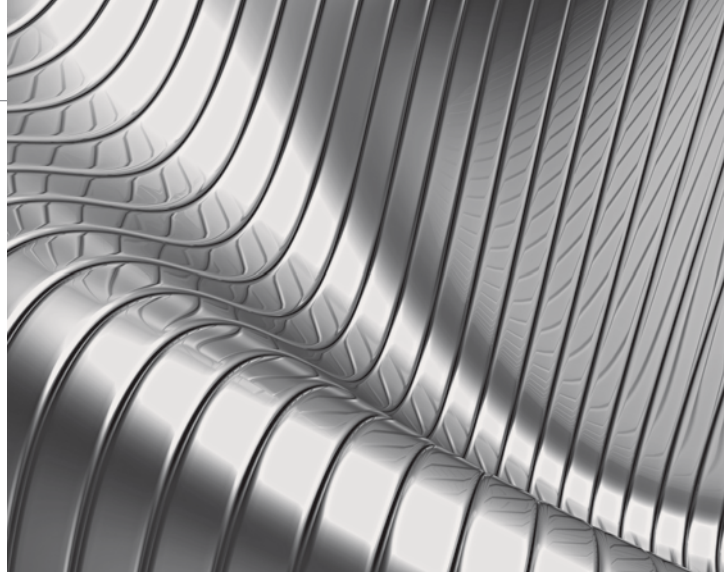
- A.1** Real Numbers and the Real Line AP-1
- A.2** Graphing with Software AP-6
- A.3** Mathematical Induction AP-10
- A.4** Lines, Circles, and Parabolas AP-13
- A.5** Proofs of Limit Theorems AP-23
- A.6** Commonly Occurring Limits AP-26
- A.7** Theory of the Real Numbers AP-27
- A.8** Probability AP-30
- A.9** The Distributive Law for Vector Cross Products AP-43
- A.10** The Mixed Derivative Theorem and the Increment Theorem AP-44

Appendix B (Online)

- B.1** Determinants
- B.2** Extreme Values and Saddle Points for Functions of More than Two Variables
- B.3** The Method of Gradient Descent

Answers to Odd-Numbered Exercises AN-1**Applications Index** AI-1**Subject Index** I-1**Credits** C-1**A Brief Table of Integrals** T-1

Preface



Thomas' Calculus: Early Transcendentals, Fifteenth Edition in SI Units, continues its tradition of clarity and precision in calculus with a modern update to the popular text. The authors have worked diligently to add exercises, revise figures and narrative for clarity, and update many applications to modern topics. *Thomas' Calculus* remains a modern and robust introduction to calculus, focusing on developing conceptual understanding of the underlying mathematical ideas. This text supports a calculus sequence typically taken by students in STEM fields over several semesters. Intuitive and precise explanations, thoughtfully chosen examples, superior figures, and time-tested exercise sets are the foundation of this text. We continue to improve this text in keeping with shifts in both the preparation and the goals of today's students, and in the applications of calculus to a changing world.

As Advanced Placement Calculus continues to grow in popularity for high school students, many instructors have communicated mixed reviews of the benefit for today's university and community college students. Some instructors report receiving students with an overconfidence in their computational abilities coupled with underlying gaps in algebra and trigonometry mastery, as well as poor conceptual understanding. In this text, we seek to meet the needs of the increasingly varied population in the calculus sequence. We have taken care to provide enough review material (in the text and appendices), detailed solutions, and a variety of examples and exercises, to support a complete understanding of calculus for students at varying levels. Additionally, the MyLab Math course that accompanies the text provides significant support to meet the needs of all students. Within the text, we present the material in a way that supports the development of mathematical maturity, going beyond memorizing formulas and routine procedures, and we show students how to generalize key concepts once they are introduced. References are made throughout, tying new concepts to related ones that were studied earlier. After studying calculus from *Thomas*, students will have developed problem-solving and reasoning abilities that will serve them well in many important aspects of their lives. Mastering this beautiful and creative subject, with its many practical applications across so many fields, is its own reward. But the real gifts of studying calculus are acquiring the ability to think logically and precisely; understanding what is defined, what is assumed, and what is deduced; and learning how to generalize conceptually. We intend this book to encourage and support those goals.

New to This Edition

We welcome to this edition a new coauthor, Przemyslaw Bogacki from Old Dominion University. Przemek joined the team for the fourth edition of *University Calculus* and now joins the *Thomas' Calculus* team. Przemek brings a keen eye for details as well as significant experience in MyLab Math. Przemek has diligently reviewed every exercise and solution in MyLab Math for mathematical accuracy, fidelity with text methods, and effectiveness for students. He has also recommended nearly 100 new Setup & Solve exercises and improved the sample assignments in MyLab. Przemek has also written the new appendix on Optimization covering determinants, extreme values, and gradient descent.

The most significant update to this 15th edition includes new online chapters on Complex Functions, Fourier Series and Wavelets, and the new appendix on Optimization. These chapters can provide material for students interested in more advanced topics. The details are outlined below in the chapter descriptions.

We have also made the following updates:

- Many updated graphics and figures to bring out clear visualization and mathematical correctness.
- Many wording clarifications and revisions.
- Many instruction clarifications for exercises, such as suggesting where the use of a calculator may be needed.
- Notation of inverse trig functions favoring arcsin notation over $\sin^{-1}$, etc.

New to MyLab Math

Pearson has continued to improve the general functionality of MyLab Math since the previous edition. Ongoing improvements to the grading algorithms, along with the development of MyLab Math for our differential equations courses allows for more sophisticated acceptance of generic constants and better parsing of mathematical expressions.

- The full suite of interactive figures has been updated for accessibility meeting WCAG standards. These figures are designed to be used in lecture as well as by students independently. The figures are editable using the freely available GeoGebra software. The figures were created by Marc Renault (Shippensburg University), Kevin Hopkins (Southwest Baptist University), Steve Phelps (University of Cincinnati), and Tim Brzezinski (Southington High School, CT).
- New! GeoGebra Exercises are gradable graphing and computational exercises that help students demonstrate their understanding. They enable students to interact directly with the graph in a manner that reflects how students would graph on paper.
- Nearly 100 additional Setup & Solve exercises have been created, selected by author Przemyslaw Bogacki. These exercises are designed to focus students on the process of problem solving by requiring them to set up their equations before moving on to the solution.
- Integrated Review quizzes and personalized homework are now built into all MyLab Math courses. No separate Integrated Review course is required.
- New online chapters and sections have exercises available, including exercises for the complex numbers and functions that many users have asked for.

Content Enhancements

Chapter 1

- Section 1.2. Revised Example 4 to clarify the distinction between vertical and horizontal scaling of a graph.
- Section 1.3. Added new Figure 1.46, illustrating a geometric proof of the angle sum identities.

Chapter 2

- Section 2.2. New Example 11, illustrating the use of the Sandwich Theorem, with corresponding new Figure 2.14.
- Section 2.4. New subsection on “Limits at Endpoints of an Interval” added. New Example 2 added, illustrating limits at a boundary point of an interval.
- Section 2.6. Exercises 41–45 on limits involving trigonometric functions moved from Chapter 3.
- Additional and Advanced Exercises. Exercises 31–40 on limits involving trigonometric functions moved from Chapter 3.

Chapter 3

- Section 3.8. Revised Figure 3.36 illustrating the relationship between slopes of graphs of inverse functions.
- Updated differentiation formulas involving exponential and logarithmic functions.
- Expanded Example 5.
- Expanded Example 7 to clarify the computation of the derivative of x^x .
- Added new Exercises 11–14 involving the derivatives of inverse functions.
- Section 3.9. Updated differentiation formulas involving inverse trigonometric functions.
- Added new Example 3 to illustrate differentiating a composition involving the arctangent function.
- Rewrote the introduction to the subsection on the derivative of $\operatorname{arcsec} x$.
- Section 3.10. Updated and improved related rates problem strategies, and correspondingly revised Examples 2–6.

Chapter 4

- Section 4.3. Added new Exercises 69–70.
- Section 4.4. Added new Exercises 107–108.
- Section 4.5. Improved the discussion of indeterminate forms.
- Expanded Example 1.
- Added new Exercises 19–20.

- Section 4.6. Updated and improved strategies for solving applied optimization problems.
- Added new Exercises 33–34.
- Section 4.8. Added Table 4.3 of integration formulas.

Chapter 5

- Section 5.1. The Midpoint Rule and the associated formula for calculating an integral numerically were given a more central role and used to introduce a numerical method.
- Section 5.3. New basic theory Exercise 89. Integrals of functions that differ at one point.
- Section 5.6. New Exercises 113–116. Compare areas using graphics and computation.

Chapter 6

Section 6.2. Discussion of cylinders in Example 1 clarified.

Chapter 7

- Clarified derivative formulas involving x versus those involving a differentiable function u .
- Section 7.1. Rewrote material on Logarithms and Laws on Exponents. Exercises 63–66 moved from Chapter 4. New Exercise 67 added.

Chapter 8

- Section 8.3. Clarified computing integrals involving powers of sines and cosines. Exercise 42 replaced. Exercises 51 and 52 added.
- Section 8.4. Ordering of exercises was updated.
- Section 8.5. Discussion of the method of partial fractions rewritten and clarified.
- Section 8.7. New subsection on the Midpoint Rule added. Discussion of Error Analysis expanded to include the Midpoint Rule. Exercises 1–10 expanded to include the Midpoint Rule.
- Section 8.8. Discussion of infinite limits of integration clarified. Material on Tests for Convergence and Divergence, including the Direct Comparison Test and the Limit Comparison Test, their proofs, and associated examples, all revised. New Exercises 69–80 added.

Chapter 9

- Section 9.2. Solution to Example 2 replaced. Solution to Example 8 replaced.
- Section 9.3. Solution to Example 5 revised.

- Section 9.5. Exercise 71 added.
- Section 9.6. Proof of Theorem 15 replaced. Discussion of Theorem 16 revised.
- Section 9.7. Discussion of absolute convergence added to the solution of Example 3. Figure 9.21 revised. New Exercises 40–41 added. Exercise 66 entirely rewritten.
- Section 9.8. Ordering of Exercises was revised. New Exercises 47 and 52 added.
- Section 9.9. Discussion of Taylor series between Examples 4 and 5 rewritten.
- Section 9.10. Exercise 9 replaced.
- Practice Exercises. New Exercises 45–46 added.
- Additional and Advanced Exercises. New Exercises 30–31 added.

Chapter 11

- Section 11.2. New subsection on Vectors in n Dimensions added, with corresponding new Figure 11.19, and new Exercises 60–65.
- Section 11.3. New subsection on The Dot Product of Two n -Dimensional Vectors added, with new Example 9, and new Exercises 53–56.
- Section 11.6. Discussion of cylinders revised.

Chapter 12

Section 12.5. New Exercises 1–2 and 5–6 added.

Chapter 13

- Section 13.2. Added a Composition Rule to Theorem 1 and expanded Example 1.
- Section 13.3. Rewrote the concept of differentiability for functions of several variables to improve clarity.
- Expanded Example 8.
- Section 13.4. Added new Exercises 62–63 on the chain rule with multiple variables.
- Section 13.5. Added a new subsection on gradients for Functions of More Than Three Variables.
- A new Example 7 illustrates a gradient of a 3-variable function.
- New Exercises 45–52 involve gradients of functions with several variables.
- Section 13.7. Added a definition of the Hessian matrix.
- Clarified Example 6.
- Section 13.8. Clarified the use of Lagrange Multipliers throughout, with a more explicit discussion of how to use them for finding maxima and minima.

Chapter 14

- Section 14.2. Added discussion of the properties of limits of iterated double integrals.
- Rewrote Exercises 1–8. Added new Exercises 19–26.
- Section 14.5. Added discussion of the properties of limits of iterated triple integrals. Revised and expanded Example 2.
- Section 14.7. Revised Figure 14.55 to clarify the shape of a spherical wedge involved in triple integration.

Chapter 16

- Section 16.2. Added Figure 16.9.
- Section 16.4. Added a new application of the logistic function showing its connection to Machine Learning and Neural Networks. Added New Exercises 21–22 on the Logistic Equation.

Appendices

Rewrote Appendix A.9 to replace the prime notation with the subscript notation.

New Online Appendix B

- B.1 Determinants
- B.2 Extreme Values and Saddle Points for Functions of More than Two Variables
- B.3 The Method of Gradient Descent

This new appendix covers many topics relevant to students interested in Machine Learning and Neural Networks.

New Online Chapter 18—Complex Functions

This new online chapter gives an introduction to complex functions. Section 1 is an introduction to complex numbers and their operations. It replaces Appendix A.9. Section 2 covers limits and continuity for complex functions. Section 3 introduces complex derivatives and Section 4 the Cauchy-Riemann Equations. Section 5 develops the theory of complex series. Section 6 studies the standard functions such as $\sin z$ and $\log z$, and Section 7 ends the chapter by introducing the theory of conformal maps.

New Online Chapter 19—Fourier Series and Wavelets

This new online chapter introduces Fourier series, and then treats wavelets as a more advanced topic.

It has sections on

- 19.1 Periodic Functions
- 19.2 Summing Sines and Cosines
- 19.3 Vectors and Approximation in Three and More Dimensions
- 19.4 Approximation of Functions
- 19.5 Advanced Topic: The Haar System and Wavelets

Continuing Features

Rigor The level of rigor is consistent with that of earlier editions. We continue to distinguish between formal and informal discussions and to point out their differences. Starting with a more intuitive, less formal approach helps students understand a new or difficult concept so they can then appreciate its full mathematical precision and outcomes. We pay attention to defining ideas carefully and to proving theorems appropriate for calculus students, while mentioning deeper or subtler issues they would study in a more advanced course. Our organization and distinctions between informal and formal discussions give the instructor a degree of flexibility in the amount and depth of coverage of the various topics. For example, while we do not prove the Intermediate Value Theorem or the Extreme Value Theorem for continuous functions on a closed finite interval, we do state these theorems precisely, illustrate their meanings in numerous examples, and use them to prove other important results. Furthermore, for those instructors who desire greater depth of coverage, in Appendix A.7 we discuss the reliance of these theorems on the completeness of the real numbers.

Writing Exercises Writing exercises placed throughout the text ask students to explore and explain a variety of calculus concepts and applications. In addition, the end of each chapter contains a list of questions for students to review and summarize what they have learned. Many of these exercises make good writing assignments.

End-of-Chapter Reviews and Projects In addition to problems appearing after each section, each chapter culminates with review questions, practice exercises covering the entire chapter, and a series of Additional and Advanced Exercises with more challenging or synthesizing problems. Most chapters also include descriptions of several **Technology Application Projects** that can be worked by individual students or groups of students over a longer period of time. These projects require the use of *Mathematica* or *Maple*, along with pre-made files that are available for download within MyLab Math.

Writing and Applications This text continues to be easy to read, conversational, and mathematically rich. Each new topic is motivated by clear, easy-to-understand examples and is then reinforced by its application to real-world problems of immediate interest to students. A hallmark of this book has been the application of calculus to science and engineering. These applied problems have been updated, improved, and extended continually over the last several editions.

Technology In a course using the text, technology can be incorporated according to the taste of the instructor. Each section contains exercises requiring the use of technology; these are marked with a **T** if suitable for calculator or computer use, or they are labeled **Computer Explorations** if a computer algebra system (CAS, such as *Maple* or *Mathematica*) is required.

MyLab Math Resources for Success

MyLab™ Math is available to accompany Pearson's market-leading text options, including *Thomas' Calculus: Early Transcendentals, 15th Edition in SI Units* (access code required).

MyLab is the teaching and learning platform that empowers you to reach every student. MyLab Math combines trusted author content—including full eText and assessment with immediate feedback—with digital tools and a flexible platform to personalize the learning experience and improve results for each student.

MyLab Math supports all learners, regardless of their ability and background, to provide an equal opportunity for success. Accessible resources support learners for a more equitable experience no matter their abilities. And options to personalize learning and address individual gaps help to provide each learner with the specific resources they need to achieve success.

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Study Slides—PowerPoint slides featuring key ideas and examples are available for students within the Video & Resource Library. These slides are compatible with screen readers.

Address Under-Preparedness—Each student learns at a different pace. Personalized learning pinpoints the precise areas where each student needs practice, giving all students the support they need—when and where they need it—to be successful.

New! Integrated Review can be used for just-in-time prerequisite review.

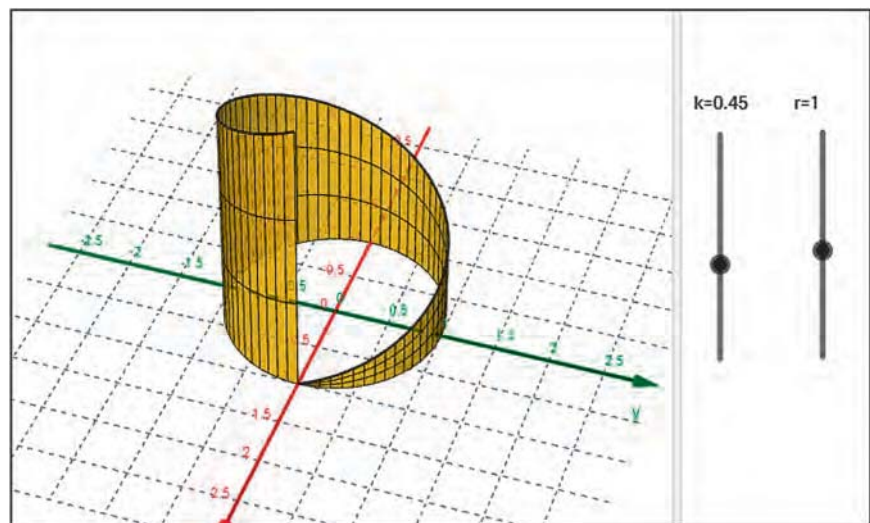
- Integrated Review at the chapter level provides a Skills Check assessment to pinpoint which prerequisite topics, if any, students need to review.
- Students who require additional review proceed to a personalized homework assignment to remediate.
- Integrated Review videos provide additional instruction.

Instructors that prefer to review at the section level can assign the Enhanced Assignments instead.

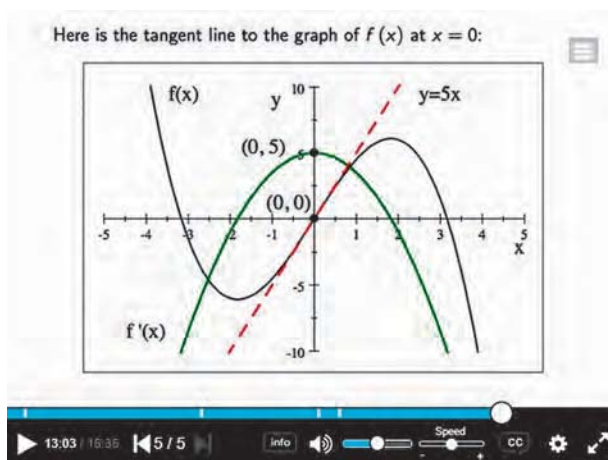
Personalized Homework—With Personalized Homework, students take a quiz or test and receive a subsequent homework assignment that is personalized based on their performance. This way, students can focus on just the topics they have not yet mastered.

Motivate Your Students—Students are motivated to succeed when they’re engaged in the learning experience and understand the relevance and power of math.

▼ **Interactive Figures** bring mathematical concepts to life, helping students see the concepts through directed explorations and purposeful manipulation. Many of the instructional videos that accompany the text include Interactive Figures to teach key concepts. These figures are assignable in MyLab Math and encourage active learning, critical thinking, and conceptual understanding. The figures were created by Marc Renault (Shippensburg University), Steve Phelps (University of Cincinnati), Kevin Hopkins (Southwest Baptist University), and Tim Brzezinski (Southington High School, CT).



▼ **Instructional videos**—Hundreds of videos are available as learning aids within exercises and for self-study under the Video and Resource Library.



Other student resources include:

- **Student's Solutions Manual** The Student's Solutions Manual provides detailed worked-out solutions to the odd-numbered exercises in *Thomas' Calculus: Early Transcendentals*, 15th Edition in SI Units. Available in MyLab Math.
- **Companion Website**

The companion Website, located at www.pearsonglobaleditions.com, offers an online appendix, three online chapters, historical biographies, and more.

Instructor Resources

Your course is unique. So, whether you'd like to build your own assignments, teach multiple sections, or set prerequisites, MyLab gives you the flexibility to easily create your course to fit your needs.

Pre-Built Assignments are designed to maximize students' performance. All assignments are *fully editable* to make your course your own.

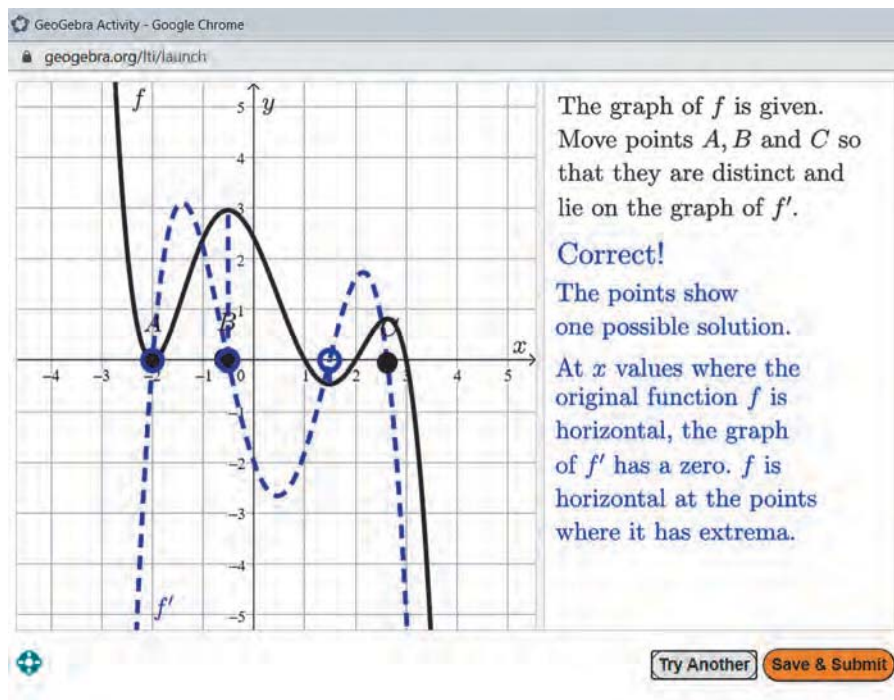
New! Video Assignments featuring short videos with corresponding MyLab Math exercises are available for each section of the textbook. These editable assignments are especially helpful for online or "flipped" classes, where some or all the learning takes place independently.

Enhanced Assignments—These section-level assignments have three unique properties:

1. They help keep skills fresh with *spaced practice* of previously learned concepts.
2. Learning aids are strategically turned off for some exercises to ensure students understand how to work the exercises independently.
3. They contain personalized prerequisite skills exercises for gaps identified in the chapter-level Skills Check Quiz.

MyLab Math Question Library is correlated to the exercises in the text, reflecting the authors' approach and learning style. They regenerate algorithmically to give students unlimited opportunity for practice and mastery. Below are a few exercise types available to assign:

▼ **New! GeoGebra Exercises** are gradable graphing and computational exercises that help students demonstrate their understanding. They enable students to interact directly with the graph in a manner that reflects how students would graph on paper.

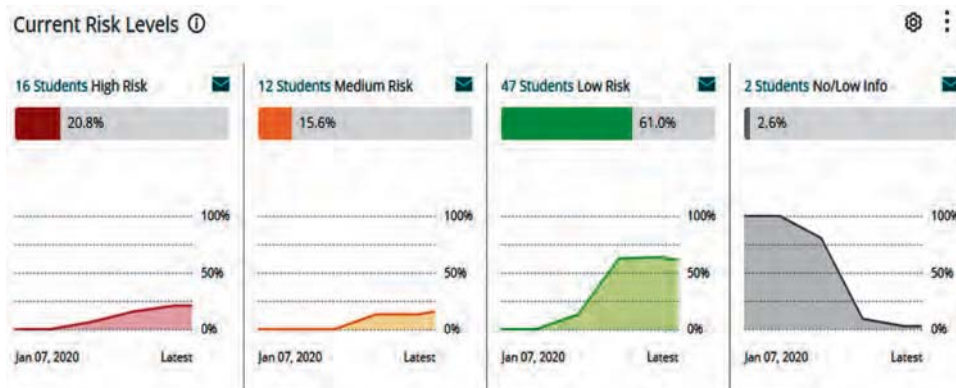


- **Nearly 100 More! Setup & Solve Exercises** require students to first describe how they will set up and approach the problem. This reinforces conceptual understanding of the process applied in approaching the problem, promotes long-term retention of the skill, and mirrors what students will be expected to do on a test. This new exercise type was widely praised by users of the 14th edition, so more were added to the 15th edition.
- **Conceptual Question Library** focuses on deeper, theoretical understanding of the key concepts in calculus. These questions were written by faculty at Cornell University under a National Science Foundation grant and are also assignable through Learning Catalytics.

Learning Catalytics—With Learning Catalytics, you'll hear from every student when it matters most. You pose a variety of questions in class (choosing from pre-loaded questions or your own) that help students recall ideas, apply concepts, and develop critical-thinking skills. Your students respond using their own smartphones, tablets, or laptops.

Performance Analytics enable instructors to see and analyze student performance across multiple courses. Based on their current course progress, individuals' performance is identified above, at, or below expectations through a variety of graphs and visualizations.

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- **PowerPoint Lecture Slides** feature editable lecture slides written and designed specifically for this text, including figures and examples.
- **TestGen** enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text. TestGen is algorithmically based, allowing instructors to create multiple but equivalent versions of the same questions or test with the click of a button. Instructors can also modify test bank questions or add new questions. The software and test bank are available for download at www.pearsonglobaleditions.com.
- **Technology Manuals and Projects**

Maple Manual and Projects

Mathematica Manual and Projects

TI-Graphing Calculator Manual

These manuals and projects cover *Maple* 2021, *Mathematica* 12, and TI-84 Plus and TI-89. Each manual provides detailed guidance for integrating a specific software package or graphing calculator throughout the course, including syntax and commands. The projects include instructions and ready-made application files for *Maple* and *Mathematica*. Available to download within MyLab Math.

Learn more at <https://www.pearsonmylabandmastering.com/global/>

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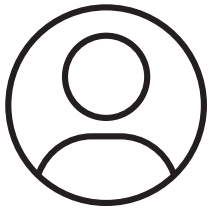
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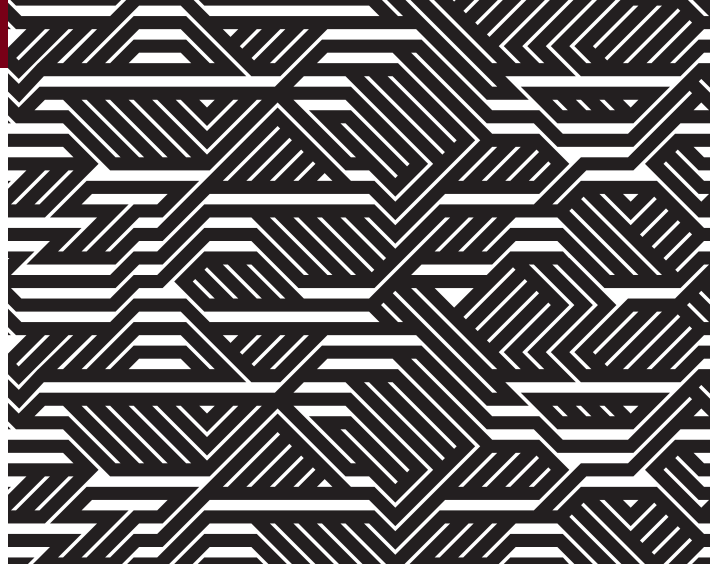
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1

Functions



OVERVIEW In this chapter we review what functions are and how they are visualized as graphs, how they are combined and transformed, and ways they can be classified.

1.1 Functions and Their Graphs

Functions are a tool for describing the real world in mathematical terms. A function can be represented by an equation, a graph, a numerical table, or a verbal description; we will use all four representations throughout this text. This section reviews these ideas.

Functions; Domain and Range

The temperature at which water boils depends on the elevation above sea level. The interest paid on a cash investment depends on the length of time the investment is held. The area of a circle depends on the radius of the circle. The distance an object travels depends on the elapsed time.

In each case, the value of one variable quantity, say y , depends on the value of another variable quantity, which we often call x . We say that “ y is a function of x ” and write this symbolically as

$$y = f(x) \quad (\text{“}y \text{ equals } f \text{ of } x\text{”}).$$

The symbol f represents the function, the letter x is the **independent variable** representing the input value to f , and y is the **dependent variable** or output value of f at x .

DEFINITION A **function** f from a set D to a set Y is a rule that assigns a single value $f(x)$ in Y to each x in D .

A rule that assigns more than one value to an input x , such as the rule that assigns to a positive number both the positive and negative square roots of the number, does not describe a function.

The set D of all possible input values is called the **domain** of the function. The domain of f will sometimes be denoted by $D(f)$. The set of all output values $f(x)$ as x varies throughout D is called the **range** of the function. The range might not include every element in the set Y . The domain and range of a function can be any sets of objects, but often in calculus they are sets of real numbers interpreted as points of a coordinate line. (In Chapters 12–15, we will encounter functions for which the elements of the sets are points in the plane, or in space.)

Often a function is given by a formula that describes how to calculate the output value from the input variable. For instance, the equation $A = \pi r^2$ is a rule that calculates the area A of a circle from its radius r . When we define a function f with a formula $y = f(x)$ and the domain is not stated explicitly or restricted by context, the domain is assumed to be

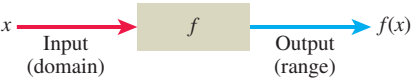


FIGURE 1.1 A diagram showing a function as a kind of machine.

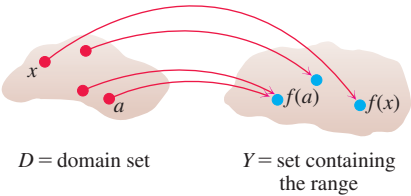


FIGURE 1.2 A function from a set D to a set Y assigns a unique element of Y to each element in D .

the largest set of real x -values for which the formula gives real y -values. This is called the **natural domain** of f . If we want to restrict the domain in some way, we must say so. The domain of $y = x^2$ is the entire set of real numbers. To restrict the domain of the function to, say, positive values of x , we would write “ $y = x^2, x > 0$.”

Changing the domain to which we apply a formula usually changes the range as well. The range of $y = x^2$ is $[0, \infty)$. The range of $y = x^2, x \geq 2$, is the set of all numbers obtained by squaring numbers greater than or equal to 2. In set notation (see Appendix A.1), the range is $\{x^2 \mid x \geq 2\}$ or $\{y \mid y \geq 4\}$ or $[4, \infty)$.

When the range of a function is a set of real numbers, the function is said to be **real-valued**. The domains and ranges of most real-valued functions we consider are intervals or combinations of intervals. Sometimes the range of a function is not easy to find.

A function f is like a machine that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain (Figure 1.1). The function keys on a calculator give an example of a function as a machine. For instance, whenever you enter a nonnegative number x and press the $\sqrt{x}$ key, the calculator gives an output value (the square root of x).

A function can also be pictured as an **arrow diagram** (Figure 1.2). Each arrow associates to an element of the domain D a single element in the set Y . In Figure 1.2, the arrows indicate that $f(a)$ is associated with a , $f(x)$ is associated with x , and so on. Notice that a function can have the same *output value* for two different input elements in the domain (as occurs with $f(a)$ in Figure 1.2), but each input element x is assigned a *single* output value $f(x)$.

EXAMPLE 1 Verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution The formula $y = x^2$ gives a real y -value for any real number x , so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root: $y = (\sqrt{y})^2$.

The formula $y = 1/x$ gives a real y -value for every x except $x = 0$. For consistency in the rules of arithmetic, we cannot divide any number by zero. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$. That is, for $y \neq 0$ the number $x = 1/y$ is the input that is assigned to the output value y .

The formula $y = \sqrt{x}$ gives a real y -value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number’s square root (namely, it is the square root of its own square).

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives nonnegative real y -values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all nonnegative numbers.

The formula $y = \sqrt{1 - x^2}$ gives a real y -value for every x in the closed interval from -1 to 1 . Outside this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is $[0, 1]$. ■

Graphs of Functions

If f is a function with domain D , its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f . In set notation, the graph is

$$\{(x, f(x)) \mid x \in D\}.$$

The graph of the function $f(x) = x + 2$ is the set of points with coordinates (x, y) for which $y = x + 2$. Its graph is the straight line sketched in Figure 1.3.

The graph of a function f is a useful picture of its behavior. If (x, y) is a point on the graph, then $y = f(x)$ is the height of the graph above (or below) the point x . The height may be positive or negative, depending on the sign of $f(x)$ (Figure 1.4).

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

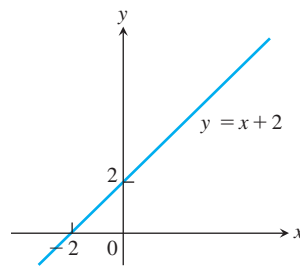


FIGURE 1.3 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

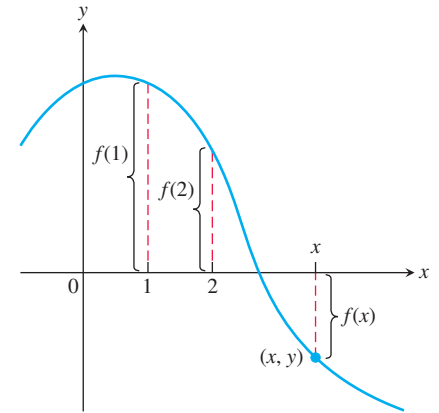


FIGURE 1.4 If (x, y) lies on the graph of f , then the value $y = f(x)$ is the height of the graph above (or below) the point x if $f(x)$ is negative).

EXAMPLE 2 Graph the function $y = x^2$ over the interval $[-2, 2]$.

Solution Make a table of xy -pairs that satisfy the equation $y = x^2$. Plot the points (x, y) whose coordinates appear in the table, and draw a *smooth* curve (labeled with its equation) through the plotted points (see Figure 1.5). ■

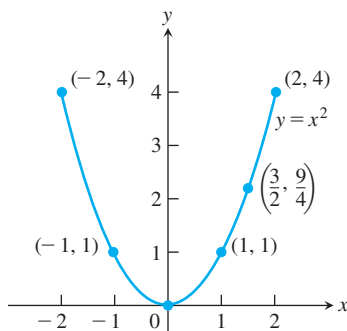
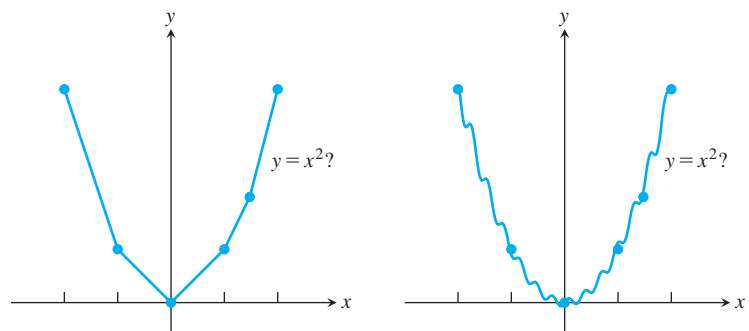


FIGURE 1.5 Graph of the function in Example 2.

How do we know that the graph of $y = x^2$ doesn't look like one of these curves?



To find out, we could plot more points. But how would we then connect *them*? The basic question still remains: How do we know for sure what the graph looks like between the points we plot? Calculus answers this question, as we will see in Chapter 4. Meanwhile, we will have to settle for plotting points and connecting them as best we can.

Time	Pressure
0.00091	−0.080
0.00108	0.200
0.00125	0.480
0.00144	0.693
0.00162	0.816
0.00180	0.844
0.00198	0.771
0.00216	0.603
0.00234	0.368
0.00253	0.099
0.00271	−0.141
0.00289	−0.309
0.00307	−0.348
0.00325	−0.248
0.00344	−0.041
0.00362	0.217
0.00379	0.480
0.00398	0.681
0.00416	0.810
0.00435	0.827
0.00453	0.749
0.00471	0.581
0.00489	0.346
0.00507	0.077
0.00525	−0.164
0.00543	−0.320
0.00562	−0.354
0.00579	−0.248
0.00598	−0.035

Representing a Function Numerically

A function may be represented algebraically by a formula and visually by a graph (Example 2). Another way to represent a function is **numerically**, through a table of values. From an appropriate table of values, a graph of the function can be obtained using the method illustrated in Example 2, possibly with the aid of a computer. The graph consisting of only the points in the table is called a **scatterplot**.

EXAMPLE 3 Musical notes are pressure waves in the air. The data associated with Figure 1.6 give recorded pressure displacement versus time in seconds of a musical note produced by a tuning fork. The table provides a representation of the pressure function (in micropascals) over time. If we first make a scatterplot and then draw a smooth curve that approximates the data points (t, p) from the table, we obtain the graph shown in the figure.

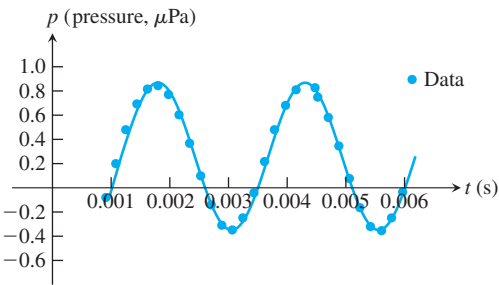


FIGURE 1.6 A smooth curve approximating the plotted points gives a graph of the pressure function represented by the accompanying tabled data (Example 3).

The Vertical Line Test for a Function

Not every curve in the coordinate plane can be the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so *no vertical line* can intersect the graph of a function at more than one point. If a is in the domain of the function f , then the vertical line $x = a$ will intersect the graph of f at the single point $(a, f(a))$.

A circle cannot be the graph of a function, since some vertical lines intersect the circle twice. The circle graphed in Figure 1.7a, however, contains the graphs of two functions of x , namely the upper semicircle defined by the function $f(x) = \sqrt{1 - x^2}$ and the lower semicircle defined by the function $g(x) = -\sqrt{1 - x^2}$ (Figures 1.7b and 1.7c).

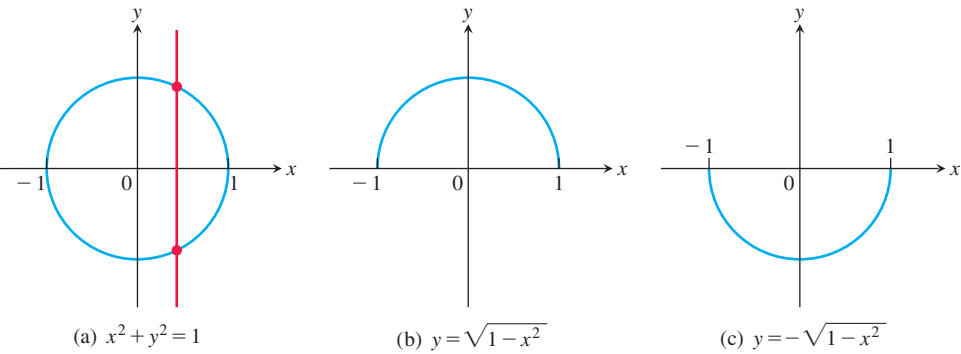


FIGURE 1.7 (a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of the function $f(x) = \sqrt{1 - x^2}$. (c) The lower semicircle is the graph of the function $g(x) = -\sqrt{1 - x^2}$.

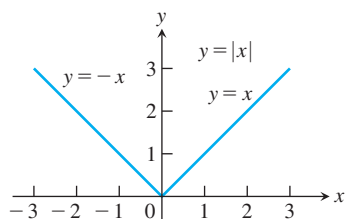


FIGURE 1.8 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

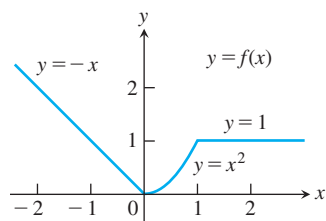


FIGURE 1.9 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 4).

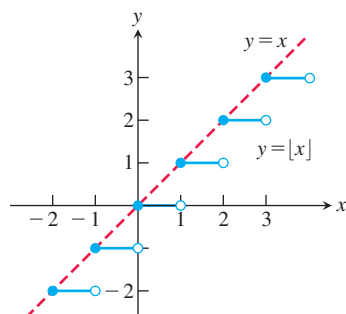


FIGURE 1.10 The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for x (Example 5).

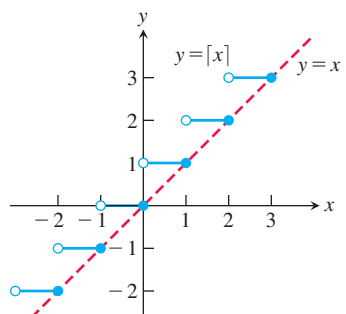


FIGURE 1.11 The graph of the least integer function $y = \lceil x \rceil$ lies on or above the line $y = x$, so it provides an integer ceiling for x (Example 6).

Piecewise-Defined Functions

Sometimes a function is described in pieces by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 & \text{First formula} \\ -x, & x < 0 & \text{Second formula} \end{cases}$$

whose graph is given in Figure 1.8. The right-hand side of the equation means that the function equals x if $x \geq 0$, and equals $-x$ if $x < 0$. Piecewise-defined functions often arise when real-world data are modeled. Here are some other examples.

EXAMPLE 4 The function

$$f(x) = \begin{cases} -x, & x < 0 & \text{First formula} \\ x^2, & 0 \leq x \leq 1 & \text{Second formula} \\ 1, & x > 1 & \text{Third formula} \end{cases}$$

is defined on the entire real line but has values given by different formulas, depending on the position of x . The values of f are given by $y = -x$ when $x < 0$, $y = x^2$ when $0 \leq x \leq 1$, and $y = 1$ when $x > 1$. The function, however, is *just one function* whose domain is the entire set of real numbers (Figure 1.9).

EXAMPLE 5 The function whose value at any number x is the *greatest integer less than or equal to* x is called the **greatest integer function** or the **integer floor function**. It is denoted $\lfloor x \rfloor$. Figure 1.10 shows the graph. Observe that

$$\begin{aligned} \lfloor 2.4 \rfloor &= 2, & \lfloor 1.9 \rfloor &= 1, & \lfloor 0 \rfloor &= 0, & \lfloor -1.2 \rfloor &= -2, \\ \lfloor 2 \rfloor &= 2, & \lfloor 0.2 \rfloor &= 0, & \lfloor -0.3 \rfloor &= -1, & \lfloor -2 \rfloor &= -2. \end{aligned}$$

EXAMPLE 6 The function whose value at any number x is the *smallest integer greater than or equal to* x is called the **least integer function** or the **integer ceiling function**. It is denoted $\lceil x \rceil$. Figure 1.11 shows the graph. For positive values of x , this function might represent, for example, the cost of parking x hours in a parking lot that charges \$1 for each hour or part of an hour.

Increasing and Decreasing Functions

If the graph of a function climbs or rises as you move from left to right, we say that the function is *increasing*. If the graph descends or falls as you move from left to right, the function is *decreasing*.

DEFINITIONS Let f be a function defined on an interval I and let x_1 and x_2 be two distinct points in I .

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I .
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I .

It is important to realize that the definitions of increasing and decreasing functions must be satisfied for *every* pair of points x_1 and x_2 in I with $x_1 < x_2$. Because we use the inequality $<$ to compare the function values, instead of $\leq$, it is sometimes said that f is *strictly increasing* or *strictly decreasing* on I . The interval I may be finite (also called *bounded*) or infinite (*unbounded*).

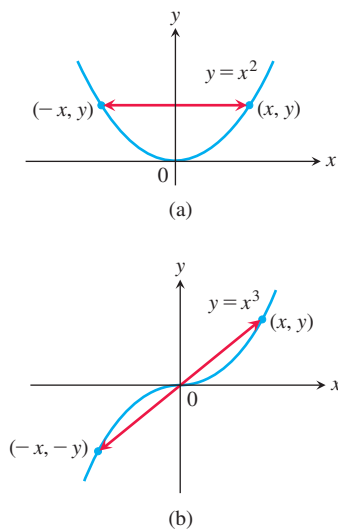


FIGURE 1.12 (a) The graph of $y = x^2$ (an even function) is symmetric about the y -axis. (b) The graph of $y = x^3$ (an odd function) is symmetric about the origin.

EXAMPLE 7 The function graphed in Figure 1.9 is decreasing on $(-\infty, 0)$ and increasing on $(0, 1)$. The function is neither increasing nor decreasing on the interval $(1, \infty)$ because the function is constant on that interval, and hence the strict inequalities in the definition of increasing or decreasing are not satisfied on $(1, \infty)$. ■

Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have special symmetry properties.

DEFINITIONS A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,

odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

The names *even* and *odd* come from powers of x . If y is an even power of x , as in $y = x^2$ or $y = x^4$, it is an even function of x because $(-x)^2 = x^2$ and $(-x)^4 = x^4$. If y is an odd power of x , as in $y = x$ or $y = x^3$, it is an odd function of x because $(-x)^1 = -x$ and $(-x)^3 = -x^3$.

The graph of an even function is **symmetric about the y -axis**. Since $f(-x) = f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph (Figure 1.12a). A reflection across the y -axis leaves the graph unchanged.

The graph of an odd function is **symmetric about the origin**. Since $f(-x) = -f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph (Figure 1.12b). Equivalently, a graph is symmetric about the origin if a rotation of 180° about the origin leaves the graph unchanged.

Notice that each of these definitions requires that both x and $-x$ be in the domain of f .

EXAMPLE 8 Here are several functions illustrating the definitions.

$f(x) = x^2$ Even function: $(-x)^2 = x^2$ for all x ; symmetry about y -axis. So $f(-3) = 9 = f(3)$. Changing the sign of x does not change the value of an even function.

$f(x) = x^2 + 1$ Even function: $(-x)^2 + 1 = x^2 + 1$ for all x ; symmetry about y -axis (Figure 1.13a).

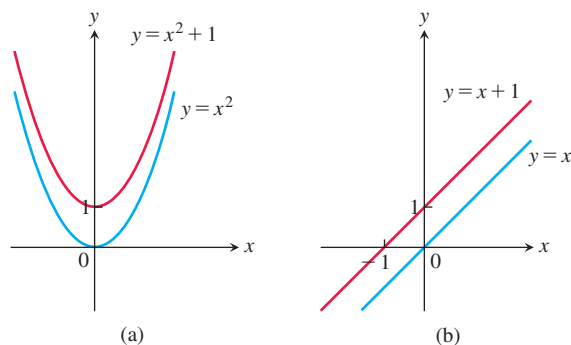


FIGURE 1.13 (a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2 + 1$ is still even and its graph is still symmetric about the y -axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd, since the symmetry about the origin is lost. The function $y = x + 1$ is also not even (Example 8).

$f(x) = x$	Odd function: $(-x) = -x$ for all x ; symmetry about the origin. So $f(-3) = -3$ while $f(3) = 3$. Changing the sign of x changes the sign of the value of an odd function.
$f(x) = x + 1$	Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal. Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.13b). ■

Common Functions

A variety of important types of functions are frequently encountered in calculus.

Linear Functions A function of the form $f(x) = mx + b$, where m and b are fixed constants, is called a **linear function**. Figure 1.14a shows an array of lines $f(x) = mx$. Each of these has $b = 0$, so these lines pass through the origin. The function $f(x) = x$, where $m = 1$ and $b = 0$, is called the **identity function**. Constant functions result when the slope is $m = 0$ (Figure 1.14b).

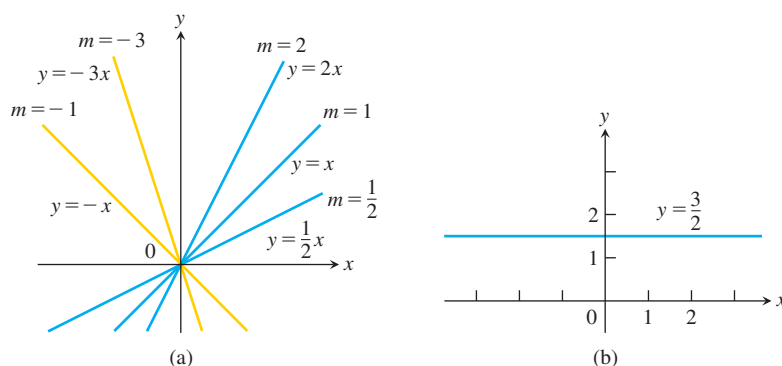


FIGURE 1.14 (a) Lines through the origin with slope m . (b) A constant function with slope $m = 0$.

DEFINITION Two variables y and x are **proportional** (to one another) if one is always a constant multiple of the other—that is, if $y = kx$ for some nonzero constant k .

If the variable y is proportional to the reciprocal $1/x$, then sometimes it is said that y is **inversely proportional** to x (because $1/x$ is the multiplicative inverse of x).

Power Functions A function $f(x) = x^a$, where a is a constant, is called a **power function**. There are several important cases to consider.

(a) $f(x) = x^a$ with $a = n$, a positive integer.

The graphs of $f(x) = x^n$, for $n = 1, 2, 3, 4, 5$, are displayed in Figure 1.15. These functions are defined for all real values of x . Notice that as the power n gets larger, the curves tend to flatten toward the x -axis on the interval $(-1, 1)$ and to rise more steeply for $|x| > 1$. Each curve passes through the point $(1, 1)$ and through the origin. The graphs of functions with even powers are symmetric about the y -axis; those with odd powers are symmetric about the origin. The even-powered functions are decreasing on the interval $(-\infty, 0]$ and increasing on $[0, \infty)$; the odd-powered functions are increasing over the entire real line $(-\infty, \infty)$.

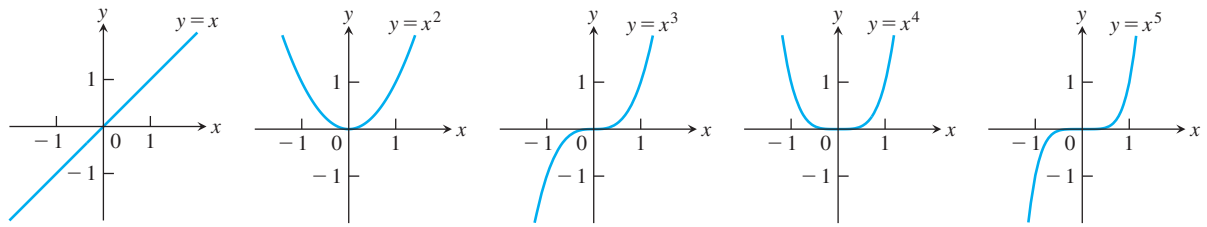


FIGURE 1.15 Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$, defined for $-\infty < x < \infty$.

(b) $f(x) = x^a$ with $a = -1$ or $a = -2$.

The graphs of the functions $f(x) = x^{-1} = 1/x$ and $f(x) = x^{-2} = 1/x^2$ are shown in Figure 1.16. Both functions are defined for all $x \neq 0$ (you can never divide by zero). The graph of $y = 1/x$ is the hyperbola $xy = 1$, which approaches the coordinate axes far from the origin. The graph of $y = 1/x^2$ also approaches the coordinate axes. The graph of the function $f(x) = 1/x$ is symmetric about the origin; this function is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$. The graph of the function $f(x) = 1/x^2$ is symmetric about the y -axis; this function is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

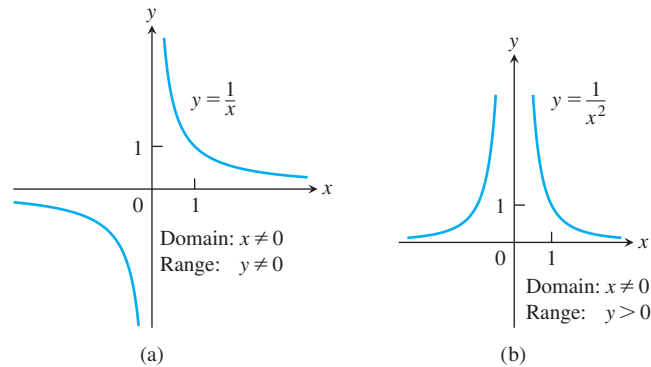


FIGURE 1.16 Graphs of the power functions $f(x) = x^a$.
(a) $a = -1$. (b) $a = -2$.

(c) $f(x) = x^a$ with $a = \frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{2}$, or $\frac{2}{3}$.

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $f(x) = x^{1/3} = \sqrt[3]{x}$ are the **square root** and **cube root** functions, respectively. The domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real x . Their graphs are displayed in Figure 1.17, along with the graphs of $y = x^{3/2}$ and $y = x^{2/3}$. (Recall that $x^{3/2} = (x^{1/2})^3$ and $x^{2/3} = (x^{1/3})^2$.)

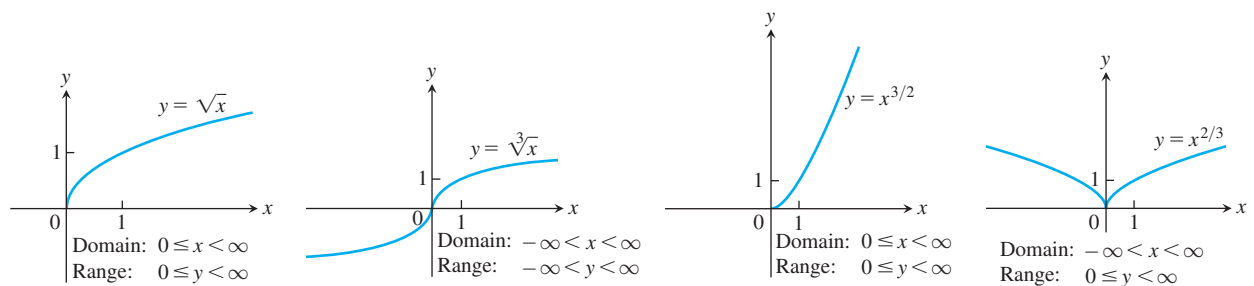


FIGURE 1.17 Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{2}$, and $\frac{2}{3}$.

Polynomials A function p is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \dots, a_n$ are real constants (called the **coefficients** of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then n is called the **degree** of the polynomial. Linear functions with $m \neq 0$ are polynomials of degree 1. Polynomials of degree 2, usually written as $p(x) = ax^2 + bx + c$, are called **quadratic functions**. Likewise, **cubic functions** are polynomials $p(x) = ax^3 + bx^2 + cx + d$ of degree 3. Figure 1.18 shows the graphs of three polynomials. Techniques to graph polynomials are studied in Chapter 4.

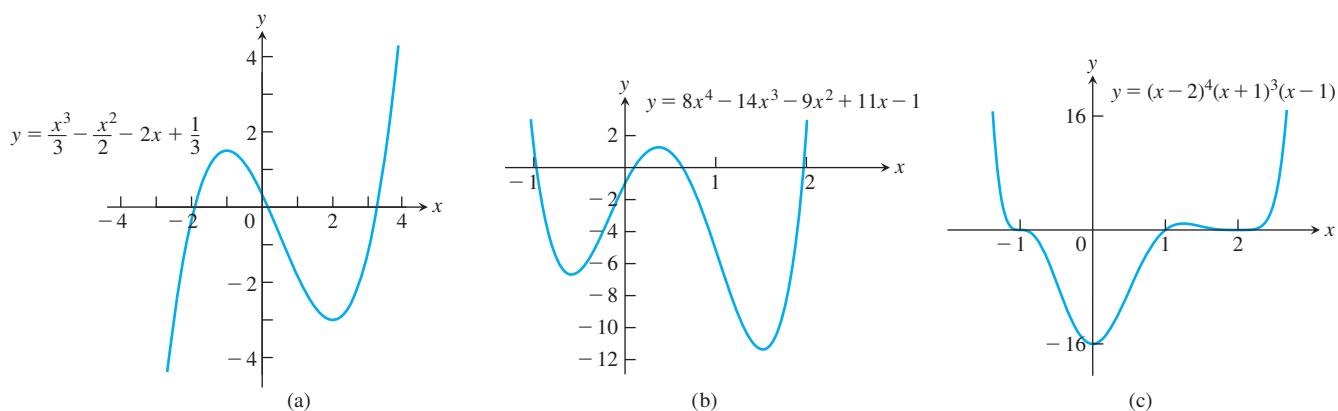


FIGURE 1.18 Graphs of three polynomial functions.

Rational Functions A **rational function** is a quotient or ratio $f(x) = p(x)/q(x)$, where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$. The graphs of three rational functions are shown in Figure 1.19.

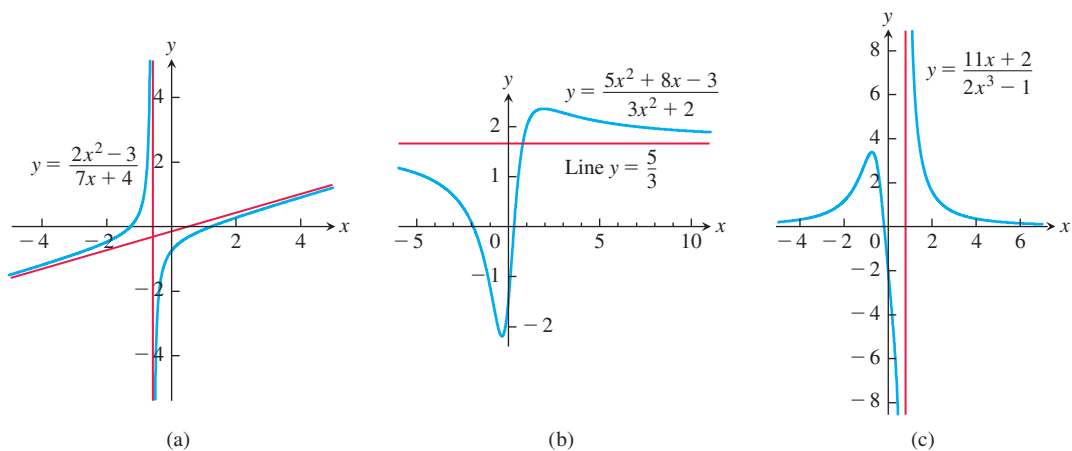


FIGURE 1.19 Graphs of three rational functions. The straight red lines approached by the graphs are called *asymptotes* and are not part of the graphs. We discuss asymptotes in Section 2.5.

Algebraic Functions Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots) lies within the class of **algebraic functions**. All rational functions are algebraic, but also included are more

complicated functions (such as those satisfying an equation like $y^3 - 9xy + x^3 = 0$, studied in Section 3.7). Figure 1.20 displays the graphs of three algebraic functions.

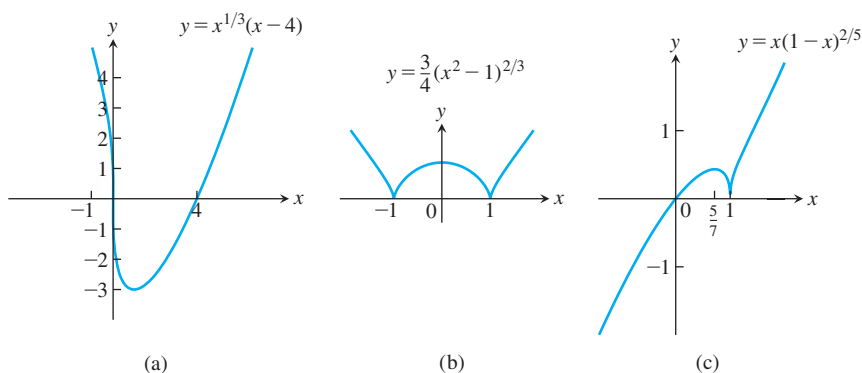


FIGURE 1.20 Graphs of three algebraic functions.

Trigonometric Functions The six basic trigonometric functions are reviewed in Section 1.3. The graphs of the sine and cosine functions are shown in Figure 1.21.

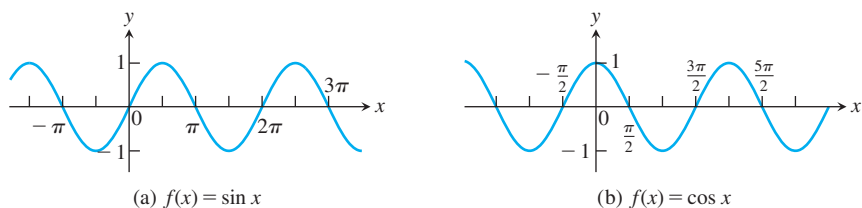


FIGURE 1.21 Graphs of the sine and cosine functions.

Exponential Functions A function of the form $f(x) = a^x$, where $a > 0$ and $a \neq 1$, is called an **exponential function** (with base a). All exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$, so an exponential function never assumes the value 0. We discuss exponential functions in Section 1.4. The graphs of some exponential functions are shown in Figure 1.22.

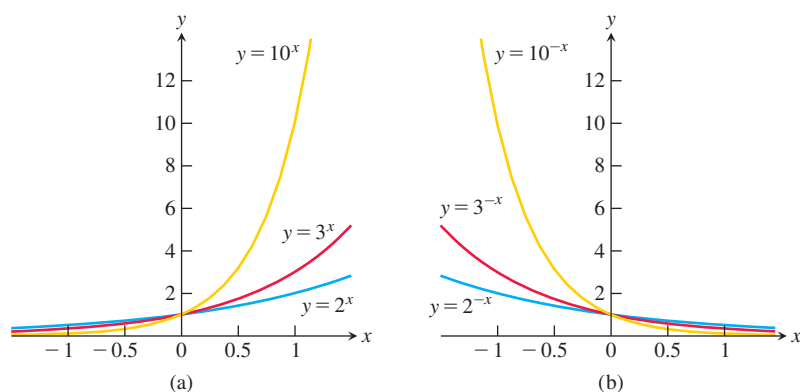


FIGURE 1.22 Graphs of exponential functions.

Logarithmic Functions These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant. They are the *inverse functions* of the exponential functions, and we discuss these functions in Section 1.5. Figure 1.23 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.

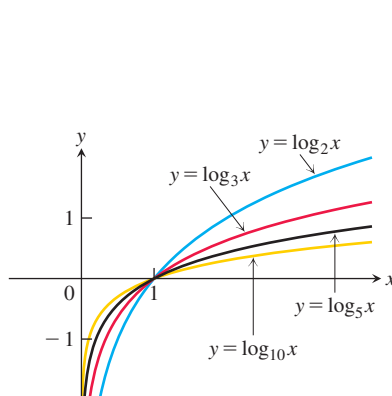


FIGURE 1.23 Graphs of four logarithmic functions.

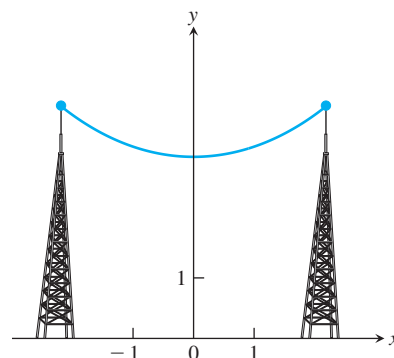


FIGURE 1.24 Graph of a catenary or hanging cable. (The Latin word *catena* means “chain.”)

Transcendental Functions These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well. The **catenary** is one example of a transcendental function. Its graph has the shape of a cable, like a telephone line or electric cable, strung from one support to another and hanging freely under its own weight (Figure 1.24). The function defining the graph is discussed in Section 7.3.

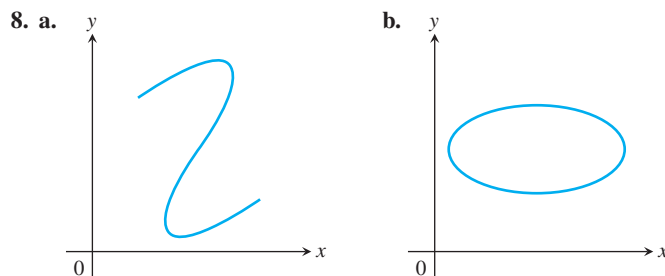
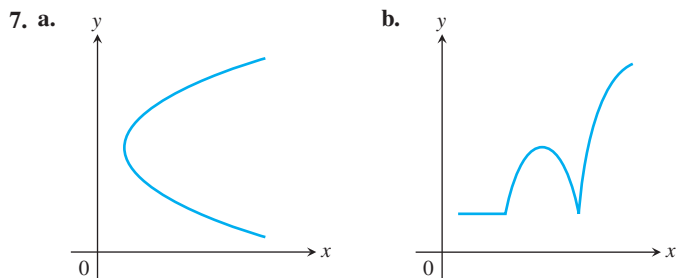
EXERCISES 1.1

Functions

In Exercises 1–6, find the domain and range of each function.

1. $f(x) = 1 + x^2$
2. $f(x) = 1 - \sqrt{x}$
3. $F(x) = \sqrt{5x + 10}$
4. $g(x) = \sqrt{x^2 - 3x}$
5. $f(t) = \frac{4}{3 - t}$
6. $G(t) = \frac{2}{t^2 - 16}$

In Exercises 7 and 8, which of the graphs are graphs of functions of x , and which are not? Give reasons for your answers.



Finding Formulas for Functions

9. Express the area and perimeter of an equilateral triangle as a function of the triangle's side length x .
10. Express the side length of a square as a function of the length d of the square's diagonal. Then express the area as a function of the diagonal length.
11. Express the edge length of a cube as a function of the cube's diagonal length d . Then express the surface area and volume of the cube as a function of the diagonal length.

12. A point P in the first quadrant lies on the graph of the function $f(x) = \sqrt{x}$. Express the coordinates of P as functions of the slope of the line joining P to the origin.
13. Consider the point (x, y) lying on the graph of the line $2x + 4y = 5$. Let L be the distance from the point (x, y) to the origin $(0, 0)$. Write L as a function of x .
14. Consider the point (x, y) lying on the graph of $y = \sqrt{x - 3}$. Let L be the distance between the points (x, y) and $(4, 0)$. Write L as a function of y .

Functions and Graphs

Find the natural domain and graph the functions in Exercises 15–20.

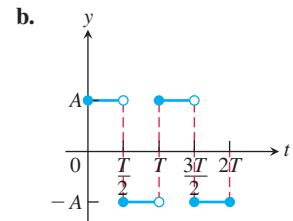
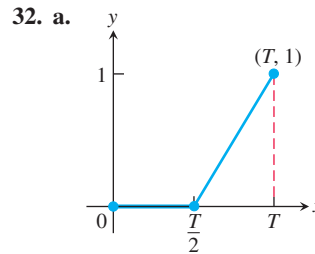
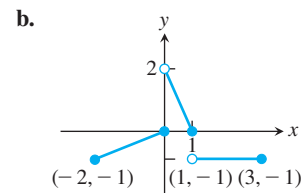
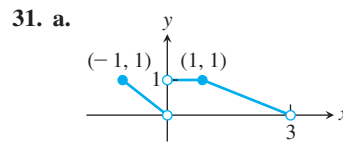
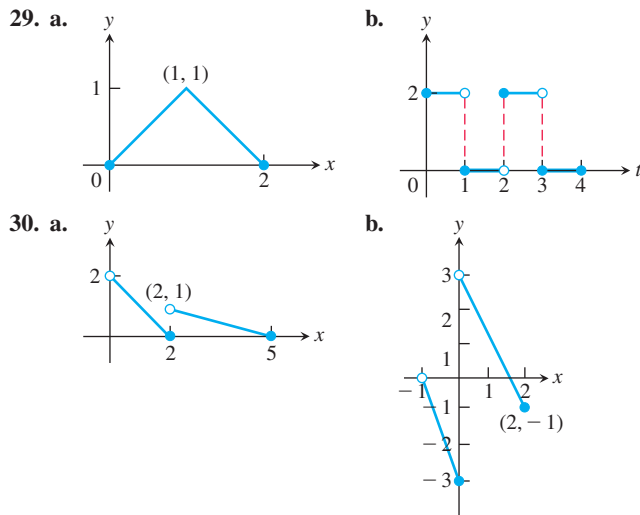
15. $f(x) = 5 - 2x$ 16. $f(x) = 1 - 2x - x^2$
 17. $g(x) = \sqrt{|x|}$ 18. $g(x) = \sqrt{-x}$
 19. $F(t) = t/|t|$ 20. $G(t) = 1/|t|$
21. Find the domain of $y = \frac{x+3}{4-\sqrt{x^2-9}}$.
22. Find the range of $y = 2 + \sqrt{9+x^2}$.
23. Graph the following equations and explain why they are not graphs of functions of x .
 a. $|y| = x$ b. $y^2 = x^2$
24. Graph the following equations and explain why they are not graphs of functions of x .
 a. $|x| + |y| = 1$ b. $|x + y| = 1$

Piecewise-Defined Functions

Graph the functions in Exercises 25–28.

25. $f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \end{cases}$
 26. $g(x) = \begin{cases} 1 - x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \end{cases}$
 27. $F(x) = \begin{cases} 4 - x^2, & x \leq 1 \\ x^2 + 2x, & x > 1 \end{cases}$
 28. $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \leq x \end{cases}$

Find a formula for each function graphed in Exercises 29–32.



The Greatest and Least Integer Functions

33. For what values of x is
 a. $\lfloor x \rfloor = 0$? b. $\lceil x \rceil = 0$?
34. What real numbers x satisfy the equation $\lfloor x \rfloor = \lceil x \rceil$?
35. Does $\lceil -x \rceil = -\lfloor x \rfloor$ for all real x ? Give reasons for your answer.
36. Graph the function

$$f(x) = \begin{cases} \lfloor x \rfloor, & x \geq 0 \\ \lceil x \rceil, & x < 0. \end{cases}$$

Why is $f(x)$ called the *integer part* of x ?

Increasing and Decreasing Functions

Graph the functions in Exercises 37–46. What symmetries, if any, do the graphs have? Specify the intervals over which the function is increasing and the intervals where it is decreasing.

37. $y = -x^3$ 38. $y = -\frac{1}{x^2}$
 39. $y = -\frac{1}{x}$ 40. $y = \frac{1}{|x|}$
 41. $y = \sqrt{|x|}$ 42. $y = \sqrt{-x}$
 43. $y = x^3/8$ 44. $y = -4\sqrt{x}$
 45. $y = -x^{3/2}$ 46. $y = (-x)^{2/3}$

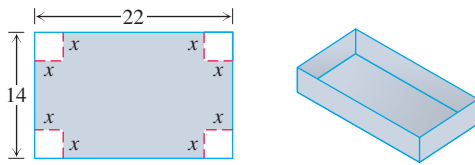
Even and Odd Functions

In Exercises 47–62, say whether the function is even, odd, or neither. Give reasons for your answer.

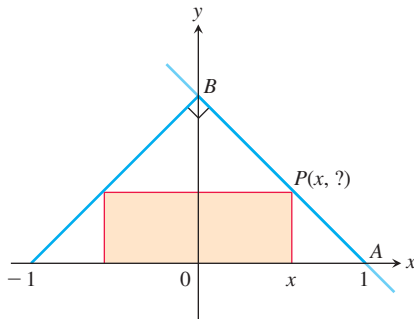
47. $f(x) = 3$ 48. $f(x) = x^{-5}$
 49. $f(x) = x^2 + 1$ 50. $f(x) = x^2 + x$
 51. $g(x) = x^3 + x$ 52. $g(x) = x^4 + 3x^2 - 1$
 53. $g(x) = \frac{1}{x^2 - 1}$ 54. $g(x) = \frac{x}{x^2 - 1}$
 55. $h(t) = \frac{1}{t - 1}$ 56. $h(t) = |t^3|$
 57. $h(t) = 2t + 1$ 58. $h(t) = 2|t| + 1$
 59. $\sin 2x$ 60. $\sin x^2$
 61. $\cos 3x$ 62. $1 + \cos x$

Theory and Examples

63. The variable s is proportional to t , and $s = 25$ when $t = 75$. Determine t when $s = 60$.
64. **Kinetic energy** The kinetic energy K of a mass is proportional to the square of its velocity v . If $K = 12,960$ joules when $v = 18$ m/s, what is K when $v = 10$ m/s?
65. The variables r and s are inversely proportional, and $r = 6$ when $s = 4$. Determine s when $r = 10$.
66. **Boyle's law** Boyle's law says that the volume V of a gas at constant temperature increases whenever the pressure P decreases, so that V and P are inversely proportional. If $P = 14.7$ N/cm² when $V = 1000$ cm³, then what is V when $P = 23.4$ N/cm²?
67. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 14 cm by 22 cm by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x .

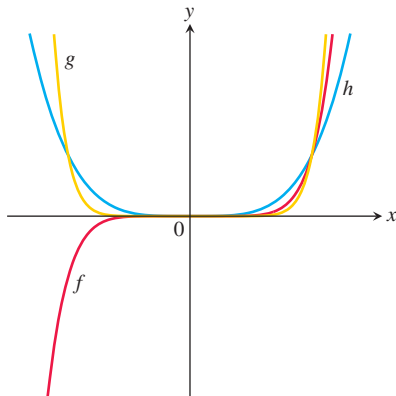


68. The accompanying figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.
- Express the y -coordinate of P in terms of x . (You might start by writing an equation for the line AB .)
 - Express the area of the rectangle in terms of x .



In Exercises 69 and 70, match each equation with its graph. Do not use a graphing device, and give reasons for your answer.

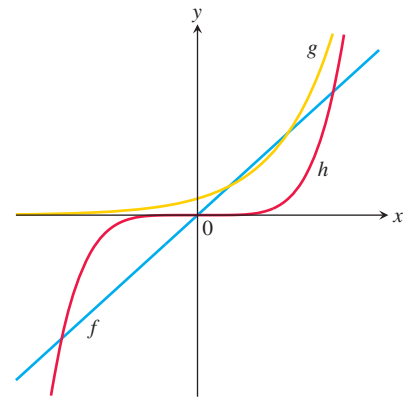
69. a. $y = x^4$ b. $y = x^7$ c. $y = x^{10}$



70. a. $y = 5x$

b. $y = 5^x$

c. $y = x^5$



- T 71. a. Graph the functions $f(x) = x/2$ and $g(x) = 1 + (4/x)$ together to identify the values of x for which

$$\frac{x}{2} > 1 + \frac{4}{x}.$$

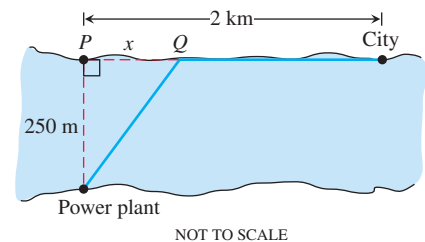
- b. Confirm your findings in part (a) algebraically.

- T 72. a. Graph the functions $f(x) = 3/(x - 1)$ and $g(x) = 2/(x + 1)$ together to identify the values of x for which

$$\frac{3}{x - 1} < \frac{2}{x + 1}.$$

- b. Confirm your findings in part (a) algebraically.

73. For a curve to be *symmetric about the x -axis*, the point (x, y) must lie on the curve if and only if the point $(x, -y)$ lies on the curve. Explain why a curve that is symmetric about the x -axis is not the graph of a function, unless the function is $y = 0$.
74. Three hundred books sell for \$40 each, resulting in a revenue of $(300)(\$40) = \$12,000$. For each \$5 increase in the price, 25 fewer books are sold. Write the revenue R as a function of the number x of \$5 increases.
75. A pen in the shape of an isosceles right triangle with legs of length x m and hypotenuse of length h m is to be built. If fencing costs \$5/m for the legs and \$10/m for the hypotenuse, write the total cost C of construction as a function of h .
76. **Industrial costs** A power plant sits next to a river where the river is 250 m wide. To lay a new cable from the plant to a location in the city 2 km downstream on the opposite side costs \$180 per meter across the river and \$100 per meter along the land.



- Suppose that the cable goes from the plant to a point Q on the opposite side that is x m from the point P directly opposite the plant. Write a function $C(x)$ that gives the cost of laying the cable in terms of the distance x .
- Generate a table of values to determine if the least expensive location for point Q is less than 300 m or greater than 300 m from point P .

1.2 Combining Functions; Shifting and Scaling Graphs

In this section we look at the main ways functions are combined or transformed to form new functions.

Sums, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions. If f and g are functions, then for every x that belongs to the domains of both f and g (that is, for $x \in D(f) \cap D(g)$), we define functions $f + g$, $f - g$, and fg by the formulas

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) \\ (f - g)(x) &= f(x) - g(x) \\ (fg)(x) &= f(x)g(x).\end{aligned}$$

Notice that the $+$ sign on the left-hand side of the first equation represents the operation of addition of *functions*, whereas the $+$ on the right-hand side of the equation means addition of the real numbers $f(x)$ and $g(x)$.

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function f/g by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad (\text{where } g(x) \neq 0).$$

Functions can also be multiplied by constants: If c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

EXAMPLE 1 The functions defined by the formulas

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \sqrt{1 - x}$$

have domains $D(f) = [0, \infty)$ and $D(g) = (-\infty, 1]$. The points common to these domains are the points in

$$[0, \infty) \cap (-\infty, 1] = [0, 1].$$

The following table summarizes the formulas and domains for the various algebraic combinations of the two functions. We also write $f \cdot g$ for the product function fg .

Function	Formula	Domain
$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1 - x}$	$[0, 1] = D(f) \cap D(g)$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1 - x}$	$[0, 1]$
$g - f$	$(g - f)(x) = \sqrt{1 - x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1 - x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1 - x}}$	$[0, 1)(x = 1 \text{ excluded})$
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1 - x}{x}}$	$(0, 1](x = 0 \text{ excluded})$

The graph of the function $f + g$ is obtained from the graphs of f and g by adding the corresponding y -coordinates $f(x)$ and $g(x)$ at each point $x \in D(f) \cap D(g)$, as in Figure 1.25. The graphs of $f + g$ and $f \cdot g$ from Example 1 are shown in Figure 1.26.

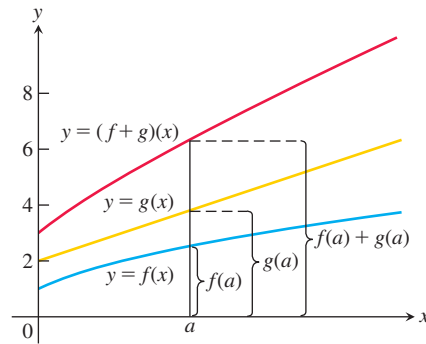


FIGURE 1.25 Graphical addition of two functions.

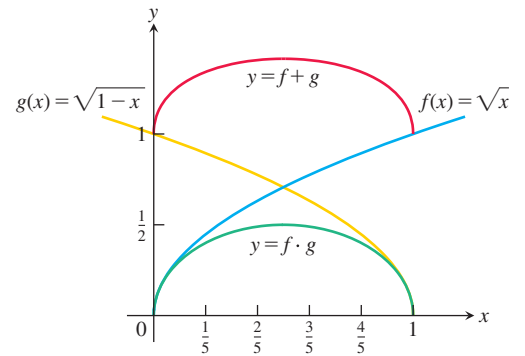


FIGURE 1.26 The domain of the function $f + g$ is the intersection of the domains of f and g , the interval $[0, 1]$ on the x -axis where these domains overlap. This interval is also the domain of the function $f \cdot g$ (Example 1).

Composing Functions

Composition is another method for combining functions. In this operation the output from one function becomes the input to a second function.

DEFINITION If f and g are functions, the function $f \circ g$ (“ f composed with g ”) is defined by

$$(f \circ g)(x) = f(g(x))$$

and called the **composition** of f and g . The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f .

To find $(f \circ g)(x)$, *first* find $g(x)$ and *second* find $f(g(x))$. Figure 1.27 pictures $f \circ g$ as a machine diagram, and Figure 1.28 shows the composition as an arrow diagram.

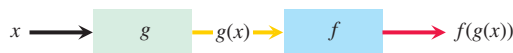


FIGURE 1.27 The composition $f \circ g$ uses the output $g(x)$ of the first function g as the input for the second function f .

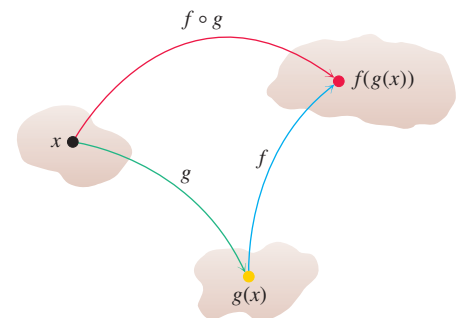


FIGURE 1.28 Arrow diagram for $f \circ g$. If x lies in the domain of g and $g(x)$ lies in the domain of f , then the functions f and g can be composed to form $(f \circ g)(x)$.

To evaluate the composition $g \circ f$ (when defined), we find $f(x)$ first and then find $g(f(x))$. The domain of $g \circ f$ is the set of numbers x in the domain of f such that $f(x)$ lies in the domain of g .

The functions $f \circ g$ and $g \circ f$ are usually quite different.

EXAMPLE 2 If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

- (a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$ (c) $(f \circ f)(x)$ (d) $(g \circ g)(x)$.

Solution

Composition	Domain
(a) $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
(b) $(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c) $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d) $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x+1) + 1 = x+2$	$(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but $g(x)$ belongs to the domain of f only if $x + 1 \geq 0$, that is to say, when $x \geq -1$. ■

Notice that if $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = (\sqrt{x})^2 = x$. However, the domain of $f \circ g$ is $[0, \infty)$, not $(-\infty, \infty)$, since $\sqrt{x}$ requires $x \geq 0$.

Shifting a Graph of a Function

A common way to obtain a new function from an existing one is by adding a constant to each output of the existing function, or to its input variable. The graph of the new function is the graph of the original function shifted vertically or horizontally, as follows.

Shift Formulas

Vertical Shifts

$y = f(x) + k$ Shifts the graph of f up k units if $k > 0$
Shifts it down $|k|$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$ Shifts the graph of f left h units if $h > 0$
Shifts it right $|h|$ units if $h < 0$

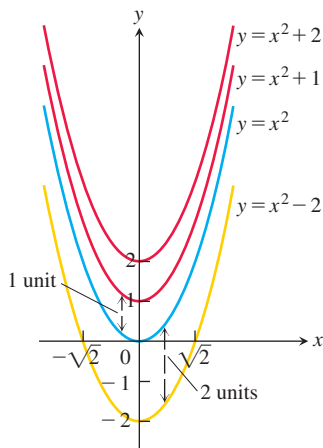


FIGURE 1.29 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f (Examples 3a and b).

EXAMPLE 3

- (a) Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Figure 1.29).
 (b) Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Figure 1.29).
 (c) Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left, while adding -2 shifts the graph 2 units to the right (Figure 1.30).
 (d) Adding -2 to x in $y = |x|$, and then adding -1 to the result, gives $y = |x - 2| - 1$ and shifts the graph 2 units to the right and 1 unit down (Figure 1.31). ■

Scaling and Reflecting a Graph of a Function

To scale the graph of a function $y = f(x)$ is to stretch or compress it, vertically or horizontally. This is accomplished by multiplying the function f , or the independent variable x , by an appropriate constant c . Reflections across the coordinate axes are special cases where $c = -1$.

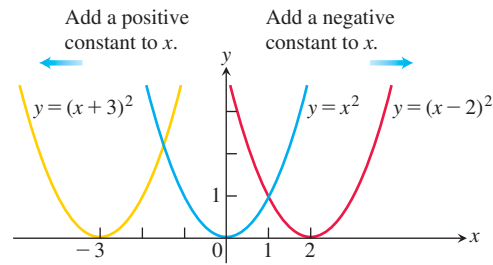


FIGURE 1.30 To shift the graph of $y = x^2$ to the left, we add a positive constant to x (Example 3c). To shift the graph to the right, we add a negative constant to x .

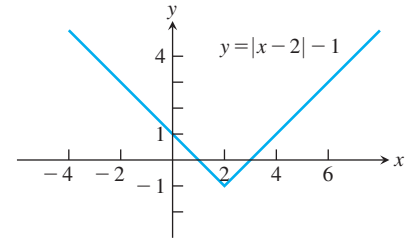


FIGURE 1.31 The graph of $y = |x|$ shifted 2 units to the right and 1 unit down (Example 3d).

Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$, the graph is scaled:

$y = cf(x)$	Stretches the graph of f vertically by a factor of c .
$y = \frac{1}{c}f(x)$	Compresses the graph of f vertically by a factor of c .
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c .
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c .

For $c = -1$, the graph is reflected:

$y = -f(x)$	Reflects the graph of f across the x -axis.
$y = f(-x)$	Reflects the graph of f across the y -axis.

EXAMPLE 4 Here we scale and reflect the graph of $y = \sqrt{x} + 1$.

- (a) **Vertical:** Multiplying the right-hand side of $y = \sqrt{x} + 1$ by 3 to get $y = 3(\sqrt{x} + 1)$ stretches the graph vertically by a factor of 3, whereas multiplying by $1/3$ compresses the graph vertically by a factor of 3 (Figure 1.32).
- (b) **Horizontal:** The graph of $y = \sqrt{3x} + 1$ is a horizontal compression of the graph of $y = \sqrt{x} + 1$ by a factor of 3, and $y = \sqrt{x/3} + 1$ is a horizontal stretching by a factor of 3 (Figure 1.33).
- (c) **Reflection:** The graph of $y = -(\sqrt{x} + 1)$ is a reflection of $y = \sqrt{x} + 1$ across the x -axis, and $y = \sqrt{-x} + 1$ is a reflection across the y -axis (Figure 1.34). ■

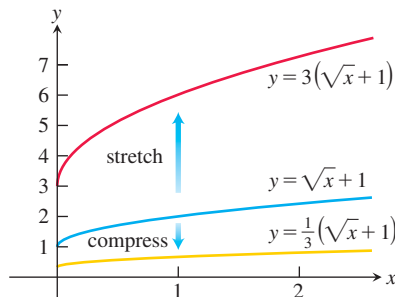


FIGURE 1.32 Vertically stretching and compressing the graph of $y = \sqrt{x} + 1$ by a factor of 3 (Example 4a).

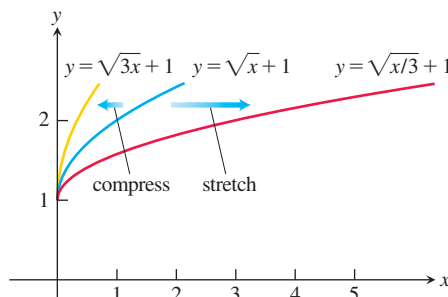


FIGURE 1.33 Horizontally stretching and compressing the graph of $y = \sqrt{x} + 1$ by a factor of 3 (Example 4b).

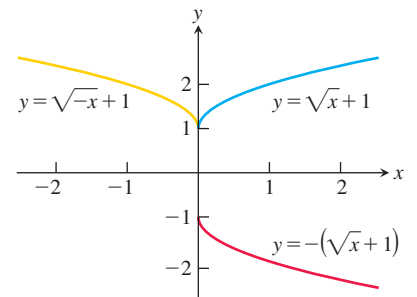


FIGURE 1.34 Reflections of the graph of $y = \sqrt{x} + 1$ across the coordinate axes (Example 4c).

EXAMPLE 5 Given the function $f(x) = x^4 - 4x^3 + 10$ (Figure 1.35a), find formulas for the graphs resulting from

- (a) horizontal compression by a factor of 2 followed by reflection across the y -axis (Figure 1.35b).
- (b) vertical compression by a factor of 2 followed by reflection across the x -axis (Figure 1.35c).

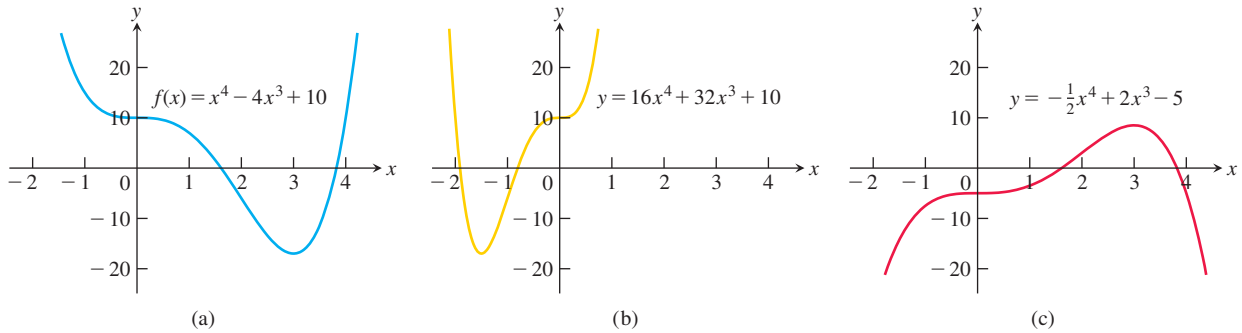


FIGURE 1.35 (a) The original graph of f . (b) The horizontal compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the y -axis. (c) The vertical compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the x -axis (Example 5).

Solution

- (a) We multiply x by 2 to get the horizontal compression, and by -1 to give reflection across the y -axis. The formula is obtained by substituting $-2x$ for x in the right-hand side of the equation for f :

$$\begin{aligned} y &= f(-2x) = (-2x)^4 - 4(-2x)^3 + 10 \\ &= 16x^4 + 32x^3 + 10. \end{aligned}$$

- (b) The formula is

$$y = -\frac{1}{2}f(x) = -\frac{1}{2}x^4 + 2x^3 - 5. \quad \blacksquare$$

EXERCISES 1.2

Algebraic Combinations

In Exercises 1 and 2, find the domains of f , g , $f + g$, and $f \cdot g$.

1. $f(x) = x$, $g(x) = \sqrt{x-1}$
2. $f(x) = \sqrt{x+1}$, $g(x) = \sqrt{x-1}$

In Exercises 3 and 4, find the domains of f , g , f/g , and g/f .

3. $f(x) = 2$, $g(x) = x^2 + 1$
4. $f(x) = 1$, $g(x) = 1 + \sqrt{x}$

Compositions of Functions

5. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following.

a. $f(g(0))$	b. $g(f(0))$
c. $f(g(x))$	d. $g(f(x))$
e. $f(f(-5))$	f. $g(g(2))$
g. $f(f(x))$	h. $g(g(x))$

6. If $f(x) = x - 1$ and $g(x) = 1/(x + 1)$, find the following.

a. $f(g(1/2))$	b. $g(f(1/2))$
c. $f(g(x))$	d. $g(f(x))$
e. $f(f(2))$	f. $g(g(2))$
g. $f(f(x))$	h. $g(g(x))$

In Exercises 7–10, write a formula for $f \circ g \circ h$.

7. $f(x) = x + 1$, $g(x) = 3x$, $h(x) = 4 - x$

8. $f(x) = 3x + 4$, $g(x) = 2x - 1$, $h(x) = x^2$

9. $f(x) = \sqrt{x+1}$, $g(x) = \frac{1}{x+4}$, $h(x) = \frac{1}{x}$

10. $f(x) = \frac{x+2}{3-x}$, $g(x) = \frac{x^2}{x^2+1}$, $h(x) = \sqrt{2-x}$

Let $f(x) = x - 3$, $g(x) = \sqrt{x}$, $h(x) = x^3$, and $j(x) = 2x$. Express each of the functions in Exercises 11 and 12 as a composition involving one or more of f , g , h , and j .

11. a. $y = \sqrt{x} - 3$

b. $y = 2\sqrt{x}$

c. $y = x^{1/4}$

d. $y = 4x$

e. $y = \sqrt{(x-3)^3}$

f. $y = (2x-6)^3$

12. a. $y = 2x - 3$

b. $y = x^{3/2}$

c. $y = x^9$

d. $y = x - 6$

e. $y = 2\sqrt{x-3}$

f. $y = \sqrt{x^3 - 3}$

13. Copy and complete the following table.

$g(x)$	$f(x)$	$(f \circ g)(x)$
a. $x - 7$	$\sqrt{x}$	?
b. $x + 2$	$3x$	?
c. ?	$\sqrt{x-5}$	$\sqrt{x^2-5}$
d. $\frac{x}{x-1}$	$\frac{x}{x-1}$	?
e. ?	$1 + \frac{1}{x}$	x
f. $\frac{1}{x}$	?	x

14. Copy and complete the following table.

$g(x)$	$f(x)$	$(f \circ g)(x)$
a. $\frac{1}{x-1}$	$ x $	?
b. ?	$\frac{x-1}{x}$	$\frac{x}{x+1}$
c. ?	$\sqrt{x}$	$ x $
d. $\sqrt{x}$	?	$ x $

15. Evaluate each expression using the given table of values:

x	-2	-1	0	1	2
$f(x)$	1	0	-2	1	2
$g(x)$	2	1	0	-1	0

a. $f(g(-1))$

b. $g(f(0))$

c. $f(f(-1))$

d. $g(g(2))$

e. $g(f(-2))$

f. $f(g(1))$

16. Evaluate each expression using the functions

$$f(x) = 2 - x, \quad g(x) = \begin{cases} -x, & -2 \leq x < 0 \\ x - 1, & 0 \leq x \leq 2. \end{cases}$$

a. $f(g(0))$

b. $g(f(3))$

c. $g(g(-1))$

d. $f(f(2))$

e. $g(f(0))$

f. $f(g(1/2))$

In Exercises 17 and 18, (a) write formulas for $f \circ g$ and $g \circ f$ and (b) find the domain of each.

17. $f(x) = \sqrt{x+1}$, $g(x) = \frac{1}{x}$

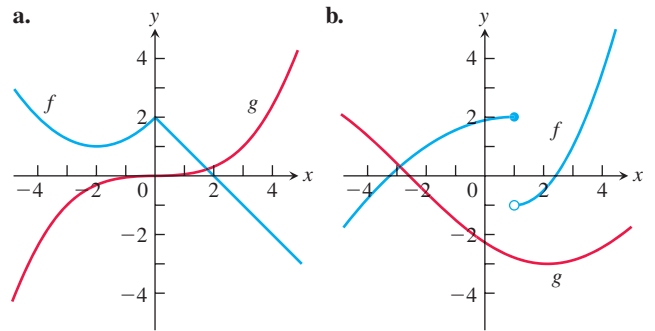
18. $f(x) = x^2$, $g(x) = 1 - \sqrt{x}$

19. Let $f(x) = \frac{x}{x-2}$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x$.

20. Let $f(x) = 2x^3 - 4$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x + 2$.

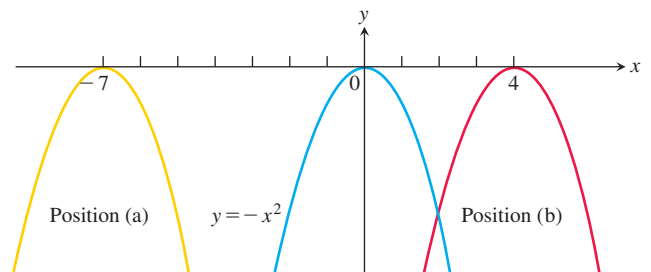
21. A balloon's volume V is given by $V = s^2 + 2s + 3 \text{ cm}^3$, where s is the ambient temperature in $^\circ\text{C}$. The ambient temperature s at time t minutes is given by $s = 2t - 3^\circ\text{C}$. Write the balloon's volume V as a function of time t .

22. Use the graphs of f and g to sketch the graph of $y = f(g(x))$.

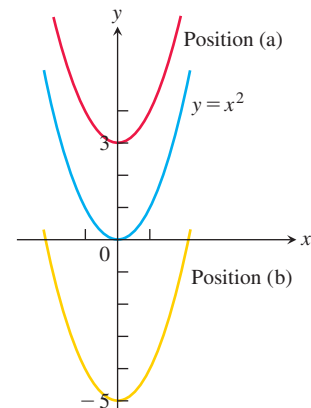


Shifting Graphs

23. The accompanying figure shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.

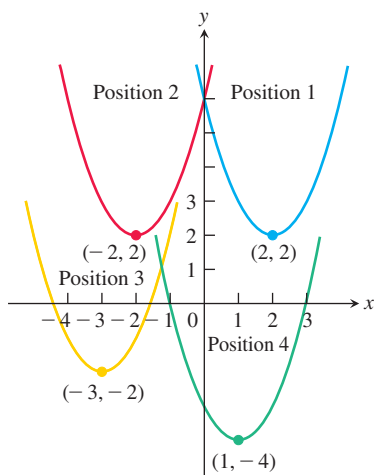


24. The accompanying figure shows the graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs.

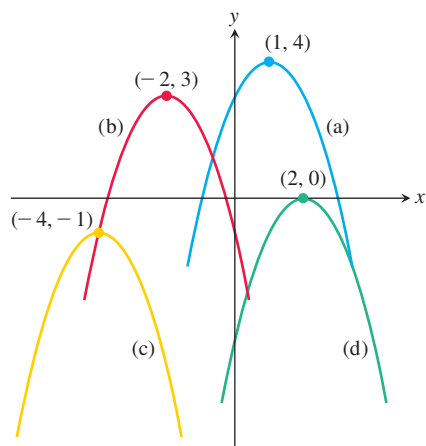


25. Match the equations listed in parts (a)–(d) to the graphs in the accompanying figure.

- a. $y = (x - 1)^2 - 4$ b. $y = (x - 2)^2 + 2$
 c. $y = (x + 2)^2 + 2$ d. $y = (x + 3)^2 - 2$



26. The accompanying figure shows the graph of $y = -x^2$ shifted to four new positions. Write an equation for each new graph.



Exercises 27–36 tell how many units and in what directions the graphs of the given equations are to be shifted. Give an equation for the shifted graph. Then sketch the original and shifted graphs together, labeling each graph with its equation.

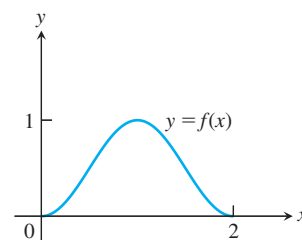
27. $x^2 + y^2 = 49$ Down 3, left 2
 28. $x^2 + y^2 = 25$ Up 3, left 4
 29. $y = x^3$ Left 1, down 1
 30. $y = x^{2/3}$ Right 1, down 1
 31. $y = \sqrt{x}$ Left 0.81
 32. $y = -\sqrt{x}$ Right 3
 33. $y = 2x - 7$ Up 7
 34. $y = \frac{1}{2}(x + 1) + 5$ Down 5, right 1
 35. $y = 1/x$ Up 1, right 1
 36. $y = 1/x^2$ Left 2, down 1

Graph the functions in Exercises 37–56.

37. $y = \sqrt{x + 4}$ 38. $y = \sqrt{9 - x}$

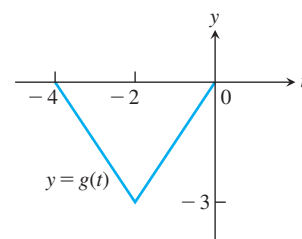
39. $y = |x - 2|$ 40. $y = |1 - x| - 1$
 41. $y = 1 + \sqrt{x - 1}$ 42. $y = 1 - \sqrt{x}$
 43. $y = (x + 1)^{2/3}$ 44. $y = (x - 8)^{2/3}$
 45. $y = 1 - x^{2/3}$ 46. $y + 4 = x^{2/3}$
 47. $y = \sqrt[3]{x - 1} - 1$ 48. $y = (x + 2)^{3/2} + 1$
 49. $y = \frac{1}{x - 2}$ 50. $y = \frac{1}{x} - 2$
 51. $y = \frac{1}{x} + 2$ 52. $y = \frac{1}{x + 2}$
 53. $y = \frac{1}{(x - 1)^2}$ 54. $y = \frac{1}{x^2} - 1$
 55. $y = \frac{1}{x^2} + 1$ 56. $y = \frac{1}{(x + 1)^2}$

57. The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions, and sketch their graphs.



- a. $f(x) + 2$ b. $f(x) - 1$
 c. $2f(x)$ d. $-f(x)$
 e. $f(x + 2)$ f. $f(x - 1)$
 g. $f(-x)$ h. $-f(x + 1) + 1$

58. The accompanying figure shows the graph of a function $g(t)$ with domain $[-4, 0]$ and range $[-3, 0]$. Find the domains and ranges of the following functions, and sketch their graphs.



- a. $g(-t)$ b. $-g(t)$
 c. $g(t) + 3$ d. $1 - g(t)$
 e. $g(-t + 2)$ f. $g(t - 2)$
 g. $g(1 - t)$ h. $-g(t - 4)$

Vertical and Horizontal Scaling

Exercises 59–68 tell in what direction and by what factor the graphs of the given functions are to be stretched or compressed. Give an equation for the stretched or compressed graph.

59. $y = x^2 - 1$, stretched vertically by a factor of 3
 60. $y = x^2 - 1$, compressed horizontally by a factor of 2
 61. $y = 1 + \frac{1}{x^2}$, compressed vertically by a factor of 2

62. $y = 1 + \frac{1}{x^2}$, stretched horizontally by a factor of 3
63. $y = \sqrt{x+1}$, compressed horizontally by a factor of 4
64. $y = \sqrt{x+1}$, stretched vertically by a factor of 3
65. $y = \sqrt{4-x^2}$, stretched horizontally by a factor of 2
66. $y = \sqrt{4-x^2}$, compressed vertically by a factor of 3
67. $y = 1 - x^3$, compressed horizontally by a factor of 3
68. $y = 1 - x^3$, stretched horizontally by a factor of 2

Graphing

In Exercises 69–76, graph each function not by plotting points, but by starting with the graph of one of the standard functions presented in Figures 1.14–1.17 and applying an appropriate transformation.

69. $y = -\sqrt{2x+1}$ 70. $y = \sqrt{1-\frac{x}{2}}$
71. $y = (x-1)^3 + 2$ 72. $y = (1-x)^3 + 2$
73. $y = \frac{1}{2x} - 1$ 74. $y = \frac{2}{x^2} + 1$

75. $y = -\sqrt[3]{x}$ 76. $y = (-2x)^{2/3}$
77. Graph the function $y = |x^2 - 1|$.
78. Graph the function $y = \sqrt{|x|}$.

Combining Functions

79. Assume that f is an even function, g is an odd function, and both f and g are defined on the entire real line $(-\infty, \infty)$. Which of the following (where defined) are even? odd?
- a. fg b. f/g c. g/f
- d. $f^2 = ff$ e. $g^2 = gg$ f. $f \circ g$
- g. $g \circ f$ h. $f \circ f$ i. $g \circ g$
80. Can a function be both even and odd? Give reasons for your answer.
- T 81. (Continuation of Example 1.) Graph the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$ together with their (a) sum, (b) product, (c) two differences, (d) two quotients.
- T 82. Let $f(x) = x - 7$ and $g(x) = x^2$. Graph f and g together with $f \circ g$ and $g \circ f$.

1.3 Trigonometric Functions

This section reviews radian measure and the basic trigonometric functions.

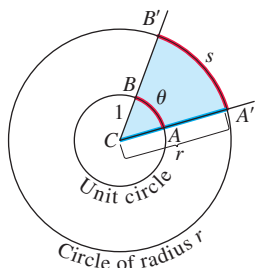


FIGURE 1.36 The radian measure of the central angle $A'CB'$ is the number $\theta = s/r$. For a unit circle of radius $r = 1$, θ is the length of arc AB that central angle ACB cuts from the unit circle.

Angles

Angles are measured in degrees or radians. The number of **radians** in the central angle $A'CB'$ within a circle of radius r is defined as the number of “radius units” contained in the arc s subtended by that central angle. If we denote this central angle by θ when measured in radians, this means that $\theta = s/r$ (Figure 1.36), or

$$s = r\theta \quad (\theta \text{ in radians}). \quad (1)$$

If the circle is a unit circle having radius $r = 1$, then from Figure 1.36 and Equation (1), we see that the central angle θ measured in radians is just the length of the arc that the angle cuts from the unit circle. Since one complete revolution of the unit circle is 360° or 2π radians, we have

$$\pi \text{ radians} = 180^\circ \quad (2)$$

and

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57.3) \text{ degrees} \quad \text{or} \quad 1 \text{ degree} = \frac{\pi}{180} (\approx 0.017) \text{ radians}.$$

Table 1.1 shows the equivalence between degree and radian measures for some basic angles.

TABLE 1.1 Angles measured in degrees and radians

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

An angle in the xy -plane is said to be in **standard position** if its vertex lies at the origin and its initial ray lies along the positive x -axis (Figure 1.37). Angles measured counterclockwise from the positive x -axis are assigned positive measures; angles measured clockwise are assigned negative measures.

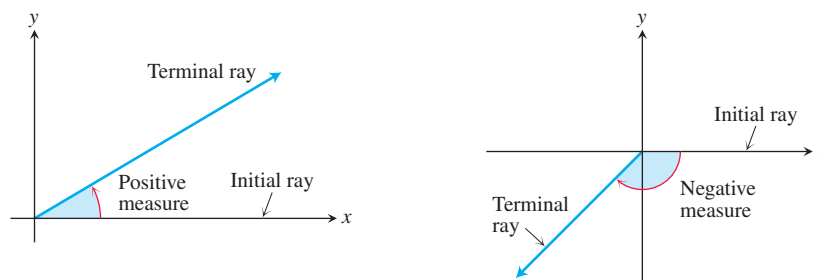


FIGURE 1.37 Angles in standard position in the xy -plane.

Angles describing counterclockwise rotations can go arbitrarily far beyond 2π radians or 360° . Similarly, angles describing clockwise rotations can have negative measures of all sizes (Figure 1.38).

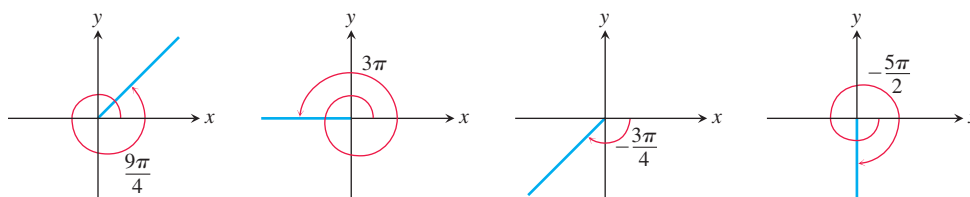


FIGURE 1.38 Nonzero radian measures can be positive or negative and can go beyond 2π .

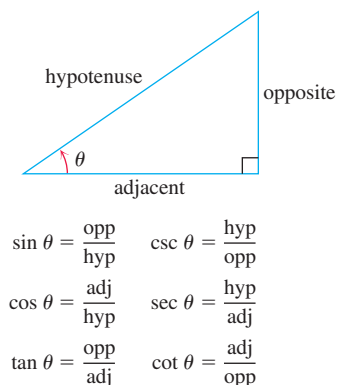


FIGURE 1.39 Trigonometric ratios of an acute angle.

Angle Convention: Use Radians From now on in this text, it is assumed that all angles are measured in radians unless degrees or some other unit is stated explicitly. When we talk about the angle $\pi/3$, we mean $\pi/3$ radians (which is 60°), not $\pi/3$ degrees. Using radians simplifies many of the operations and computations in calculus.

The Six Basic Trigonometric Functions

The trigonometric functions of an acute angle are given in terms of the sides of a right triangle (Figure 1.39). We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius r . We then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle (Figure 1.40).

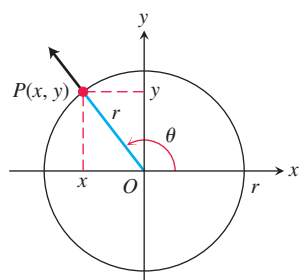


FIGURE 1.40 The trigonometric functions of a general angle θ are defined in terms of x , y , and r .

sine:	$\sin \theta = \frac{y}{r}$	cosecant:	$\csc \theta = \frac{r}{y}$
cosine:	$\cos \theta = \frac{x}{r}$	secant:	$\sec \theta = \frac{r}{x}$
tangent:	$\tan \theta = \frac{y}{x}$	cotangent:	$\cot \theta = \frac{x}{y}$

These extended definitions agree with the right-triangle definitions when the angle is acute. Notice also that whenever the quotients are defined,

$$\begin{aligned}\tan \theta &= \frac{\sin \theta}{\cos \theta} & \cot \theta &= \frac{1}{\tan \theta} \\ \sec \theta &= \frac{1}{\cos \theta} & \csc \theta &= \frac{1}{\sin \theta}\end{aligned}$$

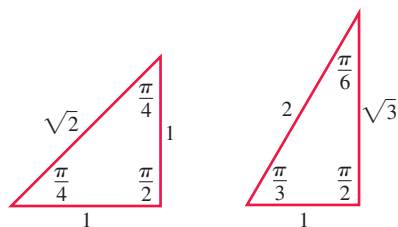


FIGURE 1.41 Radian angles and side lengths of two common triangles.

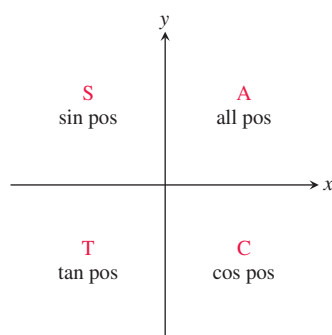


FIGURE 1.42 The ASTC rule, remembered by the statement “All Students Take Calculus,” tells which trigonometric functions are positive in each quadrant.

As you can see, $\tan \theta$ and $\sec \theta$ are not defined if $x = \cos \theta = 0$. This means they are not defined if θ is $\pm\pi/2, \pm3\pi/2, \dots$. Similarly, $\cot \theta$ and $\csc \theta$ are not defined for values of θ for which $y = 0$, namely $\theta = 0, \pm\pi, \pm2\pi, \dots$.

The exact values of these trigonometric ratios for some angles can be read from the triangles in Figure 1.41. For instance,

$$\begin{array}{lll} \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} & \sin \frac{\pi}{6} = \frac{1}{2} & \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \\ \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} & \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} & \cos \frac{\pi}{3} = \frac{1}{2} \\ \tan \frac{\pi}{4} = 1 & \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} & \tan \frac{\pi}{3} = \sqrt{3} \end{array}$$

The ASTC rule (Figure 1.42) is useful for remembering when the basic trigonometric functions are positive or negative. For instance, from the triangle in Figure 1.43, we see that

$$\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \quad \cos \frac{2\pi}{3} = -\frac{1}{2}, \quad \tan \frac{2\pi}{3} = -\sqrt{3}.$$

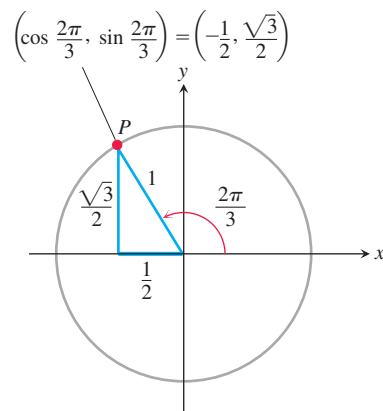


FIGURE 1.43 The triangle for calculating the sine and cosine of $2\pi/3$ radians. The side lengths come from the geometry of right triangles.

Using a similar method we obtain the values of $\sin \theta$, $\cos \theta$, and $\tan \theta$ shown in Table 1.2.

TABLE 1.2 Values of $\sin \theta$, $\cos \theta$, and $\tan \theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin \theta$	0	$-\frac{\sqrt{2}}{2}$	−1	$-\frac{\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos \theta$	−1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	−1	0	1
$\tan \theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$-\frac{\sqrt{3}}{3}$	0		0

Periodicity and Graphs of the Trigonometric Functions

When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values: $\sin(\theta + 2\pi) = \sin \theta$, $\tan(\theta + 2\pi) = \tan \theta$, and so on. Similarly, $\cos(\theta - 2\pi) = \cos \theta$, $\sin(\theta - 2\pi) = \sin \theta$, and so on. We describe this repeating behavior by saying that the six basic trigonometric functions are *periodic*.

Periods of Trigonometric Functions

Period π : $\tan(x + \pi) = \tan x$
 $\cot(x + \pi) = \cot x$

Period 2π : $\sin(x + 2\pi) = \sin x$
 $\cos(x + 2\pi) = \cos x$
 $\sec(x + 2\pi) = \sec x$
 $\csc(x + 2\pi) = \csc x$

DEFINITION A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x . The smallest such value of p is the **period** of f .

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by x instead of θ . Figure 1.44 shows that the tangent and cotangent functions have period $p = \pi$, and the other four functions have period 2π . Also, the symmetries in these graphs reveal that the cosine and secant functions are even and the other four functions are odd (although this does not prove those results).

Even

$$\cos(-x) = \cos x$$

$$\sec(-x) = \sec x$$

Odd

$$\sin(-x) = -\sin x$$

$$\tan(-x) = -\tan x$$

$$\csc(-x) = -\csc x$$

$$\cot(-x) = -\cot x$$

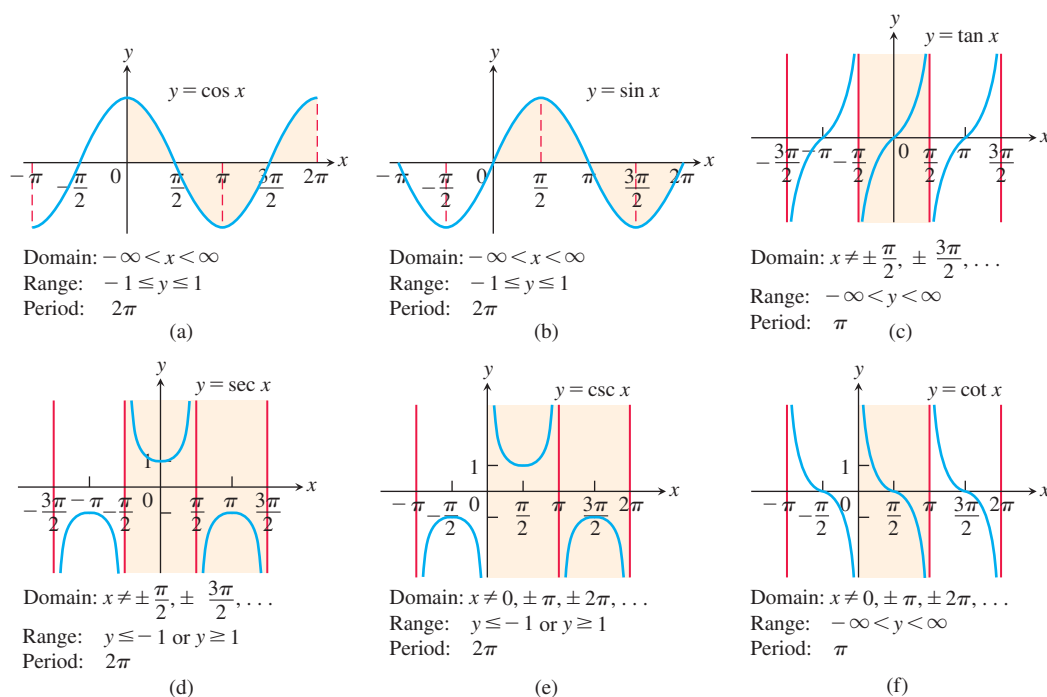


FIGURE 1.44 Graphs of the six basic trigonometric functions using radian measure. The shading for each trigonometric function indicates its periodicity.

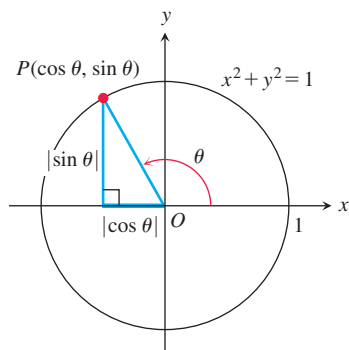


FIGURE 1.45 The reference triangle for a general angle θ .

Trigonometric Identities

The coordinates of any point $P(x, y)$ in the plane can be expressed in terms of the point's distance r from the origin and the angle θ that ray OP makes with the positive x -axis (Figure 1.40). Since $x/r = \cos \theta$ and $y/r = \sin \theta$, we have

$$x = r \cos \theta, \quad y = r \sin \theta.$$

When $r = 1$ we can apply the Pythagorean theorem to the reference right triangle in Figure 1.45 and obtain the equation

$$\cos^2 \theta + \sin^2 \theta = 1. \quad (3)$$

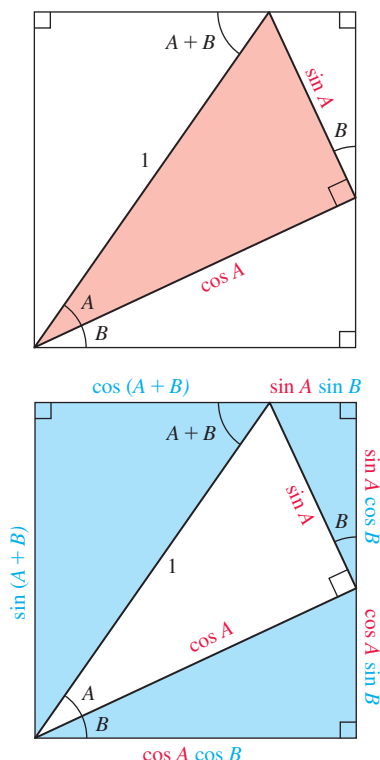


FIGURE 1.46 In a geometric proof of the angle sum identities we compare the opposite sides of the rectangle, which are equal. This assumes that A , B , and $A + B$ are acute, but the identities hold for all values of A and B .

This equation, true for all values of θ , is the most frequently used identity in trigonometry. Dividing this identity in turn by $\cos^2 \theta$ and $\sin^2 \theta$ gives

$$\begin{aligned} 1 + \tan^2 \theta &= \sec^2 \theta \\ 1 + \cot^2 \theta &= \csc^2 \theta \end{aligned}$$

The following formulas hold for all angles A and B (see Figure 1.46 and Exercise 58).

Addition Formulas

$$\begin{aligned} \cos(A + B) &= \cos A \cos B - \sin A \sin B \\ \sin(A + B) &= \sin A \cos B + \cos A \sin B \end{aligned} \quad (4)$$

There are similar formulas for $\cos(A - B)$ and $\sin(A - B)$ (Exercises 35 and 36). All the trigonometric identities needed in this text derive from Equations (3) and (4). For example, substituting θ for both A and B in the addition formulas gives

Double-Angle Formulas

$$\begin{aligned} \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ \sin 2\theta &= 2 \sin \theta \cos \theta \end{aligned} \quad (5)$$

Additional formulas come from combining the equations

$$\cos^2 \theta + \sin^2 \theta = 1, \quad \cos^2 \theta - \sin^2 \theta = \cos 2\theta.$$

We add the two equations to get $2 \cos^2 \theta = 1 + \cos 2\theta$ and subtract the second from the first to get $2 \sin^2 \theta = 1 - \cos 2\theta$. This results in the following identities, which are useful in integral calculus.

Half-Angle Formulas

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \quad (6)$$

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2} \quad (7)$$

The Law of Cosines

If a , b , and c are sides of a triangle ABC and if θ is the angle opposite c , then

$$c^2 = a^2 + b^2 - 2ab \cos \theta. \quad (8)$$

This equation is called the **law of cosines**.

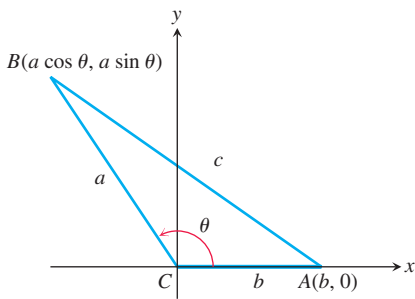


FIGURE 1.47 The square of the distance between A and B gives the law of cosines.

To see why the law holds, we position the triangle in the xy -plane with the origin at C and the positive x -axis along one side of the triangle, as in Figure 1.47. The coordinates of A are $(b, 0)$; the coordinates of B are $(a \cos \theta, a \sin \theta)$. The square of the distance between A and B is therefore

$$\begin{aligned} c^2 &= (a \cos \theta - b)^2 + (a \sin \theta)^2 \\ &= a^2(\cos^2 \theta + \sin^2 \theta) + b^2 - 2ab \cos \theta \\ &= a^2 + b^2 - 2ab \cos \theta. \end{aligned}$$

The law of cosines generalizes the Pythagorean theorem. If $\theta = \pi/2$, then $\cos \theta = 0$ and $c^2 = a^2 + b^2$.

Two Special Inequalities

For any angle θ measured in radians, the sine and cosine functions satisfy

$$-|\theta| \leq \sin \theta \leq |\theta| \quad \text{and} \quad -|\theta| \leq 1 - \cos \theta \leq |\theta|.$$

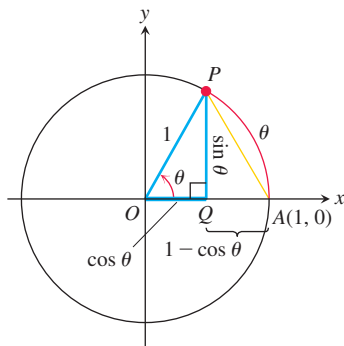


FIGURE 1.48 From the geometry of this figure, drawn for $\theta > 0$, we get the inequality $\sin^2 \theta + (1 - \cos \theta)^2 \leq \theta^2$.

To establish these inequalities, we picture θ as a nonzero angle in standard position (Figure 1.48). The circle in the figure is a unit circle, so $|\theta|$ equals the length of the circular arc AP . The length of line segment AP is therefore less than $|\theta|$.

Triangle APQ is a right triangle with sides of length

$$QP = |\sin \theta|, \quad AQ = 1 - \cos \theta.$$

From the Pythagorean theorem and the fact that $AP < |\theta|$, we get

$$\sin^2 \theta + (1 - \cos \theta)^2 = (AP)^2 \leq \theta^2. \quad (9)$$

The terms on the left-hand side of Equation (9) are both positive, so each is smaller than their sum and hence is less than or equal to θ^2 :

$$\sin^2 \theta \leq \theta^2 \quad \text{and} \quad (1 - \cos \theta)^2 \leq \theta^2.$$

By taking square roots, this is equivalent to saying that

$$|\sin \theta| \leq |\theta| \quad \text{and} \quad |1 - \cos \theta| \leq |\theta|,$$

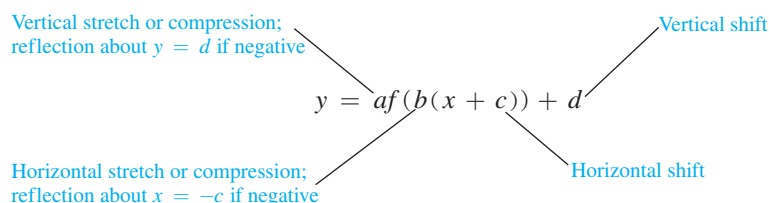
so

$$-|\theta| \leq \sin \theta \leq |\theta| \quad \text{and} \quad -|\theta| \leq 1 - \cos \theta \leq |\theta|.$$

These inequalities will be useful in the next chapter.

Transformations of Trigonometric Graphs

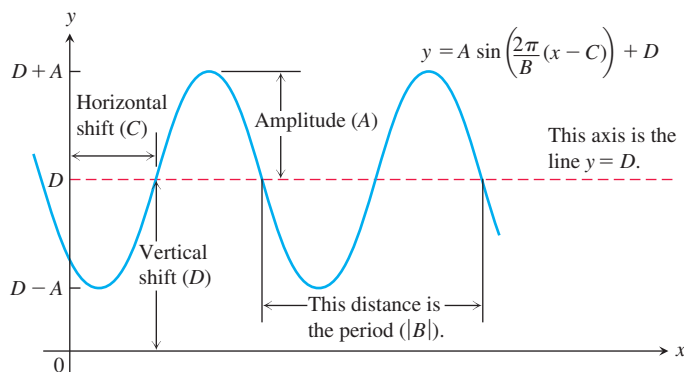
The rules for shifting, stretching, compressing, and reflecting the graph of a function summarized in the following diagram apply to the trigonometric functions we have discussed in this section.



The transformation rules applied to the sine function give the **general sine function** or **sinusoid** formula

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

where $|A|$ is the *amplitude*, $|B|$ is the *period*, C is the *horizontal shift*, and D is the *vertical shift*. A graphical interpretation of the various terms is given below.



EXERCISES 1.3

Radians and Degrees

- On a circle of radius 10 m, how long is an arc that subtends a central angle of (a) $4\pi/5$ radians? (b) 110° ?
- A central angle in a circle of radius 8 is subtended by an arc of length 10π . Find the angle's radian and degree measures.
- T** You want to make an 80° angle by marking an arc on the perimeter of a 12-cm-diameter disk and drawing lines from the ends of the arc to the disk's center. To the nearest millimeter, how long should the arc be?
- T** If you roll a 1-m-diameter wheel forward 30 cm over level ground, through what angle will the wheel turn? Answer in radians (to the nearest tenth) and degrees (to the nearest degree).

Evaluating Trigonometric Functions

- Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
$\sin \theta$					
$\cos \theta$					
$\tan \theta$					
$\cot \theta$					
$\sec \theta$					
$\csc \theta$					

- Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-3\pi/2$	$-\pi/3$	$-\pi/6$	$\pi/4$	$5\pi/6$
$\sin \theta$					
$\cos \theta$					
$\tan \theta$					
$\cot \theta$					
$\sec \theta$					
$\csc \theta$					

In Exercises 7–12, one of $\sin x$, $\cos x$, and $\tan x$ is given. Find the other two if x lies in the specified interval.

- $\sin x = \frac{3}{5}$, $x \in \left[\frac{\pi}{2}, \pi\right]$
- $\tan x = 2$, $x \in \left[0, \frac{\pi}{2}\right]$
- $\cos x = \frac{1}{3}$, $x \in \left[-\frac{\pi}{2}, 0\right]$
- $\cos x = -\frac{5}{13}$, $x \in \left[\frac{\pi}{2}, \pi\right]$
- $\tan x = \frac{1}{2}$, $x \in \left[\pi, \frac{3\pi}{2}\right]$
- $\sin x = -\frac{1}{2}$, $x \in \left[\pi, \frac{3\pi}{2}\right]$

Graphing Trigonometric Functions

Graph the functions in Exercises 13–22. What is the period of each function?

- $\sin 2x$
- $\sin(x/2)$
- $\cos \pi x$
- $\cos \frac{\pi x}{2}$
- $-\sin \frac{\pi x}{3}$
- $-\cos 2\pi x$
- $\cos\left(x - \frac{\pi}{2}\right)$
- $\sin\left(x + \frac{\pi}{6}\right)$

$$21. \sin\left(x - \frac{\pi}{4}\right) + 1 \quad 22. \cos\left(x + \frac{2\pi}{3}\right) - 2$$

Graph the functions in Exercises 23–26 in the ts -plane (t -axis horizontal, s -axis vertical). What is the period of each function? What symmetries do the graphs have?

$$23. s = \cot 2t \quad 24. s = -\tan \pi t$$

$$25. s = \sec\left(\frac{\pi t}{2}\right) \quad 26. s = \csc\left(\frac{t}{2}\right)$$

- T** 27. a. Graph $y = \cos x$ and $y = \sec x$ together for $-\pi/2 \leq x \leq 3\pi/2$. Comment on the behavior of $\sec x$ in relation to the signs and values of $\cos x$.
 b. Graph $y = \sin x$ and $y = \csc x$ together for $-\pi \leq x \leq 2\pi$. Comment on the behavior of $\csc x$ in relation to the signs and values of $\sin x$.
- T** 28. Graph $y = \tan x$ and $y = \cot x$ together for $-7 \leq x \leq 7$. Comment on the behavior of $\cot x$ in relation to the signs and values of $\tan x$.
29. Graph $y = \sin x$ and $y = |\sin x|$ together. What are the domain and range of $|\sin x|$?
30. Graph $y = \sin x$ and $y = \lceil \sin x \rceil$ together. What are the domain and range of $\lceil \sin x \rceil$?

Using the Addition Formulas

Use the addition formulas to derive the identities in Exercises 31–36.

$$31. \cos\left(x - \frac{\pi}{2}\right) = \sin x \quad 32. \cos\left(x + \frac{\pi}{2}\right) = -\sin x$$

$$33. \sin\left(x + \frac{\pi}{2}\right) = \cos x \quad 34. \sin\left(x - \frac{\pi}{2}\right) = -\cos x$$

35. $\cos(A - B) = \cos A \cos B + \sin A \sin B$ (Exercise 57 provides a different derivation.)
36. $\sin(A - B) = \sin A \cos B - \cos A \sin B$
37. What happens if you take $B = A$ in the trigonometric identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$? Does the result agree with something you already know?
38. What happens if you take $B = 2\pi$ in the addition formulas? Do the results agree with something you already know?

In Exercises 39–42, express the given quantity in terms of $\sin x$ and $\cos x$.

$$39. \cos(\pi + x) \quad 40. \sin(2\pi - x)$$

$$41. \sin\left(\frac{3\pi}{2} - x\right) \quad 42. \cos\left(\frac{3\pi}{2} + x\right)$$

$$43. \text{Evaluate } \sin \frac{7\pi}{12} \text{ as } \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right).$$

$$44. \text{Evaluate } \cos \frac{11\pi}{12} \text{ as } \cos\left(\frac{\pi}{4} + \frac{2\pi}{3}\right).$$

$$45. \text{Evaluate } \cos \frac{\pi}{12}. \quad 46. \text{Evaluate } \sin \frac{5\pi}{12}.$$

Using the Half-Angle Formulas

Find the function values in Exercises 47–50.

$$47. \cos^2 \frac{\pi}{8} \quad 48. \cos^2 \frac{5\pi}{12}$$

$$49. \sin^2 \frac{\pi}{12} \quad 50. \sin^2 \frac{3\pi}{8}$$

Solving Trigonometric Equations

For Exercises 51–54, solve for the angle θ , where $0 \leq \theta \leq 2\pi$.

$$51. \sin^2 \theta = \frac{3}{4} \quad 52. \sin^2 \theta = \cos^2 \theta$$

$$53. \sin 2\theta - \cos \theta = 0 \quad 54. \cos 2\theta + \cos \theta = 0$$

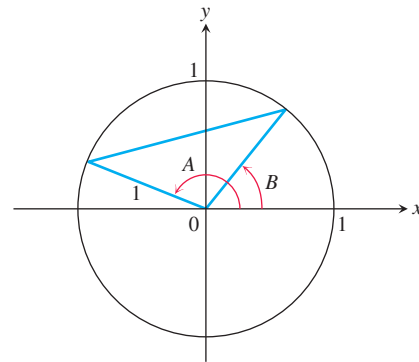
Theory and Examples

55. **The tangent sum formula** The standard formula for the tangent of the sum of two angles is

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Derive the formula.

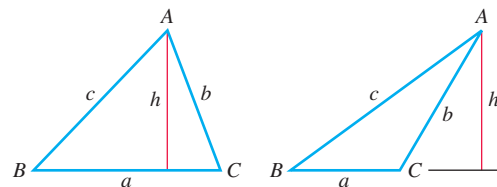
56. (Continuation of Exercise 55.) Derive a formula for $\tan(A - B)$.
57. Apply the law of cosines to the triangle in the accompanying figure to derive the formula for $\cos(A - B)$.



58. a. Apply the formula for $\cos(A - B)$ to the identity $\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$ to obtain the addition formula for $\sin(A + B)$.
 b. Derive the formula for $\cos(A + B)$ by substituting $-B$ for B in the formula for $\cos(A - B)$ from Exercise 35.
59. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60^\circ$. Find the length of side c .
60. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 40^\circ$. Find the length of side c .
61. **The law of sines** The law of sines says that if a , b , and c are the sides opposite the angles A , B , and C in a triangle, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Use the accompanying figures and the identity $\sin(\pi - \theta) = \sin \theta$, if required, to derive the law.

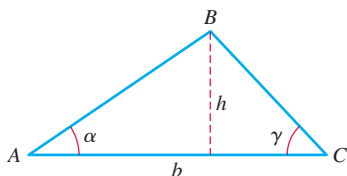


62. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60^\circ$ (as in Exercise 59). Find the sine of angle B using the law of sines.

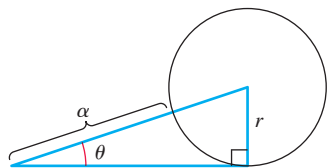
T 63. A triangle has side $c = 2$ and angles $A = \pi/4$ and $B = \pi/3$. Find the length a of the side opposite A .

64. Consider the length h of the perpendicular from point B to side b in the given triangle. Show that

$$h = \frac{b \tan \alpha \tan \gamma}{\tan \alpha + \tan \gamma}$$



65. Refer to the given figure. Write the radius r of the circle in terms of α and θ .



T 66. **The approximation $\sin x \approx x$** It is often useful to know that, when x is measured in radians, $\sin x \approx x$ for numerically small values of x . In Section 3.11, we will see why the approximation holds. The approximation error is less than 1 in 5000 if $|x| < 0.1$.

- With your grapher in radian mode, graph $y = \sin x$ and $y = x$ together in a viewing window about the origin. What do you see happening as x nears the origin?
- With your grapher in degree mode, graph $y = \sin x$ and $y = x$ together about the origin again. How is the picture different from the one obtained with radian mode?

General Sine Curves

For

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

identify A , B , C , and D for the sine functions in Exercises 67–70 and sketch their graphs.

- $y = 2 \sin(x + \pi) - 1$
- $y = \frac{1}{2} \sin(\pi x - \pi) + \frac{1}{2}$
- $y = -\frac{2}{\pi} \sin\left(\frac{\pi}{2}t\right) + \frac{1}{\pi}$
- $y = \frac{L}{2\pi} \sin \frac{2\pi t}{L}, \quad L > 0$

COMPUTER EXPLORATIONS

In Exercises 71–74, you will explore graphically the general sine function

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D$$

as you change the values of the constants A , B , C , and D . Use a CAS or computer grapher to perform the steps in the exercises.

- The period B** Set the constants $A = 3$ and $C = D = 0$.
 - Plot $f(x)$ for the values $B = 1, 3, 2\pi, 5\pi$ over the interval $-4\pi \leq x \leq 4\pi$. Describe what happens to the graph of the general sine function as the period increases.
 - What happens to the graph for negative values of B ? Try it with $B = -3$ and $B = -2\pi$.
- The horizontal shift C** Set the constants $A = 3, B = 6, D = 0$.
 - Plot $f(x)$ for the values $C = 0, 1$, and 2 over the interval $-4\pi \leq x \leq 4\pi$. Describe what happens to the graph of the general sine function as C increases through positive values.
 - What happens to the graph for negative values of C ?
 - What smallest positive value should be assigned to C so the graph exhibits no horizontal shift? Confirm your answer with a plot.
- The vertical shift D** Set the constants $A = 3, B = 6, C = 0$.
 - Plot $f(x)$ for the values $D = 0, 1$, and 3 over the interval $-4\pi \leq x \leq 4\pi$. Describe what happens to the graph of the general sine function as D increases through positive values.
 - What happens to the graph for negative values of D ?
- The amplitude A** Set the constants $B = 6$ and $C = D = 0$.
 - Describe what happens to the graph of the general sine function as A increases through positive values. Confirm your answer by plotting $f(x)$ for the values $A = 1, 5$, and 9 .
 - What happens to the graph for negative values of A ?

1.4 Exponential Functions

Exponential functions occur in a wide variety of applications, including interest rates, radioactive decay, population growth, the spread of a disease, consumption of natural resources, the earth's atmospheric pressure, temperature change of a heated object placed in a cooler environment, and the dating of fossils. In this section we introduce these functions informally, using an intuitive approach. We give a rigorous development of them in Chapter 7, based on the ideas of integral calculus.

Exponential Behavior

When a positive quantity P doubles, it increases by a factor of 2 and the quantity becomes $2P$. If it doubles again, it becomes $2(2P) = 2^2P$, and a third doubling gives $2(2^2P) = 2^3P$. Continuing to double in this fashion leads us to consider the function $f(x) = 2^x$. We call this an *exponential* function because the variable x appears in the exponent of 2^x . Functions

Don't confuse the exponential 2^x with the power function x^2 . In the exponential, the variable x is in the exponent, whereas the variable x is the base in the power function.

such as $g(x) = 10^x$ and $h(x) = (1/2)^x$ are other examples of exponential functions. In general, if $a \neq 1$ is a positive constant, the function

$$f(x) = a^x$$

is the **exponential function with base a** .

EXAMPLE 1 In 2022, \$100 is put into an investment account, where it grows by accruing interest that is compounded annually (once a year) at an interest rate of 5.5%. Assuming no additional funds are deposited to the account and no money is withdrawn, give a formula for a function describing the amount A in the account after x years have elapsed.

Solution If $P = 100$, at the end of the first year the amount in the account is the original amount plus the interest accrued, or

$$P + \left(\frac{5.5}{100}\right)P = (1 + 0.055)P = (1.055)P.$$

At the end of the second year the account earns interest again and grows to

$$(1 + 0.055) \cdot (1.055P) = (1.055)^2 P = 100 \cdot (1.055)^2. \quad P = 100$$

Continuing this process, after x years the value of the account is

$$A = 100 \cdot (1.055)^x.$$

This is a multiple of the exponential function $f(x) = (1.055)^x$ with base 1.055. Table 1.3 shows the amounts accrued over the first four years. Notice that the amount in the account each year is always 1.055 times its value in the previous year.

TABLE 1.3 Investment account growth

Year	Amount (dollars)	Yearly increase
2022	100	
2023	$100(1.055) = 105.50$	5.50
2024	$100(1.055)^2 \approx 111.30$	5.80
2025	$100(1.055)^3 \approx 117.42$	6.12
2026	$100(1.055)^4 \approx 123.88$	6.46

In general, the amount after x years is given by $P(1 + r)^x$, where P is the starting amount and r is the interest rate (expressed as a decimal). ■

For integer and rational exponents, the value of an exponential function $f(x) = a^x$ is obtained arithmetically by taking an appropriate number of products, quotients, or roots. If $x = n$ is a positive integer, the number a^n is given by multiplying a by itself n times:

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

If $x = 0$, then we set $a^0 = 1$, and if $x = -n$ for some positive integer n , then

$$a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n.$$

If $x = 1/n$ for some positive integer n , then

$$a^{1/n} = \sqrt[n]{a},$$

which is the positive number that when multiplied by itself n times gives a . If $x = p/q$ is any rational number, then

$$a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p.$$

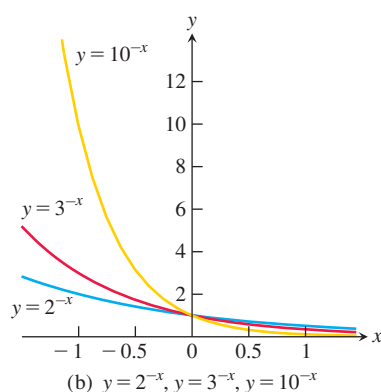
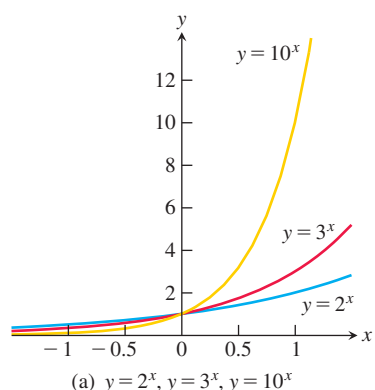


FIGURE 1.49 Graphs of exponential functions.

TABLE 1.4 Values of $2^{\sqrt{3}}$ for rational r closer and closer to $\sqrt{3}$

r	2^r
1.0	2.000000000
1.7	3.249009585
1.73	3.317278183
1.732	3.321880096
1.7320	3.321880096
1.73205	3.321995226
1.732050	3.321995226
1.7320508	3.321997068
1.73205080	3.321997068
1.732050808	3.321997086

When x is *irrational*, the meaning of a^x is not immediately apparent. The value of a^x can be approximated by raising a to rational numbers that get closer and closer to the irrational number x . We will describe this informally now and will give a rigorous definition in Chapter 7.

The graphs of several exponential functions are shown in Figure 1.49. These graphs show the values of the exponential functions for real inputs x . We choose the value of a^x when x is irrational so that there are no “holes” or “jumps” in the graph of a^x (these words are not rigorous mathematical terms, but they informally convey the underlying idea). The value of a^x when x is irrational is chosen so that the function $f(x) = a^x$ is *continuous*, a notion that will be carefully developed in Chapter 2. This choice ensures that the graph is increasing when $a > 1$ and is decreasing when $0 < a < 1$ (see Figure 1.49).

We illustrate how to define the value of an exponential function at an irrational power using the exponential function $f(x) = 2^x$. How do we make sense of the expression $2^{\sqrt{3}}$? Any particular irrational number, say $x = \sqrt{3}$, has a decimal expansion

$$\sqrt{3} = 1.732050808\dots$$

We then consider the list of powers of 2 with more and more digits in the decimal expansion,

$$2^1, 2^{1.7}, 2^{1.73}, 2^{1.732}, 2^{1.7320}, 2^{1.73205}, \dots \quad (1)$$

We know the meaning of each number in list (1) because the successive decimal approximations to $\sqrt{3}$ given by 1, 1.7, 1.73, 1.732, and so on are all *rational* numbers. As these decimal approximations get closer and closer to $\sqrt{3}$, it seems reasonable that the list of numbers in (1) gets closer and closer to some fixed number, which we specify to be $2^{\sqrt{3}}$.

Table 1.4 illustrates how taking better approximations to $\sqrt{3}$ gives better approximations to the number $2^{\sqrt{3}} \approx 3.321997086$. It is the *completeness property* of the real numbers (discussed in Appendix A.7) which guarantees that this procedure gives a single number we define to be $2^{\sqrt{3}}$ (although it is beyond the scope of this text to give a proof). In a similar way, we can identify the number 2^x (or a^x , $a > 0$) for any irrational x . By identifying the number a^x for both rational and irrational x , we eliminate any “holes” or “gaps” in the graph of a^x .

Exponential functions obey the rules of exponents listed below. It is easy to check these rules using algebra when the exponents are integers or rational numbers. We prove them for all real exponents in Chapter 7.

Rules for Exponents

If $a > 0$ and $b > 0$, the following rules hold for all real numbers x and y .

- $a^x \cdot a^y = a^{x+y}$
- $\frac{a^x}{a^y} = a^{x-y}$
- $(a^x)^y = (a^y)^x = a^{xy}$
- $a^x \cdot b^x = (ab)^x$
- $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$

EXAMPLE 2 We use the rules for exponents to simplify some numerical expressions.

- $3^{1.1} \cdot 3^{0.7} = 3^{1.1+0.7} = 3^{1.8}$ Rule 1
- $\frac{(\sqrt{10})^3}{\sqrt{10}} = (\sqrt{10})^{3-1} = (\sqrt{10})^2 = 10$ Rule 2
- $(5^{\sqrt{2}})^{\sqrt{2}} = 5^{\sqrt{2} \cdot \sqrt{2}} = 5^2 = 25$ Rule 3
- $7^\pi \cdot 8^\pi = (56)^\pi$ Rule 4
- $\left(\frac{4}{9}\right)^{1/2} = \frac{4^{1/2}}{9^{1/2}} = \frac{2}{3}$ Rule 5

The Natural Exponential Function e^x

The most important exponential function used for modeling natural, physical, and economic phenomena is the **natural exponential function**, whose base is the special number e . The number e is irrational, and its value to nine decimal places is 2.718281828. (In Section 3.8 we will see a way to calculate the value of e .) It might seem strange that we would use this number for a base rather than a simple number like 2 or 10. The advantage in using e as a base is that it greatly simplifies many of the calculations in calculus.

In Figure 1.49a you can see that for $x \geq 0$, the graphs of the exponential functions $y = a^x$ get steeper as the base a gets larger. This idea of steepness is conveyed by the slope of the tangent line to the graph at a point. Tangent lines to graphs of functions are defined precisely in the next chapter, but intuitively the tangent line to the graph at a point is the line that best approximates the graph at the point, like a tangent to a circle. Figure 1.50 shows the slope of the graph of $y = a^x$ as it crosses the y -axis for some values of a . Notice that the slope is exactly equal to 1 when a equals the number e . The slope is smaller than 1 if $a < e$, and larger than 1 if $a > e$. The graph of $y = e^x$ has slope 1 when it crosses the y -axis.

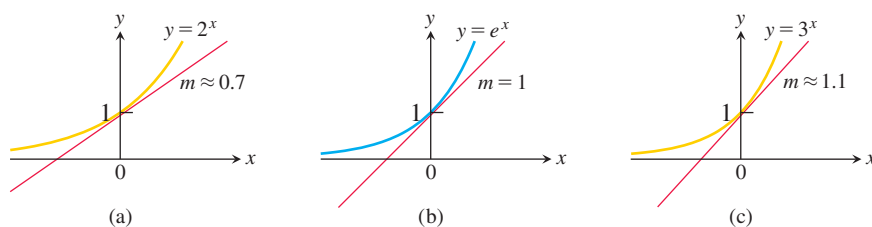


FIGURE 1.50 Among the exponential functions, the graph of $y = e^x$ has the property that the slope m of the tangent line to the graph is exactly 1 when it crosses the y -axis. The slope is smaller for a base less than e , such as 2^x , and larger for a base greater than e , such as 3^x .

Exponential Growth and Decay

The function $y = y_0 e^{kx}$, where k is a nonzero constant, is a model for **exponential growth** if $k > 0$ and a model for **exponential decay** if $k < 0$. Here y_0 is a positive constant that represents the value of the function when $x = 0$. An example of exponential growth occurs when computing interest **compounded continuously**. This is modeled by the formula $y = Pe^{rt}$, where P is the initial monetary investment, r is the interest rate as a decimal, and t is time in units consistent with r . An example of exponential decay is the model $y = Ae^{-1.2 \times 10^{-4}t}$, which represents how the radioactive isotope carbon-14 decays over time. Here A is the original amount of carbon-14 and t is the time in years. Carbon-14 decay is used to date the remains of dead organisms such as shells, seeds, and wooden artifacts. Figure 1.51 shows graphs of exponential growth and exponential decay.

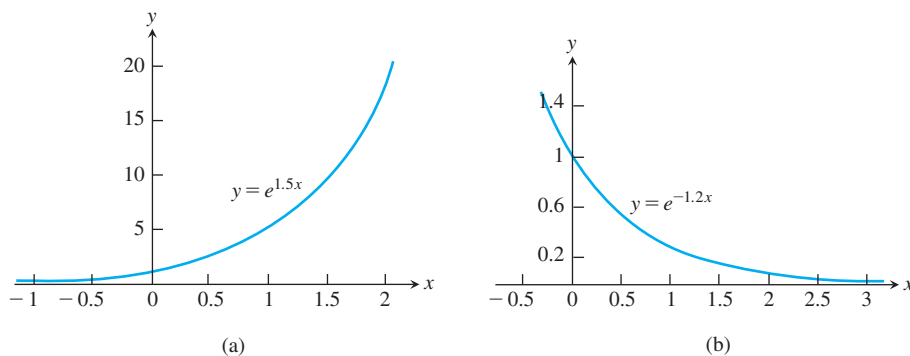


FIGURE 1.51 Graphs of (a) exponential growth, $k = 1.5 > 0$, and (b) exponential decay, $k = -1.2 < 0$.

EXAMPLE 3 Investment companies often use the model $y = Pe^{rt}$ in calculating the growth of an investment. Use this model to track the growth of \$100 invested in 2022 at an annual interest rate of 5.5%.

Solution Let $t = 0$ represent 2022, $t = 1$ represent 2023, and so on. Then the exponential growth model is $y(t) = Pe^{rt}$, where $P = 100$ (the initial investment), $r = 0.055$ (the annual interest rate expressed as a decimal), and t is time in years. To predict the amount in the account in 2026, after four years have elapsed, we take $t = 4$ and calculate

$$\begin{aligned} y(4) &= 100e^{0.055(4)} \\ &= 100e^{0.22} \\ &= 124.61. \end{aligned} \quad \text{Nearest cent using calculator}$$

This compares with \$123.88 in the account when the interest is compounded annually, as was done in Example 1. ■

EXAMPLE 4 Laboratory experiments indicate that some atoms emit a part of their mass as radiation, with the remainder of the atom re-forming to make an atom of some new element. For example, radioactive carbon-14 decays into nitrogen; radium eventually decays into lead. If y_0 is the number of radioactive nuclei present at time zero, the number still present at any later time t will be

$$y = y_0 e^{-rt}, \quad r > 0.$$

The number r is called the **decay rate** of the radioactive substance. (We will see how this formula is obtained in Section 7.2.) For carbon-14, the decay rate has been determined experimentally to be about $r = 1.2 \times 10^{-4}$ when t is measured in years. Predict the percent of carbon-14 present after 866 years have elapsed.

Solution If we start with an amount y_0 of carbon-14 nuclei, after 866 years we are left with the amount

$$\begin{aligned} y(866) &= y_0 e^{(-1.2 \times 10^{-4})(866)} \\ &\approx (0.901)y_0. \end{aligned} \quad \text{Calculator evaluation}$$

That is, after 866 years, we are left with about 90% of the original amount of carbon-14, so about 10% of the original nuclei have decayed. ■

You may wonder why we use the family of functions $y = e^{kx}$ for different values of the constant k instead of the general exponential functions $y = a^x$. In the next section, we show that the exponential function a^x is equal to e^{kx} for an appropriate value of k . So the formula $y = e^{kx}$ covers the entire range of possibilities, and it is generally easier to use.

EXERCISES 1.4

Sketching Exponential Curves

In Exercises 1–6, sketch the given curves together in the appropriate coordinate plane, and label each curve with its equation.

- $y = 2^x$, $y = 4^x$, $y = 3^{-x}$, $y = (1/5)^x$
- $y = 3^x$, $y = 8^x$, $y = 2^{-x}$, $y = (1/4)^x$
- $y = 2^{-t}$ and $y = -2^t$
- $y = 3^{-t}$ and $y = -3^t$
- $y = e^x$ and $y = 1/e^x$
- $y = -e^x$ and $y = -e^{-x}$

In each of Exercises 7–10, sketch the shifted exponential curves.

- $y = 2^x - 1$ and $y = 2^{-x} - 1$
- $y = 3^x + 2$ and $y = 3^{-x} + 2$

9. $y = 1 - e^x$ and $y = 1 - e^{-x}$

10. $y = -1 - e^x$ and $y = -1 - e^{-x}$

Applying the Laws of Exponents

Use the laws of exponents to simplify the expressions in Exercises 11–20.

- $16^2 \cdot 16^{-1.75}$
- $9^{1/3} \cdot 9^{1/6}$
- $\frac{4^{4.2}}{4^{3.7}}$
- $\frac{3^{5/3}}{3^{2/3}}$
- $(25^{1/8})^4$
- $(13^{\sqrt{2}})^{\sqrt{2}/2}$
- $2^{\sqrt{3}} \cdot 7^{\sqrt{3}}$
- $(\sqrt{3})^{1/2} \cdot (\sqrt{12})^{1/2}$
- $\left(\frac{2}{\sqrt{2}}\right)^4$
- $\left(\frac{\sqrt{6}}{3}\right)^2$

Compositions Involving Exponential Functions

Find the domain and range for each of the functions in Exercises 21–24.

21. $f(x) = \frac{1}{2 + e^x}$

22. $g(t) = \cos(e^{-t})$

23. $g(t) = \sqrt{1 + 3^{-t}}$

24. $f(x) = \frac{3}{1 - e^{2x}}$

Applications**T** In Exercises 25–28, use graphs to find approximate solutions.

25. $2^x = 5$

26. $e^x = 4$

27. $3^x - 0.5 = 0$

28. $3 - 2^{-x} = 0$

T In Exercises 29–36, use an exponential model and a graphing calculator to estimate the answer in each problem.**29. Population growth** The population of a midwestern city is 500,000 and is increasing at the rate of 3.75% each year. Approximately when will the population reach 1 million?**30. Population growth** The population of Silver Run in the year 1890 was 6250. Assume the population increased at a rate of 2.75% per year.**a.** Estimate the population in 1915 and 1940.**b.** Approximately when did the population reach 50,000?**31. Radioactive decay** The half-life of phosphorus-32 is about 14 days. There are 6.6 grams present initially.**a.** Express the amount of phosphorus-32 remaining as a function of time t .**b.** When will there be 1 gram remaining?**32.** If Jean invests \$2300 in a retirement account with a 6% interest rate compounded annually, how long will it take until Jean's account has a balance of \$4150?**33. Doubling your money** Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded annually.**34. Tripling your money** Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded continuously.**35. Cholera bacteria** Suppose that a colony of bacteria starts with 1 bacterium and doubles in number every half hour. How many bacteria will the colony contain at the end of 24 hours?**36. Eliminating a disease** Suppose that in any given year the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take**a.** to reduce the number of cases to 1000?**b.** to eliminate the disease; that is, to reduce the number of cases to less than 1?

1.5 Inverse Functions and Logarithms

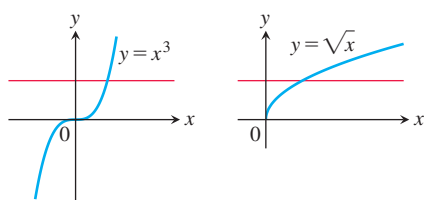
A function that undoes, or inverts, the effect of a function f is called the *inverse* of f . Many common functions, though not all, are paired with an inverse. In this section we present the natural logarithmic function $y = \ln x$ as the inverse of the exponential function $y = e^x$, and we also give examples of several inverse trigonometric functions.

One-to-One Functions

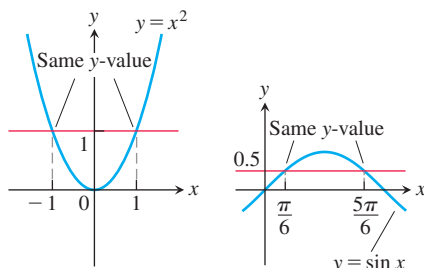
A function is a rule that assigns a value from its range to each element in its domain. Some functions assign the same range value to more than one element in the domain. The function $f(x) = x^2$ assigns the same value, 1, to both of the numbers -1 and $+1$. Similarly the sines of $\pi/3$ and $2\pi/3$ are both $\sqrt{3}/2$. Other functions assume each value in their range no more than once. The square roots and cubes of different numbers are always different. A function that has distinct values at distinct elements in its domain is called one-to-one.

DEFINITION A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D .

EXAMPLE 1 Some functions are one-to-one on their entire natural domain. Other functions are not one-to-one on their entire domain, but by restricting the function to a smaller domain we can create a function that is one-to-one. The original and restricted functions are not the same functions, because they have different domains. However, the two functions have the same values on the smaller domain.



(a) One-to-one: Graph meets each horizontal line at most once.



(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

FIGURE 1.52 (a) $y = x^3$ and $y = \sqrt{x}$ are one-to-one on their domains $(-\infty, \infty)$ and $[0, \infty)$. (b) $y = x^2$ and $y = \sin x$ are not one-to-one on their domains $(-\infty, \infty)$.

Caution

Do not confuse the inverse function f^{-1} with the reciprocal function $1/f$.

- (a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.
- (b) $g(x) = \sin x$ is *not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. In fact, for each element x_1 in the subinterval $[0, \pi/2]$ there is a corresponding element x_2 in the subinterval $(\pi/2, \pi]$ satisfying $\sin x_1 = \sin x_2$. The sine function *is* one-to-one on $[0, \pi/2]$, however, because it is an increasing function on $[0, \pi/2]$ and hence gives distinct outputs for distinct inputs in that interval. ■

The graph of a one-to-one function $y = f(x)$ can intersect a given horizontal line at most once. If the graph intersects the line more than once, then the function assumes the same y -value for at least two different x -values and is therefore not one-to-one (Figure 1.52).

The Horizontal Line Test for One-to-One Functions

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

Inverse Functions

Since each output of a one-to-one function comes from just one input, the effect of the function can be inverted to send each output back to the input from which it came.

DEFINITION Suppose that f is a one-to-one function on a domain D with range R . The **inverse function** f^{-1} is defined by

$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D .

The symbol f^{-1} for the inverse of f is read “ f inverse.” The “ -1 ” in f^{-1} is *not* an exponent; $f^{-1}(x)$ does not mean $1/f(x)$. Notice that the domains and ranges of f and f^{-1} are interchanged.

EXAMPLE 2 Suppose a one-to-one function $y = f(x)$ is given by a table of values

x	1	2	3	4	5	6	7	8
$f(x)$	3	4.5	7	10.5	15	20.5	27	34.5

A table for the values of $x = f^{-1}(y)$ can then be obtained by simply interchanging the values in each column of the table for f :

y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8

If we apply f to send an input x to the output $f(x)$ and follow by applying f^{-1} to $f(x)$, we get right back to x , just where we started. Similarly, if we take some number y in the range of f , apply f^{-1} to it, and then apply f to the resulting value $f^{-1}(y)$, we get back the value y from which we began. Composing a function and its inverse has the same effect as doing nothing.

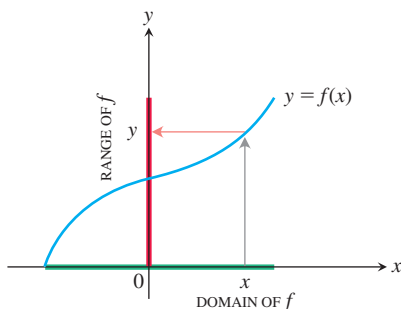
$$\begin{aligned} (f^{-1} \circ f)(x) &= x, & \text{for all } x \text{ in the domain of } f \\ (f \circ f^{-1})(y) &= y, & \text{for all } y \text{ in the domain of } f^{-1} \text{ (or range of } f) \end{aligned}$$

Only a one-to-one function can have an inverse. The reason is that if $f(x_1) = y$ and $f(x_2) = y$ for two distinct inputs x_1 and x_2 , then there is no way to assign a value to $f^{-1}(y)$ that satisfies both $f^{-1}(f(x_1)) = x_1$ and $f^{-1}(f(x_2)) = x_2$.

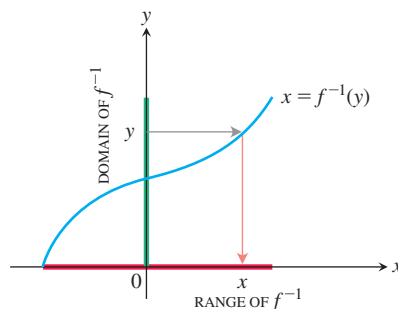
Suppose f is a function whose domain is an interval. If f is increasing, then it satisfies the inequality $f(x_2) > f(x_1)$ when $x_2 > x_1$, so it is one-to-one and has an inverse. Likewise, if f is decreasing, then it also has an inverse. Functions that are neither increasing nor decreasing may still be one-to-one and have an inverse, as with the function $f(x) = 1/x$ for $x \neq 0$ and $f(0) = 0$, defined on $(-\infty, \infty)$ and passing the horizontal line test.

Finding Inverses

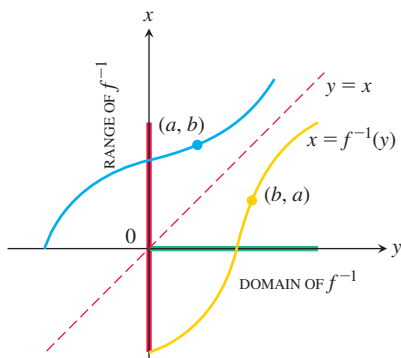
The graphs of a function and its inverse are closely related. To read the value of a function from its graph, we start at a point x on the x -axis, go vertically to the graph, and then move horizontally to the y -axis to read the value of y . The inverse function can be read from the graph by reversing this process. Start with a point y on the y -axis, go horizontally to the graph of $y = f(x)$, and then move vertically to the x -axis to read the value of $x = f^{-1}(y)$ (Figure 1.53).



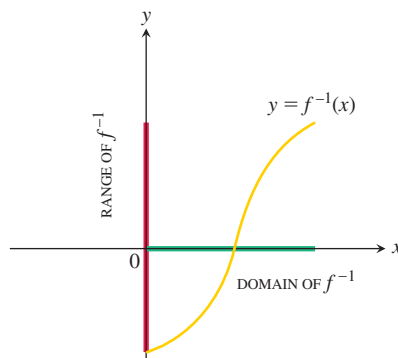
(a) To find the value of f at x , we start at x , go up to the curve, and then move to the y -axis.



(b) The graph of f^{-1} is the graph of f , but with x and y interchanged. To find the x that gave y , we start at y and go over to the curve and down to the x -axis. The domain of f^{-1} is the range of f . The range of f^{-1} is the domain of f .



(c) To draw the graph of f^{-1} in the more usual way, we reflect the system across the line $y = x$.



(d) Then we interchange the letters x and y . We now have a normal-looking graph of f^{-1} as a function of x .

FIGURE 1.53 The graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ about the line $y = x$.

We want to set up the graph of f^{-1} so that its input values lie along the x -axis, as is usually done for functions, rather than on the y -axis. To achieve this we interchange the x - and y -axes by reflecting across the 45° line $y = x$. After this reflection, we have a new graph that represents f^{-1} . The value of $f^{-1}(x)$ can now be read from the graph in the usual way, by starting with a point x on the x -axis, going vertically to the graph, and then horizontally to

the y -axis to get the value of $f^{-1}(x)$. Figure 1.53 indicates the relationship between the graphs of f and f^{-1} . The graphs are interchanged by reflection through the line $y = x$.

The process of passing from f to f^{-1} can be summarized as a two-step procedure.

1. Solve the equation $y = f(x)$ for x . This gives a formula $x = f^{-1}(y)$, where x is expressed as a function of y .
2. Interchange x and y , obtaining a formula $y = f^{-1}(x)$, where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

EXAMPLE 3 Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x .

Solution

1. Solve for x in terms of y : $y = \frac{1}{2}x + 1$
 $2y = x + 2$
 $x = 2y - 2$.

The graph satisfies the horizontal line test, so the function is one-to-one (Fig. 1.59).

2. Interchange x and y : $y = 2x - 2$.

Expresses the function in the usual form, where y is the dependent variable.

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2$. (See Figure 1.54.) To check, we verify that both compositions give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$

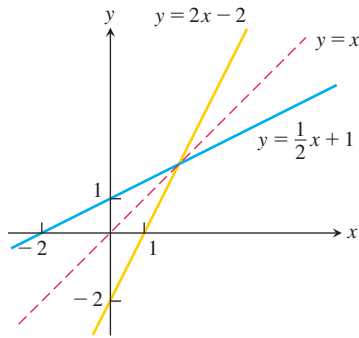


FIGURE 1.54 Graphing $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$ (Example 3).

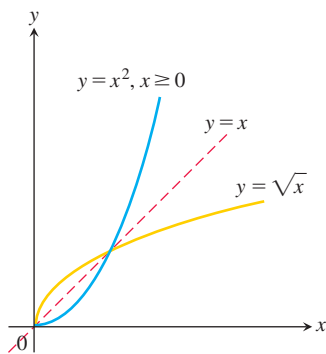


FIGURE 1.55 The functions $y = \sqrt{x}$ and $y = x^2, x \geq 0$, are inverses of one another (Example 4).

EXAMPLE 4 Find the inverse of the function $y = x^2, x \geq 0$, expressed as a function of x .

Solution For $x \geq 0$, the graph satisfies the horizontal line test, so the function is one-to-one and has an inverse. To find the inverse, we first solve for x in terms of y :

$$y = x^2$$

$$\sqrt{y} = \sqrt{x^2} = |x| = x \quad |x| = x \text{ because } x \geq 0$$

We then interchange x and y , obtaining

$$y = \sqrt{x}.$$

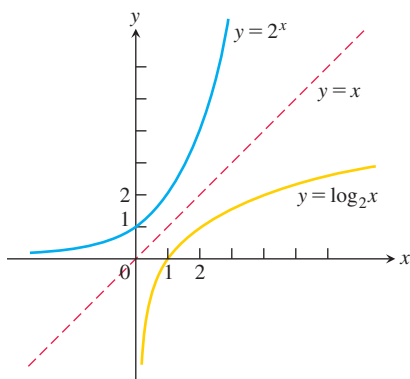
The inverse of the function $y = x^2, x \geq 0$, is the function $y = \sqrt{x}$ (Figure 1.55).

Notice that the function $y = x^2, x \geq 0$, with domain *restricted* to the nonnegative real numbers, *is* one-to-one (Figure 1.55) and has an inverse. On the other hand, the function $y = x^2$, with no domain restrictions, *is not* one-to-one (Figure 1.52b) and therefore has no inverse.

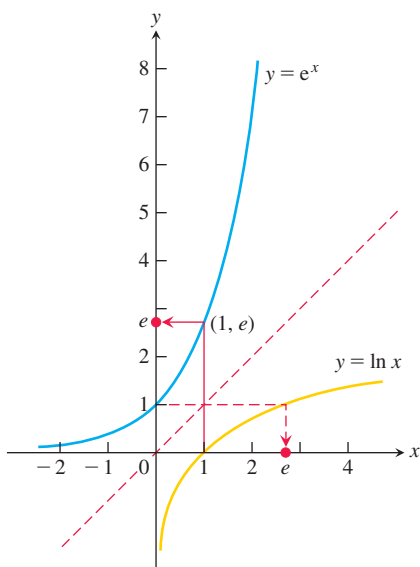
Logarithmic Functions

If a is any positive real number other than 1, then the base a exponential function $f(x) = a^x$ is one-to-one. It therefore has an inverse. Its inverse is called the *logarithm function with base a* .

DEFINITION The **logarithm function with base a** , written $y = \log_a x$, is the inverse of the base a exponential function $y = a^x$ ($a > 0, a \neq 1$).



(a)



(b)

FIGURE 1.56 (a) The graphs of 2^x and its inverse, $\log_2 x$. (b) The graphs of e^x and its inverse, $\ln x$.

The domain of $\log_a x$ is $(0, \infty)$, the same as the range of a^x . The range of $\log_a x$ is $(-\infty, \infty)$, the same as the domain of a^x .

Figure 1.56a shows the graph of $y = \log_2 x$. The graph of $y = a^x$, $a > 1$, increases rapidly for $x > 0$, so its inverse, $y = \log_a x$, increases slowly for $x > 1$.

Because we have no technique yet for solving the equation $y = a^x$ for x in terms of y , we do not have an explicit formula for computing the logarithm at a given value of x . Nevertheless, we can obtain the graph of $y = \log_a x$ by reflecting the graph of the exponential $y = a^x$ across the line $y = x$. Figure 1.56a shows the graphs for $a = 2$ and $a = e$.

Logarithms with base 2 are often used when working with binary numbers, as is common in computer science. Logarithms with base e and base 10 are so important in applications that many calculators have special keys for them. They also have their own special notation and names:

$$\begin{aligned}\log_e x &\text{ is written as } \ln x. \\ \log_{10} x &\text{ is written as } \log x.\end{aligned}$$

The function $y = \ln x$ is called the **natural logarithm function**, and $y = \log x$ is often called the **common logarithm function**. Since the logarithm is the inverse function of exponentiation, it follows that:

$$\begin{aligned}\log_a x = y &\Leftrightarrow a^y = x \\ \ln x = y &\Leftrightarrow e^y = x\end{aligned}$$

In particular, because $e^1 = e$, we obtain

$$\ln e = 1.$$

Properties of Logarithms

Logarithms, invented by John Napier, were the single most important improvement in arithmetic calculation before the modern electronic computer. The properties of logarithms reduce multiplication of positive numbers to addition of their logarithms, division of positive numbers to subtraction of their logarithms, and exponentiation of a number to multiplying its logarithm by the exponent.

We summarize these properties for the natural logarithm as a series of rules that we prove in Chapter 7.

THEOREM 1—Algebraic Properties of the Natural Logarithm

For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. **Product Rule:** $\ln bx = \ln b + \ln x$
2. **Quotient Rule:** $\ln \frac{b}{x} = \ln b - \ln x$
3. **Reciprocal Rule:** $\ln \frac{1}{x} = -\ln x$ Rule 2 with $b = 1$
4. **Power Rule:** $\ln x^r = r \ln x$

HISTORICAL BIOGRAPHY

John Napier
(1550–1617)

Scotsman John Napier went to St. Salvator's College in St. Andrews, where he studied from mathematician John Rutherford. Today, Napier is best known as the inventor of logarithms.

To know more, visit the companion Website.

EXAMPLE 5 We use the properties in Theorem 1 to rewrite three expressions.

(a) $\ln 4 + \ln \sin x = \ln(4 \sin x)$ Product Rule

(b) $\ln \frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$ Quotient Rule

(c) $\ln \frac{1}{8} = -\ln 8$ Reciprocal Rule
 $= -\ln 2^3 = -3 \ln 2$ Power Rule

Because a^x and $\log_a x$ are inverses, composing them in either order gives the identity function.

Inverse Properties for a^x and $\log_a x$

1. Base a ($a > 0, a \neq 1$):

$$a^{\log_a x} = x, \quad x > 0$$

$$\log_a a^x = x$$

2. Base e :

$$e^{\ln x} = x, \quad x > 0$$

$$\ln e^x = x$$

Substituting a^x for x in the equation $x = e^{\ln x}$ enables us to rewrite a^x as a power of e :

$$a^x = e^{\ln(a^x)} \quad \text{Substitute } a^x \text{ for } x \text{ in } x = e^{\ln x}.$$

$$= e^{x \ln a} \quad \text{Power Rule for logs}$$

$$= e^{(\ln a)x}. \quad \text{Exponent rearranged}$$

Thus, the exponential function a^x is the same as e^{kx} with $k = \ln a$.

Every exponential function is a power of the natural exponential function.

$$a^x = e^{x \ln a}$$

That is, a^x is the same as e^x raised to the power $\ln a$: $a^x = e^{kx}$ for $k = \ln a$.

For example,

$$2^x = e^{(\ln 2)x} = e^{x \ln 2}, \quad \text{and} \quad 5^{-3x} = e^{(\ln 5)(-3x)} = e^{-3x \ln 5}.$$

Returning once more to the properties of a^x and $\log_a x$, we have

$$\ln x = \ln(a^{\log_a x}) \quad \text{Inverse Property for } a^x \text{ and } \log_a x$$

$$= (\log_a x)(\ln a). \quad \text{Power Rule for logarithms, with } r = \log_a x$$

Rewriting this equation as $\log_a x = (\ln x)/(\ln a)$ shows that every logarithmic function is a constant multiple of the natural logarithm $\ln x$. This allows us to extend the algebraic properties for $\ln x$ to $\log_a x$. For instance, $\log_a bx = \log_a b + \log_a x$.

Change-of-Base Formula

Every logarithmic function is a constant multiple of the natural logarithm.

$$\log_a x = \frac{\ln x}{\ln a} \quad (a > 0, a \neq 1)$$

Applications

In Section 1.4 we looked at examples of exponential growth and decay problems. Here we use properties of logarithms to answer more questions concerning such problems.

EXAMPLE 6 If \$1000 is invested in an account that earns 5.25% interest compounded annually, how long will it take the account to reach \$2500?

Solution From Example 1, Section 1.4, with $P = 1000$ and $r = 0.0525$, the amount in the account at any time t in years is $1000(1.0525)^t$, so to find the time t when the account reaches \$2500, we need to solve the equation

$$1000(1.0525)^t = 2500.$$

Thus we have

$$(1.0525)^t = 2.5$$

Divide by 1000.

$$\ln(1.0525)^t = \ln 2.5$$

Take logarithms of both sides.

$$t \ln 1.0525 = \ln 2.5$$

Power Rule

$$t = \frac{\ln 2.5}{\ln 1.0525} \approx 17.9$$

Values obtained by calculator

The amount in the account will reach \$2500 in 18 years, when the annual interest payment is deposited for that year. ■

EXAMPLE 7 The **half-life** of a radioactive element is the time expected to pass until half of the radioactive nuclei present in a sample decay. The half-life is a constant that does not depend on the number of radioactive nuclei initially present in the sample, but only on the radioactive substance.

To compute the half-life, let y_0 be the number of radioactive nuclei initially present in the sample. Then the number y present at any later time t will be $y = y_0 e^{-kt}$. We seek the value of t at which the number of radioactive nuclei present equals half the original number:

$$y_0 e^{-kt} = \frac{1}{2} y_0$$

$$e^{-kt} = \frac{1}{2}$$

$$-kt = \ln \frac{1}{2} = -\ln 2 \quad \text{Reciprocal Rule for logarithms}$$

$$t = \frac{\ln 2}{k}. \quad (1)$$

This value of t is the half-life of the element. It depends only on the value of k ; the number y_0 does not have any effect.

The effective radioactive lifetime of polonium-210 is so short that we measure it in days rather than years. The number of radioactive atoms remaining after t days in a sample that starts with y_0 radioactive atoms is

$$y = y_0 e^{-5 \times 10^{-3} t}.$$

The element's half-life is

$$\text{Half-life} = \frac{\ln 2}{k} \quad \text{Eq. (1)}$$

$$= \frac{\ln 2}{5 \times 10^{-3}} \quad \text{The } k \text{ from polonium-210's decay equation}$$

$$\approx 139 \text{ days.}$$

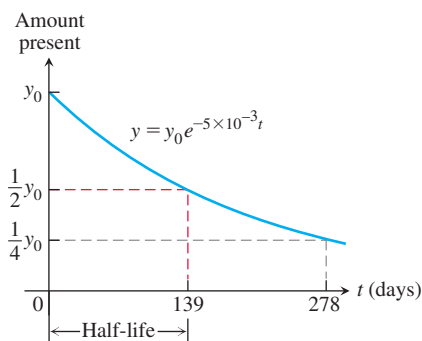


FIGURE 1.57 Amount of polonium-210 present at time t , where y_0 represents the number of radioactive atoms initially present (Example 7).

This means that after 139 days, $1/2$ of y_0 radioactive atoms remain; after another 139 days (278 days altogether) half of those remain, or $1/4$ of y_0 radioactive atoms remain, and so on (see Figure 1.57). ■

Inverse Trigonometric Functions

The six basic trigonometric functions are not one-to-one (since their values repeat periodically). However, we can restrict their domains to intervals on which they are one-to-one. The sine function increases from -1 at $x = -\pi/2$ to $+1$ at $x = \pi/2$. By restricting its

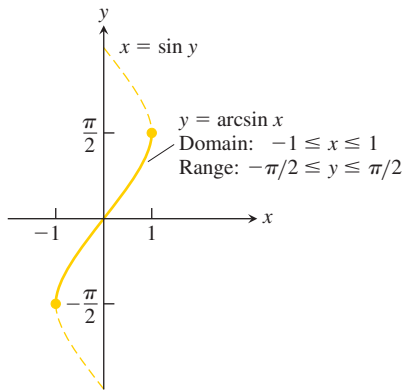
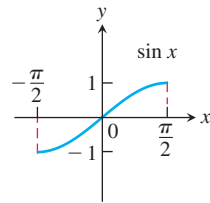


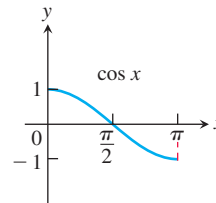
FIGURE 1.58 The graph of $y = \arcsin x$.

domain to the interval $[-\pi/2, \pi/2]$ we make it one-to-one, so that it has an inverse which is called $\arcsin x$ (Figure 1.58). Similar domain restrictions can be applied to all six trigonometric functions.

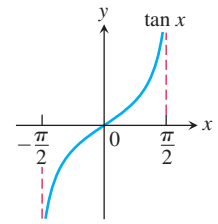
Domain restrictions that make the trigonometric functions one-to-one



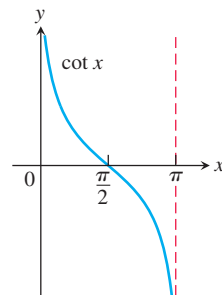
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



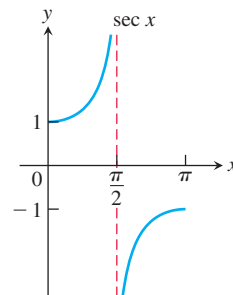
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



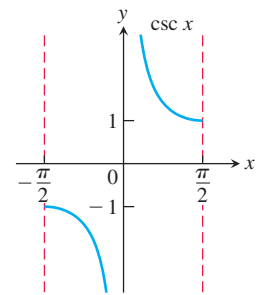
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



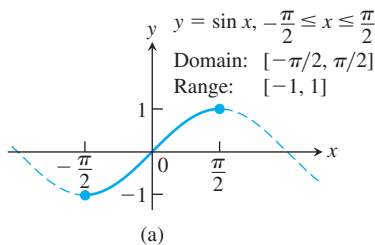
$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



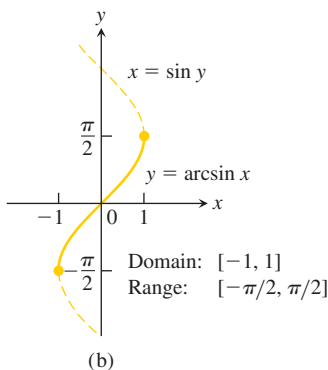
$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$



(a)



(b)

FIGURE 1.59 The graphs of (a) $y = \sin x, -\pi/2 \leq x \leq \pi/2$, and (b) its inverse, $y = \arcsin x$. The graph of $\arcsin x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$\begin{aligned} y &= \sin^{-1} x \quad \text{or} \quad y = \arcsin x, & y &= \cos^{-1} x \quad \text{or} \quad y = \arccos x \\ y &= \tan^{-1} x \quad \text{or} \quad y = \arctan x, & y &= \cot^{-1} x \quad \text{or} \quad y = \operatorname{arccot} x \\ y &= \sec^{-1} x \quad \text{or} \quad y = \operatorname{arcsec} x, & y &= \csc^{-1} x \quad \text{or} \quad y = \operatorname{arccsc} x \end{aligned}$$

These equations are read “ y equals the arcsine of x ” or “ y equals $\arcsin x$ ” and so on.

Caution The -1 in the expressions for the inverse means “inverse.” It does *not* mean reciprocal. For example, the *reciprocal* of $\sin x$ is $(\sin x)^{-1} = 1/\sin x = \csc x$. ■

The graphs of the six inverse trigonometric functions are obtained by reflecting the graphs of the restricted trigonometric functions through the line $y = x$. Figure 1.59b shows the graph of $y = \arcsin x$, and Figure 1.60 shows the graphs of all six functions. We now take a closer look at two of these functions.

The Arcsine and Arccosine Functions

We define the arcsine and arccosine as functions whose values are angles (measured in radians) that belong to restricted domains of the sine and cosine functions.

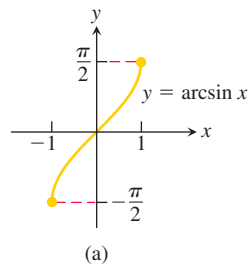
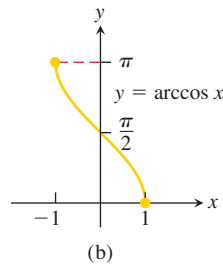
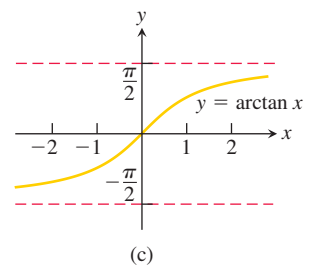
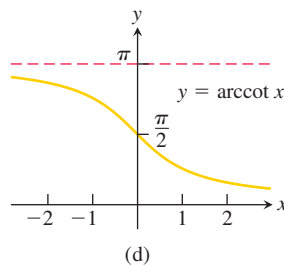
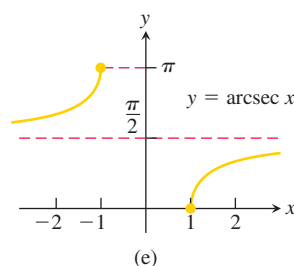
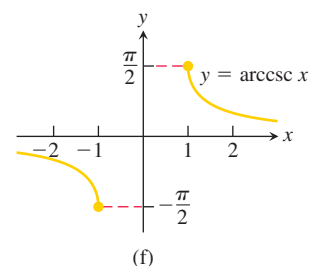
Domain: $-1 \leq x \leq 1$
 Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

 Domain: $-1 \leq x \leq 1$
 Range: $0 \leq y \leq \pi$

 Domain: $-\infty < x < \infty$
 Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

 Domain: $-\infty < x < \infty$
 Range: $0 < y < \pi$

 Domain: $x \leq -1$ or $x \geq 1$
 Range: $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$

 Domain: $x \leq -1$ or $x \geq 1$
 Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$


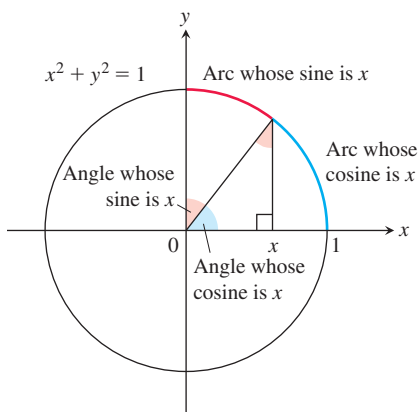
FIGURE 1.60 Graphs of the six basic inverse trigonometric functions.

DEFINITION

$y = \arcsin x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.
 $y = \arccos x$ is the number in $[0, \pi]$ for which $\cos y = x$.

The “Arc” in Arcsine and Arccosine

For a unit circle and radian angles, the arc length equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure. If $x = \sin \theta$, then, in addition to being the angle whose sine is x , θ is also the length of arc on the unit circle that subtends an angle whose sine is x . So we call θ “the arc whose sine is x .”



The graph of $y = \arcsin x$ (Figure 1.59b) is symmetric about the origin. The arcsine is therefore an odd function:

$$\arcsin(-x) = -\arcsin x. \quad (2)$$

The graph of $y = \arccos x$ (Figure 1.61b) has no such symmetry.

EXAMPLE 8 Evaluate (a) $\arcsin\left(\frac{\sqrt{3}}{2}\right)$ and (b) $\arccos\left(-\frac{1}{2}\right)$.

Solution

(a) We see that

$$\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

because $\sin(\pi/3) = \sqrt{3}/2$ and $\pi/3$ belongs to the range $[-\pi/2, \pi/2]$ of the arcsine function. See Figure 1.62a.

(b) We have

$$\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

because $\cos(2\pi/3) = -1/2$ and $2\pi/3$ belongs to the range $[0, \pi]$ of the arccosine function. See Figure 1.62b. ■

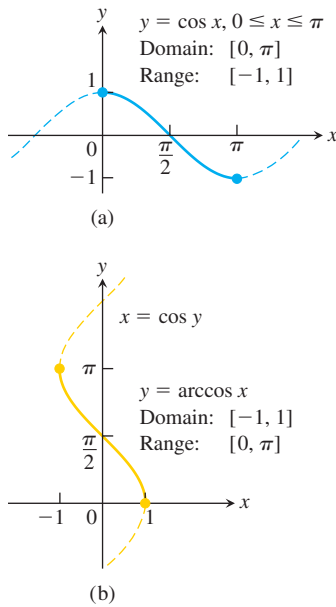


FIGURE 1.61 The graphs of (a) $y = \cos x, 0 \leq x \leq \pi$, and (b) its inverse, $y = \arccos x$. The graph of $\arccos x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

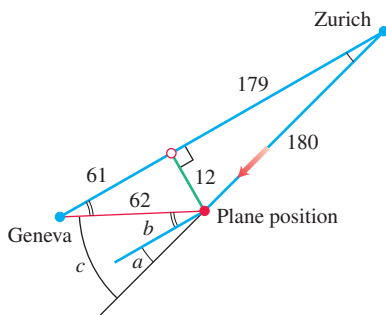


FIGURE 1.63 Diagram for drift correction (Example 9), with distances rounded to the nearest kilometer (drawing not to scale).

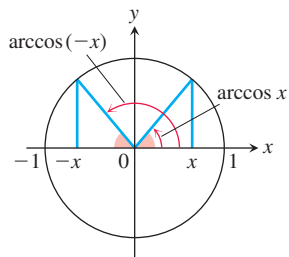


FIGURE 1.64 $\arccos x$ and $\arccos(-x)$ are supplementary angles (so their sum is π).

Using the same procedure illustrated in Example 8, we can create the following table of common values for the arcsine and arccosine functions.

x	$\arcsin x$	$\arccos x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$

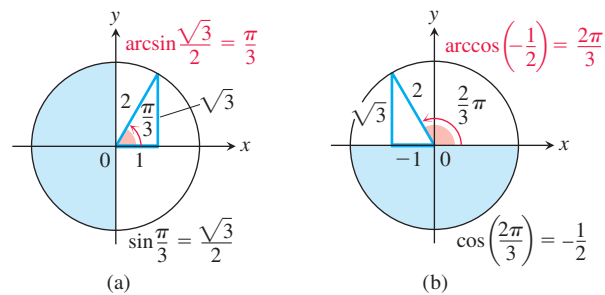


FIGURE 1.62 Values of the arcsine and arccosine functions (Example 8).

EXAMPLE 9 During a 240-km airplane flight from Zurich to Geneva, after flying 180 km the navigator determines that the plane is 12 km off course, as shown in Figure 1.63. Find the angle a for a course parallel to the original correct course, the angle b , and the drift correction angle $c = a + b$.

Solution From the Pythagorean theorem and given information, we compute an approximate hypothetical flight distance of 179 km, had the plane been flying along the original correct course (see Figure 1.63). Knowing the flight distance from Zurich to Geneva, we next calculate the remaining leg of the original course to be 61 km. Applying the Pythagorean theorem again then gives an approximate distance of 62 km from the position of the plane to Geneva. Finally, from Figure 1.63, we see that $180 \sin a = 12$ and $62 \sin b = 12$, so

$$a = \arcsin \frac{12}{180} \approx 0.067 \text{ radian} \approx 3.8^\circ$$

$$b = \arcsin \frac{12}{62} \approx 0.195 \text{ radian} \approx 11.2^\circ$$

$$c = a + b \approx 15^\circ.$$

Identities Involving Arcsine and Arccosine

As we can see from Figure 1.64, the arccosine of x satisfies the identity

$$\arccos x + \arccos(-x) = \pi, \quad (3)$$

or

$$\arccos(-x) = \pi - \arccos x. \quad (4)$$

Also, we can see from the triangle in Figure 1.65 that for $x > 0$,

$$\arcsin x + \arccos x = \pi/2. \quad (5)$$

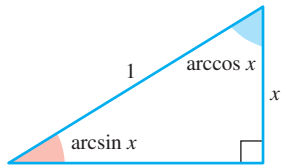


FIGURE 1.65 $\arcsin x$ and $\arccos x$ are complementary angles (so their sum is $\pi/2$).

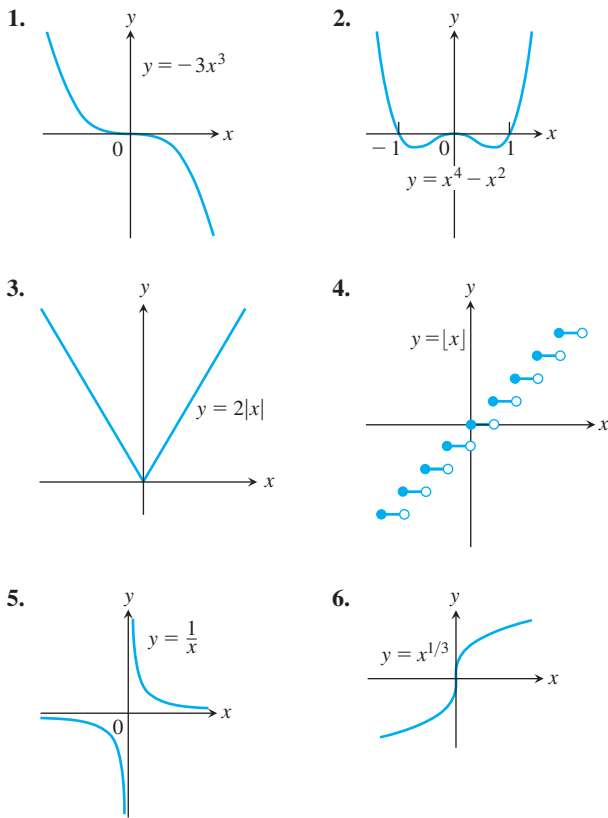
Equation (5) holds for the other values of x in $[-1, 1]$ as well, but we cannot conclude this from the triangle in Figure 1.65. It is, however, a consequence of Equations (2) and (4) (Exercise 80).

The arctangent, arccotangent, arcsecant, and arccosecant functions are defined in Section 3.9. There we develop additional properties of the inverse trigonometric functions using the identities discussed here.

EXERCISES 1.5

Identifying One-to-One Functions Graphically

Which of the functions graphed in Exercises 1–6 are one-to-one, and which are not?

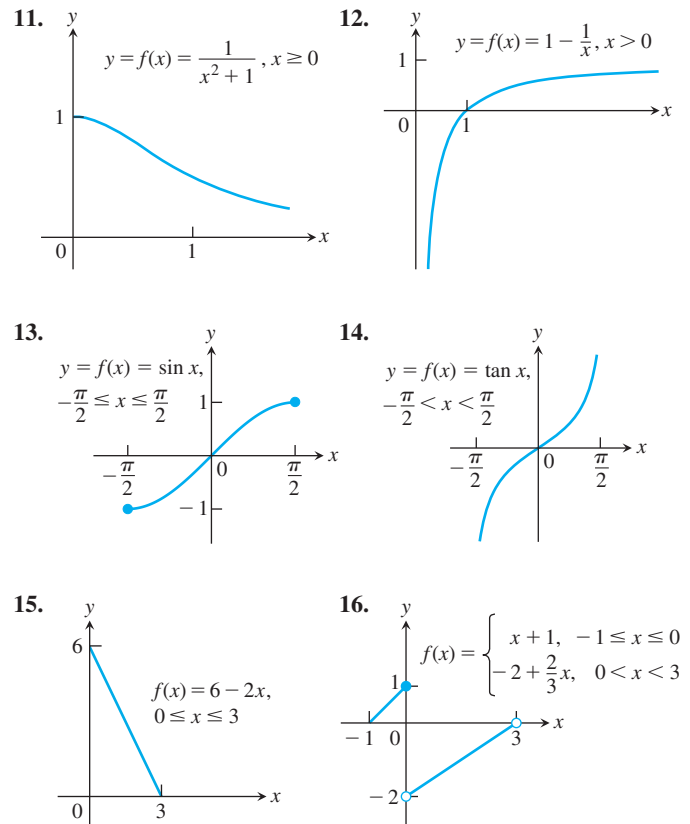


In Exercises 7–10, determine from its graph whether the function is one-to-one.

7. $f(x) = \begin{cases} 3 - x, & x < 0 \\ 3, & x \geq 0 \end{cases}$
8. $f(x) = \begin{cases} 2x + 6, & x \leq -3 \\ x + 4, & x > -3 \end{cases}$
9. $f(x) = \begin{cases} 1 - \frac{x}{2}, & x \leq 0 \\ \frac{x}{x+2}, & x > 0 \end{cases}$
10. $f(x) = \begin{cases} 2 - x^2, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

Graphing Inverse Functions

Each of Exercises 11–16 shows the graph of a function $y = f(x)$. Copy the graph and draw in the line $y = x$. Then use reflection with respect to the line $y = x$ to add the graph of f^{-1} to your sketch. (It is not necessary to find a formula for f^{-1} .) Identify the domain and range of f^{-1} .

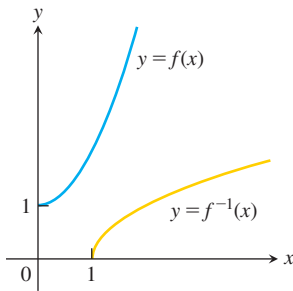


17. a. Graph the function $f(x) = \sqrt{1 - x^2}$, $0 \leq x \leq 1$. What symmetry does the graph have?
b. Show that f is its own inverse. (Remember that $\sqrt{x^2} = x$ if $x \geq 0$.)
18. a. Graph the function $f(x) = 1/x$. What symmetry does the graph have?
b. Show that f is its own inverse.

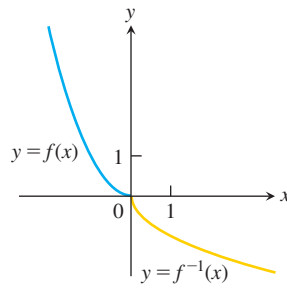
Formulas for Inverse Functions

Each of Exercises 19–24 gives a formula for a function $y = f(x)$ and shows the graphs of f and f^{-1} . Find a formula for f^{-1} in each case.

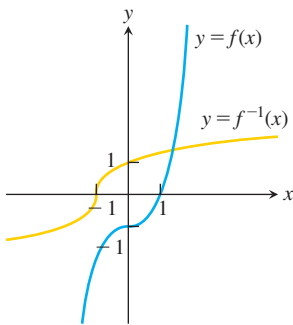
19. $f(x) = x^2 + 1, x \geq 0$



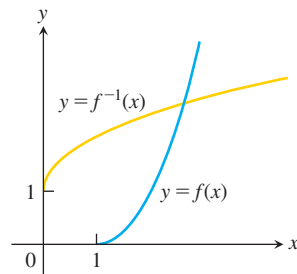
20. $f(x) = x^2, x \leq 0$



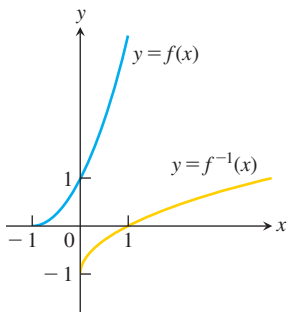
21. $f(x) = x^3 - 1$



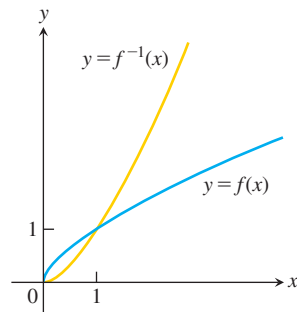
22. $f(x) = x^2 - 2x + 1, x \geq 1$



23. $f(x) = (x+1)^2, x \geq -1$



24. $f(x) = x^{2/3}, x \geq 0$



Each of Exercises 25–36 gives a formula for a function $y = f(x)$. In each case, find $f^{-1}(x)$ and identify the domain and range of f^{-1} . As a check, show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

25. $f(x) = x^5$

26. $f(x) = x^4, x \geq 0$

27. $f(x) = x^3 + 1$

28. $f(x) = (1/2)x - 7/2$

29. $f(x) = 1/x^2, x > 0$

30. $f(x) = 1/x^3, x \neq 0$

31. $f(x) = \frac{x+3}{x-2}$

32. $f(x) = \frac{\sqrt{x}}{\sqrt{x}-3}$

33. $f(x) = x^2 - 2x, x \leq 1$

34. $f(x) = (2x^3 + 1)^{1/5}$

(Hint: Complete the square.)

35. $f(x) = \frac{x+b}{x-2}, b > -2$ and constant

36. $f(x) = x^2 - 2bx, b > 0$ and constant, $x \leq b$

Inverses of Lines

37. a. Find the inverse of the function $f(x) = mx$, where m is a constant different from zero.
b. What can you conclude about the inverse of a function $y = f(x)$ whose graph is a line through the origin with a nonzero slope m ?
38. Show that the graph of the inverse of $f(x) = mx + b$, where m and b are constants and $m \neq 0$, is a line with slope $1/m$ and y -intercept $-b/m$.
39. a. Find the inverse of $f(x) = x + 1$. Graph f and its inverse together. Add the line $y = x$ to your sketch, drawing it with dashes or dots for contrast.
b. Find the inverse of $f(x) = x + b$ (b constant). How is the graph of f^{-1} related to the graph of f ?
c. What can you conclude about the inverses of functions whose graphs are lines parallel to the line $y = x$?
40. a. Find the inverse of $f(x) = -x + 1$. Graph the line $y = -x + 1$ together with the line $y = x$. At what angle do the lines intersect?
b. Find the inverse of $f(x) = -x + b$ (b constant). What angle does the line $y = -x + b$ make with the line $y = x$?
c. What can you conclude about the inverses of functions whose graphs are lines perpendicular to the line $y = x$?

Logarithms and Exponentials

41. Express the following logarithms in terms of $\ln 2$ and $\ln 3$.

a. $\ln 0.75$

b. $\ln(4/9)$

c. $\ln(1/2)$

d. $\ln \sqrt[3]{9}$

e. $\ln 3\sqrt{2}$

f. $\ln \sqrt{13.5}$

42. Express the following logarithms in terms of $\ln 5$ and $\ln 7$.

a. $\ln(1/125)$

b. $\ln 9.8$

c. $\ln 7\sqrt{7}$

d. $\ln 1225$

e. $\ln 0.056$

f. $(\ln 35 + \ln(1/7))/(\ln 25)$

Use the properties of logarithms to write the expressions in Exercises 43 and 44 as a single term.

43. a. $\ln \sin \theta - \ln\left(\frac{\sin \theta}{5}\right)$

b. $\ln(3x^2 - 9x) + \ln\left(\frac{1}{3x}\right)$

c. $\frac{1}{2}\ln(4t^4) - \ln b$

44. a. $\ln \sec \theta + \ln \cos \theta$

b. $\ln(8x + 4) - 2 \ln c$

c. $3 \ln \sqrt[3]{t^2 - 1} - \ln(t + 1)$

Find simpler expressions for the quantities in Exercises 45–48.

45. a. $e^{\ln 7.2}$

b. $e^{-\ln x^2}$

c. $e^{\ln x - \ln y}$

46. a. $e^{\ln(x^2 + y^2)}$

b. $e^{-\ln 0.3}$

c. $e^{\ln \pi x - \ln 2}$

47. a. $2 \ln \sqrt{e}$

b. $\ln(\ln e^e)$

c. $\ln(e^{-x^2 - y^2})$

48. a. $\ln(e^{\sec \theta})$

b. $\ln(e^{(e^x)})$

c. $\ln(e^{2 \ln x})$

In Exercises 49–54, solve for y in terms of t or x , as appropriate.

49. $\ln y = 2t + 4$

50. $\ln y = -t + 5$

51. $\ln(y - b) = 5t$

52. $\ln(c - 2y) = t$

53. $\ln(y - 1) - \ln 2 = x + \ln x$

54. $\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x)$

In Exercises 55 and 56, solve for k .

$$\begin{array}{lll} 55. \mathbf{a.} e^{2k} = 4 & \mathbf{b.} 100e^{10k} = 200 & \mathbf{c.} e^{k/1000} = a \\ 56. \mathbf{a.} e^{5k} = \frac{1}{4} & \mathbf{b.} 80e^k = 1 & \mathbf{c.} e^{(\ln 0.8)k} = 0.8 \end{array}$$

In Exercises 57–64, solve for t .

$$\begin{array}{lll} 57. \mathbf{a.} e^{-0.3t} = 27 & \mathbf{b.} e^{kt} = \frac{1}{2} & \mathbf{c.} e^{(\ln 0.2)t} = 0.4 \\ 58. \mathbf{a.} e^{-0.01t} = 1000 & \mathbf{b.} e^{kt} = \frac{1}{10} & \mathbf{c.} e^{(\ln 2)t} = \frac{1}{2} \\ 59. e^{\sqrt{t}} = x^2 & 60. e^{(x^2)}e^{(2x+1)} = e^t & \\ 61. e^{2t} - 3e^t = 0 & 62. e^{-2t} + 6 = 5e^{-t} & \\ 63. \ln\left(\frac{t}{t-1}\right) = 2 & 64. \ln(t-2) = \ln 8 - \ln t & \end{array}$$

Simplify the expressions in Exercises 65–68.

$$\begin{array}{lll} 65. \mathbf{a.} 5^{\log_5 7} & \mathbf{b.} 8^{\log_8 \sqrt{2}} & \mathbf{c.} 1.3^{\log_{1.3} 75} \\ \mathbf{d.} \log_4 16 & \mathbf{e.} \log_3 \sqrt{3} & \mathbf{f.} \log_4 \left(\frac{1}{4}\right) \\ 66. \mathbf{a.} 2^{\log_2 3} & \mathbf{b.} 10^{\log_{10} (1/2)} & \mathbf{c.} \pi^{\log_\pi 7} \\ \mathbf{d.} \log_{11} 121 & \mathbf{e.} \log_{121} 11 & \mathbf{f.} \log_3 \left(\frac{1}{9}\right) \\ 67. \mathbf{a.} 2^{\log_4 x} & \mathbf{b.} 9^{\log_3 x} & \mathbf{c.} \log_2 (e^{(\ln 2)(\sin x)}) \\ 68. \mathbf{a.} 25^{\log_5 (3x^2)} & \mathbf{b.} \log_e (e^x) & \mathbf{c.} \log_4 (2^{e^x \sin x}) \end{array}$$

Express the ratios in Exercises 69 and 70 as ratios of natural logarithms and simplify.

$$\begin{array}{lll} 69. \mathbf{a.} \frac{\log_2 x}{\log_3 x} & \mathbf{b.} \frac{\log_2 x}{\log_8 x} & \mathbf{c.} \frac{\log_x a}{\log_{x^2} a} \\ 70. \mathbf{a.} \frac{\log_9 x}{\log_3 x} & \mathbf{b.} \frac{\log_{\sqrt{10}} x}{\log_{\sqrt{2}} x} & \mathbf{c.} \frac{\log_a b}{\log_b a} \end{array}$$

Arcsine and Arccosine

In Exercises 71–74, find the exact value of each expression. Remember that $\sin^{-1} x$ and $\arcsin x$ are the same function, and, similarly, $\cos^{-1} x$ and $\arccos x$.

$$\begin{array}{lll} 71. \mathbf{a.} \sin^{-1}\left(-\frac{1}{2}\right) & \mathbf{b.} \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) & \mathbf{c.} \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) \\ 72. \mathbf{a.} \cos^{-1}\left(\frac{1}{2}\right) & \mathbf{b.} \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) & \mathbf{c.} \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \\ 73. \mathbf{a.} \arccos(-1) & \mathbf{b.} \arccos(0) & \\ 74. \mathbf{a.} \arcsin(-1) & \mathbf{b.} \arcsin\left(-\frac{1}{\sqrt{2}}\right) & \end{array}$$

Theory and Examples

75. If $f(x)$ is one-to-one, can anything be said about $g(x) = -f(x)$? Is it also one-to-one? Give reasons for your answer.
76. If $f(x)$ is one-to-one and $f(x)$ is never zero, can anything be said about $h(x) = 1/f(x)$? Is it also one-to-one? Give reasons for your answer.
77. Suppose that the range of g lies in the domain of f so that the composition $f \circ g$ is defined. If f and g are one-to-one, can anything be said about $f \circ g$? Give reasons for your answer.
78. If a composition $f \circ g$ is one-to-one, must g be one-to-one? Give reasons for your answer.

79. Find a formula for the inverse function f^{-1} and verify that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

$$\begin{array}{l} \mathbf{a.} f(x) = \frac{100}{1 + 2^{-x}} \\ \mathbf{b.} f(x) = \frac{50}{1 + 1.1^{-x}} \\ \mathbf{c.} f(x) = \frac{e^x - 1}{e^x + 1} \\ \mathbf{d.} f(x) = \frac{\ln x}{2 - \ln x} \end{array}$$

80. **The identity $\arcsin x + \arccos x = \pi/2$** Figure 1.65 establishes the identity for $0 < x < 1$. To establish it for the rest of $[-1, 1]$, verify by direct calculation that it holds for $x = 1, 0$, and -1 . Then, for values of x in $(-1, 0)$, let $x = -a$, $a > 0$, and apply Eqs. (3) and (5) to the sum $\arcsin(-a) + \arccos(-a)$.

81. Start with the graph of $y = \ln x$. Find an equation of the graph that results from

- shifting down 3 units.
- shifting right 1 unit.
- shifting left 1, up 3 units.
- shifting down 4, right 2 units.
- reflecting about the y -axis.
- reflecting about the line $y = x$.

82. Start with the graph of $y = \ln x$. Find an equation of the graph that results from

- vertical stretching by a factor of 2.
- horizontal stretching by a factor of 3.
- vertical compression by a factor of 4.
- horizontal compression by a factor of 2.

- T** 83. The equation $x^2 = 2^x$ has three solutions: $x = 2$, $x = 4$, and one other. Estimate the third solution as accurately as you can by graphing.

- T** 84. Could $x^{\ln 2}$ possibly be the same as $2^{\ln x}$ for $x > 0$? Graph the two functions and explain what you see.

- T** 85. **Radioactive decay** The half-life of a certain radioactive substance is 12 hours. There are 8 grams present initially.

- Express the amount of substance remaining as a function of time t .
- When will there be 1 gram remaining?

- T** 86. **Doubling your money** Determine how much time is required for a \$500 investment to double in value if interest is earned at the rate of 4.75% compounded annually.

- T** 87. **Population growth** The population of a town in California is 375,000 and is increasing at the rate of 2.25% per year. Predict when the population will be 1 million.

- T** 88. **Radon-222** The decay equation for radon-222 gas is known to be $y = y_0 e^{-0.18t}$, with t in days. About how long will it take the radon in a sealed sample of air to fall to 90% of its original value?

CHAPTER 1 Questions to Guide Your Review

1. What is a function? What is its domain? Its range? What is an arrow diagram for a function? Give examples.
2. What is the graph of a real-valued function of a real variable? What is the vertical line test?
3. What is a piecewise-defined function? Give examples.
4. What are the important types of functions frequently encountered in calculus? Give an example of each type.
5. What is meant by an increasing function? A decreasing function? Give an example of each.
6. What is an even function? An odd function? What symmetry properties do the graphs of such functions have? What advantage can we take of this? Give an example of a function that is neither even nor odd.
7. If f and g are real-valued functions, how are the domains of $f + g$, $f - g$, fg , and f/g related to the domains of f and g ? Give examples.
8. When is it possible to compose one function with another? Give examples of compositions and their values at various points. Does the order in which functions are composed ever matter?
9. How do you change the equation $y = f(x)$ to shift its graph vertically up or down by $|k|$ units? Horizontally to the left or right? Give examples.
10. How do you change the equation $y = f(x)$ to compress or stretch the graph by a factor $c > 1$? Reflect the graph across a coordinate axis? Give examples.
11. What is radian measure? How do you convert from radians to degrees? Degrees to radians?
12. Graph the six basic trigonometric functions. What symmetries do the graphs have?
13. What is a periodic function? Give examples. What are the periods of the six basic trigonometric functions?
14. Starting with the identity $\sin^2 \theta + \cos^2 \theta = 1$ and the formulas for $\cos(A + B)$ and $\sin(A + B)$, show how a variety of other trigonometric identities may be derived.
15. How does the formula for the general sine function $f(x) = A \sin((2\pi/B)(x - C)) + D$ relate to the shifting, stretching, compressing, and reflection of its graph? Give examples. Graph the general sine curve and identify the constants A , B , C , and D .
16. Name three issues that arise when functions are graphed using a calculator or computer with graphing software. Give examples.
17. What is an exponential function? Give examples. What laws of exponents does it obey? How does it differ from a simple power function like $f(x) = x^n$? What kind of real-world phenomena are modeled by exponential functions?
18. What is the number e , and how is it defined? What are the domain and range of $f(x) = e^x$? What does its graph look like? How do the values of e^x relate to x^2 , x^3 , and so on?
19. What functions have inverses? How do you know if two functions f and g are inverses of one another? Give examples of functions that are (are not) inverses of one another.
20. How are the domains, ranges, and graphs of functions and their inverses related? Give an example.
21. What procedure can you sometimes use to express the inverse of a function of x as a function of x ?
22. What is a logarithmic function? What properties does it satisfy? What is the natural logarithm function? What are the domain and range of $y = \ln x$? What does its graph look like?
23. How is the graph of $\log_a x$ related to the graph of $\ln x$? What truth is in the statement that there is really only one exponential function and one logarithmic function?
24. How are the inverse trigonometric functions defined? How can you sometimes use right triangles to find values of these functions? Give examples.

CHAPTER 1 Practice Exercises

Functions and Graphs

1. Express the area and circumference of a circle as functions of the circle's radius. Then express the area as a function of the circumference.
2. Express the radius of a sphere as a function of the sphere's surface area. Then express the surface area as a function of the volume.
3. A point P in the first quadrant lies on the parabola $y = x^2$. Express the coordinates of P as functions of the angle of inclination of the line joining P to the origin.
4. A hot-air balloon rising straight up from a level field is tracked by a range finder located 500 m from the point of liftoff. Express the balloon's height as a function of the angle the line from the range finder to the balloon makes with the ground.

In Exercises 5–8, determine whether the graph of the function is symmetric about the y -axis, the origin, or neither.

5. $y = x^{1/5}$
6. $y = x^{2/5}$
7. $y = x^2 - 2x - 1$
8. $y = e^{-x^2}$

In Exercises 9–16, determine whether the function is even, odd, or neither.

9. $y = x^2 + 1$
10. $y = x^5 - x^3 - x$
11. $y = 1 - \cos x$
12. $y = \sec x \tan x$
13. $y = \frac{x^4 + 1}{x^3 - 2x}$
14. $y = x - \sin x$
15. $y = x + \cos x$
16. $y = x \cos x$

17. Suppose that f and g are both odd functions defined on the entire real line. Which of the following (where defined) are even? odd?

- a. fg b. f^3 c. $f(\sin x)$ d. $g(\sec x)$ e. $|g|$

18. If $f(a - x) = f(a + x)$, show that $g(x) = f(x + a)$ is an even function.

In Exercises 19–32, find the (a) domain and (b) range.

19. $y = |x| - 2$ 20. $y = -2 + \sqrt{1 - x}$
 21. $y = \sqrt{16 - x^2}$ 22. $y = 3^{2-x} + 1$
 23. $y = 2e^{-x} - 3$ 24. $y = \tan(2x - \pi)$
 25. $y = 2 \sin(3x + \pi) - 1$ 26. $y = x^{2/5}$
 27. $y = \ln(x - 3) + 1$ 28. $y = -1 + \sqrt[3]{2 - x}$
 29. $y = 5 - \sqrt{x^2 - 2x - 3}$ 30. $y = 2 + \frac{3x^2}{x^2 + 4}$
 31. $y = 4 \sin\left(\frac{1}{x}\right)$ 32. $y = 3 \cos x + 4 \sin x$
 (Hint: A trig identity is required.)

33. State whether each function is increasing, decreasing, or neither.

- a. Volume of a sphere as a function of its radius
 b. Greatest integer function
 c. Height above Earth's sea level as a function of atmospheric pressure (assumed nonzero)
 d. Kinetic energy as a function of a particle's velocity

34. Find the largest interval on which the given function is increasing.

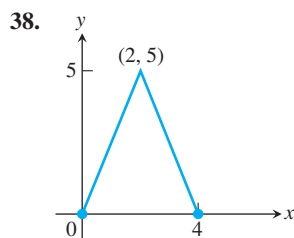
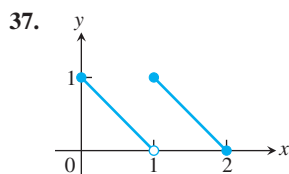
- a. $f(x) = |x - 2| + 1$ b. $f(x) = (x + 1)^4$
 c. $g(x) = (3x - 1)^{1/3}$ d. $R(x) = \sqrt{2x - 1}$

Piecewise-Defined Functions

In Exercises 35 and 36, find the (a) domain and (b) range.

35. $y = \begin{cases} \sqrt{-x}, & -4 \leq x \leq 0 \\ \sqrt{x}, & 0 < x \leq 4 \end{cases}$
 36. $y = \begin{cases} -x - 2, & -2 \leq x \leq -1 \\ x, & -1 < x \leq 1 \\ -x + 2, & 1 < x \leq 2 \end{cases}$

In Exercises 37 and 38, write a piecewise formula for the function.



Composition of Functions

In Exercises 39 and 40, find

- a. $(f \circ g)(-1)$ b. $(g \circ f)(2)$
 c. $(f \circ f)(x)$ d. $(g \circ g)(x)$
 39. $f(x) = \frac{1}{x}$, $g(x) = \frac{1}{\sqrt{x+2}}$
 40. $f(x) = 2 - x$, $g(x) = \sqrt[3]{x+1}$

In Exercises 41 and 42, (a) write formulas for $f \circ g$ and $g \circ f$ and find the (b) domain and (c) range of each.

41. $f(x) = 2 - x^2$, $g(x) = \sqrt{x+2}$
 42. $f(x) = \sqrt{x}$, $g(x) = \sqrt{1-x}$

For Exercises 43 and 44, sketch the graphs of f and $f \circ f$.

43. $f(x) = \begin{cases} -x - 2, & -4 \leq x \leq -1 \\ -1, & -1 < x \leq 1 \\ x - 2, & 1 < x \leq 2 \end{cases}$
 44. $f(x) = \begin{cases} x + 1, & -2 \leq x < 0 \\ x - 1, & 0 \leq x \leq 2 \end{cases}$

Composition with absolute values In Exercises 45–52, graph f_1 and f_2 together. Then describe how applying the absolute value function in f_2 affects the graph of f_1 .

$f_1(x)$	$f_2(x)$
45. x	$ x $
46. x^2	$ x ^2$
47. x^3	$ x^3 $
48. $x^2 + x$	$ x^2 + x $
49. $4 - x^2$	$ 4 - x^2 $
50. $\frac{1}{x}$	$\frac{1}{ x }$
51. $\sqrt{x}$	$\sqrt{ x }$
52. $\sin x$	$\sin x $

Shifting and Scaling Graphs

53. Suppose the graph of g is given. Write equations for the graphs that are obtained from the graph of g by shifting, scaling, or reflecting, as indicated.

- a. Up $\frac{1}{2}$ unit, right 3
 b. Down 2 units, left $\frac{2}{3}$
 c. Reflect about the y -axis
 d. Reflect about the x -axis
 e. Stretch vertically by a factor of 5
 f. Compress horizontally by a factor of 5

54. Describe how each graph is obtained from the graph of $y = f(x)$.

- a. $y = f(x - 5)$ b. $y = f(4x)$
 c. $y = f(-3x)$ d. $y = f(2x + 1)$
 e. $y = f\left(\frac{x}{3}\right) - 4$ f. $y = -3f(x) + \frac{1}{4}$

In Exercises 55–58, graph each function, not by plotting points, but by starting with the graph of one of the standard functions presented in Figures 1.15–1.17, and applying an appropriate transformation.

55. $y = -\sqrt{1 + \frac{x}{2}}$ 56. $y = 1 - \frac{x}{3}$
 57. $y = \frac{1}{2x^2} + 1$ 58. $y = (-5x)^{1/3}$

Trigonometry

In Exercises 59–62, sketch the graph of the given function. What is the period of the function?

59. $y = \cos 2x$

60. $y = \sin \frac{x}{2}$

61. $y = \sin \pi x$

62. $y = \cos \frac{\pi x}{2}$

63. Sketch the graph $y = 2 \cos\left(x - \frac{\pi}{3}\right)$.

64. Sketch the graph $y = 1 + \sin\left(x + \frac{\pi}{4}\right)$.

In Exercises 65–68, ABC is a right triangle with the right angle at C . The sides opposite angles A , B , and C are a , b , and c , respectively.

65. a. Find a and b if $c = 2$, $B = \pi/3$.

b. Find a and c if $b = 2$, $B = \pi/3$.

66. a. Express a in terms of A and c .

b. Express a in terms of A and b .

67. a. Express a in terms of B and b .

b. Express c in terms of A and a .

68. a. Express $\sin A$ in terms of a and c .

b. Express $\sin A$ in terms of b and c .

69. **Height of a pole** Two wires stretch from the top T of a vertical pole to points B and C on the ground, where C is 10 m closer to the base of the pole than is B . If wire BT makes an angle of 35° with the horizontal and wire CT makes an angle of 50° with the horizontal, how high is the pole?

70. **Height of a weather balloon** Observers at positions A and B 2 km apart simultaneously measure the angle of elevation of a weather balloon to be 40° and 70° , respectively. If the balloon is directly above a point on the line segment between A and B , find the height of the balloon.

T 71. a. Graph the function $f(x) = \sin x + \cos(x/2)$.

b. What appears to be the period of this function?

c. Confirm your finding in part (b) algebraically.

T 72. a. Graph $f(x) = \sin(1/x)$.

b. What are the domain and range of f ?

c. Is f periodic? Give reasons for your answer.

Transcendental Functions

In Exercises 73–76, find the domain of each function.

73. a. $f(x) = 1 + e^{-\sin x}$

b. $g(x) = e^x + \ln \sqrt{x}$

74. a. $f(x) = e^{1/x^2}$

b. $g(x) = \ln|4 - x^2|$

75. a. $h(x) = \sin^{-1}\left(\frac{x}{3}\right)$

b. $f(x) = \cos^{-1}(\sqrt{x} - 1)$

76. a. $h(x) = \ln(\cos^{-1} x)$

b. $f(x) = \sqrt{\pi - \sin^{-1} x}$

77. If $f(x) = \ln x$ and $g(x) = 4 - x^2$, find the functions $f \circ g$, $g \circ f$, $f \circ f$, $g \circ g$, and their domains.

78. Determine whether f is even, odd, or neither.

a. $f(x) = e^{-x^2}$

b. $f(x) = 1 + \sin^{-1}(-x)$

c. $f(x) = |e^x|$

d. $f(x) = e^{\ln|x|+1}$

T 79. Graph $\ln x$, $\ln 2x$, $\ln 4x$, $\ln 8x$, and $\ln 16x$ (as many as you can) together for $0 < x \leq 10$. What is going on? Explain.

T 80. Graph $y = \ln(x^2 + c)$ for $c = -4, -2, 0, 3$, and 5 . How does the graph change when c changes?

T 81. Graph $y = \ln|\sin x|$ in the window $0 \leq x \leq 22, -2 \leq y \leq 0$. Explain what you see. How could you change the formula to turn the arches upside down?

T 82. Graph the three functions $y = x^a$, $y = a^x$, and $y = \log_a x$ together on the same screen for $a = 2, 10$, and 20 . For large values of x , which of these functions has the largest values and which has the smallest values?

Theory and Examples

In Exercises 83 and 84, find the domain and range of each composite function. Then graph the compositions on separate screens. Do the graphs make sense in each case? Give reasons for your answers and comment on any differences you see.

83. a. $y = \sin^{-1}(\sin x)$

b. $y = \sin(\sin^{-1} x)$

84. a. $y = \cos^{-1}(\cos x)$

b. $y = \cos(\cos^{-1} x)$

85. Use a graph to decide whether f is one-to-one.

a. $f(x) = x^3 - \frac{x}{2}$

b. $f(x) = x^3 + \frac{x}{2}$

T 86. Use a graph to find to 3 decimal places the values of x for which $e^x > 10,000,000$.

87. a. Show that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of one another.

T b. Graph f and g over an x -interval large enough to show the graphs intersecting at $(1, 1)$ and $(-1, -1)$. Be sure the picture shows the required symmetry in the line $y = x$.

88. a. Show that $h(x) = x^3/4$ and $k(x) = (4x)^{1/3}$ are inverses of one another.

T b. Graph h and k over an x -interval large enough to show the graphs intersecting at $(2, 2)$ and $(-2, -2)$. Be sure the picture shows the required symmetry in the line $y = x$.

CHAPTER 1**Additional and Advanced Exercises****Functions and Graphs**

- Are there two functions f and g such that $f \circ g = g \circ f$? Give reasons for your answer.
- Are there two functions f and g with the following property? The graphs of f and g are not straight lines but the graph of $f \circ g$ is a straight line. Give reasons for your answer.
- If $f(x)$ is odd, can anything be said of $g(x) = f(x) - 2$? What if f is even instead? Give reasons for your answer.

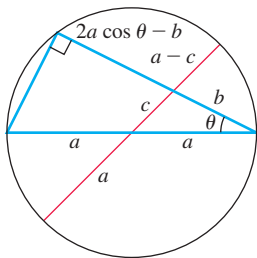
- If $g(x)$ is an odd function defined for all values of x , can anything be said about $g(0)$? Give reasons for your answer.
- Graph the equation $|x| + |y| = 1 + x$.
- Graph the equation $y + |y| = x + |x|$.

Derivations and Proofs

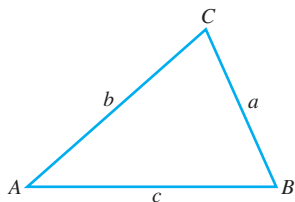
7. Prove the following identities.

a. $\frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}$ b. $\frac{1 - \cos x}{1 + \cos x} = \tan^2 \frac{x}{2}$

8. Explain the following “proof without words” of the law of cosines. (Source: Kung, Sidney H., “Proof Without Words: The Law of Cosines,” *Mathematics Magazine*, Vol. 63, no. 5, Dec. 1990, p. 342.)



9. Show that the area of triangle ABC is given by $(1/2)ab \sin C = (1/2)bc \sin A = (1/2)ca \sin B$.



10. Show that the area of triangle ABC is given by $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a+b+c)/2$ is the semiperimeter of the triangle.

11. Show that if f is both even and odd, then $f(x) = 0$ for every x in the domain of f .

12. a. **Even-odd decompositions** Let f be a function whose domain is symmetric about the origin, that is, $-x$ belongs to the domain whenever x does. Show that f is the sum of an even function and an odd function:

$$f(x) = E(x) + O(x),$$

where E is an even function and O is an odd function.

(Hint: Let $E(x) = (f(x) + f(-x))/2$. Show that $E(-x) = E(x)$, so that E is even. Then show that $O(x) = f(x) - E(x)$ is odd.)

b. **Uniqueness** Show that there is only one way to write f as the sum of an even and an odd function. (Hint: One way is given in part (a). If also $f(x) = E_1(x) + O_1(x)$ where E_1 is even and O_1 is odd, show that $E - E_1 = O_1 - O$. Then use Exercise 11 to show that $E = E_1$ and $O = O_1$.)

Effects of Parameters on Graphs

T 13. What happens to the graph of $y = ax^2 + bx + c$ as

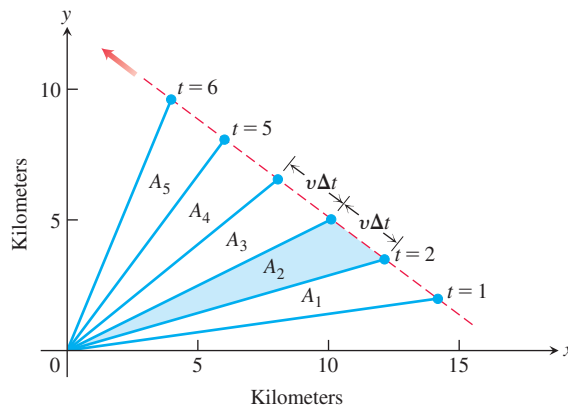
- a changes while b and c remain fixed?
- b changes (a and c fixed, $a \neq 0$)?
- c changes (a and b fixed, $a \neq 0$)?

T 14. What happens to the graph of $y = a(x+b)^3 + c$ as

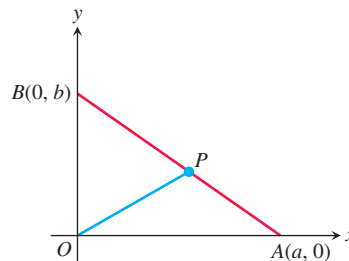
- a changes while b and c remain fixed?
- b changes (a and c fixed, $a \neq 0$)?
- c changes (a and b fixed, $a \neq 0$)?

Geometry

15. An object's center of mass moves at a constant velocity v along a straight line past the origin. The accompanying figure shows the coordinate system and the line of motion. The dots show positions that are 1 sec apart. Why are the areas $A_1, A_2, \dots, A_5$ in the figure all equal? As in Kepler's equal area law (see Section 12.6), the line that joins the object's center of mass to the origin sweeps out equal areas in equal times.



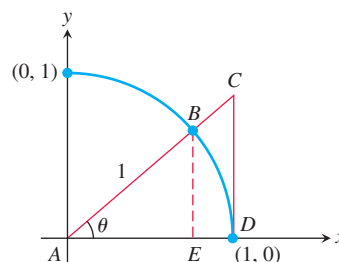
16. a. Find the slope of the line from the origin to the midpoint P of side AB in the triangle in the accompanying figure ($a, b > 0$).



b. When is OP perpendicular to AB ?

17. Consider the quarter-circle of radius 1 and right triangles ABE and ACD given in the accompanying figure. Use standard area formulas to conclude that

$$\frac{1}{2} \sin \theta \cos \theta < \frac{\theta}{2} < \frac{1}{2} \frac{\sin \theta}{\cos \theta}.$$



18. Let $f(x) = ax + b$ and $g(x) = cx + d$. What condition must be satisfied by the constants a, b, c, d in order that $(f \circ g)(x) = (g \circ f)(x)$ for every value of x ?

Theory and Examples

- 19. Domain and range** Suppose that $a \neq 0$, $b \neq 1$, and $b > 0$. Determine the domain and range of the function.

a. $y = a(b^{c-x}) + d$

b. $y = a \log_b(x - c) + d$

- 20. Inverse functions** Let

$$f(x) = \frac{ax + b}{cx + d}, \quad c \neq 0, \quad ad - bc \neq 0.$$

- a. Give a convincing argument that f is one-to-one.
b. Find a formula for the inverse of f .

- 21. Depreciation** Smith Hauling purchased an 18-wheel truck for \$100,000. The truck depreciates at the constant rate of \$10,000 per year for 10 years.

- a. Write an expression that gives the value y after x years.
b. When is the value of the truck \$55,000?

- 22. Drug absorption** A drug is administered intravenously for pain. The function

$$f(t) = 90 - 52 \ln(1 + t), \quad 0 \leq t \leq 4$$

gives the number of units of the drug remaining in the body after t hours.

- a. What was the initial number of units of the drug administered?
b. How much is present after 2 hours?
c. Draw the graph of f .

- 23. Finding investment time** If Juanita invests \$1500 in a retirement account that earns 8% compounded annually, how long will it take this single payment to grow to \$5000?

- 24. The rule of 70** If you use the approximation $\ln 2 \approx 0.70$ (in place of 0.69314 ...), you can derive a rule of thumb that says, “To estimate how many years it will take an amount of money to double when invested at r percent compounded continuously, divide r into 70.” For instance, an amount of money invested at 5% will double in about $70/5 = 14$ years. If you want it to double in 10 years instead, you have to invest it at $70/10 = 7\%$. Show how the rule of 70 is derived. (A similar “rule of 72” uses 72 instead of 70, because 72 has more integer factors.)

- 25.** For what $x > 0$ does $x^{(x^x)} = (x^x)^x$? Give reasons for your answer.

- T 26. a.** If $(\ln x)/x = (\ln 2)/2$, must $x = 2$?
b. If $(\ln x)/x = -2 \ln 2$, must $x = 1/2$?

Give reasons for your answers.

- 27.** The quotient $(\log_4 x)/(\log_2 x)$ has a constant value. What value? Give reasons for your answer.

- T 28. $\log_x(2)$ vs. $\log_2(x)$** How does $f(x) = \log_x(2)$ compare with $g(x) = \log_2(x)$? Here is one way to find out.

- a. Use the equation $\log_a b = (\ln b)/(\ln a)$ to express $f(x)$ and $g(x)$ in terms of natural logarithms.
b. Graph f and g together. Comment on the behavior of f in relation to the signs and values of g .

CHAPTER 1 Technology Application Projects

Mathematica/Maple Projects

Projects can be found within MyLab Math.

- An Overview of Mathematica**

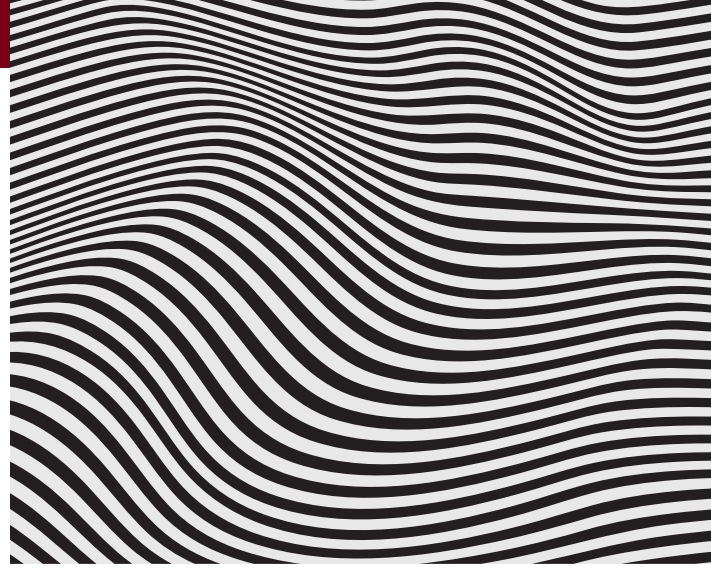
An overview of *Mathematica* sufficient to complete the *Mathematica* modules appearing on the Web site.

- Modeling Change: Springs, Driving Safety, Radioactivity, Trees, Fish, and Mammals**

Construct and interpret mathematical models, analyze and improve them, and make predictions using them.

2

Limits and Continuity



OVERVIEW In this chapter we develop the concept of a limit, first intuitively and then formally. We use limits to describe the way a function varies. Some functions vary *continuously*; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump, vary erratically, or tend to increase or decrease without bound. The notion of limit gives a precise way to distinguish among these behaviors.

2.1 Rates of Change and Tangent Lines to Curves

HISTORICAL BIOGRAPHY

Galileo Galilei
(1564–1642)

Galileo was an Italian mathematician and astronomer. He attempted to apply mathematics to his work in astronomy, physics of kinematics, and strength of materials.

To know more, visit the companion Website.

Average and Instantaneous Speed

In the late sixteenth century, Galileo discovered that a solid object dropped from rest (initially not moving) near the surface of the earth and allowed to fall freely, will fall a distance proportional to the square of the time it has been falling. This type of motion is called **free fall**. It assumes negligible air resistance to slow the object down, and it assumes that gravity is the only force acting on the falling object. If y denotes the distance fallen in meters after t seconds, then Galileo's law is

$$y = 4.9t^2 \text{ m,}$$

where 4.9 is the (approximate) constant of proportionality.

More generally, suppose that a moving object has traveled distance $f(t)$ at time t . The object's **average speed** during an interval of time $[t_1, t_2]$ is found by dividing the distance traveled $f(t_2) - f(t_1)$ by the time elapsed $t_2 - t_1$. The unit of measure is length per unit time: kilometers per hour, meters per second, or whatever is appropriate to the problem at hand.

Average Speed

When $f(t)$ measures the distance traveled at time t ,

$$\text{Average speed over } [t_1, t_2] = \frac{\text{distance traveled}}{\text{elapsed time}} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}.$$

EXAMPLE 1 A rock breaks loose from the top of a tall cliff. What is its average speed

- (a) during the first 2 seconds of fall?
- (b) during the 1-second interval between second 1 and second 2?

Δ is the capital Greek letter Delta.

Solution The average speed of the rock during a given time interval is the change in distance, Δy , divided by the length of the time interval, Δt . (Increments like Δy and Δt are reviewed in Appendix A.4, and pronounced “delta y” and “delta t.”) Measuring distance in meters and time in seconds, we have the following calculations:

$$(a) \text{ For the first 2 seconds: } \frac{\Delta y}{\Delta t} = \frac{4.9(2)^2 - 4.9(0)^2}{2 - 0} = 9.8 \text{ m/s}$$

$$(b) \text{ From second 1 to second 2: } \frac{\Delta y}{\Delta t} = \frac{4.9(2)^2 - 4.9(1)^2}{2 - 1} = 14.7 \text{ m/s} \quad \blacksquare$$

We want a way to determine the speed of a falling object at a single instant t_0 , instead of using its average speed over an interval of time. To do this, we examine what happens when we calculate the average speed over shorter and shorter time intervals starting at t_0 . The next example illustrates this process. Our discussion is informal here but will be made precise in Chapter 3.

EXAMPLE 2 Find the speed of the falling rock in Example 1 at $t = 1$ and $t = 2$ s.

Solution We can calculate the average speed of the rock over a time interval $[t_0, t_0 + h]$, having length $\Delta t = h$, as

$$\frac{\Delta y}{\Delta t} = \frac{4.9(t_0 + h)^2 - 4.9t_0^2}{h}. \quad (1)$$

We cannot use this formula to calculate the “instantaneous” speed at the exact moment t_0 by simply substituting $h = 0$, because we cannot divide by zero. But we *can* use it to calculate average speeds over increasingly short time intervals starting at $t_0 = 1$ and $t_0 = 2$. When we do so, by taking smaller and smaller values of h , we see a pattern (Table 2.1).

TABLE 2.1 Average speeds over short time intervals $[t_0, t_0 + h]$

Average speed: $\frac{\Delta y}{\Delta t} = \frac{4.9(t_0 + h)^2 - 4.9t_0^2}{h}$		
Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
1	14.7	24.5
0.1	10.29	20.09
0.01	9.849	19.649
0.001	9.8049	19.6049
0.0001	9.80049	19.60049

The average speed on intervals starting at $t_0 = 1$ seems to approach a limiting value of 9.8 as the length of the interval decreases. This suggests that the rock is falling at a speed of 9.8 m/s at $t_0 = 1$ s. Let’s confirm this algebraically.

If we set $t_0 = 1$ and then expand the numerator in Equation (1) and simplify, we find that

$$\begin{aligned}\frac{\Delta y}{\Delta t} &= \frac{4.9(1+h)^2 - 4.9(1)^2}{h} = \frac{4.9(1+2h+h^2) - 4.9}{h} \\ &= \frac{9.8h + 4.9h^2}{h} = 9.8 + 4.9h. \quad \text{Can cancel } h \text{ when } h \neq 0\end{aligned}$$

For values of h different from 0, the expressions on the right and left are equivalent and the average speed is $9.8 + 4.9h$ m/sec. We can now see why the average speed has the limiting value $9.8 + 4.9(0) = 9.8$ m/sec as h approaches 0.

Similarly, setting $t_0 = 2$ in Equation (1), for values of h different from 0 the procedure yields

$$\frac{\Delta y}{\Delta t} = 19.6 + 4.9h.$$

As h gets closer and closer to 0, the average speed has the limiting value 19.6 m/s when $t_0 = 2$ s, as suggested by Table 2.1. ■

The average speed of a falling object is an example of a more general idea, an average rate of change.

Average Rates of Change and Secant Lines

Given any function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in the value of y , $\Delta y = f(x_2) - f(x_1)$, by the length $\Delta x = x_2 - x_1 = h$ of the interval over which the change occurs. (We use the symbol h for Δx to simplify the notation here and later on.)

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0.$$

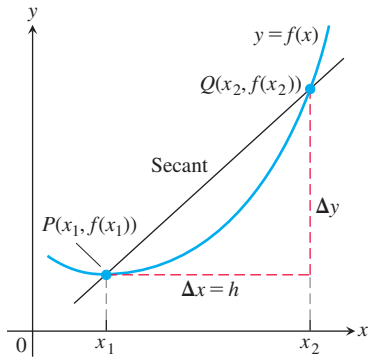


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ (Figure 2.1). In geometry, a line joining two points of a curve is called a **secant line**. Thus, the average rate of change of f from x_1 to x_2 is identical to the slope of secant line PQ . As the point Q approaches the point P along the curve, the length h of the interval over which the change occurs approaches zero. We will see that this procedure leads to the definition of the slope of a curve at a point.

Defining the Slope of a Curve

We know what is meant by the slope of a straight line, which tells us the rate at which it rises or falls—its rate of change as a linear function. But what is meant by the *slope of a curve* at a point P on the curve? If there were a *tangent line* to the curve at P —a line that grazes the curve like the tangent line to a circle—it would be reasonable to identify the slope of the tangent line as the slope of the curve at P . We will see that, among all the lines that pass through the point P , the tangent line is the one that gives the best approximation to the curve at P . We need a precise way to specify the tangent line at a point on a curve.

Specifying a tangent line to a circle is straightforward. A line L is tangent to a circle at a point P if L passes through P and is perpendicular to the radius at P (Figure 2.2). But what does it mean to say that a line L is tangent to a more general curve at a point P ?

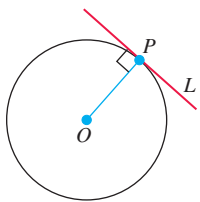


FIGURE 2.2 L is tangent to the circle at P if it passes through P perpendicular to radius OP .

HISTORICAL BIOGRAPHY**Pierre de Fermat
(1601–1665)**

Fermat was born to a prosperous family in France. He studied the classics and mastered Latin, Greek, Italian, and Spanish.

To know more, visit the companion Website.

To define tangency for general curves, we use an approach that analyzes the behavior of the secant lines that pass through P and nearby points Q as Q moves toward P along the curve (Figure 2.3). We start with what we *can* calculate, namely the slope of the secant line PQ . We then compute the limiting value of the secant line's slope as Q approaches P along the curve. (We clarify the limit idea in the next section.) If the limit exists, we take it to be the slope of the curve at P and *define* the tangent line to the curve at P to be the line through P with this slope.

The next example illustrates the geometric idea for finding the tangent line to a curve.

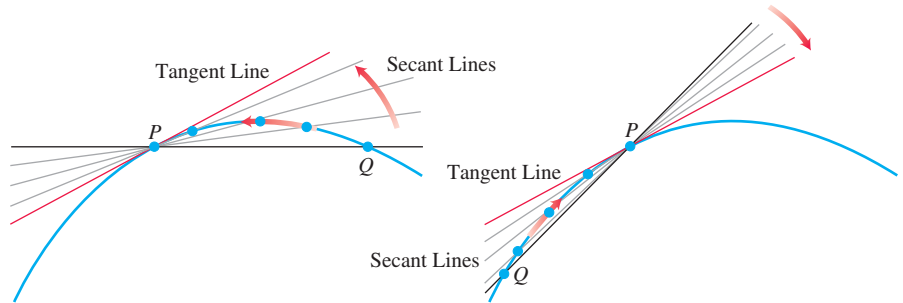


FIGURE 2.3 The tangent line to the curve at P is the line through P whose slope is the limit of the secant line slopes as $Q \rightarrow P$ from either side.

EXAMPLE 3 Find the slope of the tangent line to the parabola $y = x^2$ at the point $(2, 4)$ by analyzing the slopes of secant lines through $(2, 4)$. Write an equation for the tangent line to the parabola at this point.

Solution We begin with a secant line through $P(2, 4)$ and a nearby point $Q(2 + h, (2 + h)^2)$, as shown in Figure 2.4. We then write an expression for the slope of the secant line PQ and investigate what happens to the slope as Q approaches P along the curve:

$$\begin{aligned} \text{Secant line slope} &= \frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 2^2}{h} = \frac{h^2 + 4h + 4 - 4}{h} \\ &= \frac{h^2 + 4h}{h} = h + 4. \end{aligned}$$

If $h > 0$, then Q lies above and to the right of P , as in Figure 2.4. If $h < 0$, then Q lies to the left of P (not shown). In either case, as Q approaches P along the curve, h approaches zero and the secant line slope $h + 4$ approaches 4. We take 4 to be the parabola's slope at P .

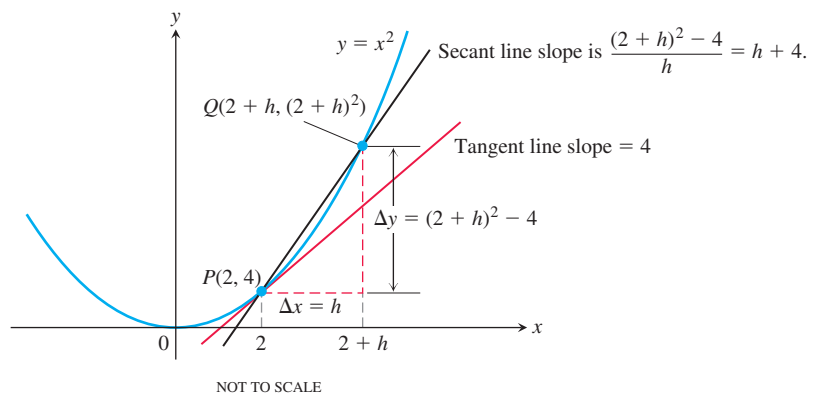


FIGURE 2.4 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ as the limit of secant line slopes (Example 3).

The tangent line to the parabola at P is the line through P with slope 4:

$$y = 4 + 4(x - 2) \quad \text{Point-slope equation}$$

$$y = 4x - 4. \quad \text{Simplify.}$$

Rates of Change and Tangent Lines

The rates at which the rock in Example 2 was falling at the instants $t = 1$ and $t = 2$ are called *instantaneous rates of change*. Instantaneous rates of change and slopes of tangent lines are closely connected, as we see in the following examples.

EXAMPLE 4 Figure 2.5 shows how a population p of fruit flies (*Drosophila*) grew in a 50-day experiment. The flies were counted at regular intervals, the counted values plotted with respect to the number of elapsed days t , and the points joined by a smooth curve (colored blue in Figure 2.5). Find the average growth rate from day 23 to day 45.

Solution There were 150 flies on day 23 and 340 flies on day 45. Thus the number of flies increased by $340 - 150 = 190$ in $45 - 23 = 22$ days. The average rate of change of the population from day 23 to day 45 was

$$\frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day.}$$

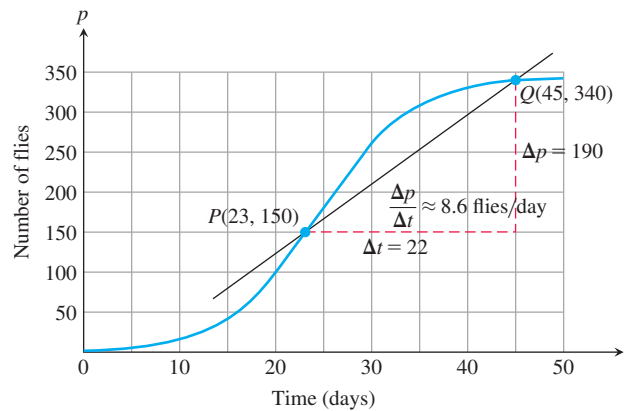


FIGURE 2.5 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p/\Delta t$ of the secant line (Example 4).

This average is the slope of the secant line through the points P and Q on the graph in Figure 2.5.

The average rate of change from day 23 to day 45 calculated in Example 4 does not tell us how fast the population was changing on day 23 itself. For that we need to examine time intervals closer to the day in question.

EXAMPLE 5 How fast was the number of flies in the population of Example 4 growing on day 23?

Solution To answer this question, we examine the average rates of change over shorter and shorter time intervals starting at day 23. In geometric terms, we find these rates by calculating the slopes of secant lines from P to Q , for a sequence of points Q approaching P along the curve (Figure 2.6).

Q	Slope of $PQ = \Delta p / \Delta t$ (flies/day)
$(45, 340)$	$\frac{340 - 150}{45 - 23} \approx 8.6$
$(40, 330)$	$\frac{330 - 150}{40 - 23} \approx 10.6$
$(35, 310)$	$\frac{310 - 150}{35 - 23} \approx 13.3$
$(30, 265)$	$\frac{265 - 150}{30 - 23} \approx 16.4$

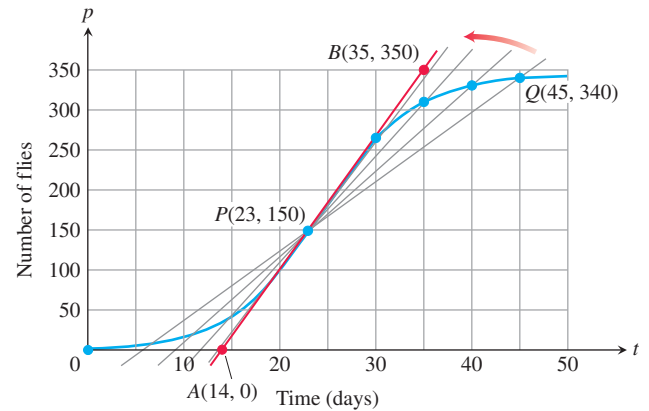


FIGURE 2.6 The positions and slopes of four secant lines through the point P on the fruit fly graph (Example 5).

The values in the table show that the secant line slopes rise from 8.6 to 16.4 as the t -coordinate of Q decreases from 45 to 30, and we would expect the slopes to rise slightly higher as t continued decreasing toward 23. Geometrically, the secant lines rotate counter-clockwise about P and seem to approach the red tangent line in the figure. Since the line appears to pass through the points $(14, 0)$ and $(35, 350)$, its slope is approximately

$$\frac{350 - 0}{35 - 14} = 16.7 \text{ flies/day.}$$

On day 23 the population was increasing at a rate of about 16.7 flies/day. ■

The instantaneous rate of change is the value the average rate of change approaches as the length h of the interval over which the change occurs approaches zero. The average rate of change corresponds to the slope of a secant line; the instantaneous rate corresponds to the slope of the tangent line at a fixed value. So instantaneous rates and slopes of tangent lines are closely connected. We give a precise definition for these terms in the next chapter, but to do so we first need to develop the concept of a *limit*.

EXERCISES 2.1

Average Rates of Change

In Exercises 1–6, find the average rate of change of the function over the given interval or intervals.

- $f(x) = x^3 + 1$
 - $[2, 3]$
 - $[-1, 1]$
- $g(x) = x^2 - 2x$
 - $[1, 3]$
 - $[-2, 4]$
- $h(t) = \cot t$
 - $[\pi/4, 3\pi/4]$
 - $[\pi/6, \pi/2]$
- $g(t) = 2 + \cos t$
 - $[0, \pi]$
 - $[-\pi, \pi]$
- $R(\theta) = \sqrt{4\theta + 1}$; $[0, 2]$
- $P(\theta) = \theta^3 - 4\theta^2 + 5\theta$; $[1, 2]$

Slope of a Curve at a Point

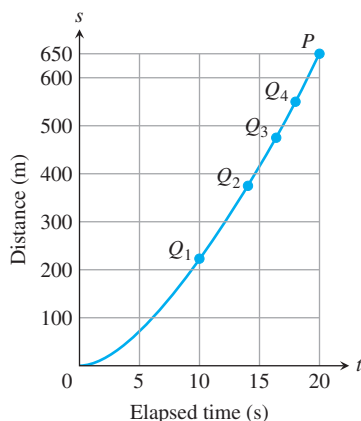
In Exercises 7–18, use the method in Example 3 to find (a) the slope of the curve at the given point P , and (b) an equation of the tangent line at P .

- $y = x^2 - 5$, $P(2, -1)$
- $y = 7 - x^2$, $P(2, 3)$
- $y = x^2 - 2x - 3$, $P(2, -3)$
- $y = x^2 - 4x$, $P(1, -3)$
- $y = x^3$, $P(2, 8)$
- $y = 2 - x^3$, $P(1, 1)$
- $y = x^3 - 12x$, $P(1, -11)$
- $y = x^3 - 3x^2 + 4$, $P(2, 0)$
- $y = \frac{1}{x}$, $P(-2, -1/2)$

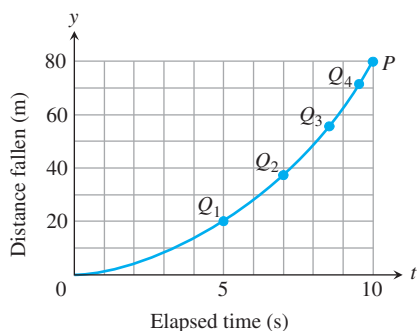
16. $y = \frac{x}{2-x}$, $P(4, -2)$
 17. $y = \sqrt{x}$, $P(4, 2)$
 18. $y = \sqrt{7-x}$, $P(-2, 3)$

Instantaneous Rates of Change

- T** 19. **Speed of a car** The accompanying figure shows the time-to-distance graph for a sports car accelerating from a standstill.



- a. Estimate the slopes of secant lines PQ_1 , PQ_2 , PQ_3 , and PQ_4 , arranging them in order in a table like the one in Figure 2.6. What are the appropriate units for these slopes?
 b. Then estimate the car's speed at time $t = 20$ s.
- T** 20. The accompanying figure shows the plot of distance fallen versus time for an object that fell from the lunar landing module a distance 80 m to the surface of the moon.
- a. Estimate the slopes of the secant lines PQ_1 , PQ_2 , PQ_3 , and PQ_4 , arranging them in a table like the one in Figure 2.6.
 b. About how fast was the object going when it hit the surface?



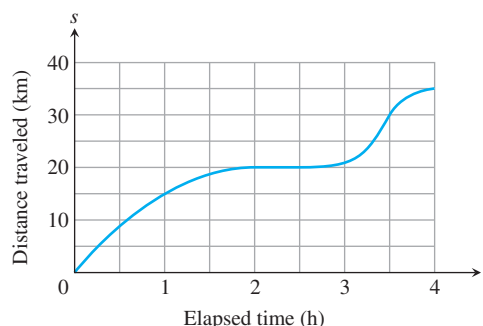
- T** 21. The profits of a small company for each of the first five years of its operation are given in the following table:

Year	Profit in \$1000s
2017	6
2018	27
2019	62
2020	111
2021	174

- a. Plot points representing the profit as a function of year, and join them by as smooth a curve as you can.

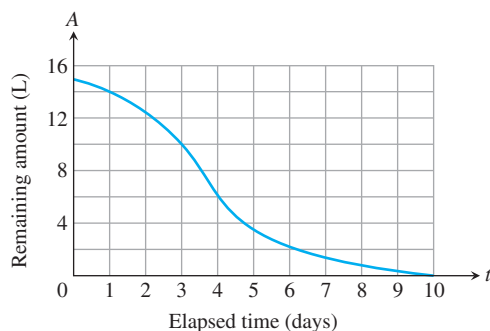
- b. What is the average rate of increase of the profits between 2019 and 2021?
 c. Use your graph to estimate the rate at which the profits were changing in 2019.
- T** 22. Make a table of values for the function $F(x) = (x+2)/(x-2)$ at the points $x = 1.2$, $x = 11/10$, $x = 101/100$, $x = 1001/1000$, $x = 10001/10000$, and $x = 1$.
- a. Find the average rate of change of $F(x)$ over the intervals $[1, x]$ for each $x \neq 1$ in your table.
 b. Extending the table if necessary, try to determine the rate of change of $F(x)$ at $x = 1$.
- T** 23. Let $g(x) = \sqrt{x}$ for $x \geq 0$.
- a. Find the average rate of change of $g(x)$ with respect to x over the intervals $[1, 2]$, $[1, 1.5]$ and $[1, 1+h]$.
 b. Make a table of values of the average rate of change of g with respect to x over the interval $[1, 1+h]$ for some values of h approaching zero, say $h = 0.1, 0.01, 0.001, 0.0001, 0.00001$, and 0.000001 .
 c. What does your table indicate is the rate of change of $g(x)$ with respect to x at $x = 1$?
- T** 24. Let $f(t) = 1/t$ for $t \neq 0$.
- a. Find the average rate of change of f with respect to t over the intervals (i) from $t = 2$ to $t = 3$, and (ii) from $t = 2$ to $t = T$.
 b. Make a table of values of the average rate of change of f with respect to t over the interval $[2, T]$, for some values of T approaching 2, say $T = 2.1, 2.01, 2.001, 2.0001, 2.00001$, and 2.000001 .
 c. What does your table indicate is the rate of change of f with respect to t at $t = 2$?

25. The accompanying graph shows the total distance s traveled by a bicyclist after t hours.



- a. Estimate the bicyclist's average speed over the time intervals $[0, 1]$, $[1, 2.5]$, and $[2.5, 3.5]$.
 b. Estimate the bicyclist's instantaneous speed at the times $t = \frac{1}{2}$, $t = 2$, and $t = 3$.
 c. Estimate the bicyclist's maximum speed and the specific time at which it occurs.

26. The accompanying graph shows the total amount of gasoline A in the gas tank of a motorcycle after being driven for t days.



- Estimate the average rate of gasoline consumption over the time intervals $[0, 3]$, $[0, 5]$, and $[7, 10]$.
- Estimate the instantaneous rate of gasoline consumption at the times $t = 1$, $t = 4$, and $t = 8$.
- Estimate the maximum rate of gasoline consumption and the specific time at which it occurs.

2.2 Limit of a Function and Limit Laws

HISTORICAL ESSAY

Limits

To read this essay, visit the companion Website.

In Section 2.1 we saw how limits arise when finding the instantaneous rate of change of a function or the tangent line to a curve. We begin this section by presenting an informal definition of the limit of a function. We then describe laws that capture the behavior of limits. These laws enable us to quickly compute limits for a variety of functions, including polynomials and rational functions. We will present the precise definition of a limit in Section 2.3.

Limits of Function Values

Frequently, when studying a function $y = f(x)$, we find ourselves interested in the function's behavior *near* a particular point c , but not *at* c itself. An important example occurs when the process of trying to evaluate a function at c leads to division by zero, which is undefined. We encountered this when seeking the instantaneous rate of change in y by considering the quotient function $\Delta y/h$ for h closer and closer to zero. In the next example we explore numerically how a function behaves near a particular point at which we cannot directly evaluate the function.

EXAMPLE 1 How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Solution The given formula defines f for all real numbers x except $x = 1$ (since we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \quad \text{for } x \neq 1.$$

The graph of f is the line $y = x + 1$ with the point $(1, 2)$ removed. This removed point is shown as a “hole” in Figure 2.7. Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ as close as we want to 2 by choosing x close enough to 1 (Table 2.2).

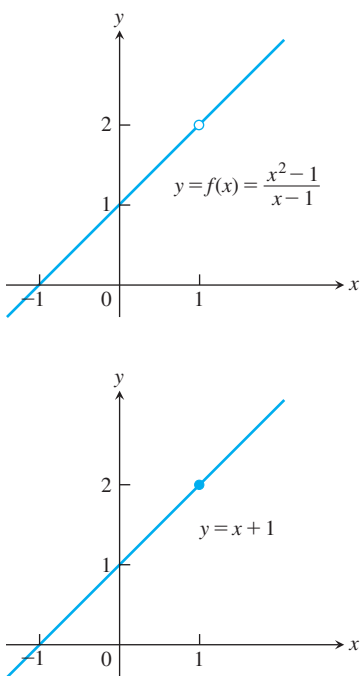


FIGURE 2.7 The graph of f is identical to the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 1).

We will illustrate some other types of behavior near a point in Example 3.

An Informal Description of the Limit of a Function

We now give an informal definition of the limit of a function f at an interior point of the domain of f . Suppose that $f(x)$ is defined on an open interval about c , *except possibly at* c .

TABLE 2.2 As x gets closer to 1, $f(x)$ gets closer to 2.

x	$f(x) = \frac{x^2 - 1}{x - 1}$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

itself. If $f(x)$ is arbitrarily close to the number L (that is, as close to L as we like) for all x sufficiently close to c , other than c itself, then we say that f approaches the **limit** L as x approaches c , and write

$$\lim_{x \rightarrow c} f(x) = L.$$

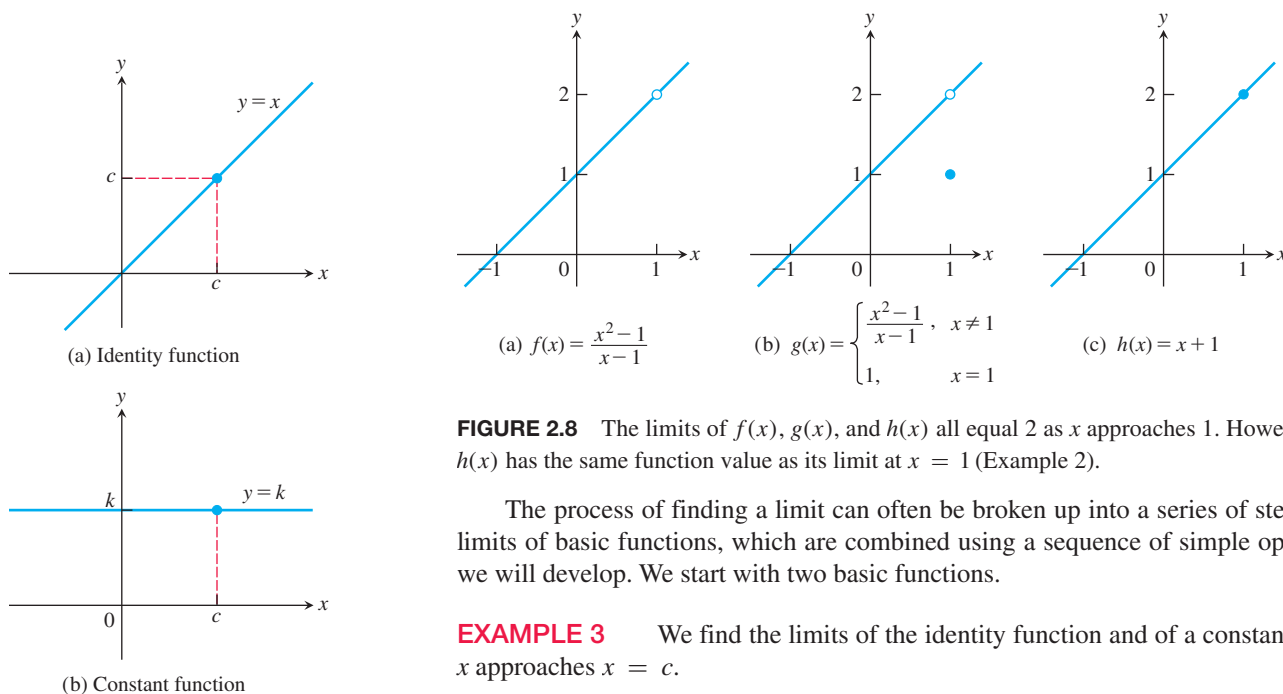
This is read “the limit of $f(x)$ as x approaches c is L .” In Example 1 we would say that $f(x)$ approaches the *limit* 2 as x approaches 1, and write

$$\lim_{x \rightarrow 1} f(x) = 2, \quad \text{or} \quad \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2.$$

Essentially, the definition says that the values of $f(x)$ are close to the number L whenever x is close to c . The value of the function at c itself is not considered.

Our definition here is informal, because phrases like *arbitrarily close* and *sufficiently close* are imprecise; their meaning depends on the context. (To a machinist manufacturing a piston, *close* may mean *within a few hundredths of a millimeter*. To an astronomer studying distant galaxies, *close* may mean *within a few thousand light-years*.) Nevertheless, the definition is clear enough to enable us to recognize and evaluate limits of many specific functions. We will need the precise definition given in Section 2.3 when we set out to prove theorems about limits or study complicated functions. Here are several more examples exploring the idea of limits.

EXAMPLE 2 The limit of a function does not depend on how the function is defined at the point being approached. It does not even matter whether the function is defined at that point. Consider the three functions in Figure 2.8. The function f has limit 2 as $x \rightarrow 1$ even though f is not defined at $x = 1$. The function g has limit 2 as $x \rightarrow 1$ even though $2 \neq g(1)$. The function h is the only one of the three functions in Figure 2.8 whose limit as $x \rightarrow 1$ equals its value at $x = 1$. For h , we have $\lim_{x \rightarrow 1} h(x) = h(1)$. This equality of limit and function value has an important meaning. As illustrated by the three examples in Figure 2.8, equality of limit and function value captures the notion of “continuity.” We study this in detail in Section 2.6. ■

**FIGURE 2.9** The functions in Example 3 have limits at all points c .**FIGURE 2.8** The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 2).

The process of finding a limit can often be broken up into a series of steps involving limits of basic functions, which are combined using a sequence of simple operations that we will develop. We start with two basic functions.

EXAMPLE 3 We find the limits of the identity function and of a constant function as x approaches $x = c$.

(a) If f is the **identity function** $f(x) = x$, then for any value of c (Figure 2.9a),

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} x = c.$$

- (b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of c (Figure 2.9b),

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} k = k.$$

For instances of each of these rules we have

$$\lim_{x \rightarrow 3} x = 3 \quad \text{Limit of identity function at } x = 3$$

and

$$\lim_{x \rightarrow -7} 4 = \lim_{x \rightarrow 2} 4 = 4. \quad \begin{array}{l} \text{Limit of constant function} \\ f(x) = 4 \text{ at } x = -7 \text{ or at } x = 2 \end{array}$$

We prove these rules in Example 3 in Section 2.3. ■

A function may not have a limit at a particular point. Some ways that limits can fail to exist are illustrated in Figure 2.10 and described in the next example.

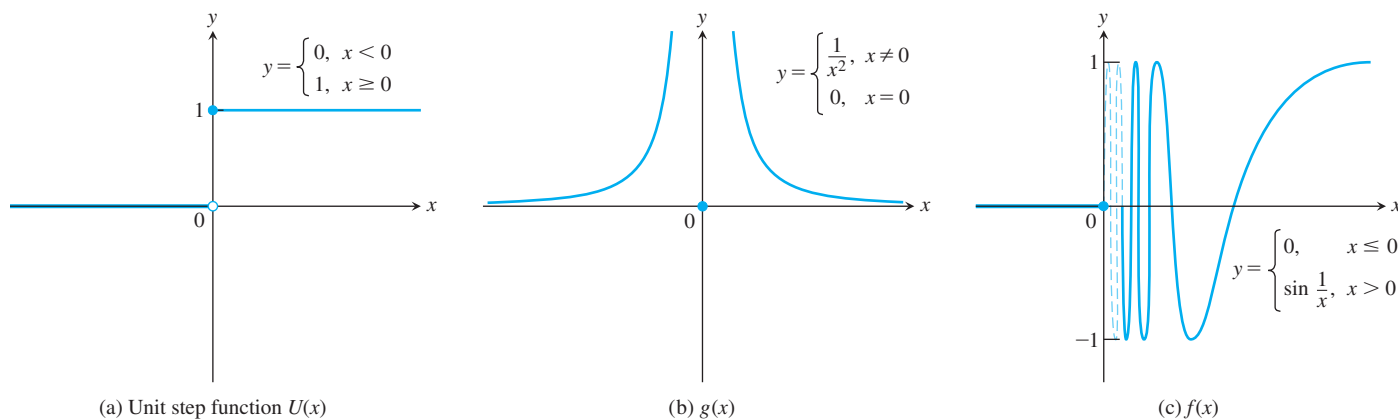


FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

EXAMPLE 4 Discuss the behavior of the following functions, explaining why they have no limit as $x \rightarrow 0$.

- (a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$
- (b) $g(x) = \begin{cases} \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
- (c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0 \end{cases}$

Solution

- (a) This function *jumps*: The **unit step function** $U(x)$ has no limit as $x \rightarrow 0$ because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no *single* value L approached by $U(x)$ as $x \rightarrow 0$ (Figure 2.10a).
- (b) This function *grows too “large” to have a limit*: $g(x)$ has no limit as $x \rightarrow 0$ because the values of g grow arbitrarily large as $x \rightarrow 0$ and therefore do not stay close to *any* fixed real number (Figure 2.10b). We say that the function g is *not bounded*.

The function U in Example 4 is known as the *Heaviside function*, after Oliver Heaviside (1850–1925).

- (c) This function *oscillates too much to have a limit*: $f(x)$ has no limit as $x \rightarrow 0$ because the function's values oscillate between $+1$ and -1 in every open interval containing 0 . The values do not stay close to any single number as $x \rightarrow 0$ (Figure 2.10c). ■

A function that oscillates may or may not have a limit. In Example 11 we will see a function that oscillates wildly, but nevertheless does have a limit. The problem with the function f discussed in Example 4 is not that it oscillates, but that it oscillates *too much* for a limit to exist.

The Limit Laws

A few basic rules allow us to break down complicated functions into simple ones when calculating limits. By using these laws, we can greatly simplify many limit computations.

THEOREM 1—Limit Laws

If L , M , c , and k are real numbers and

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = M, \quad \text{then}$$

1. <i>Sum Rule</i> :	$\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$
2. <i>Difference Rule</i> :	$\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$
3. <i>Constant Multiple Rule</i> :	$\lim_{x \rightarrow c} (k \cdot f(x)) = k \cdot L$
4. <i>Product Rule</i> :	$\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$
5. <i>Quotient Rule</i> :	$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. <i>Power Rule</i> :	$\lim_{x \rightarrow c} [f(x)]^n = L^n, \quad n \text{ a positive integer}$
7. <i>Root Rule</i> :	$\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, \quad n \text{ a positive integer}$

(If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c .)

The Sum Rule says that the limit of a sum is the sum of the limits. Similarly, the next rules say that the limit of a difference is the difference of the limits; the limit of a constant times a function is the constant times the limit of the function; the limit of a product is the product of the limits; the limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0); the limit of a positive integer power (or root) of a function is the integer power (or root) of the limit (provided that the root of the limit is a real number).

There are simple intuitive arguments for why the properties in Theorem 1 are true (although these do not constitute proofs). If x is sufficiently close to c , then $f(x)$ is close to L and $g(x)$ is close to M , from our informal definition of a limit. It is then reasonable that $f(x) + g(x)$ is close to $L + M$; $f(x) - g(x)$ is close to $L - M$; $kf(x)$ is close to kL ; $f(x)g(x)$ is close to LM ; and $f(x)/g(x)$ is close to L/M if M is not zero. We prove the Sum Rule in Section 2.3, based on a rigorous definition of the limit. Rules 2–5 are proved in Appendix A.6. Rule 6 is obtained by applying Rule 4 repeatedly. Rule 7 is proved in more advanced texts. The Sum, Difference, and Product Rules can be extended to any number of functions, not just two.

EXAMPLE 5 Use the observations $\lim_{x \rightarrow c} k = k$ and $\lim_{x \rightarrow c} x = c$ (Example 3) and the limit laws in Theorem 1 to find the following limits.

(a) $\lim_{x \rightarrow c} (x^3 + 4x^2 - 3)$ (b) $\lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5}$ (c) $\lim_{x \rightarrow -2} \sqrt{4x^2 + 3}$

Solution

$$\begin{aligned}
 \text{(a)} \quad \lim_{x \rightarrow c} (x^3 + 4x^2 - 3) &= \lim_{x \rightarrow c} x^3 + \lim_{x \rightarrow c} 4x^2 - \lim_{x \rightarrow c} 3 && \text{Sum and Difference Rules} \\
 &= c^3 + 4c^2 - 3 && \text{Power and Multiple Rules and limit of a constant function} \\
 \\
 \text{(b)} \quad \lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5} &= \frac{\lim_{x \rightarrow c} (x^4 + x^2 - 1)}{\lim_{x \rightarrow c} (x^2 + 5)} && \text{Quotient Rule: Note that } (x^2 + 5) > 0 \text{ for all } x. \\
 &= \frac{\lim_{x \rightarrow c} x^4 + \lim_{x \rightarrow c} x^2 - \lim_{x \rightarrow c} 1}{\lim_{x \rightarrow c} x^2 + \lim_{x \rightarrow c} 5} && \text{Sum and Difference Rules} \\
 &= \frac{c^4 + c^2 - 1}{c^2 + 5} && \text{Power Rule and limit of a constant function} \\
 \\
 \text{(c)} \quad \lim_{x \rightarrow -2} \sqrt{4x^2 + 3} &= \sqrt{\lim_{x \rightarrow -2} (4x^2 + 3)} && \text{Root Rule with } n = 2 (4x^2 + 3 \geq 0) \\
 &= \sqrt{\lim_{x \rightarrow -2} 4x^2 + \lim_{x \rightarrow -2} 3} && \text{Difference Rule} \\
 &= \sqrt{4(-2)^2 + 3} && \text{Power and Multiple Rules and limit of a constant function} \\
 &= \sqrt{16 + 3} \\
 &= \sqrt{19}
 \end{aligned}$$

Evaluating Limits of Polynomials and Rational Functions

Theorem 1 simplifies the task of calculating limits of polynomials and rational functions. To evaluate the limit of a polynomial function as x approaches c , just substitute c for x in the formula for the function. To evaluate the limit of a rational function as x approaches a point c at which the denominator is not zero, substitute c for x in the formula for the function. (See Examples 5a and 5b.) We state these results formally as theorems.

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \rightarrow c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 6 The following calculation illustrates Theorems 2 and 3:

$$\lim_{x \rightarrow -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

Since the denominator of this rational expression does not equal 0 when we substitute -1 for x , we can just compute the value of the expression at $x = -1$ to evaluate the limit. ■

Eliminating Common Factors from Zero Denominators

Theorem 3 applies only if the denominator of the rational function is not zero at the limit point c . If the denominator is zero, canceling common factors in the numerator and

Identifying Common Factors

If $Q(x)$ is a polynomial and $Q(c) = 0$, then $(x - c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x = c$, they have $(x - c)$ as a common factor.

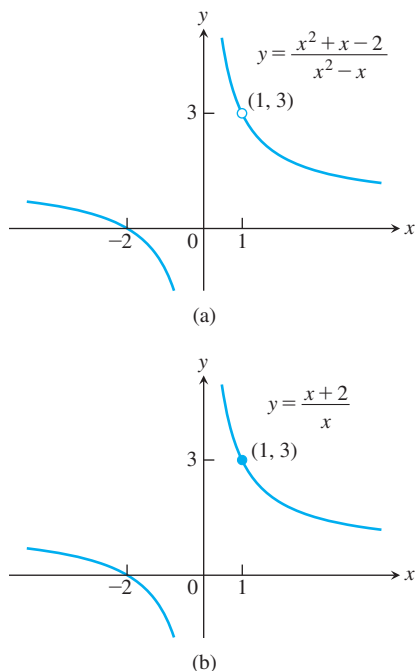


FIGURE 2.11 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \rightarrow 1$ (Example 7).

denominator may reduce the fraction to one whose denominator is no longer zero at c . If this happens, we can find the limit by substitution in the simplified fraction.

EXAMPLE 7 Evaluate

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}.$$

Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling this common factor gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \quad \text{if } x \neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \rightarrow 1$ by evaluating the function at $x = 1$, as in Theorem 3:

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \rightarrow 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

See Figure 2.11. ■

Using Calculators and Computers to Estimate Limits

We can try using a calculator or computer to guess a limit numerically. However, calculators and computers can sometimes give false values and misleading evidence about limits. Usually the problem is associated with rounding errors, as we now illustrate.

EXAMPLE 8 Estimate the value of $\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$.

Solution Table 2.3 lists values of the function obtained on a calculator for several points approaching $x = 0$. As x approaches 0 through the points $\pm 1, \pm 0.5, \pm 0.1$, and ± 0.01 , the function seems to approach the number 0.05.

As we take even smaller values of x , $\pm 0.0005, \pm 0.0001, \pm 0.00001$, and ± 0.000001 , the function appears to approach the number 0.

Is the answer 0.05 or 0, or some other value? We resolve this question in the next example. ■

Using a computer or calculator may give ambiguous results, as in Example 8. A computer cannot always keep track of enough digits to avoid rounding errors in computing the values of $f(x)$ when x is very small. We cannot substitute $x = 0$ in the problem, and the

TABLE 2.3 Computed values of $f(x) = \frac{\sqrt{x^2 + 100} - 10}{x^2}$ near $x = 0$

x	$f(x)$	
± 1	0.049876	approaches 0.05?
± 0.5	0.049969	
± 0.1	0.049999	
± 0.01	0.050000	
± 0.0005	0.050000	approaches 0?
± 0.0001	0.000000	
± 0.00001	0.000000	
± 0.000001	0.000000	

numerator and denominator have no obvious common factors (as they did in Example 7). Sometimes, however, we can create a common factor by using algebra.

EXAMPLE 9 Evaluate

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}.$$

Solution This is the limit we considered in Example 8. We can create a common factor by multiplying both numerator and denominator by the conjugate radical expression $\sqrt{x^2 + 100} + 10$ (obtained by changing the sign after the square root). The preliminary algebra rationalizes the numerator:

$$\begin{aligned} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \frac{\sqrt{x^2 + 100} - 10}{x^2} \cdot \frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10} && \text{Multiply and divide by the conjugate.} \\ &= \frac{x^2 + 100 - 100}{x^2(\sqrt{x^2 + 100} + 10)} && \text{Simplify.} \\ &= \frac{x^2}{x^2(\sqrt{x^2 + 100} + 10)} && \text{Common factor } x^2 \\ &= \frac{1}{\sqrt{x^2 + 100} + 10}. && \text{Cancel } x^2 \text{ for } x \neq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x^2 + 100} + 10} \\ &= \frac{1}{\sqrt{0^2 + 100} + 10} && \text{Use limit laws: Sum Rule, Power Rule, Root Rule, and Quotient Rule (denominator not 0).} \\ &= \frac{1}{20} = 0.05. \end{aligned}$$

This calculation provides the correct answer, resolving the ambiguous computer results in Example 8. ■

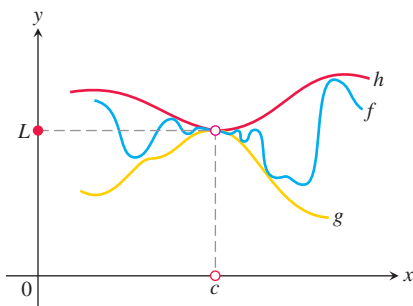


FIGURE 2.12 The graph of f is sandwiched between the graphs of g and h .

We cannot always manipulate the terms in an expression to find the limit of a quotient where the denominator becomes zero. In some cases the limit might then be found with geometric arguments (see the proof of Theorem 6 in Section 2.4), or through methods of calculus (developed in Section 4.5). The next theorem shows how to evaluate difficult limits by comparing them with functions having known limits.

The Sandwich Theorem

The following theorem enables us to calculate a variety of limits. It is called the Sandwich Theorem because it refers to a function f whose values are sandwiched between the values of two other functions g and h that have the same limit L at a point c . Being trapped between the values of two functions that approach L , the values of f must also approach L (Figure 2.12). A proof is given in Appendix A.6.

THEOREM 4—The Sandwich Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c , except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L.$$

Then $\lim_{x \rightarrow c} f(x) = L$.

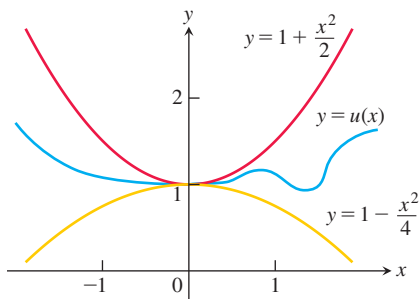


FIGURE 2.13 Any function $u(x)$ whose graph lies in the region between $y = 1 + (x^2/2)$ and $y = 1 - (x^2/4)$ has limit 1 as $x \rightarrow 0$ (Example 10).

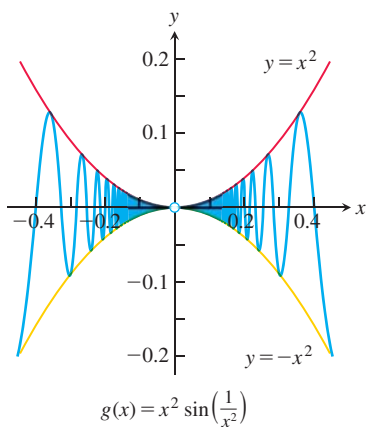


FIGURE 2.14 The graph of the function g (Example 11). It is not defined at $x = 0$. Even though the function oscillates, it has a limit as $x \rightarrow 0$. The value of $g(x)$ always lies between x^2 and $-x^2$.

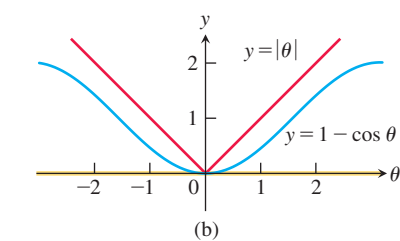
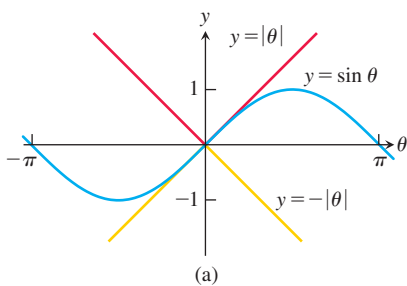


FIGURE 2.15 The Sandwich Theorem confirms the limits in Example 12.

The Sandwich Theorem is also called the Squeeze Theorem or the Pinching Theorem.

EXAMPLE 10 Given a function u that satisfies

$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2} \quad \text{for all } x \neq 0,$$

find $\lim_{x \rightarrow 0} u(x)$, no matter how complicated u is.

Solution Since

$$\lim_{x \rightarrow 0} (1 - (x^2/4)) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} (1 + (x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x \rightarrow 0} u(x) = 1$ (Figure 2.13). ■

We use the Sandwich Theorem to show that it is possible for a function that oscillates to have a limit.

EXAMPLE 11 How does the function $g(x) = x^2 \sin(1/x^2)$ behave near $x = 0$?

Solution The formula defines $g(x)$ for all real numbers x except $x = 0$. The graph of g is shown in Figure 2.14. We can see that the graph oscillates, but we can use the Sandwich Theorem to find the limit of $g(x)$ as x approaches 0. If $x \neq 0$, then $1/x^2$ is a positive real number. Since the range of the sine function is the interval $[-1, 1]$, it follows that $-1 \leq \sin(1/x^2) \leq 1$ for all $x \neq 0$. Even though we may not know the exact value of $g(x) = x^2 \sin(1/x^2)$, we do know that it lies between $-x^2$ and x^2 . Since

$$\lim_{x \rightarrow 0} x^2 = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (-x^2) = 0,$$

the Sandwich Theorem implies that $\lim_{x \rightarrow 0} g(x) = 0$. ■

EXAMPLE 12 The Sandwich Theorem helps us establish several important limit rules:

- (a) $\lim_{\theta \rightarrow 0} \sin \theta = 0$
- (b) $\lim_{\theta \rightarrow 0} \cos \theta = 1$
- (c) For any function f , $\lim_{x \rightarrow c} |f(x)| = 0$ implies $\lim_{x \rightarrow c} f(x) = 0$.

Solution

- (a) In Section 1.3 we established that $-|\theta| \leq \sin \theta \leq |\theta|$ for all θ (see Figure 2.15a). Since $\lim_{\theta \rightarrow 0} (-|\theta|) = \lim_{\theta \rightarrow 0} |\theta| = 0$, we have

$$\lim_{\theta \rightarrow 0} \sin \theta = 0.$$

- (b) From Section 1.3, $0 \leq 1 - \cos \theta \leq |\theta|$ for all θ (see Figure 2.15b). Since $\lim_{\theta \rightarrow 0} |\theta| = 0$ and $\lim_{\theta \rightarrow 0} 0 = 0$, we have $\lim_{\theta \rightarrow 0} (1 - \cos \theta) = 0$ so

$$\lim_{\theta \rightarrow 0} \cos \theta = \lim_{\theta \rightarrow 0} (1 - (1 - \cos \theta)) = 1 - \lim_{\theta \rightarrow 0} (1 - \cos \theta) = 1 - 0 = 1.$$

- (c) Since $-|f(x)| \leq f(x) \leq |f(x)|$ and $-|f(x)|$ and $|f(x)|$ have limit 0 as $x \rightarrow c$, it follows that $\lim_{x \rightarrow c} f(x) = 0$. ■

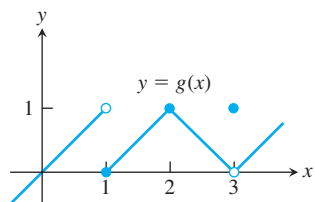
Example 12 shows that the sine and cosine functions are equal to their limits at $\theta = 0$. We have not yet established that for any c , $\lim_{\theta \rightarrow c} \sin \theta = \sin c$, and $\lim_{\theta \rightarrow c} \cos \theta = \cos c$. These limit formulas do hold, as will be shown in Section 2.6.

EXERCISES 2.2

Limits from Graphs

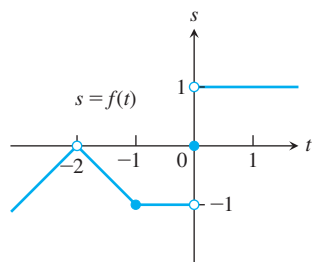
1. For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.

a. $\lim_{x \rightarrow 1} g(x)$ b. $\lim_{x \rightarrow 2} g(x)$ c. $\lim_{x \rightarrow 3} g(x)$ d. $\lim_{x \rightarrow 2.5} g(x)$



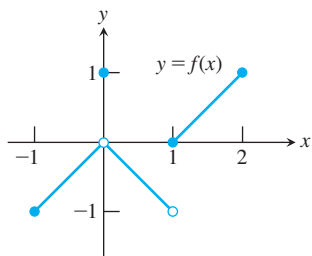
2. For the function $f(t)$ graphed here, find the following limits or explain why they do not exist.

a. $\lim_{t \rightarrow -2} f(t)$ b. $\lim_{t \rightarrow -1} f(t)$ c. $\lim_{t \rightarrow 0} f(t)$ d. $\lim_{t \rightarrow 0.5} f(t)$



3. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

- a. $\lim_{x \rightarrow 0} f(x)$ exists.
 b. $\lim_{x \rightarrow 0} f(x) = 0$
 c. $\lim_{x \rightarrow 0} f(x) = 1$
 d. $\lim_{x \rightarrow 1} f(x) = 1$
 e. $\lim_{x \rightarrow 1} f(x) = 0$
 f. $\lim_{x \rightarrow c} f(x)$ exists at every point c in $(-1, 1)$.
 g. $\lim_{x \rightarrow 1} f(x)$ does not exist.
 h. $f(0) = 0$
 i. $f(0) = 1$
 j. $f(1) = 0$
 k. $f(1) = -1$

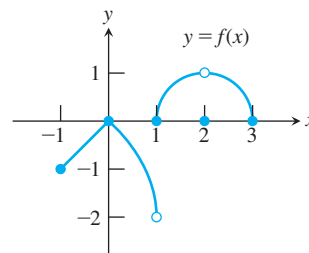


4. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

- a. $\lim_{x \rightarrow 2} f(x)$ does not exist.
 b. $\lim_{x \rightarrow 2} f(x) = 2$
 c. $\lim_{x \rightarrow 1} f(x)$ does not exist.
 d. $\lim_{x \rightarrow c} f(x)$ exists at every point c in $(-1, 1)$.

- e. $\lim_{x \rightarrow c} f(x)$ exists at every point c in $(1, 3)$.

- f. $f(1) = 0$
 g. $f(1) = -2$
 h. $f(2) = 0$
 i. $f(2) = 1$



Existence of Limits

In Exercises 5 and 6, explain why the limits do not exist.

5. $\lim_{x \rightarrow 0} \frac{x}{|x|}$

6. $\lim_{x \rightarrow 1} \frac{1}{x-1}$

7. Suppose that a function $f(x)$ is defined for all real values of x except $x = c$. Can anything be said about the existence of $\lim_{x \rightarrow c} f(x)$? Give reasons for your answer.

8. Suppose that a function $f(x)$ is defined for all x in $[-1, 1]$. Can anything be said about the existence of $\lim_{x \rightarrow 0} f(x)$? Give reasons for your answer.

9. If $\lim_{x \rightarrow 1} f(x) = 5$, must f be defined at $x = 1$? If it is, must $f(1) = 5$? Can we conclude *anything* about the values of f at $x = 1$? Explain.

10. If $f(1) = 5$, must $\lim_{x \rightarrow 1} f(x)$ exist? If it does, then must $\lim_{x \rightarrow 1} f(x) = 5$? Can we conclude *anything* about $\lim_{x \rightarrow 1} f(x)$? Explain.

Calculating Limits

Find the limits in Exercises 11–22.

11. $\lim_{x \rightarrow -3} (x^2 - 13)$

12. $\lim_{x \rightarrow 2} (-x^2 + 5x - 2)$

13. $\lim_{t \rightarrow 6} 8(t-5)(t-7)$

14. $\lim_{x \rightarrow -2} (x^3 - 2x^2 + 4x + 8)$

15. $\lim_{x \rightarrow 2} \frac{2x+5}{11-x^3}$

16. $\lim_{s \rightarrow 2/3} (8-3s)(2s-1)$

17. $\lim_{x \rightarrow -1/2} 4x(3x+4)^2$

18. $\lim_{y \rightarrow 2} \frac{y+2}{y^2+5y+6}$

19. $\lim_{y \rightarrow -3} (5-y)^{4/3}$

20. $\lim_{z \rightarrow 4} \sqrt{z^2-10}$

21. $\lim_{h \rightarrow 0} \frac{3}{\sqrt{3h+1}+1}$

22. $\lim_{h \rightarrow 0} \frac{\sqrt{5h+4}-2}{h}$

Limits of quotients Find the limits in Exercises 23–42.

23. $\lim_{x \rightarrow 5} \frac{x-5}{x^2-25}$

24. $\lim_{x \rightarrow -3} \frac{x+3}{x^2+4x+3}$

25. $\lim_{x \rightarrow -5} \frac{x^2+3x-10}{x+5}$

26. $\lim_{x \rightarrow 2} \frac{x^2-7x+10}{x-2}$

27. $\lim_{t \rightarrow 1} \frac{t^2+t-2}{t^2-1}$

28. $\lim_{t \rightarrow -1} \frac{t^2+3t+2}{t^2-t-2}$

29. $\lim_{t \rightarrow -2} \frac{-2x-4}{x^3+2x^2}$

30. $\lim_{y \rightarrow 0} \frac{5y^3+8y^2}{3y^4-16y^2}$

$$31. \lim_{x \rightarrow 1} \frac{x^{-1} - 1}{x - 1} \qquad 32. \lim_{x \rightarrow 0} \frac{\frac{1}{x-1} + \frac{1}{x+1}}{x}$$

$$33. \lim_{u \rightarrow 1} \frac{u^4 - 1}{u^3 - 1} \qquad 34. \lim_{v \rightarrow 2} \frac{v^3 - 8}{v^4 - 16}$$

$$35. \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} \qquad 36. \lim_{x \rightarrow 4} \frac{4x - x^2}{2 - \sqrt{x}}$$

$$37. \lim_{x \rightarrow 1} \frac{x - 1}{\sqrt{x} + 3 - 2} \qquad 38. \lim_{x \rightarrow -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}$$

$$39. \lim_{x \rightarrow 2} \frac{\sqrt{x^2 + 12} - 4}{x - 2} \qquad 40. \lim_{x \rightarrow -2} \frac{x + 2}{\sqrt{x^2 + 5} - 3}$$

$$41. \lim_{x \rightarrow -3} \frac{2 - \sqrt{x^2 - 5}}{x + 3} \qquad 42. \lim_{x \rightarrow 4} \frac{4 - x}{5 - \sqrt{x^2 + 9}}$$

Limits with trigonometric functions Find the limits in Exercises 43–50.

$$43. \lim_{x \rightarrow 0} (2 \sin x - 1) \qquad 44. \lim_{x \rightarrow 0} \sin^2 x$$

$$45. \lim_{x \rightarrow 0} \sec x \qquad 46. \lim_{x \rightarrow 0} \tan x$$

$$47. \lim_{x \rightarrow 0} \frac{1 + x + \sin x}{3 \cos x} \qquad 48. \lim_{x \rightarrow 0} (x^2 - 1)(2 - \cos x)$$

$$49. \lim_{x \rightarrow -\pi} \sqrt{x + 4} \cos(x + \pi) \qquad 50. \lim_{x \rightarrow 0} \sqrt{7 + \sec^2 x}$$

Using Limit Rules

51. Suppose $\lim_{x \rightarrow 0} f(x) = 1$ and $\lim_{x \rightarrow 0} g(x) = -5$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x \rightarrow 0} \frac{2f(x) - g(x)}{(f(x) + 7)^2} = \frac{\lim_{x \rightarrow 0} (2f(x) - g(x))}{\lim_{x \rightarrow 0} (f(x) + 7)^2} \quad (a)$$

(We assume the denominator is nonzero.)

$$= \frac{\lim_{x \rightarrow 0} 2f(x) - \lim_{x \rightarrow 0} g(x)}{\left(\lim_{x \rightarrow 0} (f(x) + 7)\right)^2} \quad (b)$$

$$= \frac{2 \lim_{x \rightarrow 0} f(x) - \lim_{x \rightarrow 0} g(x)}{\left(\lim_{x \rightarrow 0} f(x) + \lim_{x \rightarrow 0} 7\right)^2} \quad (c)$$

$$= \frac{(2)(1) - (-5)}{(1 + 7)^2} = \frac{7}{64}$$

52. Let $\lim_{x \rightarrow 1} h(x) = 5$, $\lim_{x \rightarrow 1} p(x) = 1$, and $\lim_{x \rightarrow 1} r(x) = 2$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x \rightarrow 1} \frac{\sqrt{5h(x)}}{p(x)(4 - r(x))} = \frac{\lim_{x \rightarrow 1} \sqrt{5h(x)}}{\lim_{x \rightarrow 1} (p(x)(4 - r(x)))} \quad (a)$$

(We assume the denominator is nonzero.)

$$= \frac{\sqrt{\lim_{x \rightarrow 1} 5h(x)}}{\left(\lim_{x \rightarrow 1} p(x)\right)\left(\lim_{x \rightarrow 1} (4 - r(x))\right)} \quad (b)$$

$$= \frac{\sqrt{5 \lim_{x \rightarrow 1} h(x)}}{\left(\lim_{x \rightarrow 1} p(x)\right)\left(\lim_{x \rightarrow 1} 4 - \lim_{x \rightarrow 1} r(x)\right)} \quad (c)$$

$$= \frac{\sqrt{(5)(5)}}{(1)(4 - 2)} = \frac{5}{2}$$

53. Suppose $\lim_{x \rightarrow c} f(x) = 5$ and $\lim_{x \rightarrow c} g(x) = -2$. Find

$$a. \lim_{x \rightarrow c} f(x)g(x) \qquad b. \lim_{x \rightarrow c} 2f(x)g(x)$$

$$c. \lim_{x \rightarrow c} (f(x) + 3g(x)) \qquad d. \lim_{x \rightarrow c} \frac{f(x)}{f(x) - g(x)}$$

54. Suppose $\lim_{x \rightarrow 4} f(x) = 0$ and $\lim_{x \rightarrow 4} g(x) = -3$. Find

$$a. \lim_{x \rightarrow 4} (g(x) + 3) \qquad b. \lim_{x \rightarrow 4} xf(x)$$

$$c. \lim_{x \rightarrow 4} (g(x))^2 \qquad d. \lim_{x \rightarrow 4} \frac{g(x)}{f(x) - 1}$$

55. Suppose $\lim_{x \rightarrow b} f(x) = 7$ and $\lim_{x \rightarrow b} g(x) = -3$. Find

$$a. \lim_{x \rightarrow b} (f(x) + g(x)) \qquad b. \lim_{x \rightarrow b} f(x) \cdot g(x)$$

$$c. \lim_{x \rightarrow b} 4g(x) \qquad d. \lim_{x \rightarrow b} f(x)/g(x)$$

56. Suppose that $\lim_{x \rightarrow -2} p(x) = 4$, $\lim_{x \rightarrow -2} r(x) = 0$, and

$$\lim_{x \rightarrow -2} s(x) = -3. \text{ Find}$$

$$a. \lim_{x \rightarrow -2} (p(x) + r(x) + s(x))$$

$$b. \lim_{x \rightarrow -2} (p(x) \cdot r(x) \cdot s(x))$$

$$c. \lim_{x \rightarrow -2} (-4p(x) + 5r(x))/s(x)$$

Limits of Average Rates of Change

Because of their connection with secant lines, tangents, and instantaneous rates, limits of the form

$$\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

occur frequently in calculus. In Exercises 57–62, evaluate this limit for the given value of x and function f .

$$57. f(x) = x^2, \quad x = 1$$

$$58. f(x) = x^2, \quad x = -2$$

$$59. f(x) = 3x - 4, \quad x = 2$$

$$60. f(x) = 1/x, \quad x = -2$$

$$61. f(x) = \sqrt{x}, \quad x = 7$$

$$62. f(x) = \sqrt{3x + 1}, \quad x = 0$$

Using the Sandwich Theorem

63. If $\sqrt{5 - 2x^2} \leq f(x) \leq \sqrt{5 - x^2}$ for $-1 \leq x \leq 1$, find $\lim_{x \rightarrow 0} f(x)$.

64. If $2 - x^2 \leq g(x) \leq 2 \cos x$ for all x , find $\lim_{x \rightarrow 0} g(x)$.

65. a. It can be shown that the inequalities

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2 \cos x} < 1$$

hold for all values of x close to zero (except for $x = 0$). What, if anything, does this tell you about

$$\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x}?$$

Give reasons for your answer.

T b. Graph $y = 1 - (x^2/6)$, $y = (x \sin x)/(2 - 2 \cos x)$, and $y = 1$ together for $-2 \leq x \leq 2$. Comment on the behavior of the graphs as $x \rightarrow 0$.

66. a. Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}$$

hold for values of x close to zero, except for $x = 0$ itself. (They do, as you will see in Section 16.9.) What, if anything, does this tell you about

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}?$$

Give reasons for your answer.

- T** b. Graph the equations $y = (1/2) - (x^2/24)$, $y = (1 - \cos x)/x^2$, and $y = 1/2$ together for $-2 \leq x \leq 2$. Comment on the behavior of the graphs as $x \rightarrow 0$.

Estimating Limits

T You will find a graphing calculator useful for Exercises 67–76.

67. Let $f(x) = (x^2 - 9)/(x + 3)$.
- Make a table of the values of f at the points $x = -3.1, -3.01, -3.001$, and so on as far as your calculator can go. Then estimate $\lim_{x \rightarrow -3} f(x)$. What estimate do you arrive at if you evaluate f at $x = -2.9, -2.99, -2.999, \dots$ instead?
 - Support your conclusions in part (a) by graphing f near $c = -3$ and using Zoom and Trace to estimate y -values on the graph as $x \rightarrow -3$.
 - Find $\lim_{x \rightarrow -3} f(x)$ algebraically, as in Example 7.
68. Let $g(x) = (x^2 - 2)/(x - \sqrt{2})$.
- Make a table of the values of g at the points $x = 1.4, 1.41, 1.414$, and so on through successive decimal approximations of $\sqrt{2}$. Estimate $\lim_{x \rightarrow \sqrt{2}} g(x)$.
 - Support your conclusion in part (a) by graphing g near $c = \sqrt{2}$ and using Zoom and Trace to estimate y -values on the graph as $x \rightarrow \sqrt{2}$.
 - Find $\lim_{x \rightarrow \sqrt{2}} g(x)$ algebraically.
69. Let $G(x) = (x + 6)/(x^2 + 4x - 12)$.
- Make a table of the values of G at $x = -5.9, -5.99, -5.999$, and so on. Then estimate $\lim_{x \rightarrow -6} G(x)$. What estimate do you arrive at if you evaluate G at $x = -6.1, -6.01, -6.001, \dots$ instead?
 - Support your conclusions in part (a) by graphing G and using Zoom and Trace to estimate y -values on the graph as $x \rightarrow -6$.
 - Find $\lim_{x \rightarrow -6} G(x)$ algebraically.
70. Let $h(x) = (x^2 - 2x - 3)/(x^2 - 4x + 3)$.
- Make a table of the values of h at $x = 2.9, 2.99, 2.999$, and so on. Then estimate $\lim_{x \rightarrow 3} h(x)$. What estimate do you arrive at if you evaluate h at $x = 3.1, 3.01, 3.001, \dots$ instead?
 - Support your conclusions in part (a) by graphing h near $c = 3$ and using Zoom and Trace to estimate y -values on the graph as $x \rightarrow 3$.
 - Find $\lim_{x \rightarrow 3} h(x)$ algebraically.
71. Let $f(x) = (x^2 - 1)/(|x| - 1)$.
- Make tables of the values of f at values of x that approach $c = -1$ from above and below. Then estimate $\lim_{x \rightarrow -1} f(x)$.

- Support your conclusion in part (a) by graphing f near $c = -1$ and using Zoom and Trace to estimate y -values on the graph as $x \rightarrow -1$.
 - Find $\lim_{x \rightarrow -1} f(x)$ algebraically.
72. Let $F(x) = (x^2 + 3x + 2)/(2 - |x|)$.
- Make tables of values of F at values of x that approach $c = -2$ from above and below. Then estimate $\lim_{x \rightarrow -2} F(x)$.
 - Support your conclusion in part (a) by graphing F near $c = -2$ and using Zoom and Trace to estimate y -values on the graph as $x \rightarrow -2$.
 - Find $\lim_{x \rightarrow -2} F(x)$ algebraically.
73. Let $g(\theta) = (\sin \theta)/\theta$.
- Make a table of the values of g at values of θ that approach $\theta_0 = 0$ from above and below. Then estimate $\lim_{\theta \rightarrow 0} g(\theta)$.
 - Support your conclusion in part (a) by graphing g near $\theta_0 = 0$.
74. Let $G(t) = (1 - \cos t)/t^2$.
- Make tables of values of G at values of t that approach $t_0 = 0$ from above and below. Then estimate $\lim_{t \rightarrow 0} G(t)$.
 - Support your conclusion in part (a) by graphing G near $t_0 = 0$.
75. Let $f(x) = x^{1/(1-x)}$.
- Make tables of values of f at values of x that approach $c = 1$ from above and below. Does f appear to have a limit as $x \rightarrow 1$? If so, what is it? If not, why not?
 - Support your conclusions in part (a) by graphing f near $c = 1$.
76. Let $f(x) = (3^x - 1)/x$.
- Make tables of values of f at values of x that approach $c = 0$ from above and below. Does f appear to have a limit as $x \rightarrow 0$? If so, what is it? If not, why not?
 - Support your conclusions in part (a) by graphing f near $c = 0$.

Theory and Examples

77. If $x^4 \leq f(x) \leq x^2$ for x in $[-1, 1]$ and $x^2 \leq f(x) \leq x^4$ for $x < -1$ and $x > 1$, at what points c do you automatically know $\lim_{x \rightarrow c} f(x)$? What can you say about the value of the limit at these points?
78. Suppose that $g(x) \leq f(x) \leq h(x)$ for all $x \neq 2$ and suppose that
- $$\lim_{x \rightarrow 2} g(x) = \lim_{x \rightarrow 2} h(x) = -5.$$
- Can we conclude anything about the values of f , g , and h at $x = 2$? Could $f(2) = 0$? Could $\lim_{x \rightarrow 2} f(x) = 0$? Give reasons for your answers.
79. If $\lim_{x \rightarrow 4} \frac{f(x) - 5}{x - 2} = 1$, find $\lim_{x \rightarrow 4} f(x)$.
80. If $\lim_{x \rightarrow -2} \frac{f(x)}{x^2} = 1$, find
- $\lim_{x \rightarrow -2} f(x)$,
 - $\lim_{x \rightarrow -2} \frac{f(x)}{x}$.
81. a. If $\lim_{x \rightarrow 2} \frac{f(x) - 5}{x - 2} = 3$, find $\lim_{x \rightarrow 2} f(x)$.
- If $\lim_{x \rightarrow 2} \frac{f(x) - 5}{x - 2} = 4$, find $\lim_{x \rightarrow 2} f(x)$.

82. If $\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 1$, find

- a. $\lim_{x \rightarrow 0} f(x)$, b. $\lim_{x \rightarrow 0} \frac{f(x)}{x}$.

T 83. a. Graph $g(x) = x \sin(1/x)$ to estimate $\lim_{x \rightarrow 0} g(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

T 84. a. Graph $h(x) = x^2 \cos(1/x^3)$ to estimate $\lim_{x \rightarrow 0} h(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

COMPUTER EXPLORATIONS

Graphical Estimates of Limits

In Exercises 85–90, use a CAS to perform the following steps:

- Plot the function near the point c being approached.
- From your plot, guess the value of the limit.

85. $\lim_{x \rightarrow 2} \frac{x^4 - 16}{x - 2}$

86. $\lim_{x \rightarrow -1} \frac{x^3 - x^2 - 5x - 3}{(x + 1)^2}$

87. $\lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x} - 1}{x}$

88. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{\sqrt{x^2 + 7} - 4}$

89. $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x \sin x}$

90. $\lim_{x \rightarrow 0} \frac{2x^2}{3 - 3 \cos x}$

2.3 The Precise Definition of a Limit

We now turn our attention to the precise definition of a limit. The early history of calculus saw controversy about the validity of the basic concepts underlying the theory. Apparent contradictions were argued over by both mathematicians and philosophers. These controversies were resolved by the precise definition, which allows us to replace vague phrases like “gets arbitrarily close to” in the informal definition with specific conditions that can be applied to any particular example. With a rigorous definition, we can avoid misunderstandings, prove the limit properties given in the preceding section, and establish many important limits.

To show that the limit of $f(x)$ as $x \rightarrow c$ equals the number L , we need to show that the gap between $f(x)$ and L can be made “as small as we choose” if x is kept “close enough” to c . Let us see what this requires if we specify the size of the gap between $f(x)$ and L .

EXAMPLE 1 Consider the function $y = 2x - 1$ near $x = 4$. Intuitively it seems clear that y is close to 7 when x is close to 4, so $\lim_{x \rightarrow 4} (2x - 1) = 7$. However, how close to $x = 4$ does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?

Solution We are asked: For what values of x is $|y - 7| < 2$? To find the answer we first express $|y - 7|$ in terms of x :

$$|y - 7| = |(2x - 1) - 7| = |2x - 8|.$$

The question then becomes: What values of x satisfy the inequality $|2x - 8| < 2$? To find out, we solve the inequality:

$$|2x - 8| < 2$$

$$-2 < 2x - 8 < 2 \quad \text{Removing absolute value gives two inequalities.}$$

$$6 < 2x < 10 \quad \text{Add 8 to each term.}$$

$$3 < x < 5 \quad \text{Solve for } x.$$

$$-1 < x - 4 < 1. \quad \text{Solve for } x - 4.$$

Keeping x within 1 unit of $x = 4$ will keep y within 2 units of $y = 7$ (Figure 2.16). ■

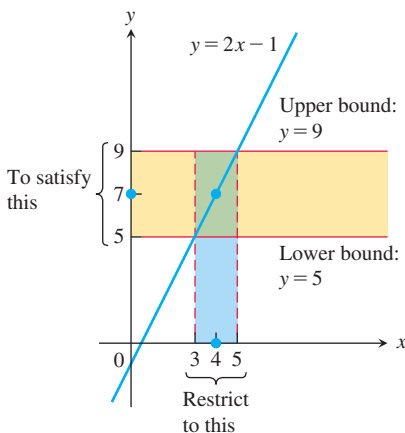


FIGURE 2.16 Keeping x within 1 unit of $x = 4$ will keep y within 2 units of $y = 7$ (Example 1).

In the previous example we determined how close x must be to a particular value c to ensure that the outputs $f(x)$ of some function lie within a prescribed interval about a limit value L . To show that the limit of $f(x)$ as $x \rightarrow c$ actually equals L , we must be able to show that the gap between $f(x)$ and L can be made less than *any prescribed error*, no matter how

δ is the Greek letter delta.
 ε is the Greek letter epsilon.

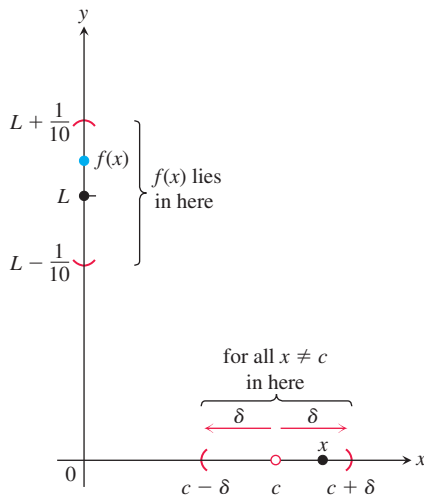


FIGURE 2.17 How should we define $\delta > 0$ so that keeping x within the interval $(c - \delta, c + \delta)$ will keep $f(x)$ within the interval $(L - \frac{1}{10}, L + \frac{1}{10})$?

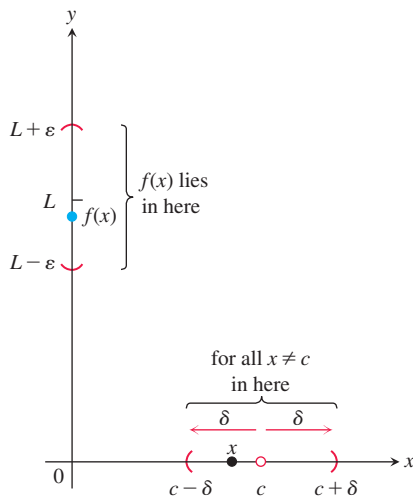


FIGURE 2.18 The relation of δ and ε in the definition of limit.

small, by holding x close enough to c . To describe arbitrary prescribed errors, we introduce two constants, δ (delta) and ε (epsilon). These Greek letters are traditionally used to represent small changes in a variable or a function.

Definition of Limit

Suppose we are watching the values of a function $f(x)$ as x approaches c (without taking on the value c itself). Certainly we want to be able to say that $f(x)$ stays within one-tenth of a unit from L as soon as x stays within some distance δ of c (Figure 2.17). But that in itself is not enough, because as x continues on its course toward c , what is to prevent $f(x)$ from jumping around within the interval from $L - (1/10)$ to $L + (1/10)$ without tending toward L ? We can be told that the error can be no more than $1/100$ or $1/1000$ or $1/100,000$. Each time, we find a new δ -interval about c so that keeping x within that interval satisfies the new error tolerance. And each time the possibility exists that $f(x)$ might jump away from L at some later stage.

The figures on the next page illustrate the problem. You can think of this as a quarrel between a skeptic and a scholar. The skeptic presents ε -challenges to show there is room for doubt that the limit exists. The scholar counters every challenge with a δ -interval around c which ensures that the function takes values within ε of L .

How do we stop this seemingly endless series of challenges and responses? We can do so by proving that for *every* error tolerance ε that the challenger can produce, we can present a matching distance δ that keeps x “close enough” to c to keep $f(x)$ within that ε -tolerance of L (Figure 2.18). This leads us to the precise definition of a limit.

DEFINITION Let $f(x)$ be defined on an open interval about c , except possibly at c itself. We say that the **limit of $f(x)$ as x approaches c is the number L** , and write

$$\lim_{x \rightarrow c} f(x) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that

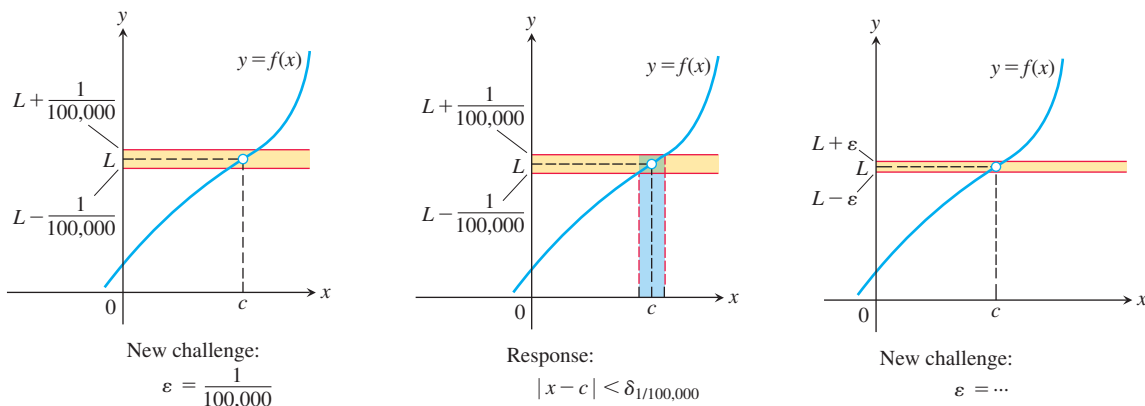
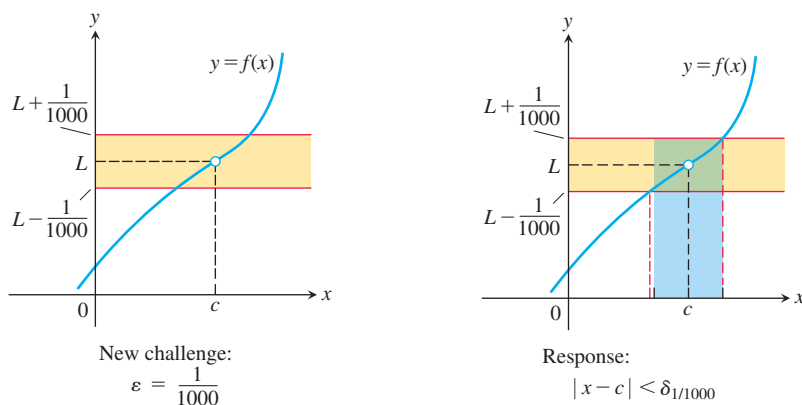
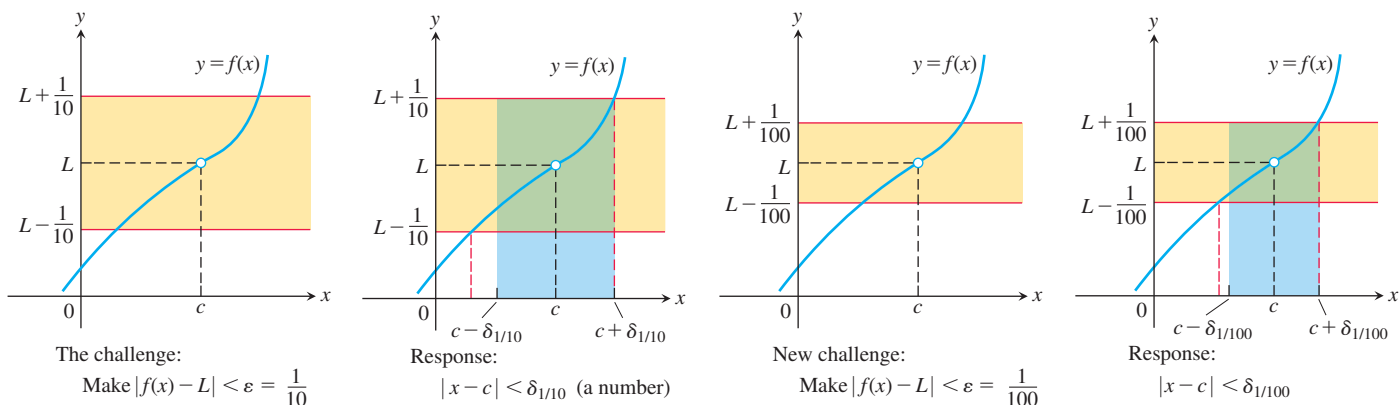
$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

To visualize the definition, imagine machining a cylindrical shaft to a close tolerance. The diameter of the shaft is determined by turning a dial to a setting measured by a variable x . We try for diameter L , but since nothing is perfect we must be satisfied with a diameter $f(x)$ somewhere between $L - \varepsilon$ and $L + \varepsilon$. The number δ is our control tolerance for the dial; it tells us how close our dial setting must be to the setting $x = c$ in order to guarantee that the diameter $f(x)$ of the shaft will be accurate to within ε of L . As the tolerance for error becomes stricter, we may have to adjust δ . The value of δ , how tight our control setting must be, depends on the value of ε , the error tolerance.

The definition of limit extends to functions on more general domains. It is only required that each open interval around c contain points in the domain of the function other than c . See Additional and Advanced Exercises 49–53 for examples of limits for functions with complicated domains. In the next section we will see how the definition of limit applies at points lying on the boundary of an interval.

Examples: Testing the Definition

The formal definition of limit does not tell how to find the limit of a function, but it does enable us to verify that a conjectured limit value is correct. The following examples show how the definition can be used to verify limit statements for specific functions. However, the real purpose of the definition is not to do calculations like this, but rather to prove general theorems so that the calculation of specific limits can be simplified, such as the theorems stated in the previous section.



EXAMPLE 2 Show that

$$\lim_{x \rightarrow 1} (5x - 3) = 2.$$

Solution Set $c = 1$, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\varepsilon > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $c = 1$, then it is true that $f(x)$ is within distance ε of $L = 2$. That is, we must show that if x satisfies

$$0 < |x - 1| < \delta,$$

then $f(x)$ will satisfy

$$|f(x) - 2| < \varepsilon.$$

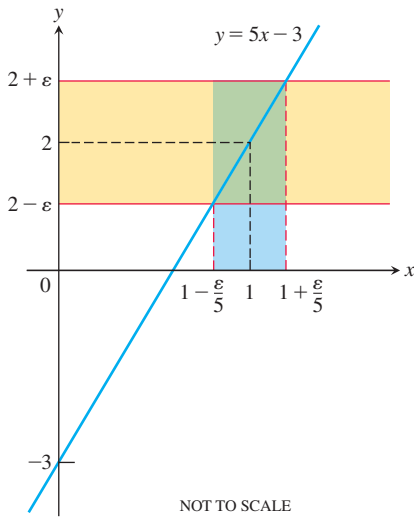


FIGURE 2.19 If $f(x) = 5x - 3$, then $0 < |x - 1| < \varepsilon/5$ guarantees that $|f(x) - 2| < \varepsilon$ (Example 2).

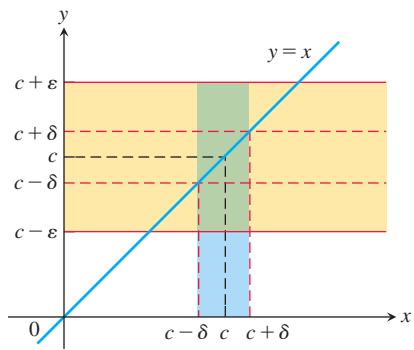


FIGURE 2.20 For the function $f(x) = x$, we find that $0 < |x - c| < \delta$ will guarantee $|f(x) - c| < \varepsilon$ whenever $\delta \leq \varepsilon$ (Example 3a).

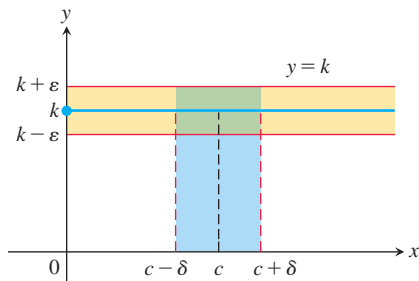


FIGURE 2.21 For the function $f(x) = k$, we find that $|f(x) - k| < \varepsilon$ for any positive δ (Example 3b).

We find δ by working backward from the ε -inequality, $|f(x) - 2| < \varepsilon$:

$$|(5x - 3) - 2| = |5x - 5| < \varepsilon \quad \text{Substitute.}$$

$$5|x - 1| < \varepsilon \quad \text{Factor.}$$

$$|x - 1| < \varepsilon/5. \quad \text{Simplify.}$$

Thus, we can take $\delta = \varepsilon/5$ (Figure 2.19). If $0 < |x - 1| < \delta = \varepsilon/5$, then

$$|(5x - 3) - 2| = |5x - 5| = 5|x - 1| < 5(\varepsilon/5) = \varepsilon,$$

which proves that $\lim_{x \rightarrow 1} (5x - 3) = 2$.

The value of $\delta = \varepsilon/5$ is not the only value that will make $0 < |x - 1| < \delta$ imply $|5x - 5| < \varepsilon$. Any smaller positive δ will do as well. The definition does not ask for the “best” positive δ , just one that will work. ■

EXAMPLE 3 Prove the following results, which were presented graphically in Section 2.2.

(a) $\lim_{x \rightarrow c} x = c$

(b) $\lim_{x \rightarrow c} k = k$ k constant

Solution

(a) Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$|x - c| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

The implication will hold if δ equals ε or any smaller positive number (Figure 2.20).

This proves that $\lim_{x \rightarrow c} x = c$.

(b) Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$|k - k| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Since $k - k = 0$, we will always have $|k - k| < \varepsilon$. Therefore we can use any positive number for δ , and the implication will hold (Figure 2.21). This proves that $\lim_{x \rightarrow c} k = k$. ■

Finding Deltas Algebraically for Given Epsilons

In Examples 2 and 3, the interval of values about c for which $|f(x) - L|$ was less than ε was symmetric about c and we could take δ to be half the length of that interval. When the interval around c on which we have $|f(x) - L| < \varepsilon$ is not symmetric about c , we can take δ to be the distance from c to the interval’s nearer endpoint.

EXAMPLE 4 For the limit $\lim_{x \rightarrow 5} \sqrt{x - 1} = 2$, find a $\delta > 0$ that works for $\varepsilon = 1$. That is, find a $\delta > 0$ such that

$$|\sqrt{x - 1} - 2| < 1 \quad \text{whenever} \quad 0 < |x - 5| < \delta$$

Solution We organize the search into two steps.

1. Solve the inequality $|\sqrt{x - 1} - 2| < 1$ to find an interval containing $x = 5$ on which the inequality holds for all $x \neq 5$.

$$|\sqrt{x - 1} - 2| < 1$$

$$-1 < \sqrt{x - 1} - 2 < 1$$

$$1 < \sqrt{x - 1} < 3$$

$$1 < x - 1 < 9$$

$$2 < x < 10$$



FIGURE 2.22 An open interval of radius 3 about $x = 5$ will lie inside the open interval $(2, 10)$.

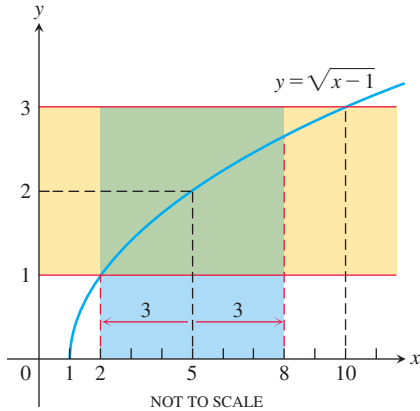


FIGURE 2.23 The function and intervals in Example 4.

The inequality holds for all x in the open interval $(2, 10)$, so it holds for all $x \neq 5$ in this interval as well.

- Find a value of $\delta > 0$ to place the centered interval $5 - \delta < x < 5 + \delta$ (centered at $x = 5$ inside the interval $(2, 10)$). The distance from 5 to the nearer endpoint of $(2, 10)$ is 3 (Figure 2.22). If we take $\delta = 3$ or any smaller positive number, then the inequality $0 < |x - 5| < \delta$ will automatically place x between 2 and 10 and imply that $|\sqrt{x-1} - 2| < 1$ (Figure 2.23):

$$|\sqrt{x-1} - 2| < 1 \quad \text{whenever} \quad 0 < |x - 5| < 3.$$

How to Find Algebraically a δ for a Given f, L, c , and $\varepsilon > 0$

The process of finding a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta$$

can be accomplished in two steps.

- Solve the inequality $|f(x) - L| < \varepsilon$ to find an open interval (a, b) , containing c on which the inequality holds for all $x \neq c$. Note that we do not require the inequality to hold at $x = c$. It may hold there or it may not, but the value of f at $x = c$ does not influence the existence of a limit.
- Find a value of $\delta > 0$ that places the open interval $(c - \delta, c + \delta)$ centered at c inside the interval (a, b) . The inequality $|f(x) - L| < \varepsilon$ will hold for all $x \neq c$ in this δ -interval.

EXAMPLE 5 Prove that $\lim_{x \rightarrow 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2. \end{cases}$$

Solution Our task is to show that given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - 4| < \varepsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

- Solve the inequality $|f(x) - 4| < \varepsilon$ to find an open interval containing $x = 2$ on which the inequality holds for all $x \neq 2$.

For $x \neq c = 2$, we have $f(x) = x^2$, and the inequality to solve is $|x^2 - 4| < \varepsilon$:

$$\begin{aligned} |x^2 - 4| &< \varepsilon \\ -\varepsilon &< x^2 - 4 < \varepsilon \\ 4 - \varepsilon &< x^2 < 4 + \varepsilon \\ \sqrt{4 - \varepsilon} &< |x| < \sqrt{4 + \varepsilon} \quad \text{Assumes } \varepsilon < 4; \text{ see below.} \\ \sqrt{4 - \varepsilon} &< x < \sqrt{4 + \varepsilon}. \quad \text{An open interval about } x = 2 \\ &\quad \text{that solves the inequality} \end{aligned}$$

The inequality $|f(x) - 4| < \varepsilon$ holds for all $x \neq 2$ in the open interval $(\sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon})$ (Figure 2.24).

- Find a value of $\delta > 0$ that places the centered interval $(2 - \delta, 2 + \delta)$ inside the interval $(\sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon})$.

Take δ to be the distance from $x = 2$ to the nearer endpoint of $(\sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon})$. In other words, take $\delta = \min\{2 - \sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon} - 2\}$, the *minimum* (the smaller) of the two numbers $2 - \sqrt{4 - \varepsilon}$ and $\sqrt{4 + \varepsilon} - 2$. If δ has this or any

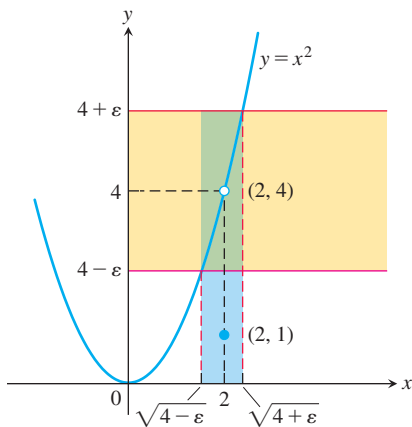


FIGURE 2.24 An interval containing $x = 2$ so that the function in Example 5 satisfies $|f(x) - 4| < \varepsilon$.

smaller positive value, the inequality $0 < |x - 2| < \delta$ will automatically place x between $\sqrt{4 - \varepsilon}$ and $\sqrt{4 + \varepsilon}$ to make $|f(x) - 4| < \varepsilon$. For all x ,

$$|f(x) - 4| < \varepsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

This completes the proof for $\varepsilon < 4$.

If $\varepsilon \geq 4$, then we take δ to be the distance from $x = 2$ to the nearer endpoint of the interval $(0, \sqrt{4 + \varepsilon})$. In other words, take $\delta = \min\{2, \sqrt{4 + \varepsilon} - 2\}$. (See Figure 2.24.) ■

Using the Definition to Prove Theorems

We do not usually rely on the formal definition of limit to verify specific limits such as those in the preceding examples. Rather, we appeal to general theorems about limits, in particular the theorems of Section 2.2. The definition is used to prove these theorems (Appendix A.6). As an example, we prove part 1 of Theorem 1, the Sum Rule.

EXAMPLE 6 Given that $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$, prove that

$$\lim_{x \rightarrow c} (f(x) + g(x)) = L + M.$$

Solution Let $\varepsilon > 0$ be given. We want to find a positive number δ such that

$$|f(x) + g(x) - (L + M)| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Regrouping terms, we get

$$\begin{aligned} |f(x) + g(x) - (L + M)| &= |(f(x) - L) + (g(x) - M)| && \text{Triangle Inequality: } |a+b| \leq |a| + |b| \\ &\leq |f(x) - L| + |g(x) - M|. \end{aligned}$$

Since $\lim_{x \rightarrow c} f(x) = L$, there exists a number $\delta_1 > 0$ such that

$$|f(x) - L| < \varepsilon/2 \quad \text{whenever} \quad 0 < |x - c| < \delta_1. \quad \text{Can find } \delta_1 \text{ since } \lim_{x \rightarrow c} f(x) = L$$

Similarly, since $\lim_{x \rightarrow c} g(x) = M$, there exists a number $\delta_2 > 0$ such that

$$|g(x) - M| < \varepsilon/2 \quad \text{whenever} \quad 0 < |x - c| < \delta_2. \quad \text{Can find } \delta_2 \text{ since } \lim_{x \rightarrow c} g(x) = M$$

Let $\delta = \min\{\delta_1, \delta_2\}$, the smaller of δ_1 and δ_2 . If $0 < |x - c| < \delta$ then $|x - c| < \delta_1$, so $|f(x) - L| < \varepsilon/2$, and $|x - c| < \delta_2$, so $|g(x) - M| < \varepsilon/2$. Therefore,

$$|f(x) + g(x) - (L + M)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$. ■

EXERCISES 2.3

Centering Intervals About a Point

In Exercises 1–6, sketch the interval (a, b) , on the x -axis with the point c inside. Then find a value of $\delta > 0$ such that $a < x < b$ whenever $0 < |x - c| < \delta$.

1. $a = 1$, $b = 7$, $c = 5$

2. $a = 1$, $b = 7$, $c = 2$

3. $a = -7/2$, $b = -1/2$, $c = -3$

4. $a = -7/2$, $b = -1/2$, $c = -3/2$

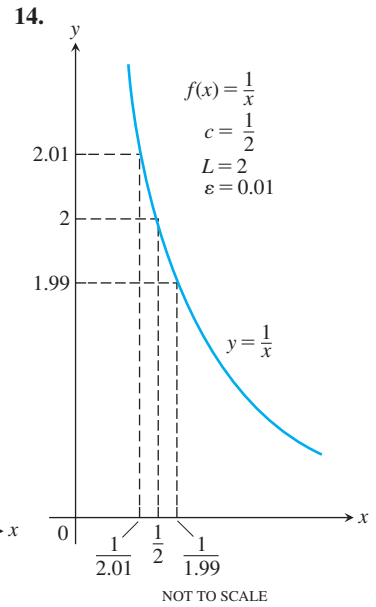
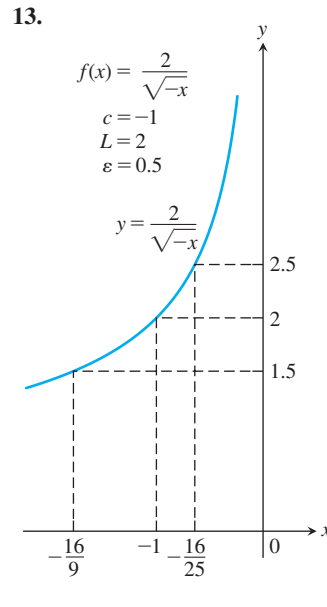
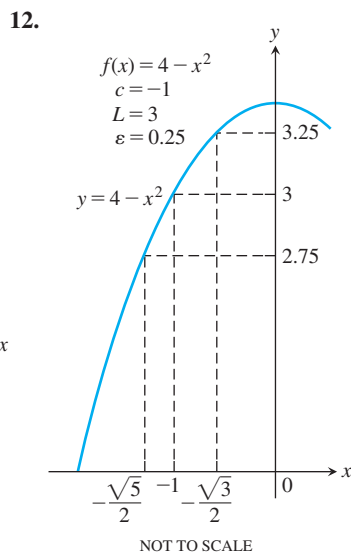
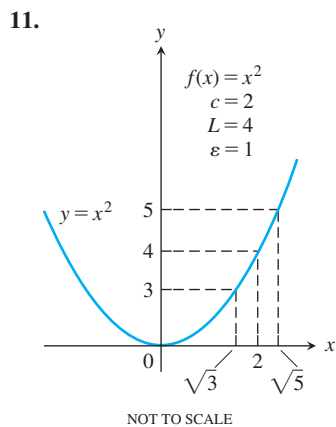
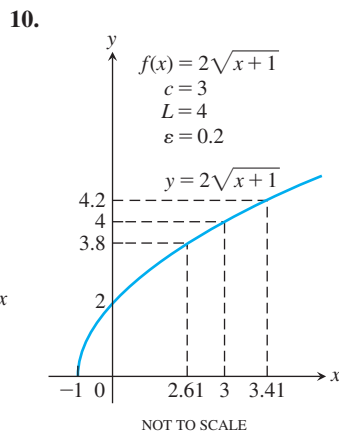
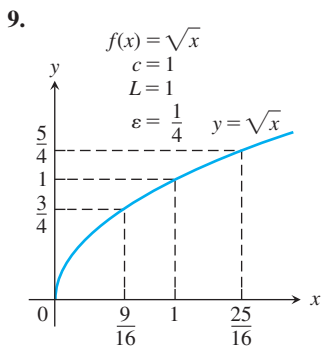
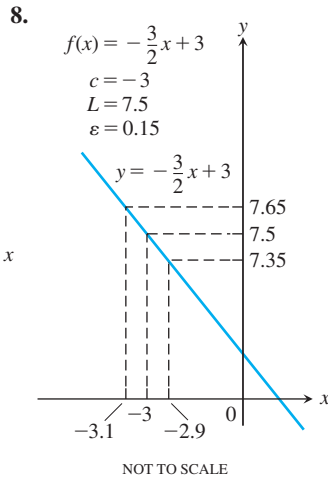
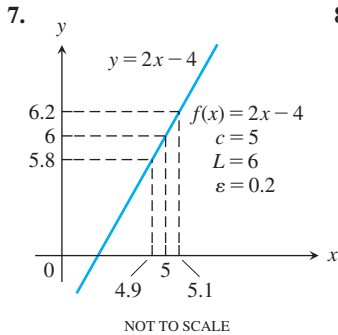
5. $a = 4/9$, $b = 4/7$, $c = 1/2$

6. $a = 2.7591$, $b = 3.2391$, $c = 3$

Finding Deltas Graphically

In Exercises 7–14, use the graphs to find a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$



Finding Deltas Algebraically

Each of Exercises 15–30 gives a function $f(x)$ and numbers L , c , and $\varepsilon > 0$. In each case, find the largest open interval about c on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, the inequality $|f(x) - L| < \varepsilon$ holds.

15. $f(x) = x + 1$, $L = 5$, $c = 4$, $\varepsilon = 0.01$
16. $f(x) = 2x - 2$, $L = -6$, $c = -2$, $\varepsilon = 0.02$
17. $f(x) = \sqrt{x+1}$, $L = 1$, $c = 0$, $\varepsilon = 0.1$
18. $f(x) = \sqrt{x}$, $L = 1/2$, $c = 1/4$, $\varepsilon = 0.1$
19. $f(x) = \sqrt{19-x}$, $L = 3$, $c = 10$, $\varepsilon = 1$
20. $f(x) = \sqrt{x-7}$, $L = 4$, $c = 23$, $\varepsilon = 1$
21. $f(x) = 1/x$, $L = 1/4$, $c = 4$, $\varepsilon = 0.05$
22. $f(x) = x^2$, $L = 3$, $c = \sqrt{3}$, $\varepsilon = 0.1$
23. $f(x) = x^2$, $L = 4$, $c = -2$, $\varepsilon = 0.5$
24. $f(x) = 1/x$, $L = -1$, $c = -1$, $\varepsilon = 0.1$
25. $f(x) = x^2 - 5$, $L = 11$, $c = 4$, $\varepsilon = 1$
26. $f(x) = 120/x$, $L = 5$, $c = 24$, $\varepsilon = 1$
27. $f(x) = mx$, $m > 0$, $L = 2m$, $c = 2$, $\varepsilon = 0.03$
28. $f(x) = mx$, $m > 0$, $L = 3m$, $c = 3$, $\varepsilon = c > 0$
29. $f(x) = mx + b$, $m > 0$, $L = (m/2) + b$, $c = 1/2$, $\varepsilon = c > 0$
30. $f(x) = mx + b$, $m > 0$, $L = m + b$, $c = 1$, $\varepsilon = 0.05$

Using the Formal Definition

Each of Exercises 31–36 gives a function $f(x)$, a point c , and a positive number ε . Find $L = \lim_{x \rightarrow c} f(x)$. Then find a number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

31. $f(x) = 3 - 2x$, $c = 3$, $\varepsilon = 0.02$
32. $f(x) = -3x - 2$, $c = -1$, $\varepsilon = 0.03$
33. $f(x) = \frac{x^2 - 4}{x - 2}$, $c = 2$, $\varepsilon = 0.05$

$$34. f(x) = \frac{x^2 + 6x + 5}{x + 5}, \quad c = -5, \quad \varepsilon = 0.05$$

$$35. f(x) = \sqrt{1 - 5x}, \quad c = -3, \quad \varepsilon = 0.5$$

$$36. f(x) = 4/x, \quad c = 2, \quad \varepsilon = 0.4$$

Prove the limit statements in Exercises 37–50.

$$37. \lim_{x \rightarrow 4} (9 - x) = 5 \qquad 38. \lim_{x \rightarrow 3} (3x - 7) = 2$$

$$39. \lim_{x \rightarrow 9} \sqrt{x - 5} = 2 \qquad 40. \lim_{x \rightarrow 0} \sqrt{4 - x} = 2$$

$$41. \lim_{x \rightarrow 1} f(x) = 1 \quad \text{if} \quad f(x) = \begin{cases} x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$$

$$42. \lim_{x \rightarrow -2} f(x) = 4 \quad \text{if} \quad f(x) = \begin{cases} x^2, & x \neq -2 \\ 1, & x = -2 \end{cases}$$

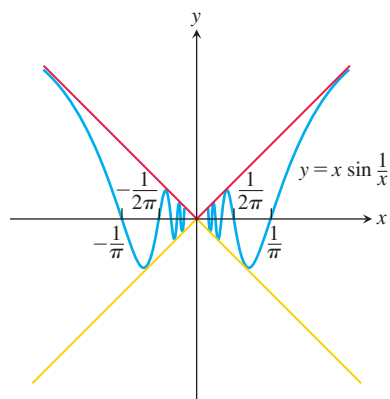
$$43. \lim_{x \rightarrow 1} \frac{1}{x} = 1 \qquad 44. \lim_{x \rightarrow \sqrt{3}} \frac{1}{x^2} = \frac{1}{3}$$

$$45. \lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} = -6 \qquad 46. \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$$

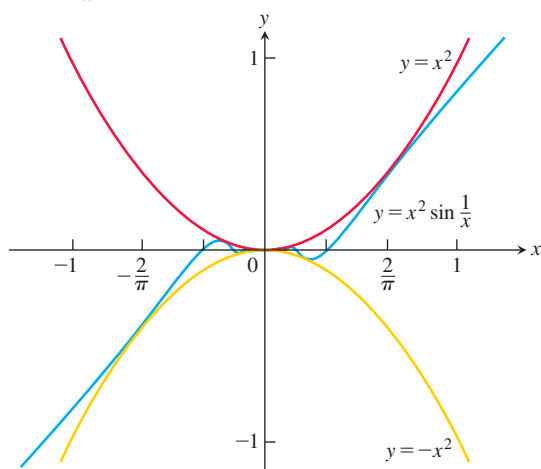
$$47. \lim_{x \rightarrow 1} f(x) = 2 \quad \text{if} \quad f(x) = \begin{cases} 4 - 2x, & x < 1 \\ 6x - 4, & x \geq 1 \end{cases}$$

$$48. \lim_{x \rightarrow 0} f(x) = 0 \quad \text{if} \quad f(x) = \begin{cases} 2x, & x < 0 \\ x/2, & x \geq 0 \end{cases}$$

$$49. \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$$



$$50. \lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$$



Theory and Examples

51. Define what it means to say that $\lim_{x \rightarrow 0} g(x) = k$.

52. Prove that $\lim_{x \rightarrow c} f(x) = L$ if and only if $\lim_{h \rightarrow 0} f(h + c) = L$.

53. **A wrong statement about limits** Show by example that the following statement is wrong.

The number L is the limit of $f(x)$ as x approaches c if $f(x)$ gets closer to L as x approaches c .

Explain why the function in your example does not have the given value of L as a limit as $x \rightarrow c$.

54. **Another wrong statement about limits** Show by example that the following statement is wrong.

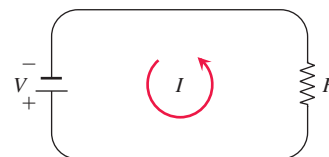
The number L is the limit of $f(x)$ as x approaches c if, given any $\varepsilon > 0$, there exists a value of x for which $|f(x) - L| < \varepsilon$.

Explain why the function in your example does not have the given value of L as a limit as $x \rightarrow c$.

T 55. Grinding engine cylinders Before contracting to grind engine cylinders to a cross-sectional area of 60 cm^2 , you need to know how much deviation from the ideal cylinder diameter of $c = 8.7404 \text{ cm}$ you can allow and still have the area come within 0.1 cm^2 of the required 60 cm^2 . To find out, you let $A = \pi(x/2)^2$ and look for the interval in which you must hold x to make $|A - 60| \leq 0.1$. What interval do you find?

56. Manufacturing electrical resistors

Ohm's law for electrical circuits like the one shown in the accompanying figure states that $V = RI$. In this equation, V is a constant voltage, I is the current in amperes, and R is the resistance in ohms. Your firm has been asked to supply the resistors for a circuit in which V will be 120 volts and I is to be 5 ± 0.1 amp. In what interval does R have to lie for I to be within 0.1 amp of the value $I_0 = 5$?



When Is a Number L Not the Limit of $f(x)$ As $x \rightarrow c$?

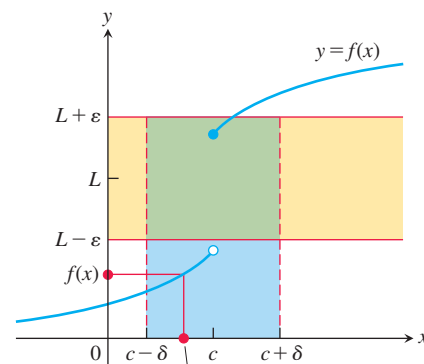
In Exercises 57–60 we will consider what it means to *not* have a limit.

Showing L is not a limit We can prove that $\lim_{x \rightarrow c} f(x) \neq L$ by providing an $\varepsilon > 0$ such that no possible $\delta > 0$ satisfies the condition

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

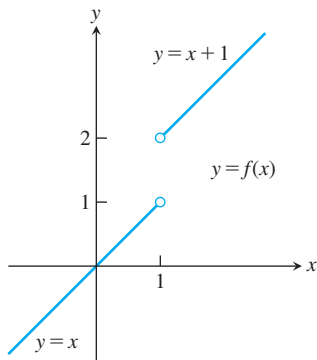
We accomplish this for our candidate ε by showing that for each $\delta > 0$ there exists a value of x such that

$$0 < |x - c| < \delta \quad \text{and} \quad |f(x) - L| \geq \varepsilon.$$



A value of x for which
 $0 < |x - c| < \delta$ and $|f(x) - L| \geq \varepsilon$

57. Let $f(x) = \begin{cases} x, & x < 1 \\ x + 1, & x > 1. \end{cases}$



- a. Let $\varepsilon = 1/2$. Show that no possible $\delta > 0$ satisfies the following condition:

$$|f(x) - 2| < 1/2 \quad \text{whenever} \quad 0 < |x - 1| < \delta.$$

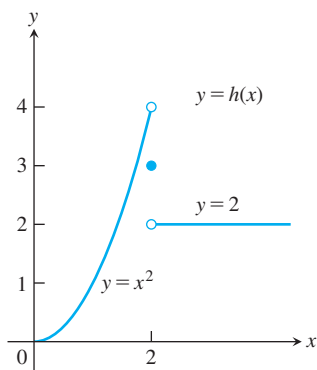
That is, show that for each $\delta > 0$, there is a value of x such that

$$0 < |x - 1| < \delta \quad \text{and} \quad |f(x) - 2| \geq 1/2.$$

This will show that $\lim_{x \rightarrow 1} f(x) \neq 2$.

- b. Show that $\lim_{x \rightarrow 1} f(x) \neq 1$.
c. Show that $\lim_{x \rightarrow 1} f(x) \neq 1.5$.

58. Let $h(x) = \begin{cases} x^2, & x < 2 \\ 3, & x = 2 \\ 2, & x > 2. \end{cases}$

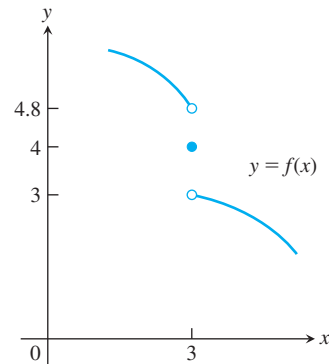


Show that

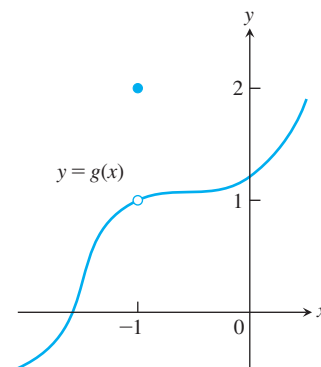
- a. $\lim_{x \rightarrow 2} h(x) \neq 4$
b. $\lim_{x \rightarrow 2} h(x) \neq 3$
c. $\lim_{x \rightarrow 2} h(x) \neq 2$

59. For the function graphed here, explain why

- a. $\lim_{x \rightarrow 3} f(x) \neq 4$
b. $\lim_{x \rightarrow 3} f(x) \neq 4.8$
c. $\lim_{x \rightarrow 3} f(x) \neq 3$



60. a. For the function graphed here, show that $\lim_{x \rightarrow -1} g(x) \neq 2$.
b. Does $\lim_{x \rightarrow -1} g(x)$ appear to exist? If so, what is the value of the limit? If not, why not?



COMPUTER EXPLORATIONS

In Exercises 61–66, you will further explore finding deltas graphically. Use a CAS to perform the following steps:

- Plot the function $y = f(x)$ near the point c being approached.
- Guess the value of the limit L and then evaluate the limit symbolically to see if you guessed correctly.
- Using the value $\varepsilon = 0.2$, graph the banding lines $y_1 = L - \varepsilon$ and $y_2 = L + \varepsilon$ together with the function f near c .
- From your graph in part (c), estimate a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Test your estimate by plotting f , y_1 , and y_2 over the interval $0 < |x - c| < \delta$. For your viewing window use $c - 2\delta \leq x \leq c + 2\delta$ and $L - 2\varepsilon \leq y \leq L + 2\varepsilon$. If any function values lie outside the interval $[L - \varepsilon, L + \varepsilon]$, your choice of δ was too large. Try again with a smaller estimate.

- e. Repeat parts (c) and (d) successively for $\varepsilon = 0.1, 0.05$, and 0.001 .

61. $f(x) = \frac{x^4 - 81}{x - 3}$, $c = 3$ 62. $f(x) = \frac{5x^3 + 9x^2}{2x^5 + 3x^2}$, $c = 0$
63. $f(x) = \frac{\sin 2x}{3x}$, $c = 0$ 64. $f(x) = \frac{x(1 - \cos x)}{x - \sin x}$, $c = 0$
65. $f(x) = \frac{\sqrt[3]{x} - 1}{x - 1}$, $c = 1$
66. $f(x) = \frac{3x^2 - (7x + 1)\sqrt{x} + 5}{x - 1}$, $c = 1$

2.4 One-Sided Limits

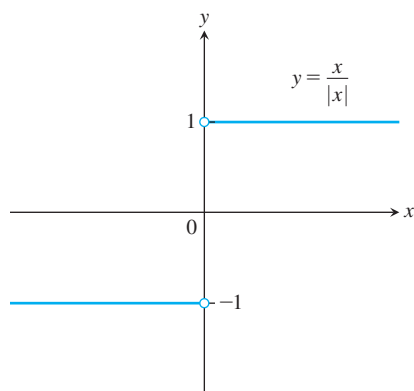


FIGURE 2.25 Different right-hand and left-hand limits at the origin.

In this section we extend the limit concept to *one-sided limits*, which are limits as x approaches the number c from the left-hand side (where $x < c$) or the right-hand side (where $x > c$) only. These allow us to describe functions that have different limits at a point, depending on whether we approach the point from the left or from the right. One-sided limits also allow us to say what it means for a function to have a limit at an endpoint of an interval.

Approaching a Limit from One Side

Suppose a function f is defined on an interval that extends to both sides of a number c . In order for f to have a limit L as x approaches c , the values of $f(x)$ must approach the value L as x approaches c from either side. Because of this, we sometimes say that the limit is **two-sided**.

If f fails to have a two-sided limit at c , it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit** or **limit from the right**. Similarly, a **left-hand limit** is also called a **limit from the left**.

The function $f(x) = x/|x|$ (Figure 2.25) has limit 1 as x approaches 0 from the right, and limit -1 as x approaches 0 from the left. Since these one-sided limit values are not the same, there is no single number that $f(x)$ approaches as x approaches 0. So $f(x)$ does not have a (two-sided) limit at 0.

Intuitively, if we consider only the values of $f(x)$ on an interval (c, b) , where $c < b$, and the values of $f(x)$ become arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c . In this case we write

$$\lim_{x \rightarrow c^+} f(x) = L.$$

The notation “ $x \rightarrow c^+$ ” means that we consider only values of $f(x)$ for x greater than c . We don’t consider values of $f(x)$ for $x \leq c$.

Similarly, if $f(x)$ is defined on an interval (a, c) , where $a < c$, and $f(x)$ approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c . We write

$$\lim_{x \rightarrow c^-} f(x) = M.$$

The symbol “ $x \rightarrow c^-$ ” means that we consider the values of f only at x -values less than c .

These informal definitions of one-sided limits are illustrated in Figure 2.26. For the function $f(x) = x/|x|$ in Figure 2.25 we have

$$\lim_{x \rightarrow 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^-} f(x) = -1.$$

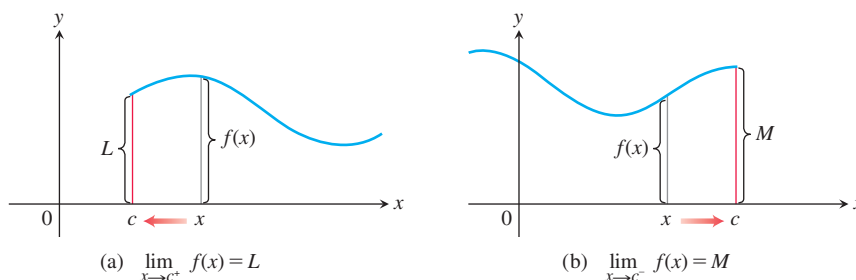


FIGURE 2.26 (a) Right-hand limit as x approaches c . (b) Left-hand limit as x approaches c .

One-sided limits have all the properties listed in Theorem 1 in Section 2.2. The right-hand limit of the sum of two functions is the sum of their right-hand limits, and so on. The theorems for limits of polynomials and rational functions hold with one-sided limits, as does the Sandwich Theorem. One-sided limits are related to limits at interior points in the following way.

THEOREM 5

Suppose that a function f is defined on an open interval containing c , except perhaps at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has both a limit from the left at c and a limit from the right at c , and these one-sided limits are equal:

$$\lim_{x \rightarrow c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \rightarrow c^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x) = L.$$

Theorem 5 applies at interior points of a function's domain. At a boundary point of an interval in its domain, a function has a limit when it has an appropriate one-sided limit.

Limits at Endpoints of an Interval

- If f is defined on an open interval (b, c) to the left of c and not defined on an open interval (c, d) to the right of c , then

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c^-} f(x).$$

- If f is defined on an open interval (c, d) to the right of c and not defined on an open interval (b, c) to the left of c , then

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c^+} f(x).$$

(The definition of a limit on an arbitrary domain is discussed in Additional and Advanced Exercises 39–42.)

EXAMPLE 1 For the function graphed in Figure 2.27,

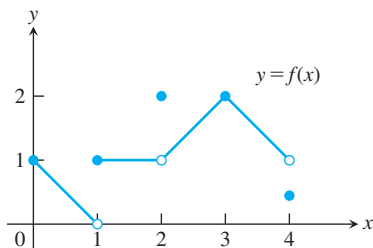


FIGURE 2.27 Graph of the function in Example 1.

- At $x = 0$: $\lim_{x \rightarrow 0^-} f(x)$ does not exist, f is not defined to the left of $x = 0$.
 $\lim_{x \rightarrow 0^+} f(x) = 1$, f has a right-hand limit at $x = 0$.
 $\lim_{x \rightarrow 0} f(x) = 1$. f has a limit at domain endpoint $x = 0$.
- At $x = 1$: $\lim_{x \rightarrow 1^-} f(x) = 0$, Even though $f(1) = 1$.
 $\lim_{x \rightarrow 1^+} f(x) = 1$,
 $\lim_{x \rightarrow 1} f(x)$ does not exist. Right- and left-hand limits are not equal.
- At $x = 2$: $\lim_{x \rightarrow 2^-} f(x) = 1$,
 $\lim_{x \rightarrow 2^+} f(x) = 1$,
 $\lim_{x \rightarrow 2} f(x) = 1$. Even though $f(2) = 2$.
- At $x = 3$: $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3} f(x) = f(3) = 2$.
- At $x = 4$: $\lim_{x \rightarrow 4^-} f(x) = 1$, Even though $f(4) \neq 1$.
 $\lim_{x \rightarrow 4^+} f(x)$ does not exist, f is not defined to the right of $x = 4$.
 $\lim_{x \rightarrow 4} f(x) = 1$. f has a limit at domain endpoint $x = 4$.

At every other point c in $[0, 4]$, $f(x)$, has limit $f(c)$. ■

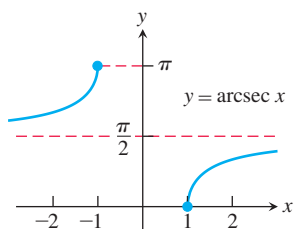


FIGURE 2.28 The arcsec function has limits at $x = \pm 1$.

EXAMPLE 2 The domain of the function $\operatorname{arcsec} x$ is a union of the intervals $(-\infty, -1]$ and $[1, \infty)$, as shown in Figure 2.28. For the boundary points of these intervals,

- At $x = -1$: $\lim_{x \rightarrow -1^-} \operatorname{arcsec} x = \pi$, $\operatorname{arcsec} x$ has a limit from the left at $x = -1$.
 $\lim_{x \rightarrow -1^+} \operatorname{arcsec} x$ does not exist, $\operatorname{arcsec} x$ is not defined on $(-1, 1)$.
 $\lim_{x \rightarrow -1} \operatorname{arcsec} x = \pi$. $\operatorname{arcsec} x$ has a limit at $x = -1$.

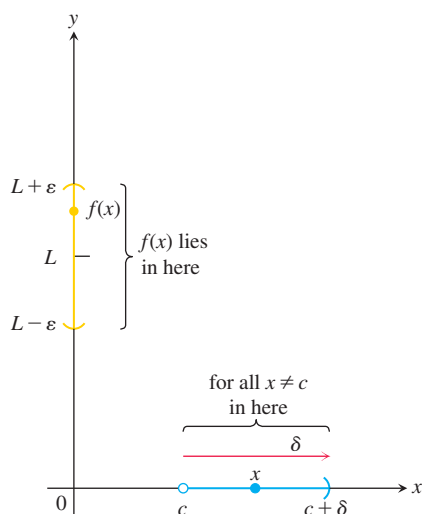


FIGURE 2.29 Intervals associated with the definition of right-hand limit.

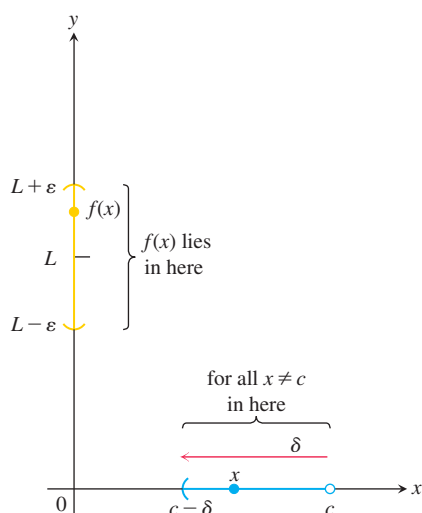


FIGURE 2.30 Intervals associated with the definition of left-hand limit.

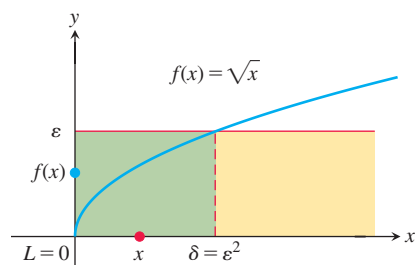


FIGURE 2.31 $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$ in Example 3.

At $x = 1$: $\lim_{x \rightarrow 1^-} \operatorname{arcsec} x$ does not exist, $\operatorname{arcsec} x$ is not defined on $(-1, 1)$.
 $\lim_{x \rightarrow 1^+} \operatorname{arcsec} x = 0$, $\operatorname{arcsec} x$ has a limit from the right at $x = 1$.
 $\lim_{x \rightarrow 1} \operatorname{arcsec} x = 0$. $\operatorname{arcsec} x$ has a limit at $x = 1$.

At every c in $(-\infty, -1)$ and every c in $(1, \infty)$, the limit of $\operatorname{arcsec} x$ as $x \rightarrow c$ is equal to $\operatorname{arcsec} c$. However, for each c in the interval $(-1, 1)$, $\lim_{x \rightarrow c} \operatorname{arcsec} x$ does not exist. ■

Precise Definitions of One-Sided Limits

The formal definition of the limit in Section 2.3 is readily modified for one-sided limits.

DEFINITIONS (a) Assume the domain of f contains an interval (c, d) to the right of c . We say that $f(x)$ has **right-hand limit** L at c , and write

$$\lim_{x \rightarrow c^+} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c < x < c + \delta.$$

(b) Assume the domain of f contains an interval (b, c) to the left of c . We say that f has **left-hand limit** L at c , and write

$$\lim_{x \rightarrow c^-} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c - \delta < x < c.$$

The definitions are illustrated in Figures 2.29 and 2.30.

EXAMPLE 3 Prove that

$$\lim_{x \rightarrow 0^+} \sqrt{x} = 0.$$

Solution Let $\varepsilon > 0$ be given. Here $c = 0$ and $L = 0$, so we want to find a $\delta > 0$ such that

$$|\sqrt{x} - 0| < \varepsilon \quad \text{whenever} \quad 0 < x < \delta,$$

or

$$\sqrt{x} < \varepsilon \quad \text{whenever} \quad 0 < x < \delta. \quad \sqrt{x} \geq 0 \text{ so } |\sqrt{x}| = \sqrt{x}$$

Squaring both sides of this last inequality gives

$$x < \varepsilon^2 \quad \text{if} \quad 0 < x < \delta.$$

If we choose $\delta = \varepsilon^2$ we have

$$\sqrt{x} < \varepsilon \quad \text{whenever} \quad 0 < x < \delta = \varepsilon^2,$$

or

$$|\sqrt{x} - 0| < \varepsilon \quad \text{whenever} \quad 0 < x < \varepsilon^2.$$

According to the definition, this shows that $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$ (Figure 2.31).

Note that since 0 is an endpoint of the domain where $\sqrt{x}$ is defined, it is also true that $\lim_{x \rightarrow 0} \sqrt{x} = 0$. ■

The functions examined so far have had some kind of limit at each point of interest. In general, that need not be the case.

EXAMPLE 4 Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side (Figure 2.32).

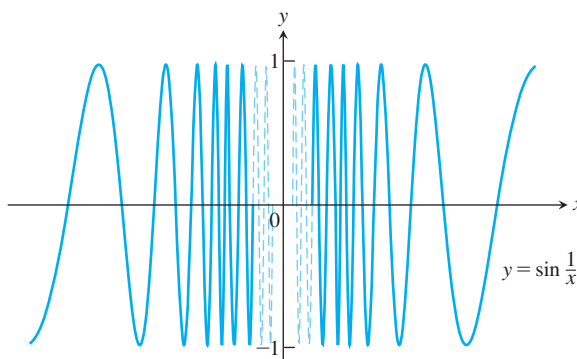


FIGURE 2.32 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 4). The graph here omits values very near the y -axis.

Solution As x approaches zero, its reciprocal, $1/x$, grows without bound, and the values of $\sin(1/x)$ cycle repeatedly from -1 to 1 . There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive values or to negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$. ■

Limits Involving $(\sin \theta)/\theta$

A central fact about $(\sin \theta)/\theta$ is that in radian measure its limit as $\theta \rightarrow 0$ is 1. We can see this in Figure 2.33 and confirm it algebraically using the Sandwich Theorem. You will see the importance of this limit in Section 3.5, where instantaneous rates of change of the trigonometric functions are studied.

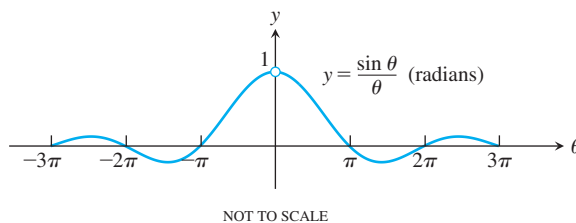


FIGURE 2.33 The graph of $f(\theta) = (\sin \theta)/\theta$ suggests that the right- and left-hand limits as θ approaches 0 are both 1.

THEOREM 6—Limit of the Ratio $\sin \theta/\theta$ as $\theta \rightarrow 0$

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad (\theta \text{ in radians}) \quad (1)$$

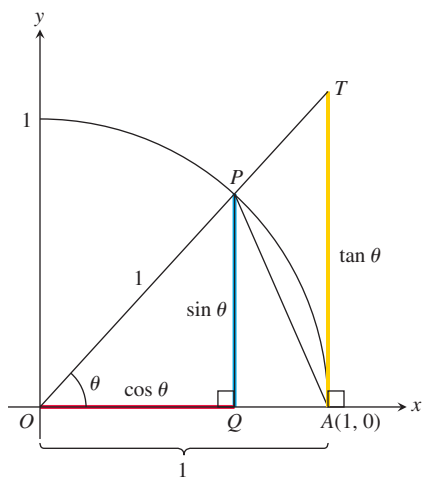


FIGURE 2.34 The ratio $TA/OA = \tan \theta$, and $OA = 1$, so $TA = \tan \theta$.

The use of radians to measure angles is essential in Equation (2): The area of sector OAP is $\theta/2$ only if θ is measured in radians.

Proof The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

To show that the right-hand limit is 1, we begin with positive values of θ less than $\pi/2$ (Figure 2.34). Notice that

$$\text{Area } \triangle OAP < \text{area sector } OAP < \text{area } \triangle OAT.$$

We can express these areas in terms of θ as follows:

$$\begin{aligned} \text{Area } \triangle OAP &= \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2}(1)(\sin \theta) = \frac{1}{2} \sin \theta \\ \text{Area sector } OAP &= \frac{1}{2} r^2 \theta = \frac{1}{2}(1)^2 \theta = \frac{\theta}{2} \\ \text{Area } \triangle OAT &= \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2}(1)(\tan \theta) = \frac{1}{2} \tan \theta. \end{aligned} \quad (2)$$

$$\frac{1}{2} \sin \theta < \frac{1}{2} \theta < \frac{1}{2} \tan \theta.$$

This last inequality goes the same way if we divide all three terms by the number $(1/2) \sin \theta$, which is positive, since $0 < \theta < \pi/2$:

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

Taking reciprocals reverses the inequalities:

$$1 > \frac{\sin \theta}{\theta} > \cos \theta.$$

Since $\lim_{\theta \rightarrow 0^+} \cos \theta = 1$ (Example 12b, Section 2.2), the Sandwich Theorem gives

$$\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1.$$

To consider the left-hand limit, we recall that $\sin \theta$ and θ are both *odd functions* (Section 1.1). Therefore, $f(\theta) = (\sin \theta)/\theta$ is an *even function*, with a graph symmetric about the y-axis (see Figure 2.33). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta \rightarrow 0^-} \frac{\sin \theta}{\theta} = 1 = \lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta},$$

so $\lim_{\theta \rightarrow 0} (\sin \theta)/\theta = 1$ by Theorem 5. ■

EXAMPLE 5 Show that (a) $\lim_{y \rightarrow 0} \frac{\cos y - 1}{y} = 0$ and (b) $\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution

(a) Using the half-angle formula $\cos y = 1 - 2 \sin^2(y/2)$, we calculate

$$\begin{aligned} \lim_{y \rightarrow 0} \frac{\cos y - 1}{y} &= \lim_{y \rightarrow 0} \frac{2 \sin^2(y/2)}{y} \\ &= -\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \sin \theta && \text{Let } \theta = y/2. \\ &= -(1)(0) = 0. && \text{Eq. (1) and Example 12a in Section 2.2} \end{aligned}$$

- (b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} &= \lim_{x \rightarrow 0} \frac{(2/5) \cdot \sin 2x}{(2/5) \cdot 5x} \\ &= \frac{2}{5} \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} && \text{Eq. (1) applies with } \theta = 2x. \\ &= \frac{2}{5}(1) = \frac{2}{5}.\end{aligned}$$

EXAMPLE 6 Find $\lim_{t \rightarrow 0} \frac{\tan t \sec 2t}{3t}$.

Solution From the definition of $\tan t$ and $\sec 2t$, we have

$$\begin{aligned}\lim_{t \rightarrow 0} \frac{\tan t \sec 2t}{3t} &= \lim_{t \rightarrow 0} \frac{1}{3} \cdot \frac{1}{t} \cdot \frac{\sin t}{\cos t} \cdot \frac{1}{\cos 2t} \\ &= \frac{1}{3} \lim_{t \rightarrow 0} \frac{\sin t}{t} \cdot \frac{1}{\cos t} \cdot \frac{1}{\cos 2t} \\ &= \frac{1}{3}(1)(1)(1) = \frac{1}{3}.\end{aligned}$$

Eq. (1) and Example 12b in Section 2.2

EXAMPLE 7 Show that for nonzero constants A and B ,

$$\lim_{\theta \rightarrow 0} \frac{\sin A\theta}{\sin B\theta} = \frac{A}{B}.$$

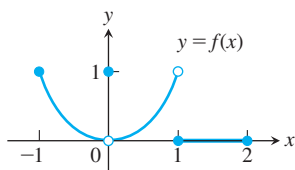
Solution

$$\begin{aligned}\lim_{\theta \rightarrow 0} \frac{\sin A\theta}{\sin B\theta} &= \lim_{\theta \rightarrow 0} \frac{\sin A\theta}{A\theta} \cdot A\theta \cdot \frac{B\theta}{\sin B\theta} \cdot \frac{1}{B\theta} && \text{Multiply and divide by } A\theta \text{ and } B\theta. \\ &= \lim_{\theta \rightarrow 0} \frac{\sin A\theta}{A\theta} \cdot \frac{B\theta}{\sin B\theta} \cdot \frac{A}{B} && \lim_{u \rightarrow 0} \frac{\sin u}{u} = 1, \text{ with } u = A\theta \\ &= \lim_{\theta \rightarrow 0} (1)(1) \cdot \frac{A}{B} && \lim_{v \rightarrow 0} \frac{v}{\sin v} = 1, \text{ with } v = B\theta \\ &= \frac{A}{B}.\end{aligned}$$

EXERCISES 2.4

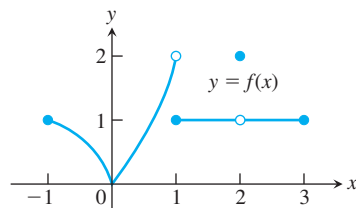
Finding Limits Graphically

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



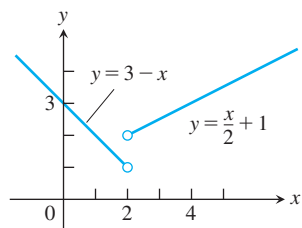
a. $\lim_{x \rightarrow -1^+} f(x) = 1$	b. $\lim_{x \rightarrow 0^-} f(x) = 0$
c. $\lim_{x \rightarrow 0^-} f(x) = 1$	d. $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$
e. $\lim_{x \rightarrow 0} f(x)$ exists.	f. $\lim_{x \rightarrow 0} f(x) = 0$
g. $\lim_{x \rightarrow 0} f(x) = 1$	h. $\lim_{x \rightarrow 1} f(x) = 1$
i. $\lim_{x \rightarrow 1} f(x) = 0$	j. $\lim_{x \rightarrow 2^-} f(x) = 2$
k. $\lim_{x \rightarrow -1^-} f(x)$ does not exist.	l. $\lim_{x \rightarrow 2^+} f(x) = 0$

2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



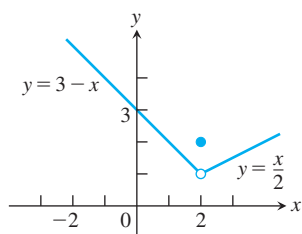
a. $\lim_{x \rightarrow -1^+} f(x) = 1$	b. $\lim_{x \rightarrow 2} f(x)$ does not exist.
c. $\lim_{x \rightarrow 2} f(x) = 2$	d. $\lim_{x \rightarrow 1^-} f(x) = 2$
e. $\lim_{x \rightarrow 1^+} f(x) = 1$	f. $\lim_{x \rightarrow 1} f(x)$ does not exist.
g. $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x)$	
h. $\lim_{x \rightarrow c} f(x)$ exists at every c in the open interval $(-1, 1)$.	
i. $\lim_{x \rightarrow c} f(x)$ exists at every c in the open interval $(1, 3)$.	
j. $\lim_{x \rightarrow -1^-} f(x) = 0$	k. $\lim_{x \rightarrow 3^+} f(x)$ does not exist.

3. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{x}{2} + 1, & x > 2. \end{cases}$



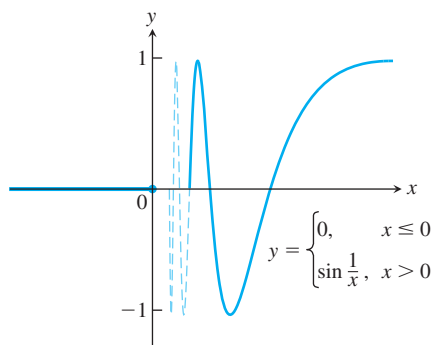
- Find $\lim_{x \rightarrow 2^-} f(x)$ and $\lim_{x \rightarrow 2^+} f(x)$.
- Does $\lim_{x \rightarrow 2} f(x)$ exist? If so, what is it? If not, why not?
- Find $\lim_{x \rightarrow 4^-} f(x)$ and $\lim_{x \rightarrow 4^+} f(x)$.
- Does $\lim_{x \rightarrow 4} f(x)$ exist? If so, what is it? If not, why not?

4. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ \frac{x}{2}, & x > 2. \end{cases}$



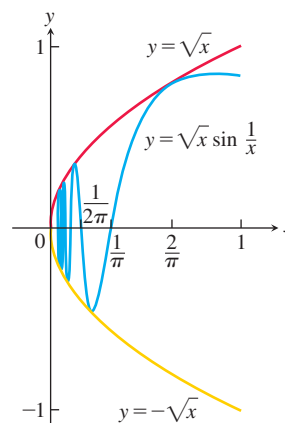
- Find $\lim_{x \rightarrow 2^-} f(x)$, $\lim_{x \rightarrow 2^+} f(x)$, and $f(2)$.
- Does $\lim_{x \rightarrow 2} f(x)$ exist? If so, what is it? If not, why not?
- Find $\lim_{x \rightarrow -1^-} f(x)$ and $\lim_{x \rightarrow -1^+} f(x)$.
- Does $\lim_{x \rightarrow -1} f(x)$ exist? If so, what is it? If not, why not?

5. Let $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0. \end{cases}$



- Does $\lim_{x \rightarrow 0^+} f(x)$ exist? If so, what is it? If not, why not?
- Does $\lim_{x \rightarrow 0^-} f(x)$ exist? If so, what is it? If not, why not?
- Does $\lim_{x \rightarrow 0} f(x)$ exist? If so, what is it? If not, why not?

6. Let $g(x) = \sqrt{x} \sin(1/x)$.



- Does $\lim_{x \rightarrow 0^+} g(x)$ exist? If so, what is it? If not, why not?
- Does $\lim_{x \rightarrow 0^-} g(x)$ exist? If so, what is it? If not, why not?
- Does $\lim_{x \rightarrow 0} g(x)$ exist? If so, what is it? If not, why not?

7. a. Graph $f(x) = \begin{cases} x^3, & x \neq 1 \\ 0, & x = 1. \end{cases}$

- Find $\lim_{x \rightarrow 1^-} f(x)$ and $\lim_{x \rightarrow 1^+} f(x)$.
- Does $\lim_{x \rightarrow 1} f(x)$ exist? If so, what is it? If not, why not?

8. a. Graph $f(x) = \begin{cases} 1 - x^2, & x \neq 1 \\ 2, & x = 1. \end{cases}$

- Find $\lim_{x \rightarrow 1^+} f(x)$ and $\lim_{x \rightarrow 1^-} f(x)$.
- Does $\lim_{x \rightarrow 1} f(x)$ exist? If so, what is it? If not, why not?

Graph the functions in Exercises 9 and 10. Then answer these questions.

- What are the domain and range of f ?
- At what points c , if any, does $\lim_{x \rightarrow c} f(x)$ exist?
- At what points does the left-hand limit exist but not the right-hand limit?
- At what points does the right-hand limit exist but not the left-hand limit?

9. $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \\ 2, & x = 2 \end{cases}$

10. $f(x) = \begin{cases} x, & -1 \leq x < 0, \text{ or } 0 < x \leq 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$

Finding One-Sided Limits Algebraically

Find the limits in Exercises 11–20.

11. $\lim_{x \rightarrow -0.5^-} \sqrt{\frac{x+2}{x+1}}$

12. $\lim_{x \rightarrow 1^+} \sqrt{\frac{x-1}{x+2}}$

13. $\lim_{x \rightarrow -2^+} \left(\frac{x}{x+1} \right) \left(\frac{2x+5}{x^2+x} \right)$

14. $\lim_{x \rightarrow 1^-} \left(\frac{1}{x+1} \right) \left(\frac{x+6}{x} \right) \left(\frac{3-x}{7} \right)$

15. $\lim_{h \rightarrow 0^+} \frac{\sqrt{h^2 + 4h + 5} - \sqrt{5}}{h}$

16. $\lim_{h \rightarrow 0^-} \frac{\sqrt{6} - \sqrt{5h^2 + 11h + 6}}{h}$

17. a. $\lim_{x \rightarrow -2^+} (x+3) \frac{|x+2|}{x+2}$ b. $\lim_{x \rightarrow -2^-} (x+3) \frac{|x+2|}{x+2}$

18. a. $\lim_{x \rightarrow 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|}$ b. $\lim_{x \rightarrow 1^-} \frac{\sqrt{2x}(x-1)}{|x-1|}$

19. a. $\lim_{x \rightarrow 0^+} \frac{|\sin x|}{x}$ b. $\lim_{x \rightarrow 0^-} \frac{|\sin x|}{x}$

20. a. $\lim_{x \rightarrow 0^+} \frac{1 - \cos x}{|\cos x - 1|}$ b. $\lim_{x \rightarrow 0^-} \frac{\cos x - 1}{|\cos x - 1|}$

Use the graph of the greatest integer function $y = \lfloor x \rfloor$, Figure 1.10 in Section 1.1, to help you find the limits in Exercises 21 and 22.

21. a. $\lim_{\theta \rightarrow 3^+} \frac{|\theta|}{\theta}$ b. $\lim_{\theta \rightarrow 3^-} \frac{|\theta|}{\theta}$

22. a. $\lim_{t \rightarrow 4^+} (t - \lfloor t \rfloor)$ b. $\lim_{t \rightarrow 4^-} (t - \lfloor t \rfloor)$

Using $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$

Find the limits in Exercises 23–46.

23. $\lim_{\theta \rightarrow 0} \frac{\sin \sqrt{2}\theta}{\sqrt{2}\theta}$

24. $\lim_{t \rightarrow 0} \frac{\sin kt}{t}$ (k constant)

25. $\lim_{y \rightarrow 0} \frac{\sin 3y}{4y}$

26. $\lim_{h \rightarrow 0^-} \frac{h}{\sin 3h}$

27. $\lim_{x \rightarrow 0} \frac{\tan 2x}{x}$

28. $\lim_{t \rightarrow 0} \frac{2t}{\tan t}$

29. $\lim_{x \rightarrow 0} \frac{x \csc 2x}{\cos 5x}$

30. $\lim_{x \rightarrow 0} 6x^2 (\cot x) (\csc 2x)$

31. $\lim_{x \rightarrow 0} \frac{x + x \cos x}{\sin x \cos x}$

32. $\lim_{x \rightarrow 0} \frac{x^2 - x + \sin x}{2x}$

33. $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin 2\theta}$

34. $\lim_{x \rightarrow 0} \frac{x - x \cos x}{\sin^2 3x}$

35. $\lim_{t \rightarrow 0} \frac{\sin(1 - \cos t)}{1 - \cos t}$

36. $\lim_{h \rightarrow 0} \frac{\sin(\sin h)}{\sin h}$

37. $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta}$

38. $\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 4x}$

39. $\lim_{\theta \rightarrow 0} \theta \cos \theta$

40. $\lim_{\theta \rightarrow 0} \sin \theta \cot \theta$

41. $\lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 8x}$

42. $\lim_{y \rightarrow 0} \frac{\sin 3y \cot 5y}{y \cot 4y}$

43. $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta^2 \cot 3\theta}$

44. $\lim_{\theta \rightarrow 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta}$

45. $\lim_{x \rightarrow 0} \frac{1 - \cos 3x}{2x}$

46. $\lim_{x \rightarrow 0} \frac{\cos^2 x - \cos x}{x^2}$

Theory and Examples

47. Once you know $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow a^-} f(x)$ at an interior point of the domain of f , do you then know $\lim_{x \rightarrow a} f(x)$? Give reasons for your answer.

48. If you know that $\lim_{x \rightarrow c} f(x)$ exists at an interior point of a domain interval of f , can you find its value by calculating $\lim_{x \rightarrow c^+} f(x)$? Give reasons for your answer.

49. Suppose that f is an odd function of x . Does knowing that $\lim_{x \rightarrow 0^+} f(x) = 3$ tell you anything about $\lim_{x \rightarrow 0^-} f(x)$? Give reasons for your answer.

50. Suppose that f is an even function of x . Does knowing that $\lim_{x \rightarrow 2^-} f(x) = 7$ tell you anything about either $\lim_{x \rightarrow -2^-} f(x)$ or $\lim_{x \rightarrow -2^+} f(x)$? Give reasons for your answer.

Formal Definitions of One-Sided Limits

51. Given $\varepsilon > 0$, find an interval $I = (5, 5 + \delta)$, $\delta > 0$, such that if x lies in I , then $\sqrt{x} - 5 < \varepsilon$. What limit is being verified and what is its value?

52. Given $\varepsilon > 0$, find an interval $I = (4 - \delta, 4)$, $\delta > 0$, such that if x lies in I , then $\sqrt{4 - x} < \varepsilon$. What limit is being verified and what is its value?

Use the definitions of right-hand and left-hand limits to prove the limit statements in Exercises 53 and 54.

53. $\lim_{x \rightarrow 0^-} \frac{x}{|x|} = -1$

54. $\lim_{x \rightarrow 2^+} \frac{x-2}{|x-2|} = 1$

55. **Greatest integer function** Find (a) $\lim_{x \rightarrow 400^+} \lfloor x \rfloor$ and (b) $\lim_{x \rightarrow 400^-} \lfloor x \rfloor$; then use limit definitions to verify your findings. (c) Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x \rightarrow 400} \lfloor x \rfloor$? Give reasons for your answer.

56. **One-sided limits** Let $f(x) = \begin{cases} x^2 \sin(1/x), & x < 0 \\ \sqrt{x}, & x > 0. \end{cases}$

Find (a) $\lim_{x \rightarrow 0^+} f(x)$ and (b) $\lim_{x \rightarrow 0^-} f(x)$; then use limit definitions to verify your findings. (c) Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x \rightarrow 0} f(x)$? Give reasons for your answer.

2.5 Limits Involving Infinity; Asymptotes of Graphs

In this section we investigate the behavior of a function when the magnitude of the independent variable x becomes increasingly large, or $x \rightarrow \pm\infty$. We further extend the concept of limit to *infinite limits*. Infinite limits provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large in magnitude. We use these ideas to analyze the graphs of functions having *horizontal* or *vertical asymptotes*.

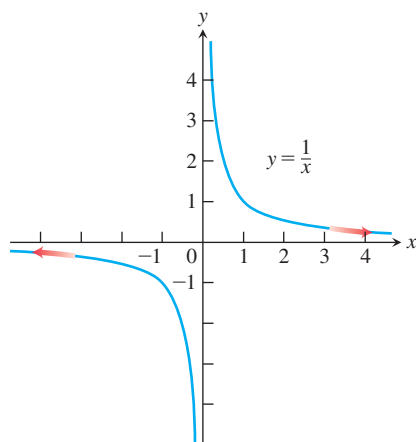


FIGURE 2.35 The graph of $y = 1/x$ approaches 0 as $x \rightarrow \infty$ or $x \rightarrow -\infty$.

Finite Limits as $x \rightarrow \pm\infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, the function $f(x) = 1/x$ is defined for all $x \neq 0$ (Figure 2.35). When x is positive and becomes increasingly large, $1/x$ becomes increasingly small. When x is negative and its magnitude becomes increasingly large, $1/x$ again becomes small. We summarize these observations by saying that $f(x) = 1/x$ has limit 0 as $x \rightarrow \infty$ or $x \rightarrow -\infty$, or that 0 is a *limit of $f(x) = 1/x$ at infinity and at negative infinity*. Here are precise definitions for the limit of a function whose domain contains positive or negative numbers of unbounded magnitude.

DEFINITIONS

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x \rightarrow \infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number M such that for all x in the domain of f

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x > M.$$

2. We say that $f(x)$ has the **limit L as x approaches negative infinity** and write

$$\lim_{x \rightarrow -\infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number N such that for all x in the domain of f

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x < N.$$

Intuitively, $\lim_{x \rightarrow \infty} f(x) = L$ if, as x moves increasingly far from the origin in the positive direction, $f(x)$ gets arbitrarily close to L . Similarly, $\lim_{x \rightarrow -\infty} f(x) = L$ if, as x moves increasingly far from the origin in the negative direction, $f(x)$ gets arbitrarily close to L .

The strategy for calculating limits of functions as $x \rightarrow +\infty$ or as $x \rightarrow -\infty$ is similar to the one for finite limits in Section 2.2. There we first found the limits of the constant and identity functions $y = k$ and $y = x$. We then extended these results to other functions by applying Theorem 1 on limits of algebraic combinations. Here we do the same thing, except that the starting functions are $y = k$ and $y = 1/x$ instead of $y = k$ and $y = x$.

The basic facts to be verified by applying the formal definition are

$$\lim_{x \rightarrow \pm\infty} k = k \quad \text{and} \quad \lim_{x \rightarrow \pm\infty} \frac{1}{x} = 0. \quad (1)$$

We prove the second result in Example 1, and leave the first to Exercises 93 and 94.

EXAMPLE 1 Show that

(a) $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$

(b) $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$.

Solution

(a) Let $\varepsilon > 0$ be given. We must find a number M such that

$$\left| \frac{1}{x} - 0 \right| = \left| \frac{1}{x} \right| < \varepsilon \quad \text{whenever} \quad x > M.$$

The implication will hold if $M = 1/\varepsilon$ or any larger positive number (Figure 2.36). This proves $\lim_{x \rightarrow \infty} (1/x) = 0$.

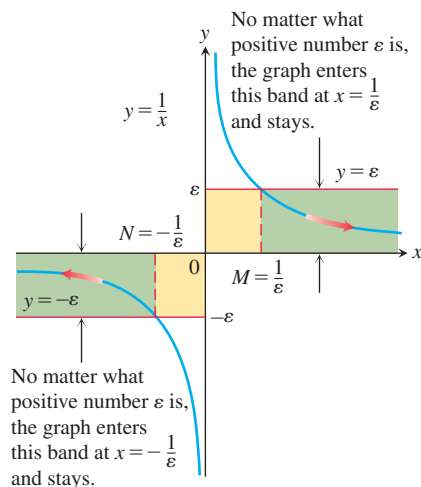


FIGURE 2.36 The geometry behind the argument in Example 1.

(b) Let $\varepsilon > 0$ be given. We must find a number N such that

$$\left| \frac{1}{x} - 0 \right| = \left| \frac{1}{x} \right| < \varepsilon \quad \text{whenever} \quad x < N.$$

The implication will hold if $N = -1/\varepsilon$ or any number less than $-1/\varepsilon$ (Figure 2.36). This proves $\lim_{x \rightarrow -\infty} (1/x) = 0$. ■

Limits at infinity have properties similar to those of finite limits.

THEOREM 7

All the Limit Laws in Theorem 1 are true when we replace $\lim$ by $\lim_{x \rightarrow c}$, $\lim_{x \rightarrow \infty}$, or $\lim_{x \rightarrow -\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

EXAMPLE 2 The properties in Theorem 7 are used to calculate limits in the same way as when x approaches a finite number c .

$$\begin{aligned} \text{(a)} \quad \lim_{x \rightarrow \infty} \left(5 + \frac{1}{x} \right) &= \lim_{x \rightarrow \infty} 5 + \lim_{x \rightarrow \infty} \frac{1}{x} && \text{Sum Rule} \\ &= 5 + 0 = 5 && \text{Known limits} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow -\infty} \frac{\pi\sqrt{3}}{x^2} &= \lim_{x \rightarrow -\infty} \pi\sqrt{3} \cdot \frac{1}{x} \cdot \frac{1}{x} \\ &= \lim_{x \rightarrow -\infty} \pi\sqrt{3} \cdot \lim_{x \rightarrow -\infty} \frac{1}{x} \cdot \lim_{x \rightarrow -\infty} \frac{1}{x} && \text{Product Rule} \\ &= \pi\sqrt{3} \cdot 0 \cdot 0 = 0 && \text{Known limits} \end{aligned}$$

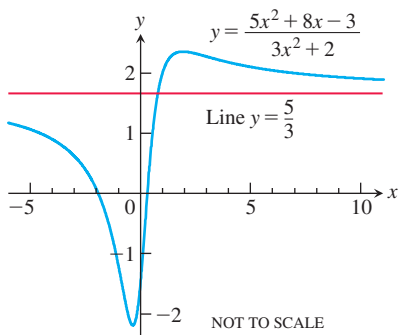


FIGURE 2.37 The graph of the function in Example 3a. The graph approaches the line $y = 5/3$ as $|x|$ increases.

Limits at Infinity of Rational Functions

To determine the limit of a rational function as $x \rightarrow \pm\infty$, we first divide the numerator and denominator by the highest power of x in the denominator. The result then depends on the degrees of the polynomials involved.

EXAMPLE 3 These examples illustrate what happens when the degree of the numerator is less than or equal to the degree of the denominator.

$$\begin{aligned} \text{(a)} \quad \lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} &= \lim_{x \rightarrow \infty} \frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)} && \text{Divide numerator and denominator by } x^2. \\ &= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3} && \text{See Fig. 2.37.} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow -\infty} \frac{11x + 2}{2x^3 - 1} &= \lim_{x \rightarrow -\infty} \frac{(11/x^2) + (2/x^3)}{2 - (1/x^3)} && \text{Divide numerator and denominator by } x^3. \\ &= \frac{0 + 0}{2 - 0} = 0 && \text{See Fig. 2.38.} \end{aligned}$$

Cases for which the degree of the numerator is greater than the degree of the denominator are illustrated in Examples 10 and 14.

Horizontal Asymptotes

If the distance between the graph of a function and some fixed line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an *asymptote* of the graph.

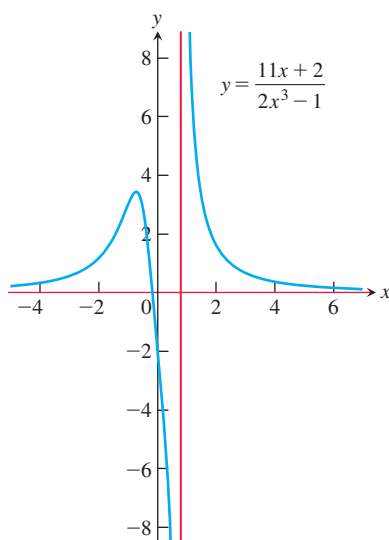


FIGURE 2.38 The graph of the function in Example 3b. The graph approaches the x -axis as $|x|$ increases.

Looking at $f(x) = 1/x$ (see Figure 2.35), we observe that the x -axis is an asymptote of the curve on the right because

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

and on the left because

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

We say that the x -axis is a *horizontal asymptote* of the graph of $f(x) = 1/x$.

DEFINITION A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b.$$

The graph of a function can have zero, one, or two horizontal asymptotes, depending on whether the function has limits as $x \rightarrow \infty$ and as $x \rightarrow -\infty$.

The graph of the function

$$f(x) = \frac{5x^2 + 8x - 3}{3x^2 + 2}$$

sketched in Figure 2.37 (Example 3a) has the line $y = 5/3$ as a horizontal asymptote on both the right and the left because

$$\lim_{x \rightarrow \infty} f(x) = \frac{5}{3} \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x) = \frac{5}{3}.$$

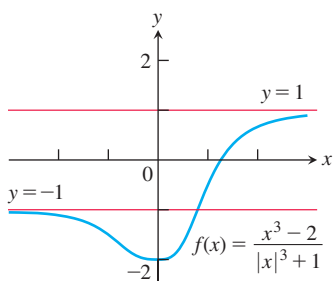


FIGURE 2.39 The graph of the function in Example 4 has two horizontal asymptotes.

EXAMPLE 4 Find the horizontal asymptotes of the graph of

$$f(x) = \frac{x^3 - 2}{|x|^3 + 1}.$$

Solution We calculate the limits as $x \rightarrow \pm\infty$.

$$\text{For } x > 0: \quad \lim_{x \rightarrow \infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow \infty} \frac{x^3 - 2}{x^3 + 1} = \lim_{x \rightarrow \infty} \frac{1 - (2/x^3)}{1 + (1/x^3)} = 1.$$

$$\text{For } x < 0: \quad \lim_{x \rightarrow -\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow -\infty} \frac{x^3 - 2}{(-x)^3 + 1} = \lim_{x \rightarrow -\infty} \frac{1 - (2/x^3)}{-1 + (1/x^3)} = -1.$$

The horizontal asymptotes are $y = -1$ and $y = 1$. The graph is displayed in Figure 2.39. Notice that the graph crosses the horizontal asymptote $y = -1$ for a positive value of x . ■

EXAMPLE 5 The x -axis (the line $y = 0$) is a horizontal asymptote of the graph of $y = e^x$ because

$$\lim_{x \rightarrow -\infty} e^x = 0.$$

To see this, we use the definition of a limit as x approaches $-\infty$. So let $\varepsilon > 0$ be given, but arbitrary. We must find a constant N such that

$$|e^x - 0| < \varepsilon \quad \text{whenever} \quad x < N.$$

Now $|e^x - 0| = e^x$, so the condition that needs to be satisfied whenever $x < N$ is

$$e^x < \varepsilon.$$

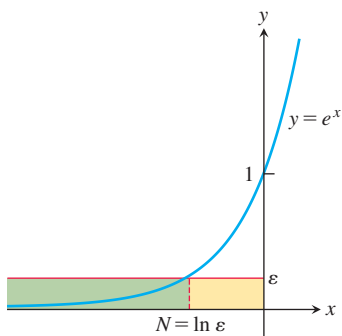


FIGURE 2.40 The graph of $y = e^x$ approaches the x -axis as $x \rightarrow -\infty$ (Example 5).

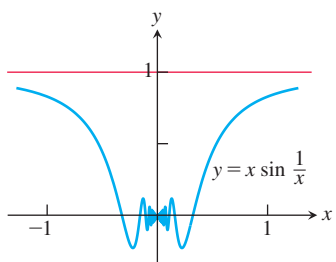


FIGURE 2.41 The line $y = 1$ is a horizontal asymptote of the function graphed here (Example 6b).

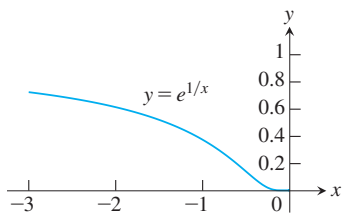


FIGURE 2.42 The graph of $y = e^{1/x}$ for $x < 0$ shows $\lim_{x \rightarrow 0^-} e^{1/x} = 0$ (Example 7).

Let $x = N$ be the number where $e^x = \varepsilon$. Since e^x is an increasing function, if $x < N$, then $e^x < \varepsilon$. We find N by taking the natural logarithm of both sides of the equation $e^N = \varepsilon$, so $N = \ln \varepsilon$ (see Figure 2.40). With this value of N the condition is satisfied, and we conclude that $\lim_{x \rightarrow -\infty} e^x = 0$. ■

Sometimes it is helpful to transform a limit in which x approaches ∞ to a new limit by setting $t = 1/x$ and seeing what happens as t approaches 0.

EXAMPLE 6 Find (a) $\lim_{x \rightarrow \infty} \sin(1/x)$ and (b) $\lim_{x \rightarrow \pm\infty} x \sin(1/x)$.

Solution

(a) We introduce the new variable $t = 1/x$. From Example 1, we know that $t \rightarrow 0^+$ as $x \rightarrow \infty$ (see Figure 2.35). Therefore,

$$\lim_{x \rightarrow \infty} \sin \frac{1}{x} = \lim_{t \rightarrow 0^+} \sin t = 0.$$

(b) We calculate the limits as $x \rightarrow \infty$ and $x \rightarrow -\infty$:

$$\lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{t \rightarrow 0^+} \frac{\sin t}{t} = 1 \quad \text{and} \quad \lim_{x \rightarrow -\infty} x \sin \frac{1}{x} = \lim_{t \rightarrow 0^-} \frac{\sin t}{t} = 1.$$

The graph is shown in Figure 2.41, and we see that the line $y = 1$ is a horizontal asymptote. ■

Similarly, we can investigate the behavior of $y = f(1/x)$ as $x \rightarrow 0$ by investigating $y = f(t)$ as $t \rightarrow \pm\infty$, where $t = 1/x$.

EXAMPLE 7 Find $\lim_{x \rightarrow 0^-} e^{1/x}$.

Solution We let $t = 1/x$. From Figure 2.35, we can see that $t \rightarrow -\infty$ as $x \rightarrow 0^-$. (We make this idea more precise further on.) Therefore,

$$\lim_{x \rightarrow 0^-} e^{1/x} = \lim_{t \rightarrow -\infty} e^t = 0 \quad \text{Example 5}$$

(Figure 2.42). ■

The Sandwich Theorem also holds for limits as $x \rightarrow \pm\infty$. You must be sure, though, that the function whose limit you are trying to find stays between the bounding functions for all very large x in the positive direction (if $x \rightarrow \infty$) or all very large x in the negative direction (if $x \rightarrow -\infty$).

EXAMPLE 8 Using the Sandwich Theorem, find the horizontal asymptote of the curve

$$y = 2 + \frac{\sin x}{x}.$$

Solution We are interested in the behavior as $x \rightarrow \pm\infty$. Since

$$0 \leq \left| \frac{\sin x}{x} \right| \leq \left| \frac{1}{x} \right|$$

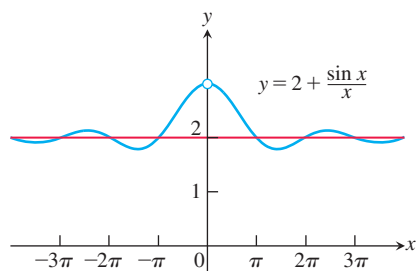


FIGURE 2.43 A curve may cross one of its asymptotes infinitely often (Example 8).

and $\lim_{x \rightarrow \pm\infty} |1/x| = 0$, we have $\lim_{x \rightarrow \pm\infty} (\sin x)/x = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x \rightarrow \pm\infty} \left(2 + \frac{\sin x}{x} \right) = 2 + 0 = 2,$$

and the line $y = 2$ is a horizontal asymptote of the curve on both left and right (Figure 2.43).

This example illustrates that a curve may cross one of its horizontal asymptotes many times. ■

EXAMPLE 9 Find $\lim_{x \rightarrow \infty} (x - \sqrt{x^2 + 16})$.

Solution Both of the terms x and $\sqrt{x^2 + 16}$ approach infinity as $x \rightarrow \infty$, so what happens to the difference in the limit is unclear (we cannot subtract ∞ from ∞ because the symbol does not represent a real number). In this situation we can multiply the numerator and the denominator by the conjugate radical expression to obtain an equivalent algebraic expression:

$$\begin{aligned} \lim_{x \rightarrow \infty} (x - \sqrt{x^2 + 16}) &= \lim_{x \rightarrow \infty} (x - \sqrt{x^2 + 16}) \frac{x + \sqrt{x^2 + 16}}{x + \sqrt{x^2 + 16}} && \text{Multiply and divide by the conjugate.} \\ &= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + 16)}{x + \sqrt{x^2 + 16}} = \lim_{x \rightarrow \infty} \frac{-16}{x + \sqrt{x^2 + 16}}. \end{aligned}$$

As $x \rightarrow \infty$, the denominator in this last expression becomes arbitrarily large, while the numerator remains constant, so we see that the limit is 0. We can also obtain this result by a direct calculation using the Limit Laws:

$$\lim_{x \rightarrow \infty} \frac{-16}{x + \sqrt{x^2 + 16}} = \lim_{x \rightarrow \infty} \frac{-16}{1 + \sqrt{\frac{x^2}{x^2} + \frac{16}{x^2}}} = \frac{0}{1 + \sqrt{1 + 0}} = 0.$$

Oblique Asymptotes

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant line asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \rightarrow \pm\infty$.

EXAMPLE 10 Find the oblique asymptote of the graph of

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

in Figure 2.44.

Solution We are interested in the behavior as $x \rightarrow \pm\infty$. We divide $(2x - 4)$ into $(x^2 - 3)$:

$$\begin{array}{r} \frac{x}{2} + 1 \\ 2x - 4 \overline{) x^2 + 0x - 3} \\ \underline{x^2 - 2x} \\ 2x - 3 \\ \underline{2x - 4} \\ 1 \end{array}$$

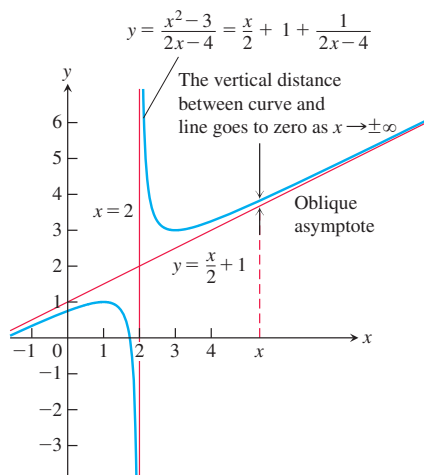


FIGURE 2.44 The graph of the function in Example 10 has an oblique asymptote.

This tells us that

$$f(x) = \frac{x^2 - 3}{2x - 4} = \underbrace{\left(\frac{x}{2} + 1\right)}_{\text{linear } g(x)} + \underbrace{\left(\frac{1}{2x - 4}\right)}_{\text{remainder}}$$

As $x \rightarrow \pm\infty$, the remainder, whose magnitude gives the vertical distance between the graphs of f and g , goes to zero, making the slanted line

$$g(x) = \frac{x}{2} + 1$$

an asymptote of the graph of f (Figure 2.44). The line $y = g(x)$ is an asymptote both to the right and to the left. ■

Notice in Example 10 that if the degree of the numerator in a rational function is greater than the degree of the denominator, then the limit as $|x|$ becomes large is $+\infty$ or $-\infty$, depending on the signs assumed by the numerator and denominator.

Infinite Limits

Let us look again at the function $f(x) = 1/x$. As $x \rightarrow 0^+$, the values of f grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number B , however large, the values of f become larger still (Figure 2.45).

Thus, f has no limit as $x \rightarrow 0^+$. It is nevertheless convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \rightarrow 0^+$. We write

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{1}{x} = \infty.$$

In writing this equation, we are *not* saying that the limit exists. Nor are we saying that there is a real number ∞ , for there is no such number. Rather, this expression is just a concise way of saying that $\lim_{x \rightarrow 0^+} (1/x)$ *does not exist because $1/x$ becomes arbitrarily large and positive as $x \rightarrow 0^+$* .

As $x \rightarrow 0^-$, the values of $f(x) = 1/x$ become arbitrarily large and negative. Given any negative real number $-B$, the values of f eventually lie below $-B$. (See Figure 2.45.) We write

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty.$$

Again, we are not saying that the limit exists and equals the number $-\infty$. There *is* no real number $-\infty$. We are describing the behavior of a function whose limit as $x \rightarrow 0^-$ *does not exist because its values become arbitrarily large and negative*.

EXAMPLE 11 Find $\lim_{x \rightarrow 1^+} \frac{1}{x-1}$ and $\lim_{x \rightarrow 1^-} \frac{1}{x-1}$.

Geometric Solution The graph of $y = 1/(x-1)$ is the graph of $y = 1/x$ shifted 1 unit to the right (Figure 2.46). Therefore, $y = 1/(x-1)$ behaves near 1 exactly the way $y = 1/x$ behaves near 0:

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1} = \infty \quad \text{and} \quad \lim_{x \rightarrow 1^-} \frac{1}{x-1} = -\infty.$$

Analytic Solution Think about the number $x-1$ and its reciprocal. As $x \rightarrow 1^+$, we have $(x-1) \rightarrow 0^+$ and $1/(x-1) \rightarrow \infty$. As $x \rightarrow 1^-$, we have $(x-1) \rightarrow 0^-$ and $1/(x-1) \rightarrow -\infty$. ■

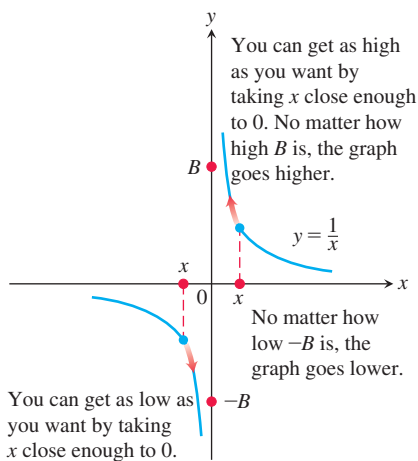


FIGURE 2.45 One-sided infinite limits:

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty.$$

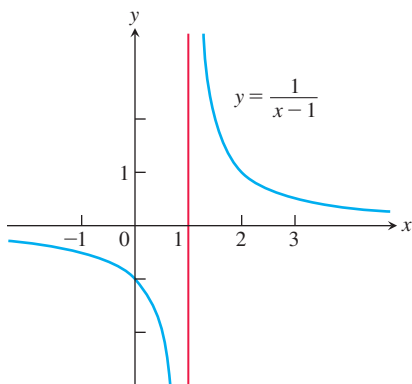


FIGURE 2.46 Near $x = 1$, the function $y = 1/(x-1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right (Example 11).

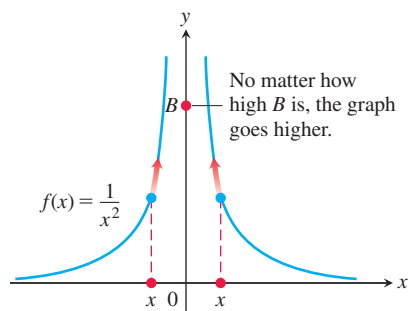


FIGURE 2.47 The graph of $f(x)$ in Example 12 approaches infinity as $x \rightarrow 0$.

EXAMPLE 12 Discuss the behavior of

$$f(x) = \frac{1}{x^2} \quad \text{as } x \rightarrow 0.$$

Solution As x approaches zero from either side, the values of $1/x^2$ are positive and become arbitrarily large (Figure 2.47). This means that

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{1}{x^2} = \infty.$$

The function $y = 1/x$ shows no consistent behavior as $x \rightarrow 0$. We have $1/x \rightarrow \infty$ if $x \rightarrow 0^+$, but $1/x \rightarrow -\infty$ if $x \rightarrow 0^-$. All we can say about $\lim_{x \rightarrow 0} (1/x)$ is that it does not exist. The function $y = 1/x^2$ is different. Its values approach infinity as x approaches zero from either side, so we can say that $\lim_{x \rightarrow 0} (1/x^2) = \infty$. ■

EXAMPLE 13 These examples illustrate that rational functions can behave in various ways near zeros of the denominator.

$$(a) \lim_{x \rightarrow 2} \frac{(x-2)^2}{x^2-4} = \lim_{x \rightarrow 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{x-2}{x+2} = 0$$

Can substitute 2 for x after algebraic manipulation eliminates division by 0.

$$(b) \lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \lim_{x \rightarrow 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{1}{x+2} = \frac{1}{4}$$

Again substitute 2 for x after algebraic manipulation eliminates division by 0.

$$(c) \lim_{x \rightarrow 2^+} \frac{x-3}{x^2-4} = \lim_{x \rightarrow 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$$

The values are negative for $x > 2$, x near 2.

$$(d) \lim_{x \rightarrow 2^-} \frac{x-3}{x^2-4} = \lim_{x \rightarrow 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$$

The values are positive for $x < 2$, x near 2.

$$(e) \lim_{x \rightarrow 2} \frac{x-3}{x^2-4} = \lim_{x \rightarrow 2} \frac{x-3}{(x-2)(x+2)} \text{ does not exist.}$$

Limits from left and from right differ.

$$(f) \lim_{x \rightarrow 2} \frac{2-x}{(x-2)^3} = \lim_{x \rightarrow 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x \rightarrow 2} \frac{-1}{(x-2)^2} = -\infty$$

Denominator is positive, so values are negative near $x = 2$.

In parts (a) and (b), the effect of the zero in the denominator at $x = 2$ is canceled because the numerator is zero there also. Thus a finite limit exists. This is not true in part (f), where cancellation still leaves a zero factor in the denominator. ■

EXAMPLE 14 Find $\lim_{x \rightarrow -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7}$.

Solution We are asked to find the limit of a rational function as $x \rightarrow -\infty$, so we divide the numerator and denominator by x^2 , the highest power of x in the denominator:

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7} &= \lim_{x \rightarrow -\infty} \frac{2x^3 - 6x^2 + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\ &= \lim_{x \rightarrow -\infty} \frac{2x^2(x-3) + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\ &= -\infty, \end{aligned} \quad x^{-n} \rightarrow 0, x-3 \rightarrow -\infty$$

because the numerator tends to $-\infty$ while the denominator approaches 3 as $x \rightarrow -\infty$. ■

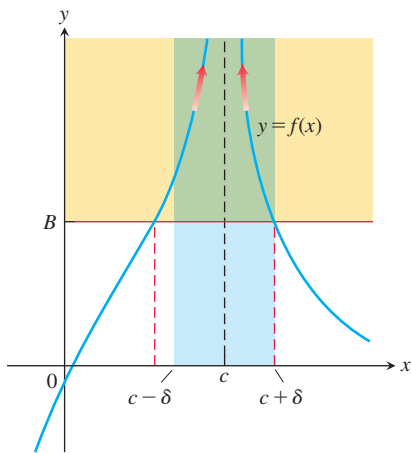


FIGURE 2.48 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies above the line $y = B$.

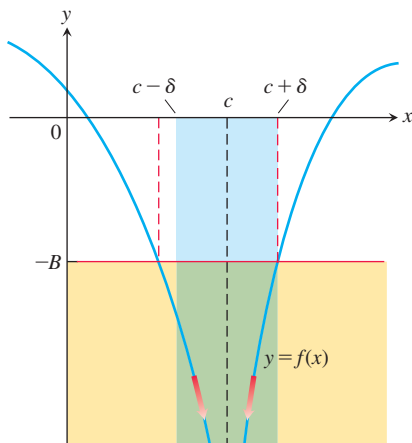


FIGURE 2.49 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

Precise Definitions of Infinite Limits

Instead of requiring $f(x)$ to lie arbitrarily close to a finite number L for all x sufficiently close to c , the definitions of infinite limits require $f(x)$ to lie arbitrarily far from zero. Except for this change, the language is very similar to what we have seen before. Figures 2.48 and 2.49 accompany these definitions.

DEFINITIONS

1. We say that $f(x)$ approaches infinity as x approaches c , and write

$$\lim_{x \rightarrow c} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that

$$f(x) > B \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

2. We say that $f(x)$ approaches negative infinity as x approaches c , and write

$$\lim_{x \rightarrow c} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that

$$f(x) < -B \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

The precise definitions of one-sided infinite limits at c are similar and are stated in the exercises.

EXAMPLE 15 Prove that $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$.

Solution Given $B > 0$, we want to find $\delta > 0$ such that

$$\frac{1}{x^2} > B \quad \text{whenever} \quad 0 < |x - 0| < \delta.$$

Now,

$$\frac{1}{x^2} > B \quad \text{if and only if} \quad x^2 < \frac{1}{B}$$

or, equivalently,

$$|x| < \frac{1}{\sqrt{B}}.$$

Thus, choosing $\delta = 1/\sqrt{B}$ (or any smaller positive number), we see that

$$\text{if } 0 < |x| < \delta \quad \text{then} \quad \frac{1}{x^2} > \frac{1}{\delta^2} \geq B.$$

Therefore, by definition,

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty.$$



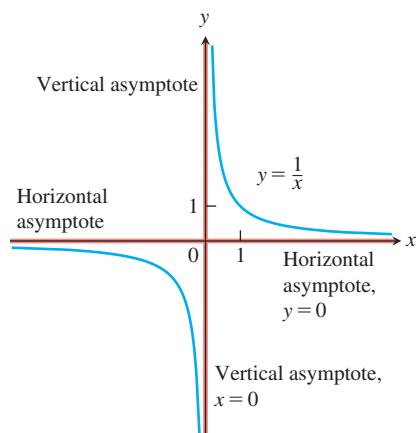


FIGURE 2.50 The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

Vertical Asymptotes

Notice that the distance between a point on the graph of $f(x) = 1/x$ and the y -axis approaches zero as the point moves nearly vertically along the graph and away from the origin (Figure 2.50). The function $f(x) = 1/x$ is unbounded as x approaches 0 because

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty.$$

We say that the line $x = 0$ (the y -axis) is a *vertical asymptote* of the graph of $f(x) = 1/x$. Observe that the denominator is zero at $x = 0$ and the function is undefined there.

DEFINITION A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) = \pm\infty.$$

EXAMPLE 16 Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x+3}{x+2}.$$

Solution We are interested in the behavior as $x \rightarrow \pm\infty$ and the behavior as $x \rightarrow -2$, where the denominator is zero.

The asymptotes are revealed if we recast the rational function as a polynomial with a remainder, by dividing $(x+3)$ into $(x+2)$:

$$\begin{array}{r} x+2 \overline{) x+3} \\ \underline{x+2} \\ 1 \end{array}$$

This result enables us to rewrite y as

$$y = 1 + \frac{1}{x+2}.$$

We see that the curve in question is the graph of $f(x) = 1/x$ shifted 1 unit up and 2 units left (Figure 2.51). The asymptotes, instead of being the coordinate axes, are now the lines $y = 1$ and $x = -2$. As $x \rightarrow \pm\infty$, the curve approaches the horizontal asymptote $y = 1$; as $x \rightarrow -2$, the curve approaches the vertical asymptote $x = -2$. ■

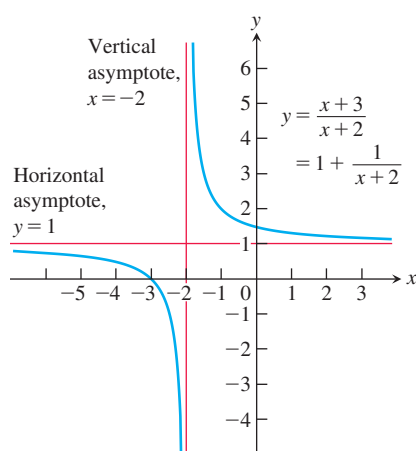


FIGURE 2.51 The lines $y = 1$ and $x = -2$ are asymptotes of the curve in Example 16.

EXAMPLE 17 Find the horizontal and vertical asymptotes of the graph of

$$f(x) = -\frac{8}{x^2 - 4}.$$

Solution We are interested in the behavior as $x \rightarrow \pm\infty$ and as $x \rightarrow \pm 2$, where the denominator is zero. Notice that f is an even function of x , so its graph is symmetric with respect to the y -axis.

(a) *The behavior as $x \rightarrow \pm\infty$.* Since $\lim_{x \rightarrow \infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of the graph to the right. By symmetry it is an asymptote to the left as well

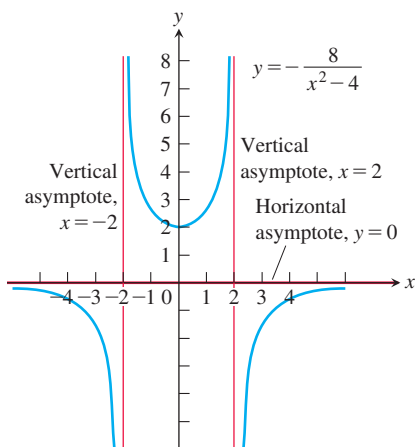


FIGURE 2.52 Graph of the function in Example 17. Notice that the curve approaches the x -axis from only one side. Asymptotes do not have to be two-sided.

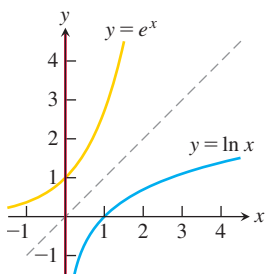


FIGURE 2.53 The vertical line $x = 0$ is a vertical asymptote of the natural logarithm function (Example 18).

(Figure 2.52). Notice that the curve approaches the x -axis from only the negative side (or from below). Also, $f(0) = 2$.

(b) The behavior as $x \rightarrow \pm 2$. Since

$$\lim_{x \rightarrow 2^+} f(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow 2^-} f(x) = \infty,$$

the line $x = 2$ is a vertical asymptote both from the right and from the left. By symmetry, the line $x = -2$ is also a vertical asymptote.

There are no other asymptotes because f has a finite limit at all other points. ■

EXAMPLE 18 The graph of the natural logarithm function has the y -axis (the line $x = 0$) as a vertical asymptote. We see this from the graph sketched in Figure 2.53 (which is the reflection of the graph of the natural exponential function across the line $y = x$) and the fact that the x -axis is a horizontal asymptote of $y = e^x$ (Example 5). Thus,

$$\lim_{x \rightarrow 0^+} \ln x = -\infty.$$

The same result is true for $y = \log_a x$ whenever $a > 1$. ■

EXAMPLE 19 The curves

$$y = \sec x = \frac{1}{\cos x} \quad \text{and} \quad y = \tan x = \frac{\sin x}{\cos x}$$

both have vertical asymptotes at odd-integer multiples of $\pi/2$, which are the points where $\cos x = 0$ (Figure 2.54).

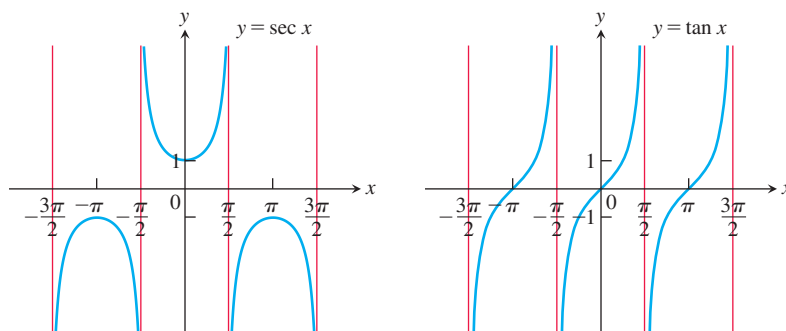


FIGURE 2.54 The graphs of $\sec x$ and $\tan x$ have infinitely many vertical asymptotes (Example 19). ■

Dominant Terms

In Example 10 we saw that by using long division, we can rewrite the function

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

as a linear function plus a remainder term:

$$f(x) = \left(\frac{x}{2} + 1\right) + \left(\frac{1}{2x - 4}\right).$$

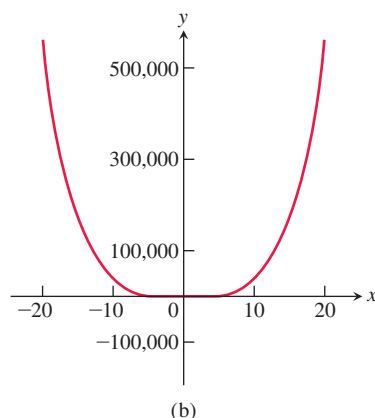
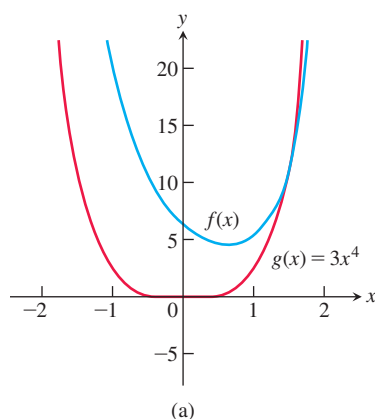


FIGURE 2.55 The graphs of f and g are (a) distinct for $|x|$ small, and (b) nearly identical for $|x|$ large (Example 20).

This tells us immediately that

For x large: $f(x) \approx \frac{x}{2} + 1$ $\frac{1}{2x-4}$ is near 0.

For x near 2: $f(x) \approx \frac{1}{2x-4}$ This term is very large in absolute value.

If we want to know how f behaves, this is one possible way to find out. It behaves like $y = (x/2) + 1$ when $|x|$ is large and the contribution of $1/(2x-4)$ to the total value of f is insignificant. It behaves like $1/(2x-4)$ when x is so close to 2 that $1/(2x-4)$ makes the dominant contribution.

We say that $(x/2) + 1$ **dominates** when x approaches ∞ or $-\infty$, and we say that $1/(2x-4)$ dominates when x approaches 2. **Dominant terms** like these help us predict a function's behavior.

EXAMPLE 20 Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$. Show that although f and g are quite different for numerically small values of x , they behave similarly for very large $|x|$, in the sense that their ratios approach 1 as $x \rightarrow \infty$ or $x \rightarrow -\infty$.

Solution The graphs of f and g behave quite differently near the origin (Figure 2.55a), but appear as virtually identical on a larger scale (Figure 2.55b).

We can test that the term $3x^4$ in f , represented graphically by g , dominates the polynomial f for numerically large values of x by examining the ratio of the two functions as $x \rightarrow \pm\infty$. We find that

$$\begin{aligned} \lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow \pm\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4} \\ &= \lim_{x \rightarrow \pm\infty} \left(1 - \frac{2}{3x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{2}{x^4} \right) \\ &= 1, \end{aligned}$$

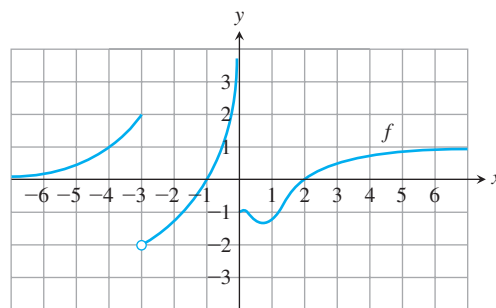
which means that f and g appear nearly identical when $|x|$ is large. ■

EXERCISES 2.5

Finding Limits

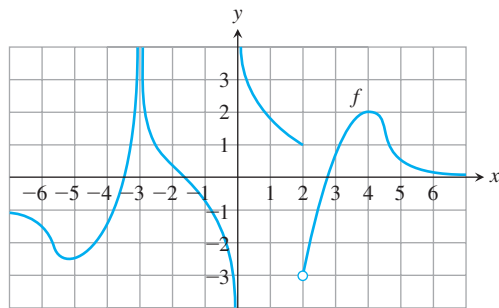
1. For the function f whose graph is given, determine the following limits. Write ∞ or $-\infty$ where appropriate.

a. $\lim_{x \rightarrow 2} f(x)$	b. $\lim_{x \rightarrow -3^+} f(x)$
c. $\lim_{x \rightarrow -3^-} f(x)$	d. $\lim_{x \rightarrow -3} f(x)$
e. $\lim_{x \rightarrow 0^+} f(x)$	f. $\lim_{x \rightarrow 0^-} f(x)$
g. $\lim_{x \rightarrow 0} f(x)$	h. $\lim_{x \rightarrow \infty} f(x)$
i. $\lim_{x \rightarrow -\infty} f(x)$	



2. For the function f whose graph is given, determine the following limits. Write ∞ or $-\infty$ where appropriate.

a. $\lim_{x \rightarrow 4} f(x)$	b. $\lim_{x \rightarrow 2^+} f(x)$	c. $\lim_{x \rightarrow 2^-} f(x)$
d. $\lim_{x \rightarrow 2} f(x)$	e. $\lim_{x \rightarrow -3^+} f(x)$	f. $\lim_{x \rightarrow -3^-} f(x)$
g. $\lim_{x \rightarrow -3} f(x)$	h. $\lim_{x \rightarrow 0^+} f(x)$	i. $\lim_{x \rightarrow 0^-} f(x)$
j. $\lim_{x \rightarrow 0} f(x)$	k. $\lim_{x \rightarrow \infty} f(x)$	l. $\lim_{x \rightarrow -\infty} f(x)$



In Exercises 3–8, find the limit of each function (a) as $x \rightarrow \infty$ and (b) as $x \rightarrow -\infty$. (You may wish to visualize your answer with a graphing calculator or computer.)

3. $f(x) = \frac{2}{x} - 3$	4. $f(x) = \pi - \frac{2}{x^2}$
5. $g(x) = \frac{1}{2 + (1/x)}$	6. $g(x) = \frac{1}{8 - (5/x^2)}$
7. $h(x) = \frac{-5 + (7/x)}{3 - (1/x^2)}$	8. $h(x) = \frac{3 - (2/x)}{4 + (\sqrt{2}/x^2)}$

Find the limits in Exercises 9–12.

9. $\lim_{x \rightarrow \infty} \frac{\sin 2x}{x}$	10. $\lim_{\theta \rightarrow -\infty} \frac{\cos \theta}{3\theta}$
11. $\lim_{t \rightarrow -\infty} \frac{2 - t + \sin t}{t + \cos t}$	12. $\lim_{r \rightarrow \infty} \frac{r + \sin r}{2r + 7 - 5 \sin r}$

Limits of Rational Functions

In Exercises 13–22, find the limit of each rational function (a) as $x \rightarrow \infty$ and (b) as $x \rightarrow -\infty$. Write ∞ or $-\infty$ where appropriate.

13. $f(x) = \frac{2x + 3}{5x + 7}$	14. $f(x) = \frac{2x^3 + 7}{x^3 - x^2 + x + 7}$
15. $f(x) = \frac{x + 1}{x^2 + 3}$	16. $f(x) = \frac{3x + 7}{x^2 - 2}$
17. $h(x) = \frac{7x^3}{x^3 - 3x^2 + 6x}$	18. $h(x) = \frac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$
19. $g(x) = \frac{10x^5 + x^4 + 31}{x^6}$	20. $g(x) = \frac{x^3 + 7x^2 - 2}{x^2 - x + 1}$
21. $f(x) = \frac{3x^7 + 5x^2 - 1}{6x^3 - 7x + 3}$	22. $h(x) = \frac{5x^8 - 2x^3 + 9}{3 + x - 4x^5}$

Limits as $x \rightarrow \infty$ or $x \rightarrow -\infty$

The process by which we determine limits of rational functions applies equally well to ratios containing noninteger or negative powers of x .

Divide numerator and denominator by the highest power of x in the denominator and proceed from there. Find the limits in Exercises 23–36. Write ∞ or $-\infty$ where appropriate.

23. $\lim_{x \rightarrow \infty} \sqrt{\frac{8x^2 - 3}{2x^2 + x}}$	24. $\lim_{x \rightarrow -\infty} \left(\frac{x^2 + x - 1}{8x^2 - 3} \right)^{1/3}$
25. $\lim_{x \rightarrow -\infty} \left(\frac{1 - x^3}{x^2 + 7x} \right)^5$	26. $\lim_{x \rightarrow \infty} \sqrt{\frac{x^2 - 5x}{x^3 + x - 2}}$
27. $\lim_{x \rightarrow \infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7}$	28. $\lim_{x \rightarrow \infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}}$
29. $\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$	30. $\lim_{x \rightarrow \infty} \frac{x^{-1} + x^{-4}}{x^{-2} - x^{-3}}$
31. $\lim_{x \rightarrow \infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$	32. $\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$
33. $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x + 1}$	34. $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1}}{x + 1}$
35. $\lim_{x \rightarrow \infty} \frac{x - 3}{\sqrt{4x^2 + 25}}$	36. $\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$

Infinite Limits

Find the limits in Exercises 37–48. Write ∞ or $-\infty$ where appropriate.

37. $\lim_{x \rightarrow 0^+} \frac{1}{3x}$	38. $\lim_{x \rightarrow 0^-} \frac{5}{2x}$
39. $\lim_{x \rightarrow 2^-} \frac{3}{x - 2}$	40. $\lim_{x \rightarrow 3^+} \frac{1}{x - 3}$
41. $\lim_{x \rightarrow -8^+} \frac{2x}{x + 8}$	42. $\lim_{x \rightarrow -5^-} \frac{3x}{2x + 10}$
43. $\lim_{x \rightarrow 7} \frac{4}{(x - 7)^2}$	44. $\lim_{x \rightarrow 0} \frac{-1}{x^2(x + 1)}$
45. a. $\lim_{x \rightarrow 0^+} \frac{2}{3x^{1/3}}$	b. $\lim_{x \rightarrow 0^-} \frac{2}{3x^{1/3}}$
46. a. $\lim_{x \rightarrow 0^+} \frac{2}{x^{1/5}}$	b. $\lim_{x \rightarrow 0^-} \frac{2}{x^{1/5}}$
47. $\lim_{x \rightarrow 0} \frac{4}{x^{2/5}}$	48. $\lim_{x \rightarrow 0} \frac{1}{x^{2/3}}$

Find the limits in Exercises 49–52. Write ∞ or $-\infty$ where appropriate.

49. $\lim_{x \rightarrow (\pi/2)^-} \tan x$	50. $\lim_{x \rightarrow (-\pi/2)^+} \sec x$
51. $\lim_{\theta \rightarrow 0^-} (1 + \csc \theta)$	52. $\lim_{\theta \rightarrow 0} (2 - \cot \theta)$

Find the limits in Exercises 53–58. Write ∞ or $-\infty$ where appropriate.

53. $\lim_{x \rightarrow 2} \frac{1}{x^2 - 4}$ as	a. $x \rightarrow 2^+$	b. $x \rightarrow 2^-$
	c. $x \rightarrow -2^+$	d. $x \rightarrow -2^-$
54. $\lim_{x \rightarrow 1} \frac{x}{x^2 - 1}$ as	a. $x \rightarrow 1^+$	b. $x \rightarrow 1^-$
	c. $x \rightarrow -1^+$	d. $x \rightarrow -1^-$

55. $\lim \left(\frac{x^2}{2} - \frac{1}{x} \right)$ as

a. $x \rightarrow 0^+$

b. $x \rightarrow 0^-$

c. $x \rightarrow \sqrt[3]{2}$

d. $x \rightarrow -1$

56. $\lim \frac{x^2 - 1}{2x + 4}$ as

a. $x \rightarrow -2^+$

b. $x \rightarrow -2^-$

c. $x \rightarrow 1^+$

d. $x \rightarrow 0^-$

57. $\lim \frac{x^2 - 3x + 2}{x^3 - 2x^2}$ as

a. $x \rightarrow 0^+$

b. $x \rightarrow 2^+$

c. $x \rightarrow 2^-$

d. $x \rightarrow 2$

e. What, if anything, can be said about the limit as $x \rightarrow 0$?

58. $\lim \frac{x^2 - 3x + 2}{x^3 - 4x}$ as

a. $x \rightarrow 2^+$

b. $x \rightarrow -2^+$

c. $x \rightarrow 0^-$

d. $x \rightarrow 1^+$

e. What, if anything, can be said about the limit as $x \rightarrow 0$?Find the limits in Exercises 59–62. Write ∞ or $-\infty$ where appropriate.

59. $\lim \left(2 - \frac{3}{t^{1/3}} \right)$ as

a. $t \rightarrow 0^+$

b. $t \rightarrow 0^-$

60. $\lim \left(\frac{1}{t^{3/5}} + 7 \right)$ as

a. $t \rightarrow 0^+$

b. $t \rightarrow 0^-$

61. $\lim \left(\frac{1}{x^{2/3}} + \frac{2}{(x-1)^{2/3}} \right)$ as

a. $x \rightarrow 0^+$

b. $x \rightarrow 0^-$

c. $x \rightarrow 1^+$

d. $x \rightarrow 1^-$

62. $\lim \left(\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}} \right)$ as

a. $x \rightarrow 0^+$

b. $x \rightarrow 0^-$

c. $x \rightarrow 1^+$

d. $x \rightarrow 1^-$

Graphing Simple Rational Functions

Graph the rational functions in Exercises 63–68. Include the graphs and equations of the asymptotes and dominant terms.

63. $y = \frac{1}{x-1}$

64. $y = \frac{1}{x+1}$

65. $y = \frac{1}{2x+4}$

66. $y = \frac{-3}{x-3}$

67. $y = \frac{x+3}{x+2}$

68. $y = \frac{2x}{x+1}$

Domains and Asymptotes

Determine the domain of each function in Exercises 69–74. Then use various limits to find the asymptotes.

69. $y = 4 + \frac{3x^2}{x^2 + 1}$

70. $y = \frac{2x}{x^2 - 1}$

71. $y = \frac{8 - e^x}{2 + e^x}$

72. $y = \frac{4e^x + e^{2x}}{e^x + e^{2x}}$

73. $y = \frac{\sqrt{x^2 + 4}}{x}$

74. $y = \frac{x^3}{x^3 - 8}$

Inventing Graphs and FunctionsIn Exercises 75–78, sketch the graph of a function $y = f(x)$ that satisfies the given conditions. No formulas are required—just label the coordinate axes and sketch an appropriate graph. (The answers are not unique, so your graphs may not be exactly like those in the answer section.)

75. $f(0) = 0, f(1) = 2, f(-1) = -2, \lim_{x \rightarrow -\infty} f(x) = -1$, and $\lim_{x \rightarrow \infty} f(x) = 1$

76. $f(0) = 0, \lim_{x \rightarrow \pm\infty} f(x) = 0, \lim_{x \rightarrow 0^+} f(x) = 2$, and $\lim_{x \rightarrow 0^-} f(x) = -2$

77. $f(0) = 0, \lim_{x \rightarrow \pm\infty} f(x) = 0, \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow -1^+} f(x) = \infty$, $\lim_{x \rightarrow 1^+} f(x) = -\infty$, and $\lim_{x \rightarrow -1^-} f(x) = -\infty$

78. $f(2) = 1, f(-1) = 0, \lim_{x \rightarrow \infty} f(x) = 0, \lim_{x \rightarrow 0^+} f(x) = \infty$, $\lim_{x \rightarrow 0^-} f(x) = -\infty$, and $\lim_{x \rightarrow -\infty} f(x) = 1$

In Exercises 79–82, find a function that satisfies the given conditions and sketch its graph. (The answers here are not unique. Any function that satisfies the conditions is acceptable. Feel free to use formulas defined in pieces if that will help.)

79. $\lim_{x \rightarrow \pm\infty} f(x) = 0, \lim_{x \rightarrow 2^-} f(x) = \infty$, and $\lim_{x \rightarrow 2^+} f(x) = \infty$

80. $\lim_{x \rightarrow \pm\infty} g(x) = 0, \lim_{x \rightarrow 3^-} g(x) = -\infty$, and $\lim_{x \rightarrow 3^+} g(x) = \infty$

81. $\lim_{x \rightarrow -\infty} h(x) = -1, \lim_{x \rightarrow \infty} h(x) = 1, \lim_{x \rightarrow 0^-} h(x) = -1$, and $\lim_{x \rightarrow 0^+} h(x) = 1$

82. $\lim_{x \rightarrow \pm\infty} k(x) = 1, \lim_{x \rightarrow 1^-} k(x) = \infty$, and $\lim_{x \rightarrow 1^+} k(x) = -\infty$

83. Suppose that $f(x)$ and $g(x)$ are polynomials in x and that $\lim_{x \rightarrow \infty} (f(x)/g(x)) = 2$. Can you conclude anything about $\lim_{x \rightarrow -\infty} (f(x)/g(x))$? Give reasons for your answer.84. Suppose that $f(x)$ and $g(x)$ are polynomials in x . Can the graph of $f(x)/g(x)$ have an asymptote if $g(x)$ is never zero? Give reasons for your answer.

85. How many horizontal asymptotes can the graph of a given rational function have? Give reasons for your answer.

Finding Limits of Differences When $x \rightarrow \pm\infty$

Find the limits in Exercises 86–92. (Hint: Try multiplying and dividing by the conjugate.)

86. $\lim_{x \rightarrow \infty} (\sqrt{x+9} - \sqrt{x+4})$

87. $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 25} - \sqrt{x^2 - 1})$

88. $\lim_{x \rightarrow -\infty} (\sqrt{x^2 + 3} + x)$

89. $\lim_{x \rightarrow -\infty} (2x + \sqrt{4x^2 + 3x - 2})$

90. $\lim_{x \rightarrow \infty} (\sqrt{9x^2 - x} - 3x)$

$$91. \lim_{x \rightarrow \infty} (\sqrt{x^2 + 3x} - \sqrt{x^2 - 2x})$$

$$92. \lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - \sqrt{x^2 - x})$$

Using the Formal Definitions

Use the formal definitions of limits as $x \rightarrow \pm\infty$ to establish the limits in Exercises 93 and 94.

93. If f has the constant value $f(x) = k$, then $\lim_{x \rightarrow \infty} f(x) = k$.

94. If f has the constant value $f(x) = k$, then $\lim_{x \rightarrow -\infty} f(x) = k$.

Use formal definitions to prove the limit statements in Exercises 95–98.

$$95. \lim_{x \rightarrow 0} \frac{-1}{x^2} = -\infty$$

$$96. \lim_{x \rightarrow 0} \frac{1}{|x|} = \infty$$

$$97. \lim_{x \rightarrow 3} \frac{-2}{(x-3)^2} = -\infty$$

$$98. \lim_{x \rightarrow -5} \frac{1}{(x+5)^2} = \infty$$

99. Here is the definition of **infinite right-hand limit**.

Suppose that an interval (c, d) lies in the domain of f . We say that $f(x)$ approaches infinity as x approaches c from the right, and write

$$\lim_{x \rightarrow c^+} f(x) = \infty,$$

if, for every positive real number B , there exists a corresponding number $\delta > 0$ such that

$$f(x) > B \quad \text{whenever} \quad c < x < c + \delta.$$

Modify the definition to cover the following cases.

a. $\lim_{x \rightarrow c^-} f(x) = \infty$

b. $\lim_{x \rightarrow c^+} f(x) = -\infty$

c. $\lim_{x \rightarrow c^-} f(x) = -\infty$

Use the formal definitions from Exercise 99 to prove the limit statements in Exercises 100–104.

$$100. \lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$$

$$101. \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

$$102. \lim_{x \rightarrow 2^-} \frac{1}{x-2} = -\infty$$

$$103. \lim_{x \rightarrow 2^+} \frac{1}{x-2} = \infty$$

$$104. \lim_{x \rightarrow 1^-} \frac{1}{1-x^2} = \infty$$

Oblique Asymptotes

Graph the rational functions in Exercises 105–110. Include the graphs and equations of the asymptotes.

$$105. y = \frac{x^2}{x-1}$$

$$106. y = \frac{x^2+1}{x-1}$$

$$107. y = \frac{x^2-4}{x-1}$$

$$108. y = \frac{x^2-1}{2x+4}$$

$$109. y = \frac{x^2-1}{x}$$

$$110. y = \frac{x^3+1}{x^2}$$

Additional Graphing Exercises

T Graph the curves in Exercises 111–114. Explain the relationship between the curve's formula and what you see.

$$111. y = \frac{x}{\sqrt{4-x^2}}$$

$$112. y = \frac{-1}{\sqrt{4-x^2}}$$

$$113. y = x^{2/3} + \frac{1}{x^{1/3}}$$

$$114. y = \sin\left(\frac{\pi}{x^2+1}\right)$$

T Graph the functions in Exercises 115 and 116. Then answer the following questions.

a. How does the graph behave as $x \rightarrow 0^+$?

b. How does the graph behave as $x \rightarrow \pm\infty$?

c. How does the graph behave near $x = 1$ and $x = -1$?

Give reasons for your answers.

$$115. y = \frac{3}{2}\left(x - \frac{1}{x}\right)^{2/3}$$

$$116. y = \frac{3}{2}\left(\frac{x}{x-1}\right)^{2/3}$$

2.6 Continuity

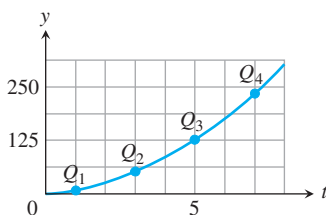


FIGURE 2.56 Connecting plotted points.

When we plot function values generated in a laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the points we did not measure (Figure 2.56). In doing so, we are assuming that we are working with a *continuous function*, so its outputs vary regularly and consistently with the inputs, and do not jump abruptly from one value to another without taking on the values in between. Intuitively, any function $y = f(x)$ whose graph can be sketched over its domain in one unbroken motion is an example of a continuous function. Such functions play an important role in the study of calculus and its applications.

Continuity at a Point

To understand continuity, it helps to consider a function like that in Figure 2.57, whose limits we investigated in Example 1 in the last section.

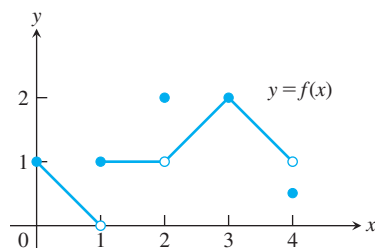


FIGURE 2.57 The function is not continuous at $x = 1$, $x = 2$, and $x = 4$ (Example 1).

EXAMPLE 1 At which numbers does the function f in Figure 2.57 appear to be not continuous? Explain why. What occurs at other numbers in the domain?

Solution First we observe that the domain of the function is the closed interval $[0, 4]$, so we will be considering the numbers x within that interval. From the figure, we notice right away that there are breaks in the graph at the numbers $x = 1$, $x = 2$, and $x = 4$. The break at $x = 1$ appears as a jump, which we identify later as a “jump discontinuity.” The break at $x = 2$ is called a “removable discontinuity” since by changing the function definition at that one point, we can create a new function that is continuous at $x = 2$. Similarly, $x = 4$ is a removable discontinuity.

Numbers at which the graph of f has breaks:

At the interior point $x = 1$, the function fails to have a limit. It does have both a left-hand limit, $\lim_{x \rightarrow 1^-} f(x) = 0$, as well as a right-hand limit, $\lim_{x \rightarrow 1^+} f(x) = 1$, but the limit values are different, resulting in a jump in the graph. The function is not continuous at $x = 1$. However, the function value $f(1) = 1$ is equal to the limit from the right, so the function is continuous from the right at $x = 1$.

At $x = 2$, the function does have a limit, $\lim_{x \rightarrow 2} f(x) = 1$, but the value of the function is $f(2) = 2$. The limit and function values are not the same, so there is a break in the graph and f is not continuous at $x = 2$.

At $x = 4$, the function does have a left-hand limit at this right endpoint, $\lim_{x \rightarrow 4^-} f(x) = 1$, but again the value of the function $f(4) = \frac{1}{2}$ differs from the value of the limit. We see again a break in the graph of the function at this endpoint and the function is not continuous from the left.

Numbers at which the graph of f has no breaks:

At $x = 3$, the function has a limit, $\lim_{x \rightarrow 3} f(x) = 2$. Moreover, the limit is the same value as the function there, $f(3) = 2$. The function is continuous at $x = 3$.

At $x = 0$, the function has a right-hand limit at this left endpoint, $\lim_{x \rightarrow 0^+} f(x) = 1$, and the value of the function is the same, $f(0) = 1$. The function is continuous from the right at $x = 0$. Because $x = 0$ is a left endpoint of the function’s domain, we have that $\lim_{x \rightarrow 0} f(x) = 1$ and so f is continuous at $x = 0$.

At all other numbers $x = c$ in the domain, the function has a limit equal to the value of the function, so $\lim_{x \rightarrow c} f(x) = f(c)$. For example, $\lim_{x \rightarrow 5/2} f(x) = f(5/2) = 3/2$. No breaks appear in the graph of the function at any of these numbers and the function is continuous at each of them. ■

The following definitions capture the continuity ideas we observed in Example 1.

DEFINITIONS Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f .

The function f is **continuous at c** if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

The function f is **right-continuous at c** (or **continuous from the right**) if

$$\lim_{x \rightarrow c^+} f(x) = f(c).$$

The function f is **left-continuous at c** (or **continuous from the left**) if

$$\lim_{x \rightarrow c^-} f(x) = f(c).$$

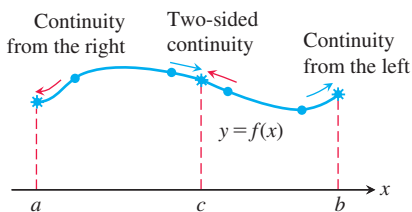


FIGURE 2.58 Continuity at points a , b , and c .

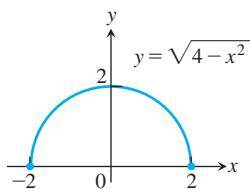


FIGURE 2.59 A function that is continuous over its domain (Example 2).

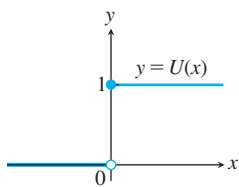


FIGURE 2.60 A function that has a jump discontinuity at the origin (Example 3).

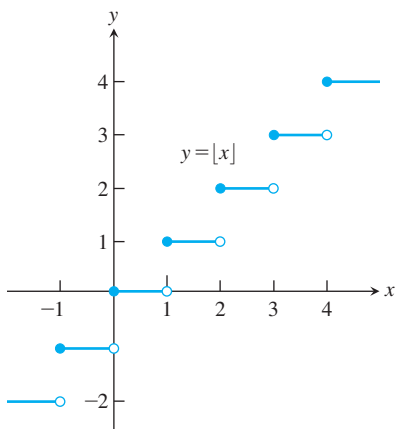


FIGURE 2.61 The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).

The function f in Example 1 is continuous at every x in $[0, 4]$ except $x = 1$, 2 , and 4 . It is right-continuous but not left-continuous at $x = 1$, neither right- nor left-continuous at $x = 2$, and not left-continuous at $x = 4$.

From Theorem 5, it follows immediately that a function f is continuous at an *interior* point c of an interval in its domain if and only if it is both right-continuous and left-continuous at c (Figure 2.58). We say that a function is **continuous over a closed interval** $[a, b]$ if it is right-continuous at a , left-continuous at b , and continuous at all interior points of the interval. This definition applies to the infinite closed intervals $[a, \infty)$ and $(-\infty, b]$ as well, but only one endpoint is involved. If a function is not continuous at point c of its domain, we say that f is **discontinuous at c** and that f has a discontinuity at c . Note that a function f can be continuous, right-continuous, or left-continuous only at a point c for which $f(c)$ is defined.

EXAMPLE 2 The function $f(x) = \sqrt{4 - x^2}$ is continuous over its domain $[-2, 2]$ (Figure 2.59). It is continuous at all points of this interval, including the endpoints $x = -2$ and $x = 2$. ■

EXAMPLE 3 The unit step function $U(x)$, graphed in Figure 2.60, is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. It has a jump discontinuity at $x = 0$. ■

At an interior point or an endpoint of an interval in its domain, a function is continuous at points where it passes the following test.

Continuity Test

A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \rightarrow c} f(x)$ exists (f has a limit as $x \rightarrow c$).
3. $\lim_{x \rightarrow c} f(x) = f(c)$ (the limit equals the function value).

For one-sided continuity, the limits in parts 2 and 3 of the test should be replaced by the appropriate one-sided limits.

EXAMPLE 4 The function $y = \lfloor x \rfloor$ introduced in Section 1.1 is graphed in Figure 2.61. It is discontinuous at every integer n , because the left-hand and right-hand limits are not equal as $x \rightarrow n$:

$$\lim_{x \rightarrow n^-} \lfloor x \rfloor = n - 1 \quad \text{and} \quad \lim_{x \rightarrow n^+} \lfloor x \rfloor = n.$$

Since $\lfloor n \rfloor = n$, the greatest integer function is right-continuous at every integer n (but not left-continuous).

The greatest integer function is continuous at every real number other than the integers. For example,

$$\lim_{x \rightarrow 1.5} \lfloor x \rfloor = 1 = \lfloor 1.5 \rfloor.$$

In general, if $n - 1 < c < n$, n an integer, then

$$\lim_{x \rightarrow c} \lfloor x \rfloor = n - 1 = \lfloor c \rfloor. \quad \blacksquare$$

Figure 2.62 displays several common ways in which a function can fail to be continuous. The function in Figure 2.62a is continuous at $x = 0$. The function in Figure 2.62b does not contain $x = 0$ in its domain. It would be continuous if its domain were extended so

that $f(0) = 1$. The function in Figure 2.62c would be continuous if $f(0)$ were 1 instead of 2. The discontinuity in Figure 2.62c is **removable**. The function has a limit as $x \rightarrow 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in Figure 2.62d through f are more serious: $\lim_{x \rightarrow 0} f(x)$ does not exist, and there is no way to improve the situation by appropriately defining f at 0. The step function in Figure 2.62d has a **jump discontinuity**: The one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in Figure 2.62e has an **infinite discontinuity**. The function in Figure 2.62f has an **oscillating discontinuity**: It oscillates so much that its values approach each number in $[-1, 1]$ as $x \rightarrow 0$. Since it does not approach a single number, it does not have a limit as x approaches 0.

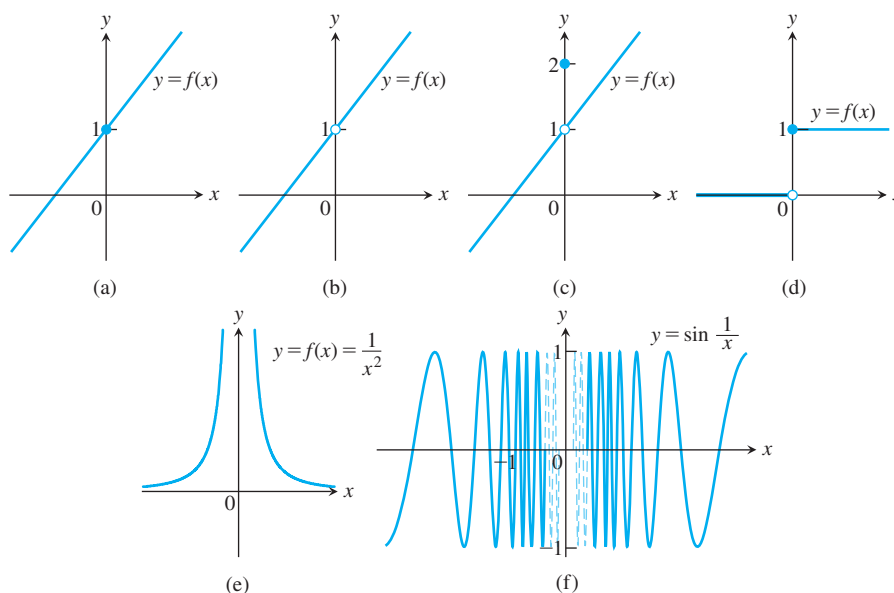


FIGURE 2.62 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

Continuous Functions

We now describe the continuity behavior of a function throughout its entire domain, not only at a single point. We define a **continuous function** to be one that is continuous at every point in its domain. This is a property of the *function*. A function always has a specified domain, so if we change the domain, then we change the function, and this may change its continuity property as well. If a function is discontinuous at one or more points of its domain, we say it is a **discontinuous function**.

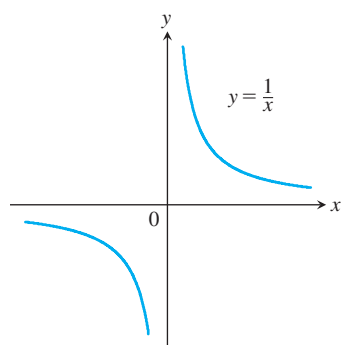


FIGURE 2.63 The function $f(x) = 1/x$ is continuous over its natural domain. It is not defined at the origin, so it is not continuous on any interval containing $x = 0$ (Example 5).

EXAMPLE 5

- (a) The function $f(x) = 1/x$ (Figure 2.63) is a continuous function because it is continuous at every point of its domain. The point $x = 0$ is not in the domain of the function f , so f is not continuous on any interval containing $x = 0$. Moreover, there is no way to extend f to a new function that is defined and continuous at $x = 0$. The function f does not have a removable discontinuity at $x = 0$.
- (b) The identity function $f(x) = x$ and constant functions are continuous everywhere by Example 3, Section 2.3. ■

Algebraic combinations of continuous functions are continuous wherever they are defined.

THEOREM 8—Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

1. <i>Sums:</i>	$f + g$
2. <i>Differences:</i>	$f - g$
3. <i>Constant multiples:</i>	$k \cdot f$, for any number k
4. <i>Products:</i>	$f \cdot g$
5. <i>Quotients:</i>	f/g , provided $g(c) \neq 0$
6. <i>Powers:</i>	f^n , n a positive integer
7. <i>Roots:</i>	$\sqrt[n]{f}$, provided it is defined on an interval containing c , where n is a positive integer

Most of the results in Theorem 8 follow from the limit rules in Theorem 1, Section 2.2. For instance, to prove the sum property we have

$$\begin{aligned}
 \lim_{x \rightarrow c} (f + g)(x) &= \lim_{x \rightarrow c} (f(x) + g(x)) \\
 &= \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) && \text{Sum Rule, Theorem 1} \\
 &= f(c) + g(c) && \text{Continuity of } f, g \text{ at } c \\
 &= (f + g)(c).
 \end{aligned}$$

This shows that $f + g$ is continuous.

EXAMPLE 6

- (a) Every polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is continuous because $\lim_{x \rightarrow c} P(x) = P(c)$ by Theorem 2, Section 2.2.
- (b) If $P(x)$ and $Q(x)$ are polynomials, then the rational function $P(x)/Q(x)$ is continuous wherever it is defined ($Q(c) \neq 0$) by Theorem 3, Section 2.2. ■

EXAMPLE 7 The function $f(x) = |x|$ is continuous. If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin, $\lim_{x \rightarrow 0} |x| = 0 = |0|$. ■

The functions $y = \sin x$ and $y = \cos x$ are continuous at $x = 0$ by Example 12 of Section 2.2. Both functions are continuous everywhere (see Exercise 76). It follows from Theorem 8 that all six trigonometric functions are continuous wherever they are defined. For example, $y = \tan x$ is continuous on $\cdots \cup (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2) \cup \cdots$.

Inverse Functions and Continuity

When a continuous function defined on an interval has an inverse, the inverse function is itself a continuous function over its own domain. This result is suggested by the observation that the graph of f^{-1} , being the reflection of the graph of f across the line $y = x$, cannot have any breaks in it when the graph of f has no breaks. A rigorous proof that f^{-1} is continuous whenever f is continuous on an interval is given in more advanced texts. As an example, the inverse trigonometric functions are all continuous over their domains.

We defined the exponential function $y = a^x$ in Section 1.4 informally. The graph was obtained from the graph of $y = a^x$ for x , a rational number, by “filling in the holes” at the irrational points x , so as to make the function $y = a^x$ continuous over the entire real line. The inverse function $y = \log_a x$ is also continuous. In particular, the natural exponential function $y = e^x$ and the natural logarithm function $y = \ln x$ are both continuous over their domains. Proofs of continuity for these functions will be given in Chapter 7.

Continuity of Compositions of Functions

Functions obtained by composing continuous functions are continuous. If $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is also continuous at $x = c$ (Figure 2.64). In this case, the limit of $g \circ f$ as $x \rightarrow c$ is $g(f(c))$.

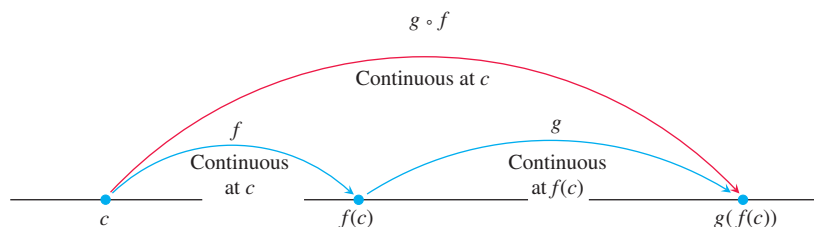


FIGURE 2.64 Compositions of continuous functions are continuous.

THEOREM 9—Compositions of Continuous Functions

If f is continuous at c , and g is continuous at $f(c)$, then the composition $g \circ f$ is continuous at c .

Intuitively, Theorem 9 is reasonable because if x is close to c , then $f(x)$ is close to $f(c)$, and since g is continuous at $f(c)$, it follows that $g(f(x))$ is close to $g(f(c))$.

The continuity of compositions holds for any finite number of compositions of functions. The only requirement is that each function be continuous where it is applied. An outline of a proof of Theorem 9 is given in Exercise 6 in Appendix A.6.

EXAMPLE 8 Show that the following functions are continuous on their natural domains.

(a) $y = \sqrt{x^2 - 2x - 5}$

(b) $y = \frac{x^{2/3}}{1 + x^4}$

(c) $y = \left| \frac{x - 2}{x^2 - 2} \right|$

(d) $y = \left| \frac{x \sin x}{x^2 + 2} \right|$

Solution

- (a) The square root function is continuous on $[0, \infty)$ because it is a root of the continuous identity function $f(x) = x$ (Part 7, Theorem 8). The given function is then the composition of the polynomial $f(x) = x^2 - 2x - 5$ with the square root function $g(t) = \sqrt{t}$, and is continuous on its natural domain.
- (b) The numerator is the cube root of the identity function squared; the denominator is an everywhere-positive polynomial. Therefore, the quotient is continuous.
- (c) The quotient $(x - 2)/(x^2 - 2)$ is continuous for all $x \neq \pm\sqrt{2}$, and the function is the composition of this quotient with the continuous absolute value function (Example 7).

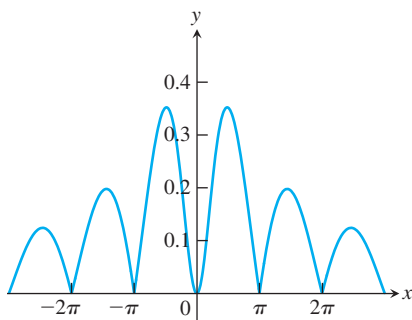


FIGURE 2.65 The graph suggests that $y = |(x \sin x)/(x^2 + 2)|$ is continuous (Example 8d).

(d) Because the sine function is everywhere-continuous (Exercise 76), the numerator term $x \sin x$ is the product of continuous functions, and the denominator term $x^2 + 2$ is an everywhere-positive polynomial. The given function is the composition of a quotient of continuous functions with the continuous absolute value function (Figure 2.65). ■

Theorem 9 is actually a consequence of a more general result, which we now prove. It states that if the limit of $f(x)$ as x approaches c is equal to b , then the limit of the composition function $g \circ f$ as x approaches c is equal to $g(b)$.

THEOREM 10—Limits of Continuous Functions

If $\lim_{x \rightarrow c} f(x) = b$ and g is continuous at the point b , then

$$\lim_{x \rightarrow c} g(f(x)) = g(b).$$

Proof Let $\varepsilon > 0$ be given. Since g is continuous at b , there exists a number $\delta_1 > 0$ such that

$$|g(y) - g(b)| < \varepsilon \quad \text{whenever} \quad 0 < |y - b| < \delta_1. \quad \begin{array}{l} \text{lim } g(y) = g(b) \text{ since } g \text{ is} \\ \text{continuous at } y = b. \end{array}$$

Note that if $|y - b| = 0$, so that $y = b$, then the inequality $|g(y) - g(b)| < \varepsilon$ holds for any positive ε , and therefore we have

$$|g(y) - g(b)| < \varepsilon \quad \text{whenever} \quad |y - b| < \delta_1. \quad (1)$$

Since $\lim_{x \rightarrow c} f(x) = b$, there exists a $\delta > 0$ such that

$$|f(x) - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta. \quad \begin{array}{l} \text{Definition of } \lim_{x \rightarrow c} f(x) = b \end{array}$$

If we let $y = f(x)$, we then have that

$$|y - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta,$$

which implies from Equation (1) that $|g(y) - g(b)| = |g(f(x)) - g(b)| < \varepsilon$ whenever $0 < |x - c| < \delta$. From the definition of limit, it follows that $\lim_{x \rightarrow c} g(f(x)) = g(b)$. This gives the proof for the case where c is an interior point of the domain of f . The case where c is an endpoint of the domain is entirely similar, using an appropriate one-sided limit in place of a two-sided limit. ■

EXAMPLE 9 As an application of Theorem 10, we have the following calculations.

$$\begin{aligned} \text{(a)} \quad \lim_{x \rightarrow \pi/2} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) &= \cos\left(\lim_{x \rightarrow \pi/2} 2x + \lim_{x \rightarrow \pi/2} \sin\left(\frac{3\pi}{2} + x\right)\right) \\ &= \cos(\pi + \sin 2\pi) = \cos \pi = -1. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow 1} \sin^{-1}\left(\frac{1-x}{1-x^2}\right) &= \sin^{-1}\left(\lim_{x \rightarrow 1} \frac{1-x}{1-x^2}\right) && \text{Arcsine is continuous.} \\ &= \sin^{-1}\left(\lim_{x \rightarrow 1} \frac{1}{1+x}\right) && \text{Cancel common factor } (1-x). \\ &= \sin^{-1}\frac{1}{2} = \frac{\pi}{6}. \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \lim_{x \rightarrow 0} \sqrt{x+1} e^{\tan x} &= \lim_{x \rightarrow 0} \sqrt{x+1} \cdot \exp\left(\lim_{x \rightarrow 0} \tan x\right) && \text{exp is continuous.} \\ &= 1 \cdot e^0 = 1. \end{aligned}$$

We sometimes denote e^u by $\exp(u)$ when u is a complicated mathematical expression.

Intermediate Value Theorem for Continuous Functions

A function is said to have the **Intermediate Value Property** if whenever it takes on two values, it also takes on all the values in between.

THEOREM 11 – The Intermediate Value Theorem for Continuous Functions

If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

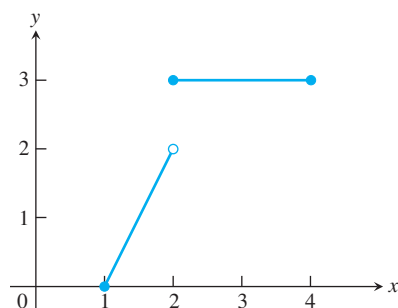
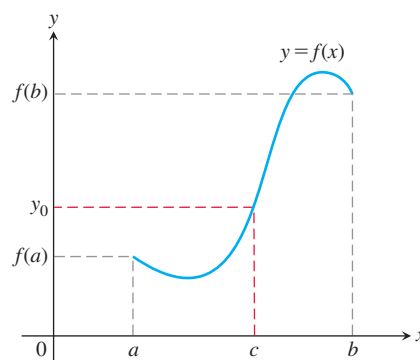


FIGURE 2.66 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \leq x < 2 \\ 3, & 2 \leq x \leq 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

Theorem 11 says that continuous functions over *finite closed* intervals have the Intermediate Value Property. Geometrically, the Intermediate Value Theorem says that any horizontal line $y = y_0$ crossing the y -axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.

The proof of the Intermediate Value Theorem depends on the completeness property of the real number system. The completeness property implies that the real numbers have no holes or gaps. In contrast, the rational numbers do not satisfy the completeness property, and a function defined only on the rationals would not satisfy the Intermediate Value Theorem. See Appendix A.9 for a discussion and examples.

The continuity of f on the interval is essential to Theorem 11. If f fails to be continuous at even one point of the interval, the theorem's conclusion may fail, as it does for the function graphed in Figure 2.66 (choose y_0 as any number between 2 and 3).

A Consequence for Graphing: Connectedness Theorem 11 implies that the graph of a function that is continuous on an interval cannot have any breaks over the interval. It will be **connected**—a single, unbroken curve. It will not have jumps such as the ones found in the graph of the greatest integer function (Figure 2.61), or separate branches as found in the graph of $1/x$ (Figure 2.63).

A Consequence for Root Finding We call a solution of the equation $f(x) = 0$ a **root** of the equation or a **zero** of the function f . The Intermediate Value Theorem tells us that if f is continuous, then any interval on which f changes sign contains a zero of the function. Somewhere between a point where a continuous function is positive and a second point where it is negative, the function must be equal to zero.

In practical terms, when we see the graph of a continuous function cross the horizontal axis on a computer screen, we know it is not stepping across. There really is a point where the function's value is zero.

EXAMPLE 10 Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

Solution Let $f(x) = x^3 - x - 1$. Since $f(1) = 1 - 1 - 1 = -1 < 0$ and $f(2) = 2^3 - 2 - 1 = 5 > 0$, we see that $y_0 = 0$ is a value between $f(1)$ and $f(2)$. Since f is a polynomial, it is continuous, and the Intermediate Value Theorem says there is a zero of f between 1 and 2. Figure 2.67 shows the result of zooming in to locate a root near $x = 1.32$. ■

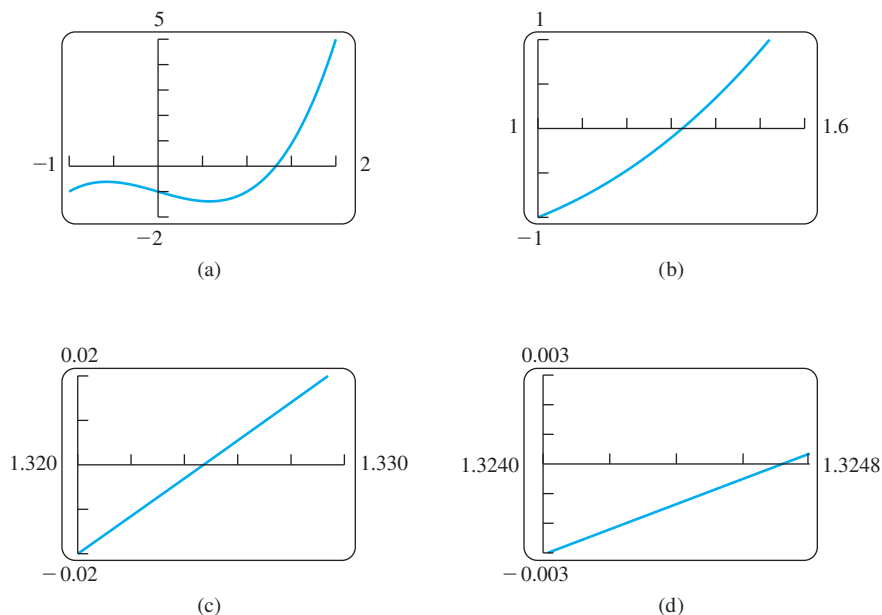


FIGURE 2.67 Zooming in on a zero of the function $f(x) = x^3 - x - 1$. The zero is near $x = 1.3247$ (Example 10).

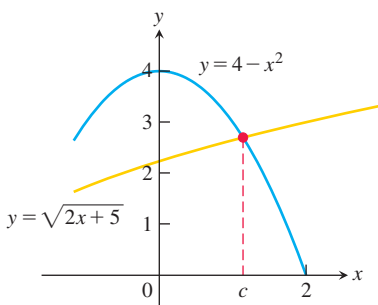


FIGURE 2.68 The curves $y = \sqrt{2x + 5}$ and $y = 4 - x^2$ have the same value at the number $x = c$ where $\sqrt{2x + 5} + x^2 - 4 = 0$ (Example 11).

EXAMPLE 11 Use the Intermediate Value Theorem to prove that the equation

$$\sqrt{2x + 5} = 4 - x^2$$

has a solution (Figure 2.68).

Solution We rewrite the equation as

$$\sqrt{2x + 5} + x^2 - 4 = 0,$$

and set $f(x) = \sqrt{2x + 5} + x^2 - 4$. Now $g(x) = \sqrt{2x + 5}$ is continuous on the interval $[-5/2, \infty)$ since it is formed as the composition of two continuous functions, the square root function with the nonnegative linear function $y = 2x + 5$. Then f is the sum of the function g and the quadratic function $y = x^2 - 4$, and the quadratic function is continuous for all values of x . It follows that $f(x) = \sqrt{2x + 5} + x^2 - 4$ is continuous on the interval $[-5/2, \infty)$. By trial and error, we find the function values $f(0) = \sqrt{5} - 4 \approx -1.76$ and $f(2) = \sqrt{9} = 3$. Note that f is continuous on the finite closed interval $[0, 2]$, which is a subset of the domain $[-5/2, \infty)$. Since the value $y_0 = 0$ is between the numbers $f(0) = -1.76$ and $f(2) = 3$, by the Intermediate Value Theorem there is a number $c \in [0, 2]$ such that $f(c) = 0$. We have found a number c that solves the original equation. ■

Continuous Extension to a Point

Sometimes the formula that describes a function f does not make sense at a point $x = c$. It might nevertheless be possible to extend the domain of f to include $x = c$, creating a new function that is continuous at $x = c$. For example, the function $y = f(x) = (\sin x)/x$ is continuous at every point except $x = 0$, since $x = 0$ is not in its domain. Since $y = (\sin x)/x$ has a finite limit as $x \rightarrow 0$ (Theorem 6), we can extend the function's domain to include the point $x = 0$ in such a way that the extended function is continuous at $x = 0$. We define the new function

$$F(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \quad \text{Same as original function for } x \neq 0 \\ 1, & x = 0. \quad \text{Value at domain point } x = 0 \end{cases}$$

The new function $F(x)$ is continuous at $x = 0$ because

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = F(0),$$

so it meets the requirements for continuity (Figure 2.69).

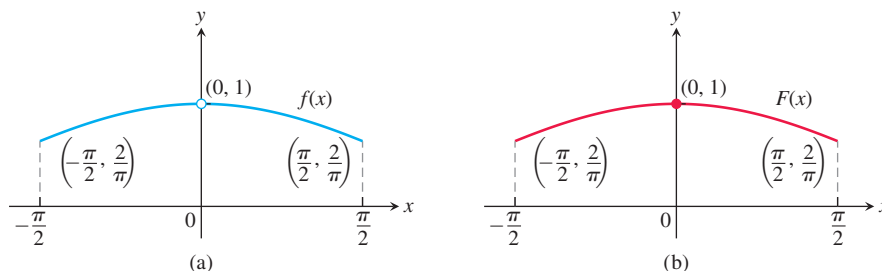


FIGURE 2.69 (a) The graph of $f(x) = (\sin x)/x$ for $-\pi/2 \leq x \leq \pi/2$ does not include the point $(0, 1)$ because the function is not defined at $x = 0$. (b) We can extend the domain to include $x = 0$ by defining the new function $F(x)$ with $F(0) = 1$ and $F(x) = f(x)$ everywhere else. Note that $F(0) = \lim_{x \rightarrow 0} f(x)$ and $F(x)$ is a continuous function at $x = 0$.

More generally, a function (such as a rational function) may have a limit at a point where it is not defined. If $f(c)$ is not defined, but $\lim_{x \rightarrow c} f(x) = L$ exists, we can define a new function $F(x)$ by the rule

$$F(x) = \begin{cases} f(x), & \text{if } x \text{ is in the domain of } f \\ L, & \text{if } x = c. \end{cases}$$

The function F is continuous at $x = c$. It is called the **continuous extension of f** to $x = c$. For rational functions f , continuous extensions are often found by canceling common factors in the numerator and denominator.

EXAMPLE 12 Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, \quad x \neq 2$$

has a continuous extension to $x = 2$, and find that extension.

Solution Although $f(2)$ is not defined, if $x \neq 2$ we have

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}.$$

The new function

$$F(x) = \frac{x+3}{x+2}$$

is equal to $f(x)$ for $x \neq 2$, but is continuous at $x = 2$, having there the value of $5/4$. Thus F is the continuous extension of f to $x = 2$, and

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \rightarrow 2} f(x) = \frac{5}{4}.$$

The graph of f is shown in Figure 2.70. The continuous extension F has the same graph except with no hole at $(2, 5/4)$. Effectively, F is the function f extended across the missing domain point at $x = 2$ so as to give a continuous function over the larger domain. ■

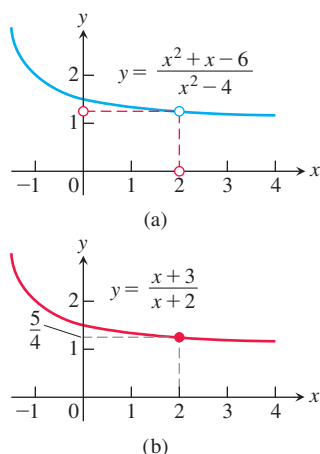
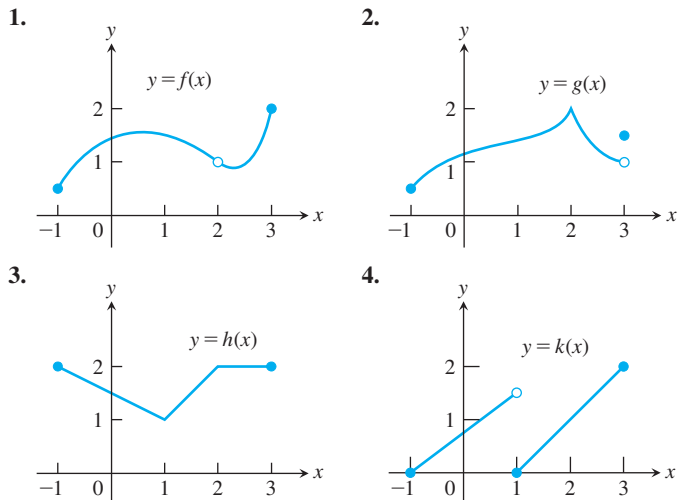


FIGURE 2.70 (a) The graph of $f(x)$ and (b) the graph of its continuous extension $F(x)$ (Example 12).

EXERCISES 2.6

Continuity from Graphs

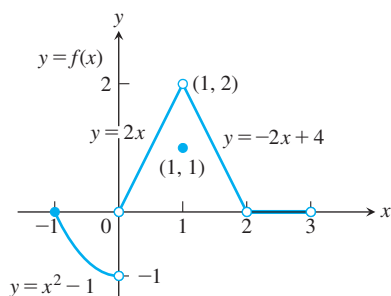
In Exercises 1–4, say whether the function graphed is continuous on $[-1, 3]$. If not, where does it fail to be continuous and why?



Exercises 5–10 refer to the function

$$f(x) = \begin{cases} x^2 - 1, & -1 \leq x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$

graphed in the accompanying figure.



5. a. Does $f(-1)$ exist?
b. Does $\lim_{x \rightarrow -1^+} f(x)$ exist?
c. Does $\lim_{x \rightarrow -1^+} f(x) = f(-1)$?
d. Is f continuous at $x = -1$?
6. a. Does $f(1)$ exist?
b. Does $\lim_{x \rightarrow 1} f(x)$ exist?
c. Does $\lim_{x \rightarrow 1} f(x) = f(1)$?
d. Is f continuous at $x = 1$?
7. a. Is f defined at $x = 2$? (Look at the definition of f).
b. Is f continuous at $x = 2$?
8. At what values of x is f continuous?

9. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?
10. To what new value should $f(1)$ be changed to remove the discontinuity?

Applying the Continuity Test

At which points do the functions in Exercises 11 and 12 fail to be continuous? At which points, if any, are the discontinuities removable? Not removable? Give reasons for your answers.

11. The function defined in Exercise 1, Section 2.4
12. The function defined in Exercise 2, Section 2.4

At what points are the functions in Exercises 13–32 continuous?

13. $y = \frac{1}{x-2} - 3x$
14. $y = \frac{1}{(x+2)^2} + 4$
15. $y = \frac{x+1}{x^2-4x+3}$
16. $y = \frac{x+3}{x^2-3x-10}$
17. $y = |x-1| + \sin x$
18. $y = \frac{1}{|x|+1} - \frac{x^2}{2}$
19. $y = \frac{\cos x}{x}$
20. $y = \frac{x+2}{\cos x}$
21. $y = \csc 2x$
22. $y = \tan \frac{\pi x}{2}$
23. $y = \frac{x \tan x}{x^2+1}$
24. $y = \frac{\sqrt{x^4+1}}{1+\sin^2 x}$
25. $y = \sqrt{2x+3}$
26. $y = \sqrt[4]{3x-1}$
27. $y = (2x-1)^{1/3}$
28. $y = (2-x)^{1/5}$

29. $g(x) = \begin{cases} \frac{x^2-x-6}{x-3}, & x \neq 3 \\ 5, & x = 3 \end{cases}$
30. $f(x) = \begin{cases} \frac{x^3-8}{x^2-4}, & x \neq 2, x \neq -2 \\ 3, & x = 2 \\ 4, & x = -2 \end{cases}$
31. $f(x) = \begin{cases} 1-x, & x < 0 \\ e^x, & 0 \leq x \leq 1 \\ x^2+2, & x > 1 \end{cases}$
32. $f(x) = \frac{x+3}{2-e^x}$

Limits Involving Trigonometric Functions

Find the limits in Exercises 33–40. Are the functions continuous at the point being approached?

33. $\lim_{x \rightarrow \pi} \sin(x - \sin x)$
34. $\lim_{t \rightarrow 0} \sin\left(\frac{\pi}{2} \cos(\tan t)\right)$
35. $\lim_{y \rightarrow 1} \sec(y \sec^2 y - \tan^2 y - 1)$
36. $\lim_{x \rightarrow 0} \tan\left(\frac{\pi}{4} \cos(\sin x^{1/3})\right)$
37. $\lim_{t \rightarrow 0} \cos\left(\frac{\pi}{\sqrt{19-3 \sec 2t}}\right)$
38. $\lim_{x \rightarrow \pi/6} \sqrt{\csc^2 x + 5\sqrt{3} \tan x}$

$$39. \lim_{x \rightarrow 0^+} \sin\left(\frac{\pi}{2}e^{\sqrt{x}}\right) \qquad 40. \lim_{x \rightarrow 1} \cos^{-1}(\ln \sqrt{x})$$

$$41. \lim_{x \rightarrow 0} \sec\left[e^x + \pi \tan\left(\frac{\pi}{4 \sec x}\right) - 1\right]$$

$$42. \lim_{x \rightarrow 0} \sin\left(\frac{\pi + \tan x}{\tan x - 2 \sec x}\right)$$

$$43. \lim_{t \rightarrow 0} \tan\left(1 - \frac{\sin t}{t}\right) \qquad 44. \lim_{\theta \rightarrow 0} \cos\left(\frac{\pi\theta}{\sin \theta}\right)$$

Continuous Extensions

45. Define $g(3)$ in a way that extends $g(x) = (x^2 - 9)/(x - 3)$ to be continuous at $x = 3$.

46. Define $h(2)$ in a way that extends $h(t) = (t^2 + 3t - 10)/(t - 2)$ to be continuous at $t = 2$.

47. Define $f(1)$ in a way that extends $f(s) = (s^3 - 1)/(s^2 - 1)$ to be continuous at $s = 1$.

48. Define $g(4)$ in a way that extends

$$g(x) = (x^2 - 16)/(x^2 - 3x - 4)$$

to be continuous at $x = 4$.

49. For what value of a is

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at every x ?

50. For what value of b is

$$g(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \geq -2 \end{cases}$$

continuous at every x ?

51. For what values of a is

$$f(x) = \begin{cases} a^2x - 2a, & x \geq 2 \\ 12, & x < 2 \end{cases}$$

continuous at every x ?

52. For what values of b is

$$g(x) = \begin{cases} \frac{x-b}{b+1}, & x \leq 0 \\ x^2 + b, & x > 0 \end{cases}$$

continuous at every x ?

53. For what values of a and b is

$$f(x) = \begin{cases} -2, & x \leq -1 \\ ax - b, & -1 < x < 1 \\ 3, & x \geq 1 \end{cases}$$

continuous at every x ?

54. For what values of a and b is

$$g(x) = \begin{cases} ax + 2b, & x \leq 0 \\ x^2 + 3a - b, & 0 < x \leq 2 \\ 3x - 5, & x > 2 \end{cases}$$

continuous at every x ?

T In Exercises 55–58, graph the function f to see whether it appears to have a continuous extension to $x = 0$. If it does, use Trace and Zoom to find a good candidate for the extended function's value at $x = 0$. If the function does not appear to have a continuous extension, can it be extended to be continuous at $x = 0$ from the right or from the left? If so, what do you think the extended function's value(s) should be?

$$55. f(x) = \frac{10^x - 1}{x}$$

$$56. f(x) = \frac{10^{|x|} - 1}{x}$$

$$57. f(x) = \frac{\sin x}{|x|}$$

$$58. f(x) = (1 + 2x)^{1/x}$$

Theory and Examples

59. A continuous function $y = f(x)$ is known to be negative at $x = 0$ and positive at $x = 1$. Why does the equation $f(x) = 0$ have at least one solution between $x = 0$ and $x = 1$? Illustrate with a sketch.

60. Explain why the equation $\cos x = x$ has at least one solution.

61. **Roots of a cubic** Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

62. **A function value** Show that the function $F(x) = (x - a)^2 \cdot (x - b)^2 + x$ takes on the value $(a + b)/2$ for some value of x .

63. **Solving an equation** If $f(x) = x^3 - 8x + 10$, show that there are values c for which $f(c)$ equals (a) π ; (b) $-\sqrt{3}$; (c) 5,000,000.

64. Explain why the following five statements ask for the same information.

a. Find the roots of $f(x) = x^3 - 3x - 1$.

b. Find the x -coordinates of the points where the curve $y = x^3$ crosses the line $y = 3x + 1$.

c. Find all the values of x for which $x^3 - 3x = 1$.

d. Find the x -coordinates of the points where the cubic curve $y = x^3 - 3x$ crosses the line $y = 1$.

e. Solve the equation $x^3 - 3x - 1 = 0$.

65. **Removable discontinuity** Give an example of a function $f(x)$ that is continuous for all values of x except $x = 2$, where it has a removable discontinuity. Explain how you know that f is discontinuous at $x = 2$, and how you know the discontinuity is removable.

66. **Nonremovable discontinuity** Give an example of a function $g(x)$ that is continuous for all values of x except $x = -1$, where it has a nonremovable discontinuity. Explain how you know that g is discontinuous there and why the discontinuity is not removable.

67. **A function discontinuous at every point**

a. Use the fact that every nonempty interval of real numbers contains both rational and irrational numbers to show that the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every point.

b. Is f right-continuous or left-continuous at any point?

68. If functions $f(x)$ and $g(x)$ are continuous for $0 \leq x \leq 1$, could $f(x)/g(x)$ possibly be discontinuous at a point of $[0, 1]$? Give reasons for your answer.

69. If the product function $h(x) = f(x) \cdot g(x)$ is continuous at $x = 0$, must $f(x)$ and $g(x)$ be continuous at $x = 0$? Give reasons for your answer.

- 70. Discontinuous compositions of continuous functions** Give an example of functions f and g , both continuous at $x = 0$, for which the composition $f \circ g$ is discontinuous at $x = 0$. Does this contradict Theorem 9? Give reasons for your answer.
- 71. Never-zero continuous functions** Is it true that a continuous function that is never zero on an interval never changes sign on that interval? Give reasons for your answer.
- 72. Stretching a rubber band** Is it true that if you stretch a rubber band by moving one end to the right and the other to the left, some point of the band will end up in its original position? That is, if x is a position on the rubber band before stretching and $f(x)$ is the position of that point after stretching, must there be some x such that $f(x) = x$? Give reasons for your answer.
- 73. A fixed point theorem** Suppose that a function f is continuous on the closed interval $[0, 1]$ and that $0 \leq f(x) \leq 1$ for every x in $[0, 1]$. Show that there must exist a number c in $[0, 1]$ such that $f(c) = c$ (c is called a **fixed point** of f).
- 74. The sign-preserving property of continuous functions** Let f be defined on an interval (a, b) and suppose that $f(c) \neq 0$ at some c where f is continuous. Show that there is an interval $(c - \delta, c + \delta)$ about c where f has the same sign as $f(c)$.

- 75.** Prove that f is continuous at c if and only if

$$\lim_{h \rightarrow 0} f(c + h) = f(c).$$

- 76.** Use Exercise 75 together with the identities

$$\sin(h + c) = \sin h \cos c + \cos h \sin c,$$

$$\cos(h + c) = \cos h \cos c - \sin h \sin c$$

to prove that both $f(x) = \sin x$ and $g(x) = \cos x$ are continuous at every point $x = c$.

Solving Equations Graphically

T Use the Intermediate Value Theorem in Exercises 77–84 to prove that each equation has a solution. Then use a graphing calculator or computer grapher to solve the equations.

77. $x^3 - 3x - 1 = 0$

78. $2x^3 - 2x^2 - 2x + 1 = 0$

79. $x(x - 1)^2 = 1$ (one root) **80.** $x^x = 2$

81. $\sqrt{x} + \sqrt{1 + x} = 4$

82. $x^3 - 15x + 1 = 0$ (three roots)

83. $\cos x = x$ (one root). Make sure you are using radian mode.

84. $2 \sin x = x$ (three roots). Make sure you are using radian mode.

CHAPTER 2 Questions to Guide Your Review

- What is the average rate of change of the function $g(t)$ over the interval from $t = a$ to $t = b$? How is it related to a secant line?
- What limit must be calculated to find the rate of change of a function $g(t)$ at $t = t_0$?
- Give an informal or intuitive definition of the limit

$$\lim_{x \rightarrow c} f(x) = L.$$

Why is the definition “informal”? Give examples.

- Does the existence and value of the limit of a function $f(x)$ as x approaches c ever depend on what happens at $x = c$? Explain and give examples.
- What function behaviors might occur for which the limit may fail to exist? Give examples.
- What theorems are available for calculating limits? Give examples of how the theorems are used.
- How are one-sided limits related to limits? How can this relationship sometimes be used to calculate a limit or prove it does not exist? Give examples.
- What is the value of $\lim_{\theta \rightarrow 0} ((\sin \theta)/\theta)$? Does it matter whether θ is measured in degrees or radians? Explain.
- What exactly does $\lim_{x \rightarrow c} f(x) = L$ mean? Give an example in which you find $\delta > 0$ for a given f, L, c , and $\varepsilon > 0$ in the precise definition of limit.
- Give precise definitions of the following statements.
 - $\lim_{x \rightarrow 2^-} f(x) = 5$
 - $\lim_{x \rightarrow 2^+} f(x) = 5$
 - $\lim_{x \rightarrow 2} f(x) = \infty$
 - $\lim_{x \rightarrow 2} f(x) = -\infty$

- What conditions must be satisfied by a function if it is to be continuous at an interior point of its domain? At an endpoint?
- How can looking at the graph of a function help you tell where the function is continuous?
- What does it mean for a function to be right-continuous at a point? Left-continuous? How are continuity and one-sided continuity related?
- What does it mean for a function to be continuous on an interval? Give examples to illustrate the fact that a function that is not continuous on its entire domain may still be continuous on selected intervals within the domain.
- What are the basic types of discontinuity? Give an example of each. What is a removable discontinuity? Give an example.
- What does it mean for a function to have the Intermediate Value Property? What conditions guarantee that a function has this property over an interval? What are the consequences for graphing and solving the equation $f(x) = 0$?
- Under what circumstances can you extend a function $f(x)$ to be continuous at a point $x = c$? Give an example.
- What exactly do $\lim_{x \rightarrow \infty} f(x) = L$ and $\lim_{x \rightarrow -\infty} f(x) = L$ mean? Give examples.
- What are $\lim_{x \rightarrow \pm\infty} k$ (k a constant) and $\lim_{x \rightarrow \pm\infty} (1/x)$? How do you extend these results to other functions? Give examples.
- How do you find the limit of a rational function as $x \rightarrow \pm\infty$? Give examples.
- What are horizontal and vertical asymptotes? Give examples.

CHAPTER 2 Practice Exercises

Limits and Continuity

1. Graph the function

$$f(x) = \begin{cases} 1, & x \leq -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \geq 1. \end{cases}$$

Then discuss, in detail, limits, one-sided limits, continuity, and one-sided continuity of f at $x = -1, 0$, and 1 . Are any of the discontinuities removable? Explain.

2. Repeat the instructions of Exercise 1 for

$$f(x) = \begin{cases} 0, & x \leq -1 \\ 1/x, & 0 < |x| < 1 \\ 0, & x = 1 \\ 1, & x > 1. \end{cases}$$

3. Suppose that $f(t)$ and $g(t)$ are defined for all t and that $\lim_{t \rightarrow t_0} f(t) = -7$ and $\lim_{t \rightarrow t_0} g(t) = 0$. Find the limit as $t \rightarrow t_0$ of the following functions.

a. $3f(t)$	b. $(f(t))^2$
c. $f(t) \cdot g(t)$	d. $\frac{f(t)}{g(t) - 7}$
e. $\cos(g(t))$	f. $ f(t) $
g. $f(t) + g(t)$	h. $1/f(t)$

4. Suppose the functions $f(x)$ and $g(x)$ are defined for all x and that $\lim_{x \rightarrow 0} f(x) = 1/2$ and $\lim_{x \rightarrow 0} g(x) = \sqrt{2}$. Find the limits as $x \rightarrow 0$ of the following functions.

a. $-g(x)$	b. $g(x) \cdot f(x)$
c. $f(x) + g(x)$	d. $1/f(x)$
e. $x + f(x)$	f. $\frac{f(x) \cdot \cos x}{x - 1}$

In Exercises 5 and 6, find the value that $\lim_{x \rightarrow 0} g(x)$ must have if the given limit statements hold.

5. $\lim_{x \rightarrow 0} \left(\frac{4 - g(x)}{x} \right) = 1$ 6. $\lim_{x \rightarrow -4} \left(x \lim_{x \rightarrow 0} g(x) \right) = 2$

7. On what intervals are the following functions continuous?

a. $f(x) = x^{1/3}$	b. $g(x) = x^{3/4}$
c. $h(x) = x^{-2/3}$	d. $k(x) = x^{-1/6}$

8. On what intervals are the following functions continuous?

a. $f(x) = \tan x$	b. $g(x) = \csc x$
c. $h(x) = \frac{\cos x}{x - \pi}$	d. $k(x) = \frac{\sin x}{x}$

Finding Limits

In Exercises 9–28, find the limit or explain why it does not exist.

9. $\lim_{x \rightarrow 0} \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x}$
 a. as $x \rightarrow 0$ b. as $x \rightarrow 2$

10. $\lim_{x \rightarrow 1} \frac{x^2 + x}{x^5 + 2x^4 + x^3}$

- a. as $x \rightarrow 0$

- b. as $x \rightarrow -1$

11. $\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x}$

12. $\lim_{x \rightarrow a} \frac{x^2 - a^2}{x^4 - a^4}$

13. $\lim_{h \rightarrow 0} \frac{(x + h)^2 - x^2}{h}$

14. $\lim_{x \rightarrow 0} \frac{(x + h)^2 - x^2}{h}$

15. $\lim_{x \rightarrow 0} \frac{1}{\frac{2}{2+x} - \frac{1}{2}}$

17. $\lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{\sqrt{x} - 1}$

19. $\lim_{x \rightarrow 0} \frac{\tan(2x)}{\tan(\pi x)}$

21. $\lim_{x \rightarrow \pi} \sin\left(\frac{x}{2} + \sin x\right)$

23. $\lim_{x \rightarrow 0} \frac{8x}{3 \sin x - x}$

25. $\lim_{t \rightarrow 3^+} \ln(t - 3)$

27. $\lim_{\theta \rightarrow 0^+} \sqrt{\theta} e^{\cos(\pi/\theta)}$

16. $\lim_{x \rightarrow 0} \frac{(2 + x)^3 - 8}{x}$

18. $\lim_{x \rightarrow 64} \frac{x^{2/3} - 16}{\sqrt{x} - 8}$

20. $\lim_{x \rightarrow \pi^-} \csc x$

22. $\lim_{x \rightarrow \pi} \cos^2(x - \tan x)$

24. $\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\sin x}$

26. $\lim_{t \rightarrow 1} t^2 \ln(2 - \sqrt{t})$

28. $\lim_{z \rightarrow 0^+} \frac{2e^{1/z}}{e^{1/z} + 1}$

In Exercises 29–32, find the limit of $g(x)$ as x approaches the indicated value.

29. $\lim_{x \rightarrow 0^+} (4g(x))^{1/3} = 2$

30. $\lim_{x \rightarrow \sqrt{5}} \frac{1}{x + g(x)} = 2$

31. $\lim_{x \rightarrow 1} \frac{3x^2 + 1}{g(x)} = \infty$

32. $\lim_{x \rightarrow -2} \frac{5 - x^2}{\sqrt{g(x)}} = 0$

T Roots

33. Let $f(x) = x^3 - x - 1$.

- Use the Intermediate Value Theorem to show that f has a zero between -1 and 2 .
- Solve the equation $f(x) = 0$ graphically with an error of magnitude at most 10^{-8} .
- It can be shown that the exact value of the solution in part (b) is

$$\left(\frac{1}{2} + \frac{\sqrt{69}}{18} \right)^{1/3} + \left(\frac{1}{2} - \frac{\sqrt{69}}{18} \right)^{1/3}.$$

Evaluate this exact answer and compare it with the value you found in part (b).

34. Let $f(\theta) = \theta^3 - 2\theta + 2$.

- Use the Intermediate Value Theorem to show that f has a zero between -2 and 0 .

- b. Solve the equation $f(\theta) = 0$ graphically with an error of magnitude at most 10^{-4} .
- c. It can be shown that the exact value of the solution in part (b) is

$$\left(\sqrt{\frac{19}{27}} - 1\right)^{1/3} - \left(\sqrt{\frac{19}{27}} + 1\right)^{1/3}.$$

Evaluate this exact answer and compare it with the value you found in part (b).

Continuous Extension

35. Can $f(x) = x(x^2 - 1)/|x^2 - 1|$ be extended to be continuous at $x = 1$ or -1 ? Give reasons for your answers. (Graph the function—you will find the graph interesting.)
36. Explain why the function $f(x) = \sin(1/x)$ has no continuous extension to $x = 0$.

T In Exercises 37–40, graph the function to see whether it appears to have a continuous extension to the given point a . If it does, use Trace and Zoom to find a good candidate for the extended function's value at a . If the function does not appear to have a continuous extension, can it be extended to be continuous from the right or left? If so, what do you think the extended function's value should be?

37. $f(x) = \frac{x-1}{x - \sqrt[4]{x}}$, $a = 1$

38. $g(\theta) = \frac{5 \cos \theta}{4\theta - 2\pi}$, $a = \pi/2$

39. $h(t) = (1 + |t|)^{1/t}$, $a = 0$

40. $k(x) = \frac{x}{1 - 2^{|x|}}$, $a = 0$

Limits at Infinity

Find the limits in Exercises 41–54.

41. $\lim_{x \rightarrow \infty} \frac{2x+3}{5x+7}$

42. $\lim_{x \rightarrow -\infty} \frac{2x^2+3}{5x^2+7}$

43. $\lim_{x \rightarrow -\infty} \frac{x^2 - 4x + 8}{3x^3}$

44. $\lim_{x \rightarrow \infty} \frac{1}{x^2 - 7x + 1}$

45. $\lim_{x \rightarrow -\infty} \frac{x^2 - 7x}{x + 1}$

46. $\lim_{x \rightarrow \infty} \frac{x^4 + x^3}{12x^3 + 128}$

47. $\lim_{x \rightarrow \infty} \frac{\sin x}{|x|}$ (If you have a grapher, try graphing the function for $-5 \leq x \leq 5$.)

48. $\lim_{\theta \rightarrow \infty} \frac{\cos \theta - 1}{\theta}$ (If you have a grapher, try graphing $f(x) = x(\cos(1/x) - 1)$ near the origin to “see” the limit at infinity.)

49. $\lim_{x \rightarrow \infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x}$

50. $\lim_{x \rightarrow \infty} \frac{x^{2/3} + x^{-1}}{x^{2/3} + \cos^2 x}$

51. $\lim_{x \rightarrow \infty} e^{1/x} \cos \frac{1}{x}$

52. $\lim_{t \rightarrow \infty} \ln\left(1 + \frac{1}{t}\right)$

53. $\lim_{x \rightarrow -\infty} \tan^{-1} x$

54. $\lim_{t \rightarrow -\infty} e^{3t} \sin^{-1}\left(\frac{1}{t}\right)$

Horizontal and Vertical Asymptotes

55. Use limits to determine the equations for all vertical asymptotes.

a. $y = \frac{x^2 + 4}{x - 3}$

b. $f(x) = \frac{x^2 - x - 2}{x^2 - 2x + 1}$

c. $y = \frac{x^2 + x - 6}{x^2 + 2x - 8}$

56. Use limits to determine the equations for all horizontal asymptotes.

a. $y = \frac{1 - x^2}{x^2 + 1}$

b. $f(x) = \frac{\sqrt{x} + 4}{\sqrt{x} + 4}$

c. $g(x) = \frac{\sqrt{x^2 + 4}}{x}$

d. $y = \sqrt{\frac{x^2 + 9}{9x^2 + 1}}$

57. Determine the domain and range of $y = \frac{\sqrt{16 - x^2}}{x - 2}$.

58. Assume that constants a and b are positive. Find equations for all horizontal and vertical asymptotes for the graph of $y = \frac{\sqrt{ax^2 + 4}}{x - b}$.

CHAPTER 2 Additional and Advanced Exercises

- T** 1. **Assigning a value to 0^0** The rules of exponents tell us that $a^0 = 1$ if a is any number different from zero. They also tell us that $0^n = 0$ if n is any positive number.

If we tried to extend these rules to include the case 0^0 , we would get conflicting results. The first rule would say $0^0 = 1$, whereas the second would say $0^0 = 0$.

We are not dealing with a question of right or wrong here. Neither rule applies as it stands, so there is no contradiction. We could, in fact, define 0^0 to have any value we wanted as long as we could persuade others to agree.

What value would you like 0^0 to have? Here is an example that might help you to decide. (See Exercise 2 below for another example.)

- a. Calculate x^x for $x = 0.1, 0.01, 0.001$, and so on as far as your calculator can go. Record the values you get. What pattern do you see?

- b. Graph the function $y = x^x$ for $0 < x \leq 1$. Even though the function is not defined for $x \leq 0$, the graph will approach the y -axis from the right. Toward what y -value does it seem to be headed? Zoom in to further support your idea.

- T** 2. **A reason you might want 0^0 to be something other than 0 or 1** As the number x increases through positive values, the numbers $1/x$ and $1/(\ln x)$ both approach zero. What happens to the number

$$f(x) = \left(\frac{1}{x}\right)^{1/(\ln x)}$$

as x increases? Here are two ways to find out.

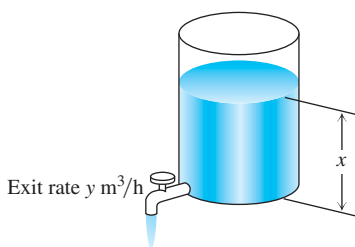
- a. Evaluate f for $x = 10, 100, 1000$, and so on as far as your calculator can reasonably go. What pattern do you see?
- b. Graph f in a variety of graphing windows, including windows that contain the origin. What do you see? Trace the y -values along the graph. What do you find?

- 3. Lorentz contraction** In relativity theory, the length of an object, say a rocket, appears to an observer to depend on the speed at which the object is traveling with respect to the observer. If the observer measures the rocket's length as L_0 at rest, then at speed v the length will appear to be

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}}.$$

This equation is the Lorentz contraction formula. Here, c is the speed of light in a vacuum, about 3×10^8 m/s. What happens to L as v increases? Find $\lim_{v \rightarrow c^-} L$. Why was the left-hand limit needed?

- 4. Controlling the flow from a draining tank** Torricelli's law says that if you drain a tank like the one in the figure shown, the rate y at which water runs out is a constant times the square root of the water's depth x . The constant depends on the size and shape of the exit valve.



Suppose that $y = \sqrt{x}/2$ for a certain tank. You are trying to maintain a fairly constant exit rate by adding water to the tank with a hose from time to time. How deep must you keep the water if you want to maintain the exit rate

- within $0.2 \text{ m}^3/\text{h}$ of the rate $y_0 = 1 \text{ m}^3/\text{h}$?
- within $0.1 \text{ m}^3/\text{h}$ of the rate $y_0 = 1 \text{ m}^3/\text{h}$?

- 5. Thermal expansion in precise equipment** As you may know, most metals expand when heated and contract when cooled. The dimensions of a piece of laboratory equipment are sometimes so critical that the shop where the equipment is made must be held at the same temperature as the laboratory where the equipment is to be used. A typical aluminum bar that is 10 cm wide at 20°C will be

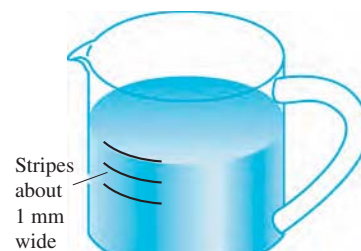
$$y = 10 + 2(t - 20) \times 10^{-4}$$

Centimeters wide at a nearby temperature t . Suppose that you are using a bar like this in a gravity wave detector, where its width must stay within 0.0005 cm of the ideal 10 cm. How close to $t_0 = 20^\circ\text{C}$ must you maintain the temperature to ensure that this tolerance is not exceeded?

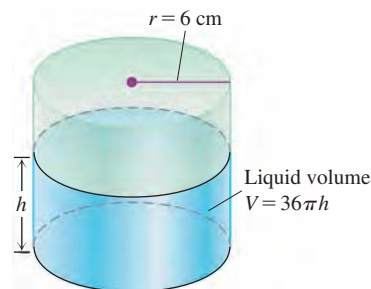
- 6. Stripes on a measuring cup** The interior of a typical 1-L measuring cup is a right circular cylinder of radius 6 cm (see accompanying figure). The volume of water we put in the cup is therefore a function of the level h to which the cup is filled, the formula being

$$V = \pi 6^2 h = 36\pi h.$$

How closely must we measure h to measure out 1 L of water (1000 cm^3) with an error of no more than 1% (10 cm^3)?



(a)



(b)

A 1-L measuring cup (a), modeled as a right circular cylinder (b) of radius $r = 6 \text{ cm}$

Precise Definition of Limit

In Exercises 7–10, use the formal definition of limit to prove that the function is continuous at c .

7. $f(x) = x^2 - 7$, $c = 1$ 8. $g(x) = 1/(2x)$, $c = 1/4$

9. $h(x) = \sqrt{2x - 3}$, $c = 2$ 10. $F(x) = \sqrt{9 - x}$, $c = 5$

- 11. Uniqueness of limits** Show that a function cannot have two different limits at the same point. That is, if $\lim_{x \rightarrow c} f(x) = L_1$ and $\lim_{x \rightarrow c} f(x) = L_2$, then $L_1 = L_2$.

- 12. Prove the limit Constant Multiple Rule:**

$$\lim_{x \rightarrow c} kf(x) = k \lim_{x \rightarrow c} f(x) \quad \text{for any constant } k.$$

- 13. One-sided limits** If $\lim_{x \rightarrow 0^+} f(x) = A$ and $\lim_{x \rightarrow 0^-} f(x) = B$, find

- $\lim_{x \rightarrow 0^+} f(x^3 - x)$
- $\lim_{x \rightarrow 0^-} f(x^3 - x)$
- $\lim_{x \rightarrow 0^+} f(x^2 - x^4)$
- $\lim_{x \rightarrow 0^-} f(x^2 - x^4)$

- 14. Limits and continuity** Which of the following statements are true, and which are false? If true, say why; if false, give a counterexample (that is, an example confirming the falsehood).

- If $\lim_{x \rightarrow c} f(x)$ exists but $\lim_{x \rightarrow c} g(x)$ does not exist, then $\lim_{x \rightarrow c} (f(x) + g(x))$ does not exist.
- If neither $\lim_{x \rightarrow c} f(x)$ nor $\lim_{x \rightarrow c} g(x)$ exists, then $\lim_{x \rightarrow c} (f(x) + g(x))$ does not exist.

- c. If f is continuous at x , then so is $|f|$.
 d. If $|f|$ is continuous at c , then so is f .

In Exercises 15 and 16, use the formal definition of limit to prove that the function has a continuous extension to the given value of x .

15. $f(x) = \frac{x^2 - 1}{x + 1}, \quad x = -1$

16. $g(x) = \frac{x^2 - 2x - 3}{2x - 6}, \quad x = 3$

17. **A function continuous at only one point** Let

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

- a. Show that f is continuous at $x = 0$.
 b. Use the fact that every nonempty open interval of real numbers contains both rational and irrational numbers to show that f is not continuous at any nonzero value of x .
 18. **The Dirichlet ruler function** If x is a rational number, then x can be written in a unique way as a quotient of integers m/n , where $n > 0$ and m and n have no common factors greater than 1. (We say that such a fraction is in *lowest terms*. For example, $6/4$ written in lowest terms is $3/2$.) Let $f(x)$ be defined for all x in the interval $[0, 1]$ by

$$f(x) = \begin{cases} 1/n & \text{if } x = m/n \text{ is a rational number in lowest terms} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

For instance, $f(0) = f(1) = 1$, $f(1/2) = 1/2$, $f(1/3) = f(2/3) = 1/3$, $f(1/4) = f(3/4) = 1/4$, and so on.

- a. Show that f is discontinuous at every rational number in $[0, 1]$.
 b. Show that f is continuous at every irrational number in $[0, 1]$.
 (Hint: If ε is a given positive number, show that there are only finitely many rational numbers r in $[0, 1]$ such that $f(r) \geq \varepsilon$.)
 c. Sketch the graph of f . Why do you think f is called the “ruler function”?
 19. **Antipodal points** Is there any reason to believe that there is always a pair of antipodal (diametrically opposite) points on Earth’s equator where the temperatures are the same? Explain.
 20. If $\lim_{x \rightarrow c} (f(x) + g(x)) = 3$ and $\lim_{x \rightarrow c} (f(x) - g(x)) = -1$, find $\lim_{x \rightarrow c} f(x)g(x)$.
 21. **Roots of a quadratic equation that is almost linear** The equation $ax^2 + 2x - 1 = 0$, where a is a constant, has two roots if $a > -1$ and $a \neq 0$, one positive and one negative:

$$r_+(a) = \frac{-1 + \sqrt{1+a}}{a}, \quad r_-(a) = \frac{-1 - \sqrt{1+a}}{a}.$$

- a. What happens to $r_+(a)$ as $a \rightarrow 0$? As $a \rightarrow -1^+$?
 b. What happens to $r_-(a)$ as $a \rightarrow 0$? As $a \rightarrow -1^+$?
 c. Support your conclusions by graphing $r_+(a)$ and $r_-(a)$ as functions of a . Describe what you see.
 d. For added support, graph $f(x) = ax^2 + 2x - 1$ simultaneously for $a = 1, 0.5, 0.2, 0.1$, and 0.05 .
 22. **Root of an equation** Show that the equation $x + 2 \cos x = 0$ has at least one solution.
 23. **Bounded functions** A real-valued function f is **bounded from above** on a set D if there exists a number N such that $f(x) \leq N$ for all x in D . We call N , when it exists, an **upper bound** for f on D and say that f is **bounded from above** by N . In a similar manner, we say that f is **bounded from below** on D if there exists a number M such that $f(x) \geq M$ for all x in D . We call M , when it

exists, a **lower bound** for f on D and say that f is **bounded** from below by M . We say that f is **bounded** on D if it is bounded from both above and below.

- a. Show that f is bounded on D if and only if there exists a number B such that $|f(x)| \leq B$ for all x in D .
 b. Suppose that f is bounded from above by N . Show that if $\lim_{x \rightarrow c} f(x) = L$, then $L \leq N$.
 c. Suppose that f is bounded from below by M . Show that if $\lim_{x \rightarrow c} f(x) = L$, then $L \geq M$.

24. **Max $\{a, b\}$ and min $\{a, b\}$**

- a. Show that the expression

$$\max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$$

equals a if $a \geq b$ and equals b if $b \geq a$. In other words, $\max\{a, b\}$ gives the larger of the two numbers a and b .

- b. Find a similar expression for $\min\{a, b\}$, the smaller of a and b .

Generalized Limits Involving $\frac{\sin \theta}{\theta}$

The formula $\lim_{\theta \rightarrow 0} (\sin \theta)/\theta = 1$ can be generalized. If $\lim_{x \rightarrow c} f(x) = 0$ and $f(x)$ is never zero in an open interval containing the point $x = c$, except possibly at c itself, then

$$\lim_{x \rightarrow c} \frac{\sin f(x)}{f(x)} = 1.$$

Here are several examples.

- a. $\lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} = 1$
 b. $\lim_{x \rightarrow 0} \frac{\sin x^2}{x} = \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} \lim_{x \rightarrow 0} \frac{x^2}{x} = 1 \cdot 0 = 0$
 c. $\lim_{x \rightarrow -1} \frac{\sin(x^2 - x - 2)}{x + 1}$
 $= \lim_{x \rightarrow -1} \frac{\sin(x^2 - x - 2)}{(x^2 - x - 2)} \cdot \lim_{x \rightarrow -1} \frac{(x^2 - x - 2)}{x + 1}$
 $= 1 \cdot \lim_{x \rightarrow -1} \frac{(x+1)(x-2)}{x+1} = -3$
 d. $\lim_{x \rightarrow 1} \frac{\sin(1 - \sqrt{x})}{x - 1} = \lim_{x \rightarrow 1} \frac{\sin(1 - \sqrt{x})}{1 - \sqrt{x}} \cdot \frac{1 - \sqrt{x}}{x - 1}$
 $= \lim_{x \rightarrow 1} \frac{(1 - \sqrt{x})(1 + \sqrt{x})}{(x - 1)(1 + \sqrt{x})}$
 $= \lim_{x \rightarrow 1} \frac{1 - x}{(x - 1)(1 + \sqrt{x})} = -\frac{1}{2}$

Find the limits in Exercises 25–30.

25. $\lim_{x \rightarrow 0} \frac{\sin(1 - \cos x)}{x}$ 26. $\lim_{x \rightarrow 0^+} \frac{\sin x}{\sin \sqrt{x}}$
 27. $\lim_{x \rightarrow 0} \frac{\sin(\sin x)}{x}$ 28. $\lim_{x \rightarrow 0} \frac{\sin(x^2 + x)}{x}$
 29. $\lim_{x \rightarrow 2} \frac{\sin(x^2 - 4)}{x - 2}$ 30. $\lim_{x \rightarrow 9} \frac{\sin(\sqrt{x} - 3)}{x - 9}$

Trigonometric Limits

Find the limits in Exercises 31–38.

31. $\lim_{x \rightarrow 0} \frac{\sin x}{2x^2 - x}$ 32. $\lim_{x \rightarrow 0} \frac{3x - \tan 7x}{2x}$

33. $\lim_{r \rightarrow 0} \frac{\sin r}{\tan 2r}$

34. $\lim_{\theta \rightarrow 0} \frac{\sin(\sin \theta)}{\theta}$

35. $\lim_{\theta \rightarrow (\pi/2)^-} \frac{4 \tan^2 \theta + \tan \theta + 1}{\tan^2 \theta + 5}$

36. $\lim_{\theta \rightarrow 0^+} \frac{1 - 2 \cot^2 \theta}{5 \cot^2 \theta - 7 \cot \theta - 8}$

37. $\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x}$

38. $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2}$

Show how to extend the functions in Exercises 39 and 40 to be continuous at the origin.

39. $g(x) = \frac{\tan(\tan x)}{\tan x}$

40. $f(x) = \frac{\tan(\tan x)}{\sin(\sin x)}$

Oblique Asymptotes

Find all possible oblique asymptotes in Exercises 41–44.

41. $y = \frac{2x^{3/2} + 2x - 3}{\sqrt{x} + 1}$

42. $y = x + x \sin \frac{1}{x}$

43. $y = \sqrt{x^2 + 1}$

44. $y = \sqrt{x^2 + 2x}$

Showing an Equation Is Solvable

45. Assume that $1 < a < b$ and $\frac{a}{x} + x = \frac{1}{x - b}$. Show that this equation is solvable for x .

More Limits

46. Find constants a and b so that each of the following limits is true.

a. $\lim_{x \rightarrow 0} \frac{\sqrt{a + bx} - 1}{x} = 2$

b. $\lim_{x \rightarrow 1} \frac{\tan(ax - a) + b - 2}{x - 1} = 3$

47. Evaluate $\lim_{x \rightarrow 1} \frac{x^{2/3} - 1}{1 - \sqrt{x}}$.

48. Evaluate $\lim_{x \rightarrow 0} \frac{|3x + 4| - |x| - 4}{x}$.

Limits on Arbitrary Domains

The definition of the limit of a function at $x = c$ extends to functions whose domains near c are more complicated than intervals.

General Definition of Limit

Suppose every open interval containing c contains a point other than c in the domain of f . We say that $\lim_{x \rightarrow c} f(x) = L$ if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x in the domain of f , $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$.

For the functions in Exercises 49–52,

- Find the domain.
- Show that at $c = 0$ the domain has the property described above.
- Evaluate $\lim_{x \rightarrow 0} f(x)$.

49. The function f is defined as follows: $f(x) = x$ if $x = 1/n$ where n is a positive integer, and $f(0) = 1$.

50. The function f is defined as follows: $f(x) = 1 - x$ if $x = 1/n$ where n is a positive integer, and $f(0) = 1$.

51. $f(x) = \sqrt{x \sin(1/x)}$

52. $f(x) = \sqrt{\ln(\sin(1/x))}$

53. Let g be a function with domain the rational numbers, defined by $g(x) = \frac{2}{x - \sqrt{2}}$ for rational x .

- Sketch the graph of g as well as you can, keeping in mind that g is defined only at rational points.
- Use the general definition of a limit to prove that $\lim_{x \rightarrow 0} g(x) = -\sqrt{2}$.
- Prove that g is continuous at the point $x = 0$ by showing that the limit in part (b) equals $g(0)$.
- Is g continuous at other points of its domain?

CHAPTER 2 Technology Application Projects

Mathematica/Maple Projects

Projects can be found within MyLab Math.

• Take It to the Limit

Part I

Part II (Zero Raised to the Power Zero: What Does It Mean?)

Part III (One-Sided Limits)

Visualize and interpret the limit concept through graphical and numerical explorations.

Part IV (What a Difference a Power Makes)

See how sensitive limits can be with various powers of x .

• Going to Infinity

Part I (Exploring Function Behavior as $x \rightarrow \infty$ or $x \rightarrow -\infty$)

This module provides four examples to explore the behavior of a function as $x \rightarrow \infty$ or $x \rightarrow -\infty$.

Part II (Rates of Growth)

Observe graphs that *appear* to be continuous, yet the function is not continuous. Several issues of continuity are explored to obtain results that you may find surprising.

3

Derivatives



OVERVIEW In Chapter 2 we discussed how to determine the slope of a curve at a point and how to measure the rate at which a function changes. Now that we have studied limits, we can make these notions precise and see that both are interpretations of the *derivative* of a function at a point. We then extend this concept from a single point to the *derivative function*, and we develop rules for finding this derivative function easily, without having to calculate limits directly. These rules are used to find derivatives of most of the common functions reviewed in Chapter 1, as well as combinations of them.

3.1 Tangent Lines and the Derivative at a Point

In this section we define the slope and tangent line to a curve at a point, and the derivative of a function at a point. The derivative gives a way to find both the slope of a graph and the instantaneous rate of change of a function.

Finding a Tangent Line to the Graph of a Function

To find a tangent line to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we use the procedure introduced in Section 2.1. We calculate the slope of the secant line through P and a nearby point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \rightarrow 0$ (Figure 3.1). If the limit exists, we call it the slope of the curve at P and define the tangent line at P to be the line through P having this slope.

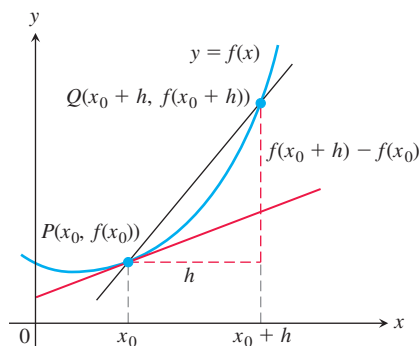


FIGURE 3.1 The slope of the tangent line at P is $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad (\text{provided the limit exists}).$$

The **tangent line** to the curve at P is the line through P with this slope.

In Section 2.1, Example 3, we applied these definitions to find the slope of the parabola $f(x) = x^2$ at the point $P(2, 4)$ and the tangent line to the parabola at P . Let's look at another example.

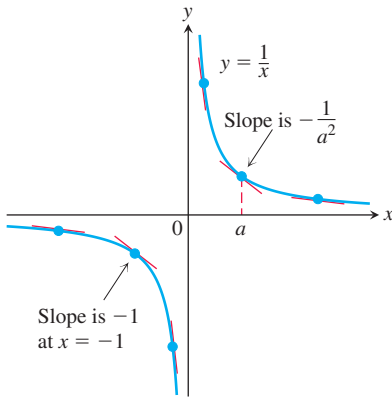


FIGURE 3.2 The tangent lines are steep when x is close to 0, and they become less steep as the point of tangency moves away (Example 1).

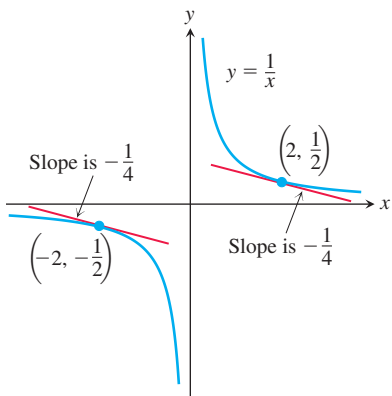


FIGURE 3.3 The two tangent lines to $y = 1/x$ having slope $-1/4$ (Example 1).

EXAMPLE 1

- (a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?
- (b) Where does the slope equal $-1/4$?
- (c) What happens to the tangent line to the curve at the point $(a, 1/a)$ as a changes?

Solution

- (a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \frac{a - (a+h)}{a(a+h)} \\ &= \lim_{h \rightarrow 0} \frac{-h}{ha(a+h)} = \lim_{h \rightarrow 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}. \end{aligned}$$

Notice how we had to keep writing “lim” before each fraction until the stage at which we could evaluate the limit by substituting $h = 0$. The number a may be positive or negative, but not 0. When $a = -1$, the slope is $-1/(-1)^2 = -1$ (Figure 3.2).

- (b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be $-1/4$, provided that

$$-\frac{1}{a^2} = -\frac{1}{4}.$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure 3.3).

- (c) The slope $-1/a^2$ is always negative if $a \neq 0$. As $a \rightarrow 0^+$, the slope approaches $-\infty$ and the tangent line becomes increasingly steep (Figure 3.2). We see this situation again as $a \rightarrow 0^-$. As a moves away from $x = 0$ in either direction, the slope approaches 0 and the tangent line levels off, becoming closer and closer to a horizontal line. ■

Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}, \quad h \neq 0$$

is called the **difference quotient of f at x_0 with increment h** . If the difference quotient has a limit as h approaches zero, that limit is given a special name and notation.

DEFINITION The **derivative of a function f at a point x_0** , denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided this limit exists.

The notation $f'(x_0)$ is read “ f prime of x_0 .”

The derivative has more than one meaning, depending on what problem we are considering. The formula for the derivative is the same as the formula for the slope of the curve $y = f(x)$ at a point. If we interpret the difference quotient as the slope of a secant line, then the derivative gives the slope of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$. If we interpret the difference quotient as an average rate of change (Section 2.1), then the derivative

gives the function's instantaneous rate of change with respect to x at the point $x = x_0$. We study this interpretation in Section 3.4.

EXAMPLE 2 In Examples 1 and 2 in Section 2.1, we studied the speed of a rock falling freely from rest near the surface of the earth. We knew that the rock fell $y = 4.9t^2$ meters during the first t seconds, and we used a sequence of average rates over increasingly short intervals to estimate the rock's speed at the instant $t = 1$. What was the rock's *exact* speed at this time?

Solution We let $f(t) = 4.9t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ seconds, for $h > 0$, was found to be

$$\frac{f(1+h) - f(1)}{h} = \frac{4.9(1+h)^2 - 4.9(1)^2}{h} = \frac{4.9(h^2 + 2h)}{h} = 4.9(h + 2).$$

The rock's speed at the instant $t = 1$ is then

$$f'(1) = \lim_{h \rightarrow 0} 4.9(h + 2) = 4.9(0 + 2) = 9.8 \text{ m/s}.$$

Our original estimate of 9.8 m/s in Section 2.1 was right. ■

Summary

We have been discussing slopes of curves, lines tangent to a curve, the rate of change of a function, and the derivative of a function at a point. All of these ideas are based on the same limit.

All of the following are interpretations for the limit of the difference quotient

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent line to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative $f'(x_0)$ at $x = x_0$

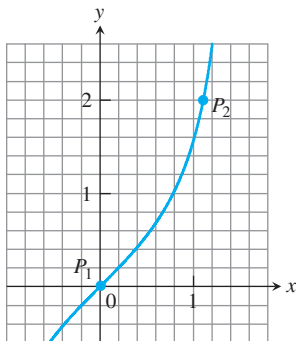
In the next sections, we allow the point x_0 to vary across the domain of the function f .

EXERCISES 3.1

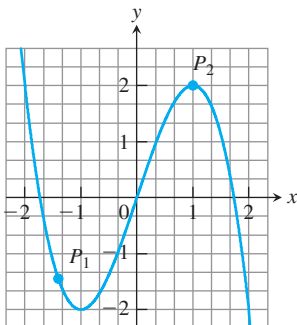
Slopes and Tangent Lines

In Exercises 1–4, use the grid and a straight edge to make a rough estimate of the slope of the curve (in y -units per x -unit) at the points P_1 and P_2 .

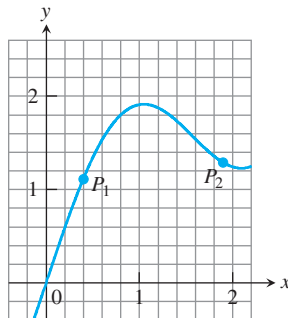
1.



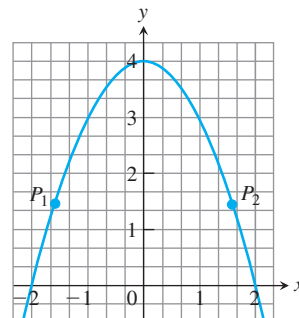
2.



3.



4.



In Exercises 5–10, find an equation for the tangent line to the curve at the given point. Then sketch the curve and tangent line together.

5. $y = 4 - x^2$, $(-1, 3)$ 6. $y = (x - 1)^2 + 1$, $(1, 1)$

7. $y = 2\sqrt{x}$, $(1, 2)$ 8. $y = \frac{1}{x^2}$, $(-1, 1)$

9. $y = x^3$, $(-2, -8)$ 10. $y = \frac{1}{x^3}$, $(-2, -\frac{1}{8})$

In Exercises 11–18, find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

11. $f(x) = x^2 + 1$, $(2, 5)$ 12. $f(x) = x - 2x^2$, $(1, -1)$

13. $g(x) = \frac{x}{x-2}$, $(3, 3)$ 14. $g(x) = \frac{8}{x^2}$, $(2, 2)$

15. $h(t) = t^3$, $(2, 8)$ 16. $h(t) = t^3 + 3t$, $(1, 4)$

17. $f(x) = \sqrt{x}$, $(4, 2)$ 18. $f(x) = \sqrt{x+1}$, $(8, 3)$

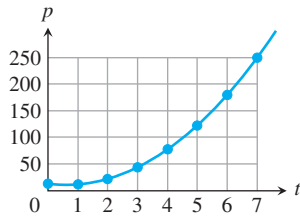
In Exercises 19–22, find the slope of the curve at the point indicated.

19. $y = 5x - 3x^2$, $x = 1$ 20. $y = x^3 - 2x + 7$, $x = -2$

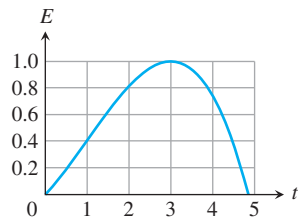
21. $y = \frac{1}{x-1}$, $x = 3$ 22. $y = \frac{x-1}{x+1}$, $x = 0$

Interpreting Derivative Values

23. **Growth of yeast cells** In a controlled laboratory experiment, yeast cells are grown in an automated cell culture system that counts the number P of cells present at hourly intervals. The number after t hours is shown in the accompanying figure.



- Explain what is meant by the derivative $P'(5)$. What are its units?
 - Which is larger, $P'(2)$ or $P'(3)$? Give a reason for your answer.
 - The quadratic curve capturing the trend of the data points (see Appendix A.2) is given by $P(t) = 6.10t^2 - 9.28t + 16.43$. Find the instantaneous rate of growth when $t = 5$ hours.
24. **Effectiveness of a drug** On a scale from 0 to 1, the effectiveness E of a pain-killing drug t hours after entering the bloodstream is displayed in the accompanying figure.



- At what times does the effectiveness appear to be increasing? What is true about the derivative at those times?
- At what time would you estimate that the drug reaches its maximum effectiveness? What is true about the derivative at that time? What is true about the derivative as time increases in the 1 hour before your estimated time?

At what points do the graphs of the functions in Exercises 25 and 26 have horizontal tangent lines?

25. $f(x) = x^2 + 4x - 1$ 26. $g(x) = x^3 - 3x$

27. Find equations of all lines having slope -1 that are tangent to the curve $y = 1/(x-1)$.

28. Find an equation of the straight line having slope $1/4$ that is tangent to the curve $y = \sqrt{x}$.

Rates of Change

29. **Object dropped from a tower** An object is dropped from the top of a 100-m-high tower. Its height above ground after t s is $100 - 4.9t^2$ m. How fast is it falling 2 s after it is dropped?

30. **Speed of a rocket** At t seconds after liftoff, the height of a rocket is $3t^2$ m. How fast is the rocket climbing 10 s after liftoff?

31. **Disk's changing area** What is the rate of change of the area of a disk ($A = \pi r^2$) with respect to the radius when the radius is $r = 3$?

32. **Ball's changing volume** What is the rate of change of the volume of a ball ($V = (4/3)\pi r^3$) with respect to the radius when the radius is $r = 2$?

33. Show that the line $y = mx + b$ is its own tangent line at any point $(x_0, mx_0 + b)$.

34. Find the slope of the tangent line to the curve $y = 1/\sqrt{x}$ at the point where $x = 4$.

Testing for Tangent Lines

35. Does the graph of

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent line at the origin? Give reasons for your answer.

36. Does the graph of

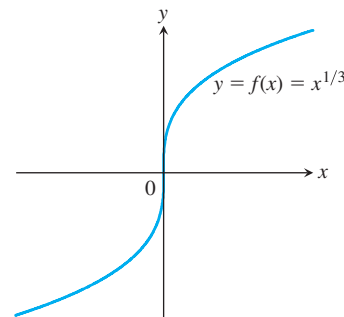
$$g(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent line at the origin? Give reasons for your answer.

Vertical Tangent Lines

We say that a continuous curve $y = f(x)$ has a **vertical tangent line** at the point where $x = x_0$ if the limit of the difference quotient is ∞ or $-\infty$. For example, $y = x^{1/3}$ has a vertical tangent line at $x = 0$ (see accompanying figure):

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0} \frac{h^{1/3} - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h^{2/3}} = \infty. \end{aligned}$$

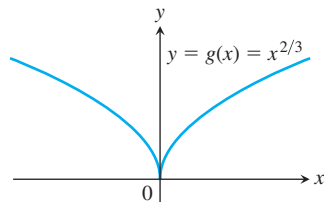


VERTICAL TANGENT LINE AT ORIGIN

However, $y = x^{2/3}$ has *no* vertical tangent line at $x = 0$ (see next figure):

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{g(0+h) - g(0)}{h} &= \lim_{h \rightarrow 0} \frac{h^{2/3} - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h^{1/3}}\end{aligned}$$

does not exist, because the limit is ∞ from the right and $-\infty$ from the left.



NO VERTICAL TANGENT LINE AT ORIGIN

37. Does the graph of

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

have a vertical tangent line at the origin? Give reasons for your answer.

38. Does the graph of

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

have a vertical tangent line at the point $(0, 1)$? Give reasons for your answer.

T Graph the curves in Exercises 39–48.

a. Where do the graphs appear to have vertical tangent lines?

b. Confirm your findings in part (a) with limit calculations. But before you do, read the introduction to Exercises 37 and 38.

39. $y = x^{2/5}$

40. $y = x^{4/5}$

41. $y = x^{1/5}$

42. $y = x^{3/5}$

43. $y = 4x^{2/5} - 2x$

44. $y = x^{5/3} - 5x^{2/3}$

45. $y = x^{2/3} - (x-1)^{1/3}$

46. $y = x^{1/3} + (x-1)^{1/3}$

47. $y = \begin{cases} -\sqrt{|x|}, & x \leq 0 \\ \sqrt{x}, & x > 0 \end{cases}$

48. $y = \sqrt{|4-x|}$

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for the functions in Exercises 49–52:

a. Plot $y = f(x)$ over the interval $(x_0 - 1/2) \leq x \leq (x_0 + 3)$.

b. Holding x_0 fixed, the difference quotient

$$q(h) = \frac{f(x_0 + h) - f(x_0)}{h}$$

at x_0 becomes a function of the step size h . Enter this function into your CAS workspace.

c. Find the limit of q as $h \rightarrow 0$.

d. Define the secant lines $y = f(x_0) + q \cdot (x - x_0)$ for $h = 3, 2$, and 1 . Graph them, together with f and the tangent line, over the interval in part (a).

49. $f(x) = x^3 + 2x$, $x_0 = 0$

50. $f(x) = x + \frac{5}{x}$, $x_0 = 1$

51. $f(x) = x + \sin(2x)$, $x_0 = \pi/2$

52. $f(x) = \cos x + 4 \sin(2x)$, $x_0 = \pi$

3.2 The Derivative as a Function

HISTORICAL ESSAY

The Derivative

To read this essay, visit the companion Website.

In the last section we defined the derivative of $y = f(x)$ at the point $x = x_0$ to be the limit

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We now investigate the derivative as a *function* derived from f by considering the limit at each point x in the domain of f .

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

We use the notation $f'(x)$ in the definition, rather than $f'(x_0)$ as before, to emphasize that f' is a function of the independent variable x with respect to which the derivative function $f'(x)$ is being defined. The domain of f' is the set of points in the domain of f for

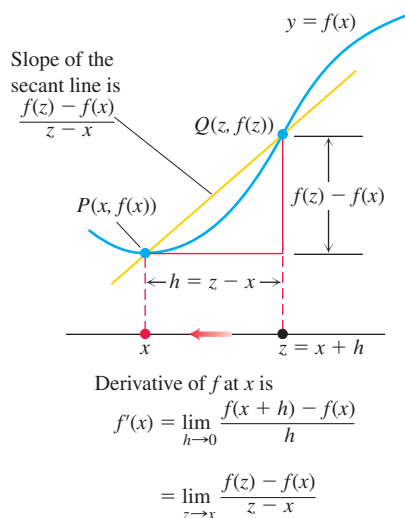


FIGURE 3.4 Two forms for the difference quotient.

which the limit exists, which means that the domain may be the same as or smaller than the domain of f . If f' exists at a particular x , we say that f is **differentiable (has a derivative) at x** . If f' exists at every point in the domain of f , we call f **differentiable**.

If we write $z = x + h$, then $h = z - x$ and h approaches 0 if and only if z approaches x . Therefore, an equivalent definition of the derivative is as follows (see Figure 3.4). This formula is sometimes more convenient to use when finding a derivative function, and it focuses on the point z that approaches x .

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$

Calculating Derivatives from the Definition

The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation

$$\frac{d}{dx} f(x)$$

as another way to denote the derivative $f'(x)$. Example 1 of Section 3.1 illustrated the differentiation process for the function $y = 1/x$ when $x = a$. For x representing any point in the domain, we get the formula

$$\frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2}.$$

Here are two more examples in which we allow x to be any point in the domain of f .

EXAMPLE 1 Differentiate $f(x) = \frac{x}{x-1}$.

Solution We use the definition of derivative, which requires us to calculate $f(x+h)$ and then subtract $f(x)$ to obtain the numerator in the difference quotient. We have

$$f(x) = \frac{x}{x-1} \quad \text{and} \quad f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} && \text{Definition} \\ &= \lim_{h \rightarrow 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} && \text{Substitute.} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} && \frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} && \text{Simplify.} \\ &= \lim_{h \rightarrow 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}. && \text{Cancel } h \neq 0 \text{ and evaluate.} \quad \blacksquare \end{aligned}$$

EXAMPLE 2

- (a) Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.
- (b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Derivative of the Square Root Function

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}, \quad x > 0$$

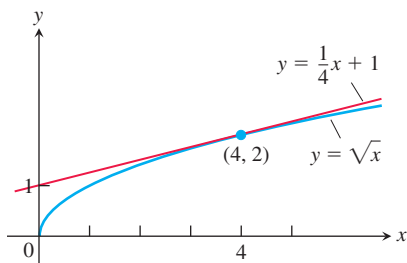


FIGURE 3.5 The curve $y = \sqrt{x}$ and its tangent line at $(4, 2)$. The tangent line's slope is found by evaluating the derivative at $x = 4$ (Example 2).

Solution

(a) We use the alternative formula to calculate f' :

$$\begin{aligned} f'(x) &= \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} \\ &= \lim_{z \rightarrow x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\ &= \lim_{z \rightarrow x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})} \quad \frac{1}{a^2 - b^2} = \frac{1}{(a - b)(a + b)} \\ &= \lim_{z \rightarrow x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \quad \text{Cancel and evaluate.} \end{aligned}$$

(b) The slope of the curve at $x = 4$ is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The tangent line is the line through the point $(4, 2)$ with slope $1/4$ (Figure 3.5):

$$y = 2 + \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1.$$

Notation

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y . Some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x).$$

The symbols d/dx and D indicate the operation of differentiation. We read dy/dx as “the derivative of y with respect to x ,” and df/dx and $(d/dx)f(x)$ as “the derivative of f with respect to x .” The “prime” notations y' and f' originate with Newton. The d/dx notations are similar to those used by Leibniz. The symbol dy/dx should not be regarded as a ratio; it simply denotes a derivative.

To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \left. \frac{dy}{dx} \right|_{x=a} = \left. \frac{df}{dx} \right|_{x=a} = \left. \frac{d}{dx}f(x) \right|_{x=a}.$$

For instance, in Example 2,

$$f'(4) = \left. \frac{d}{dx}\sqrt{x} \right|_{x=4} = \left. \frac{1}{2\sqrt{x}} \right|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Graphing the Derivative

We can often make an approximate plot of the derivative of $y = f(x)$ by estimating the slopes on the graph of f . That is, we plot the points $(x, f'(x))$ in the xy -plane and connect them with a curve that represents $y = f'(x)$.

EXAMPLE 3 Graph the derivative of the function $y = f(x)$ in Figure 3.6a.

Solution We sketch the tangent lines to the graph of f at frequent intervals and use their slopes to estimate the values of $f'(x)$ at these points. We plot the corresponding $(x, f'(x))$ pairs and connect them with a curve as sketched in Figure 3.6b. ■

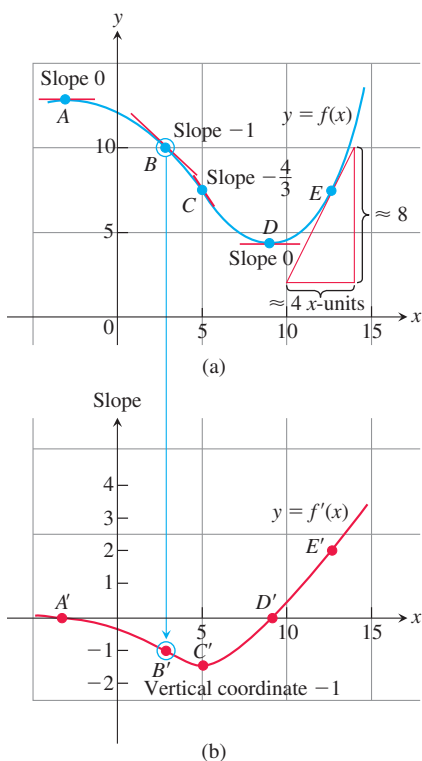


FIGURE 3.6 We made the graph of $y = f'(x)$ in (b) by plotting slopes from the graph of $y = f(x)$ in (a). The vertical coordinate of B' is the slope at B , and so on. The slope at E is approximately $8/4 = 2$. In (b) we see that the rate of change of f is negative for x between A' and D' ; the rate of change is positive for x to the right of D' .

What can we learn from the graph of $y = f'(x)$? At a glance we can see

1. where the rate of change of f is positive, negative, or zero;
2. the rough size of the growth rate at any x ;
3. where the rate of change itself is increasing or decreasing.

Differentiability on an Interval; One-Sided Derivatives

A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it has a derivative at each point of the interval. It is **differentiable on a closed interval** $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h} \quad \text{Right-hand derivative at } a$$

$$\lim_{h \rightarrow 0^-} \frac{f(b+h) - f(b)}{h} \quad \text{Left-hand derivative at } b$$

exist at the endpoints (Figure 3.7).

Right-hand and left-hand derivatives may or may not be defined at any point of a function's domain. Because of Theorem 5, Section 2.4, a function has a derivative at an interior point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

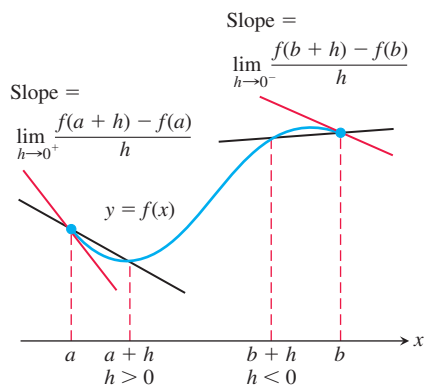


FIGURE 3.7 Derivatives at endpoints of a closed interval are one-sided limits.

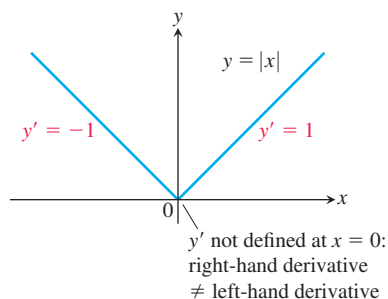


FIGURE 3.8 The function $y = |x|$ is not differentiable at the origin where the graph has a “corner” (Example 4).

EXAMPLE 4 Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and on $(0, \infty)$ but has no derivative at $x = 0$.

Solution The graph of the function $y = mx + b$ is a straight line with slope m . Thus, to the right of the origin, when $x > 0$,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x) = 1. \quad |x| = x \text{ since } x > 0, \quad \frac{d}{dx}(mx+b) = m$$

To the left, when $x < 0$,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x) = -1 \quad |x| = -x \text{ since } x < 0$$

(Figure 3.8). The two branches of the graph come together at an angle at the origin, forming a non-smooth corner. There is no derivative at the origin because the one-sided derivatives differ there:

$$\begin{aligned} \text{Right-hand derivative of } |x| \text{ at zero} &= \lim_{h \rightarrow 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \rightarrow 0^+} \frac{|h|}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{h}{h} \quad |h| = h \text{ when } h > 0 \\ &= \lim_{h \rightarrow 0^+} 1 = 1 \end{aligned}$$

$$\begin{aligned} \text{Left-hand derivative of } |x| \text{ at zero} &= \lim_{h \rightarrow 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \rightarrow 0^-} \frac{|h|}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{-h}{h} \quad |h| = -h \text{ when } h < 0 \\ &= \lim_{h \rightarrow 0^-} -1 = -1. \end{aligned}$$

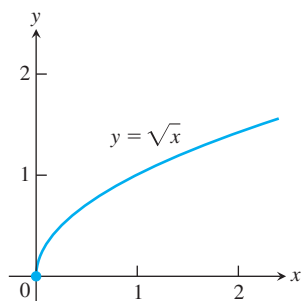


FIGURE 3.9 The square root function is not differentiable at $x = 0$, where the graph of the function has a vertical tangent line.

EXAMPLE 5 In Example 2 we found that for $x > 0$,

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

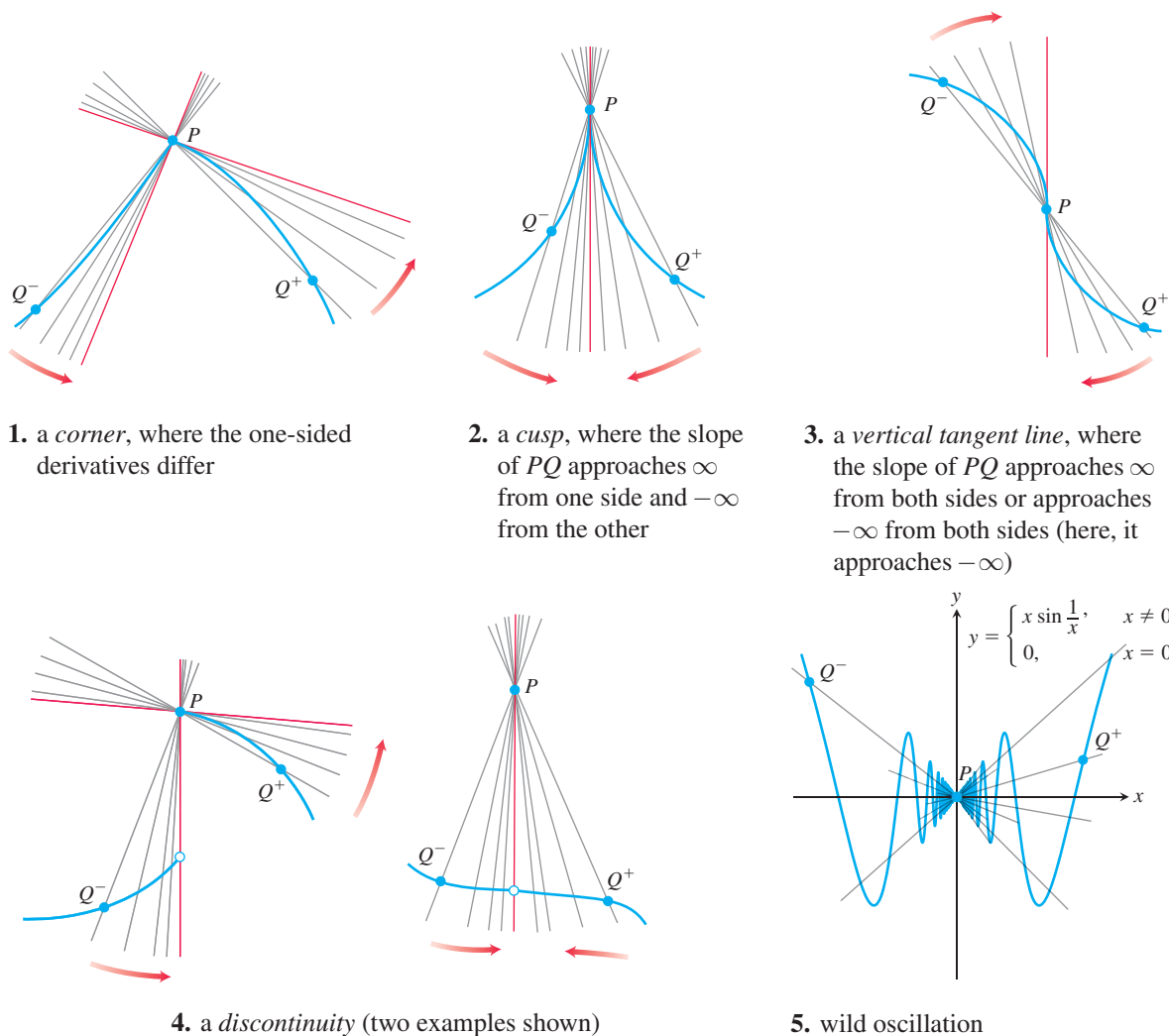
We apply the definition to examine whether the derivative exists at $x = 0$:

$$\lim_{h \rightarrow 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h \rightarrow 0^+} \frac{1}{\sqrt{h}} = \infty.$$

Since the (right-hand) limit is not finite, there is no derivative at $x = 0$. Since the slopes of the secant lines joining the origin to the points $(h, \sqrt{h})$ on a graph of $y = \sqrt{x}$ approach ∞ , the graph has a *vertical tangent line* at the origin. (See Figure 3.9 and Exercises 37 and 38 in Section 3.1.)

When Does a Function Not Have a Derivative at a Point?

A function has a derivative at a point x_0 if the slopes of the secant lines through $P(x_0, f(x_0))$ and a nearby point Q on the graph approach a finite limit as Q approaches P . Thus differentiability is a “smoothness” condition on the graph of f . A function can fail to have a derivative at a point for many reasons, including the existence of points where the graph has



The last example shows a function that is continuous at $x = 0$, but whose graph oscillates wildly up and down as it approaches $x = 0$. The slopes of the secant lines through 0 oscillate between -1 and 1 as x approaches 0 , and do not have a limit at $x = 0$.

Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

THEOREM 1—Differentiable Implies Continuous If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof Given that $f'(c)$ exists, we must show that $\lim_{x \rightarrow c} f(x) = f(c)$, or, equivalently, that $\lim_{h \rightarrow 0} f(c + h) = f(c)$. If $h \neq 0$, then

$$\begin{aligned} f(c + h) &= f(c) + (f(c + h) - f(c)) && \text{Add and subtract } f(c). \\ &= f(c) + \frac{f(c + h) - f(c)}{h} \cdot h. && \text{Divide and multiply by } h. \end{aligned}$$

Now take limits as $h \rightarrow 0$. By Theorem 1 of Section 2.2,

$$\begin{aligned} \lim_{h \rightarrow 0} f(c + h) &= \lim_{h \rightarrow 0} f(c) + \lim_{h \rightarrow 0} \frac{f(c + h) - f(c)}{h} \cdot \lim_{h \rightarrow 0} h \\ &= f(c) + f'(c) \cdot 0 \\ &= f(c) + 0 \\ &= f(c). \end{aligned}$$

Similar arguments with one-sided limits show that if f has a derivative from one side (right or left) at $x = c$, then f is continuous from that side at $x = c$.

Theorem 1 says that if a function has a discontinuity at a point (for instance, a jump discontinuity), then it cannot be differentiable there. The greatest integer function $y = \lfloor x \rfloor$ fails to be differentiable at every integer $x = n$ (Example 4, Section 2.6).

Caution The converse of Theorem 1 is false. A function need not have a derivative at a point where it is continuous, as we saw with the absolute value function in Example 4. ●

EXERCISES 3.2

Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions in Exercises 1–6. Then find the values of the derivatives as specified.

1. $f(x) = 4 - x^2$; $f'(-3)$, $f'(0)$, $f'(1)$

2. $F(x) = (x - 1)^2 + 1$; $F'(-1)$, $F'(0)$, $F'(2)$

3. $g(t) = \frac{1}{t^2}$; $g'(-1)$, $g'(2)$, $g'(\sqrt{3})$

4. $k(z) = \frac{1-z}{2z}$; $k'(-1)$, $k'(1)$, $k'(\sqrt{2})$

5. $p(\theta) = \sqrt{3\theta}$; $p'(1)$, $p'(3)$, $p'(2/3)$

6. $r(s) = \sqrt{2s + 1}$; $r'(0)$, $r'(1)$, $r'(1/2)$

In Exercises 7–12, find the indicated derivatives.

7. $\frac{dy}{dx}$ if $y = 2x^3$ 8. $\frac{dr}{ds}$ if $r = s^3 - 2s^2 + 3$

9. $\frac{ds}{dt}$ if $s = \frac{t}{2t + 1}$ 10. $\frac{dv}{dt}$ if $v = t - \frac{1}{t}$

11. $\frac{dp}{dq}$ if $p = q^{3/2}$ 12. $\frac{dz}{dw}$ if $z = \frac{1}{\sqrt{w^2 - 1}}$

Slopes and Tangent Lines

In Exercises 13–16, differentiate the functions and find the slope of the tangent line at the given value of the independent variable.

13. $f(x) = x + \frac{9}{x}$, $x = -3$ 14. $k(x) = \frac{1}{2 + x}$, $x = 2$

15. $s = t^3 - t^2$, $t = -1$ 16. $y = \frac{x + 3}{1 - x}$, $x = -2$

In Exercises 17–18, differentiate the functions. Then find an equation of the tangent line at the indicated point on the graph of the function.

17. $y = f(x) = \frac{8}{\sqrt{x - 2}}$, $(x, y) = (6, 4)$

18. $w = g(z) = 1 + \sqrt{4 - z}$, $(z, w) = (3, 2)$

In Exercises 19–22, find the values of the derivatives.

19. $\left. \frac{ds}{dt} \right|_{t=-1}$ if $s = 1 - 3t^2$ 20. $\left. \frac{dy}{dx} \right|_{x=\sqrt{3}}$ if $y = 1 - \frac{1}{x}$

21. $\left. \frac{dr}{d\theta} \right|_{\theta=0}$ if $r = \frac{2}{\sqrt{4 - \theta}}$ 22. $\left. \frac{dw}{dz} \right|_{z=4}$ if $w = z + \sqrt{z}$

Using the Alternative Formula for Derivatives

Use the formula

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$

to find the derivative of the functions in Exercises 23–26.

23. $f(x) = \frac{1}{x+2}$

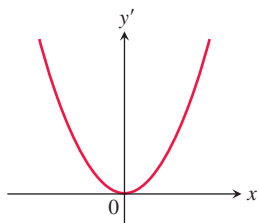
24. $f(x) = x^2 - 3x + 4$

25. $g(x) = \frac{x}{x-1}$

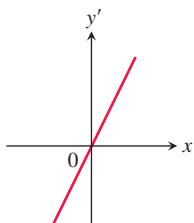
26. $g(x) = 1 + \sqrt{x}$

Graphs

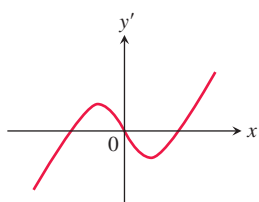
Match the functions graphed in Exercises 27–30 with the derivatives graphed in the accompanying figures (a)–(d).



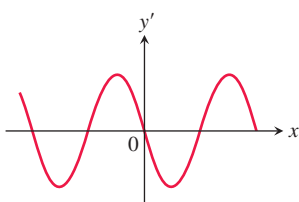
(a)



(b)

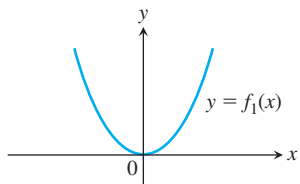


(c)

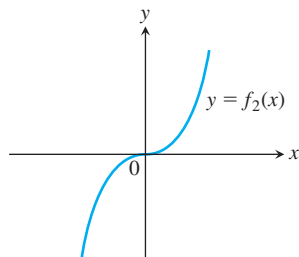


(d)

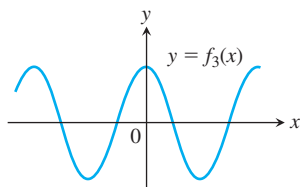
27.



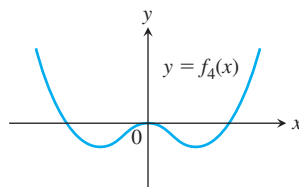
28.



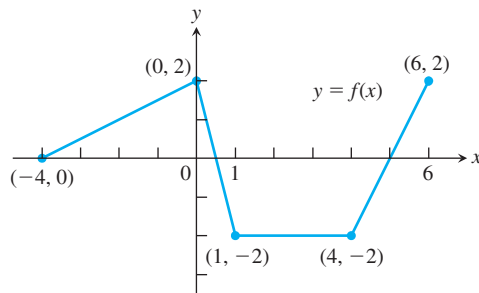
29.



30.



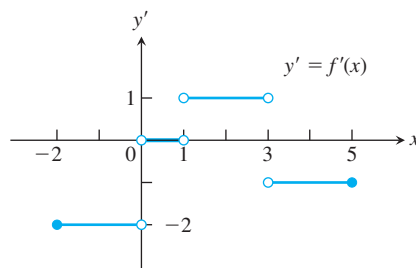
31. Consider the function f graphed here. The domain of f is the interval $[-4, 6]$ and its graph is made of line segments joined end to end.



- At which points of the domain interval is f' not defined? Give reasons for your answer.
- Graph the derivative of f . The graph should show a step function.

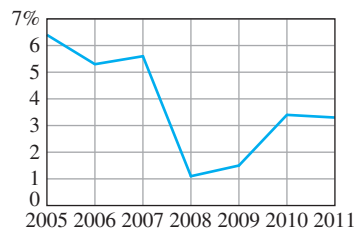
32. Recovering a function from its derivative

- Use the following information to graph the function f over the closed interval $[-2, 5]$.
 - The graph of f is made of closed line segments joined end to end.
 - The graph starts at the point $(-2, 3)$.
 - The derivative of f is the step function in the figure shown here.



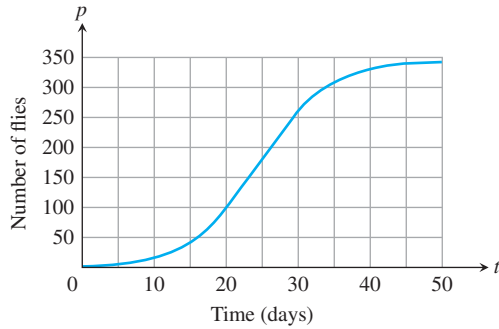
- Repeat part (a), assuming that the graph starts at $(-2, 0)$ instead of $(-2, 3)$.

33. **Growth in the economy** The graph in the accompanying figure shows the average annual percentage change $y = f(t)$ in the U.S. gross national product (GNP) for the years 2005–2011. Graph dy/dt (where defined).



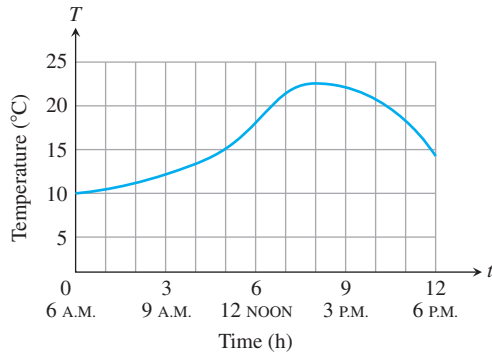
34. **Fruit flies** (Continuation of Example 4, Section 2.1.) Populations starting out in closed environments grow slowly at first, when there are relatively few members, then more rapidly as the number of reproducing individuals increases and resources are still abundant, then slowly again as the population reaches the carrying capacity of the environment.

- a. Use the graphical technique of Example 3 to graph the derivative of the fruit fly population as a function of time (in days). The graph of the population is reproduced here.



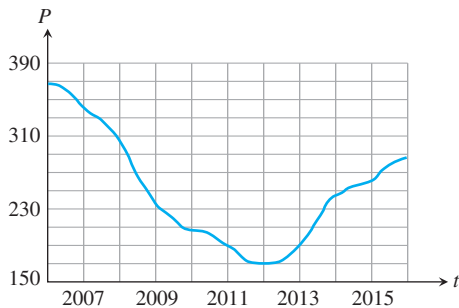
- b. During what days does the population seem to be increasing fastest? Slowest?

35. **Temperature** The given graph shows the temperature T in $^{\circ}\text{C}$ between 6 A.M. and 6 P.M.



- a. Estimate the rate of temperature change at the times
i) 7 A.M. ii) 9 A.M. iii) 2 P.M. iv) 4 P.M.
b. At what time does the temperature increase most rapidly? Decrease most rapidly? What is the rate for each of those times?
c. Use the graphical technique of Example 3 to graph the derivative of temperature T versus time t .

36. Average single-family home prices P (in thousands of dollars) in Sacramento, California, are shown in the accompanying figure from the beginning of 2006 through the end of 2015.



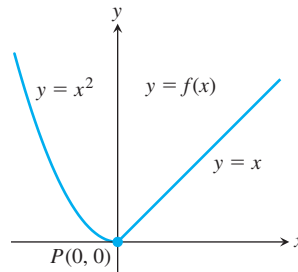
- a. During what years did home prices decrease? increase?
b. Estimate home prices at the end of
i) 2007 ii) 2012 iii) 2015

- c. Estimate the rate of change of home prices at the beginning of
i) 2007 ii) 2010 iii) 2014
d. During what year did home prices drop most rapidly and what is an estimate of this rate?
e. During what year did home prices rise most rapidly and what is an estimate of this rate?
f. Use the graphical technique of Example 3 to graph the derivative of home price P versus time t .

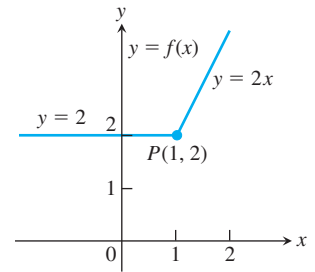
One-Sided Derivatives

Compute the right-hand and left-hand derivatives as limits to show that the functions in Exercises 37–40 are not differentiable at the point P .

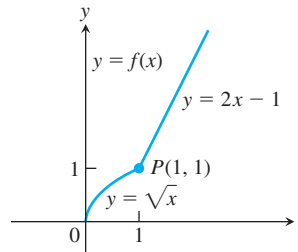
37.



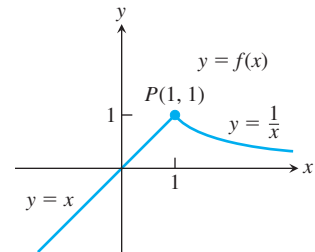
38.



39.



40.



In Exercises 41–44, determine whether the piecewise-defined function is differentiable at $x = 0$.

$$41. f(x) = \begin{cases} 2x - 1, & x \geq 0 \\ x^2 + 2x + 7, & x < 0 \end{cases}$$

$$42. g(x) = \begin{cases} x^{2/3}, & x \geq 0 \\ x^{1/3}, & x < 0 \end{cases}$$

$$43. f(x) = \begin{cases} 2x + \tan x, & x \geq 0 \\ x^2, & x < 0 \end{cases}$$

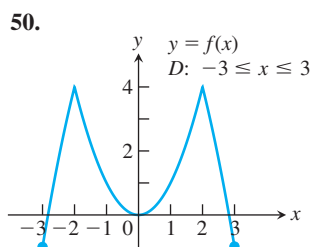
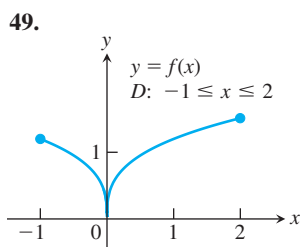
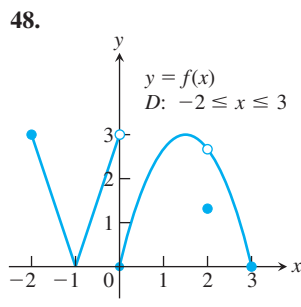
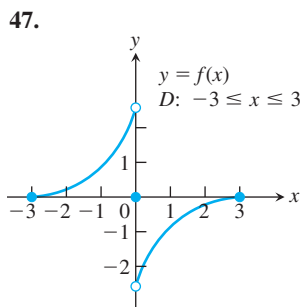
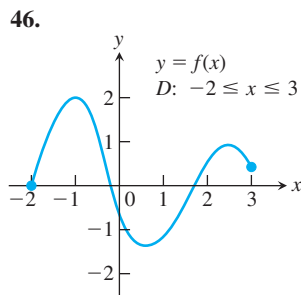
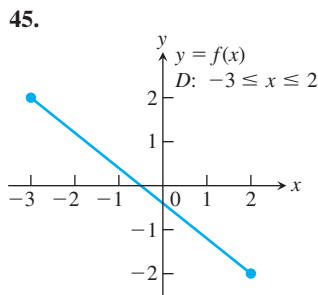
$$44. g(x) = \begin{cases} 2x - x^3 - 1, & x \geq 0 \\ x - \frac{1}{x+1}, & x < 0 \end{cases}$$

Differentiability and Continuity on an Interval

Each figure in Exercises 45–50 shows the graph of a function over a closed interval D . At what domain points does the function appear to be

- a. differentiable?
b. continuous but not differentiable?
c. neither continuous nor differentiable?

Give reasons for your answers.



Theory and Examples

In Exercises 51–54,

- Find the derivative $f'(x)$ of the given function $y = f(x)$.
 - Graph $y = f(x)$ and $y = f'(x)$ side by side using separate sets of coordinate axes, and answer the following questions.
 - For what values of x , if any, is f' positive? Zero? Negative?
 - Over what intervals of x -values, if any, does the function $y = f(x)$ increase as x increases? Decrease as x increases? How is this related to what you found in part (c)? (We will say more about this relationship in Section 4.3.)
51. $y = -x^2$ 52. $y = -1/x$
 53. $y = x^3/3$ 54. $y = x^4/4$
55. **Tangent line to a parabola** Does the parabola $y = 2x^2 - 13x + 5$ have a tangent line whose slope is -1 ? If so, find an equation for the line and the point of tangency. If not, why not?
56. **Tangent line to $y = \sqrt{x}$** Does any tangent line to the curve $y = \sqrt{x}$ cross the x -axis at $x = -1$? If so, find an equation for the line and the point of tangency. If not, why not?
57. **Derivative of $-f$** Does knowing that a function $f(x)$ is differentiable at $x = x_0$ tell you anything about the differentiability of the function $-f$ at $x = x_0$? Give reasons for your answer.
58. **Derivative of multiples** Does knowing that a function $g(t)$ is differentiable at $t = 7$ tell you anything about the differentiability of the function $3g$ at $t = 7$? Give reasons for your answer.

59. **Limit of a quotient** Suppose that functions $g(t)$ and $h(t)$ are defined for all values of t and $g(0) = h(0) = 0$. Can $\lim_{t \rightarrow 0} g(t)/h(t)$ exist? If it does exist, must it equal zero? Give reasons for your answers.

60. a. Let $f(x)$ be a function satisfying $|f(x)| \leq x^2$ for $-1 \leq x \leq 1$. Show that f is differentiable at $x = 0$ and find $f'(0)$.
 b. Show that

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at $x = 0$ and find $f'(0)$.

- T 61. Graph $y = 1/(2\sqrt{x})$ in a window that has $0 \leq x \leq 2$. Then, on the same screen, graph

$$y = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

for $h = 1, 0.5, 0.1$. Then try $h = -1, -0.5, -0.1$. Explain what is going on.

- T 62. Graph $y = 3x^2$ in a window that has $-2 \leq x \leq 2, 0 \leq y \leq 3$. Then, on the same screen, graph

$$y = \frac{(x+h)^3 - x^3}{h}$$

for $h = 2, 1, 0.2$. Then try $h = -2, -1, -0.2$. Explain what is going on.

63. **Derivative of $y = |x|$** Graph the derivative of $f(x) = |x|$. Then graph $y = (|x| - 0)/(x - 0) = |x|/x$. What can you conclude?

- T 64. **Weierstrass's nowhere differentiable continuous function** The sum of the first eight terms of the Weierstrass function $f(x) = \sum_{n=0}^{\infty} (2/3)^n \cos(9^n \pi x)$ is

$$g(x) = \cos(\pi x) + (2/3)^1 \cos(9\pi x) + (2/3)^2 \cos(9^2 \pi x) + (2/3)^3 \cos(9^3 \pi x) + \cdots + (2/3)^7 \cos(9^7 \pi x).$$

Graph this sum. Zoom in several times. How wiggly and bumpy is this graph? Specify a viewing window in which the displayed portion of the graph is smooth.

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for the functions in Exercises 65–70.

- Plot $y = f(x)$ to see that function's global behavior.
- Define the difference quotient q at a general point x , with general step size h .
- Take the limit as $h \rightarrow 0$. What formula does this give?
- Substitute the value $x = x_0$ and plot the function $y = f(x)$ together with its tangent line at that point.
- Substitute various values for x larger and smaller than x_0 into the formula obtained in part (c). Do the numbers make sense with your picture?
- Graph the formula obtained in part (c). What does it mean when its values are negative? Zero? Positive? Does this make sense with your plot from part (a)? Give reasons for your answer.

65. $f(x) = x^3 + x^2 - x, \quad x_0 = 1$

66. $f(x) = x^{1/3} + x^{2/3}, \quad x_0 = 1$

67. $f(x) = \frac{4x}{x^2 + 1}, \quad x_0 = 2$

68. $f(x) = \frac{x-1}{3x^2+1}, \quad x_0 = -1$

69. $f(x) = \sin 2x, \quad x_0 = \pi/2$

70. $f(x) = x^2 \cos x, \quad x_0 = \pi/4$

3.3 Differentiation Rules

This section introduces several rules that allow us to differentiate constant functions, power functions, polynomials, exponential functions, rational functions, and certain combinations of them, simply and directly, without having to take limits each time.

Powers, Multiples, Sums, and Differences

A basic rule of differentiation is that the derivative of every constant function is zero.

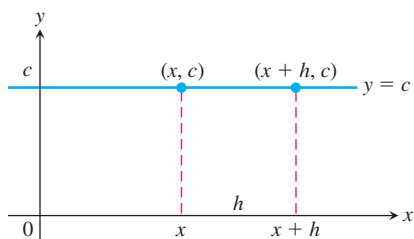


FIGURE 3.10 The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Proof We apply the definition of the derivative to $f(x) = c$, the function whose outputs have the constant value c (Figure 3.10). At every value of x , we find that

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0. \quad \blacksquare$$

We now consider powers of x . From Section 3.1, we know that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}, \quad \text{or} \quad \frac{d}{dx}(x^{-1}) = -x^{-2}.$$

From Example 2 of the last section we also know that

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}, \quad \text{or} \quad \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2}.$$

These two examples illustrate a general rule for differentiating a power x^n . We first prove the rule when n is a positive integer.

Derivative of a Positive Integer Power

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Proof of the Positive Integer Power Rule The formula

$$z^n - x^n = (z - x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})$$

can be verified by multiplying out the right-hand side. Then, from the alternative formula for the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} = \lim_{z \rightarrow x} \frac{z^n - x^n}{z - x} \\ &= \lim_{z \rightarrow x} (z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1}) \quad n \text{ terms} \\ &= nx^{n-1}. \quad \blacksquare \end{aligned}$$

HISTORICAL BIOGRAPHY

Richard Courant
(1888–1972)

Courant obtained a PhD from Göttingen in 1910. He founded Göttingen's Mathematics Institute and was its director from 1920 until 1933. His research work focused on mathematical physics.

To know more, visit the companion Website.

The Power Rule is actually valid for all real numbers n , not just for positive integers. We have seen examples for a negative integer and fractional power, but n could be an irrational number as well. Here we state the general version of the rule, but postpone its proof until Section 3.8.

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

EXAMPLE 1 Differentiate the following powers of x .

(a) x^3 (b) $x^{2/3}$ (c) $x^{\sqrt{2}}$ (d) $\frac{1}{x^4}$ (e) $x^{-4/3}$ (f) $\sqrt{x^{2+\pi}}$

Solution

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx}(x^3) &= 3x^{3-1} = 3x^2 \\ \text{(b)} \quad \frac{d}{dx}(x^{2/3}) &= \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3} \\ \text{(c)} \quad \frac{d}{dx}(x^{\sqrt{2}}) &= \sqrt{2}x^{\sqrt{2}-1} \\ \text{(d)} \quad \frac{d}{dx}\left(\frac{1}{x^4}\right) &= \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5} \\ \text{(e)} \quad \frac{d}{dx}(x^{-4/3}) &= -\frac{4}{3}x^{-(4/3)-1} = -\frac{4}{3}x^{-7/3} \\ \text{(f)} \quad \frac{d}{dx}(\sqrt{x^{2+\pi}}) &= \frac{d}{dx}(x^{1+(\pi/2)}) = \left(1 + \frac{\pi}{2}\right)x^{1+(\pi/2)-1} = \frac{1}{2}(2 + \pi)\sqrt{x^\pi} \end{aligned}$$

The next rule says that when a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant.

Derivative Constant Multiple Rule

If u is a differentiable function of x , and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Proof

$$\begin{aligned} \frac{d}{dx}cu &= \lim_{h \rightarrow 0} \frac{cu(x+h) - cu(x)}{h} && \text{Derivative definition} \\ &&& \text{with } f(x) = cu(x) \\ &= c \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} && \text{Constant Multiple Rule for Limits} \\ &= c \frac{du}{dx} && u \text{ is differentiable.} \end{aligned}$$

Applying the Power Rule

Subtract 1 from the exponent and multiply the result by the original exponent.

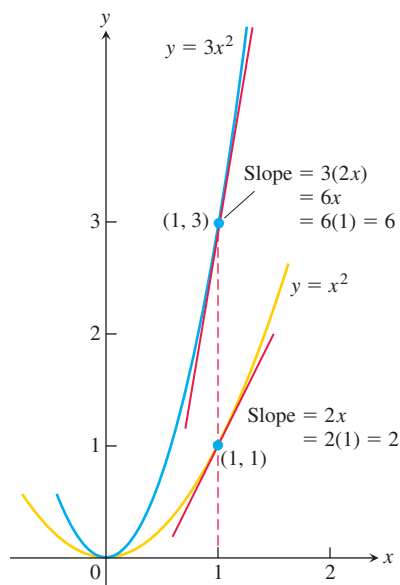


FIGURE 3.11 The graphs of $y = x^2$ and $y = 3x^2$. Tripling the y-coordinate triples the slope (Example 2).

EXAMPLE 2

(a) The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each y-coordinate by 3, then we multiply the slope at each point by 3 (Figure 3.11).

(b) Negative of a function

The derivative of the negative of a differentiable function u is the negative of the function's derivative. The Constant Multiple Rule with $c = -1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}.$$

The next rule says that the derivative of the sum of two differentiable functions is the sum of their derivatives.

Derivative Sum Rule

If u and v are differentiable functions of x , then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Denoting Functions by u and v

The functions we are working with when we need a differentiation formula are likely to be denoted by letters like f and g . We do not want to use these same letters when stating general differentiation rules, so instead we use letters like u and v that are not likely to be already in use.

Proof We apply the definition of the derivative to $f(x) = u(x) + v(x)$:

$$\begin{aligned} \frac{d}{dx}[u(x) + v(x)] &= \lim_{h \rightarrow 0} \frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h} \right] \\ &= \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} + \lim_{h \rightarrow 0} \frac{v(x+h) - v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}. \end{aligned}$$

Combining the Sum Rule with the Constant Multiple Rule gives the **Difference Rule**, which says that the derivative of a *difference* of differentiable functions is the difference of their derivatives:

$$\frac{d}{dx}(u - v) = \frac{d}{dx}[u + (-1)v] = \frac{du}{dx} + (-1)\frac{dv}{dx} = \frac{du}{dx} - \frac{dv}{dx}.$$

The Sum Rule also extends to finite sums of more than two functions. If $u_1, u_2, \dots, u_n$ are differentiable at x , then so is $u_1 + u_2 + \dots + u_n$, and

$$\frac{d}{dx}(u_1 + u_2 + \dots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \dots + \frac{du_n}{dx}.$$

A proof by mathematical induction for any finite number of terms is given in Appendix A.3.

EXAMPLE 3 Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

Solution $\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$ Sum and Difference Rules

$$= 3x^2 + \frac{4}{3} \cdot 2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5$$

We can differentiate any polynomial term by term, the way we differentiated the polynomial in Example 3. All polynomials are differentiable at all values of x .

EXAMPLE 4 Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangent lines? If so, where?

Solution The horizontal tangent lines, if any, occur where the slope dy/dx is zero. We have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x.$$

Now solve the equation $\frac{dy}{dx} = 0$ for x :

$$\begin{aligned} 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \\ x &= 0, 1, -1. \end{aligned}$$

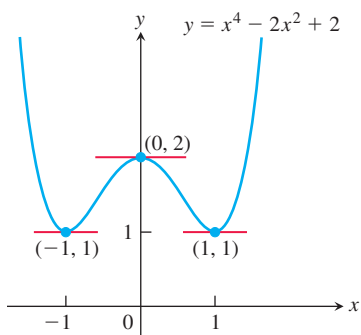


FIGURE 3.12 The curve in Example 4 and its horizontal tangent lines.

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangent lines at $x = 0, 1$, and -1 . The corresponding points on the curve are $(0, 2)$, $(1, 1)$, and $(-1, 1)$. See Figure 3.12.

Derivatives of Exponential Functions

We briefly reviewed exponential functions in Section 1.4. When we apply the definition of the derivative to $f(x) = a^x$, we get

$$\begin{aligned} \frac{d}{dx}(a^x) &= \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} && \text{Derivative definition} \\ &= \lim_{h \rightarrow 0} \frac{a^x \cdot a^h - a^x}{h} && a^{x+h} = a^x \cdot a^h \\ &= \lim_{h \rightarrow 0} a^x \cdot \frac{a^h - 1}{h} && \text{Factoring out } a^x \\ &= a^x \cdot \lim_{h \rightarrow 0} \frac{a^h - 1}{h} && a^x \text{ is constant as } h \rightarrow 0. \\ &= \left(\lim_{h \rightarrow 0} \frac{a^h - 1}{h} \right) \cdot a^x. && (1) \end{aligned}$$

a fixed number L

Thus we see that the derivative of a^x is a constant multiple L of a^x . The constant L is a limit we have not encountered before. Note, however, that it equals the derivative of $f(x) = a^x$ at $x = 0$:

$$f'(0) = \lim_{h \rightarrow 0} \frac{a^h - a^0}{h} = \lim_{h \rightarrow 0} \frac{a^h - 1}{h} = L.$$

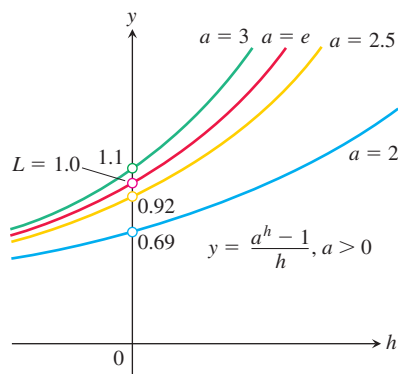


FIGURE 3.13 The position of the curve $y = (a^h - 1)/h$, $a > 0$, varies continuously with a . The limit L of y as $h \rightarrow 0$ changes with different values of a . The number for which $L = 1$ as $h \rightarrow 0$ is the number e between $a = 2$ and $a = 3$.

The limit L is therefore the slope of the graph of $f(x) = a^x$ where it crosses the y -axis. In Chapter 7, where we carefully develop the logarithmic and exponential functions, we prove that the limit L exists and has the value $\ln a$. For now we investigate values of L by graphing the function $y = (a^h - 1)/h$ and studying its behavior as h approaches 0.

Figure 3.13 shows the graphs of $y = (a^h - 1)/h$ for four different values of a . The limit L is approximately 0.69 if $a = 2$, about 0.92 if $a = 2.5$, and about 1.1 if $a = 3$. It appears that the value of L is 1 at some number a chosen between 2.5 and 3. That number is given by $a = e \approx 2.718281828$. With this choice of base we obtain the natural exponential function $f(x) = e^x$ as in Section 1.4, and see that it satisfies the property

$$f'(0) = \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1 \quad (2)$$

because it is the exponential function whose graph has slope 1 when it crosses the y -axis. That the limit is 1 implies an important relationship between the natural exponential function e^x and its derivative:

$$\begin{aligned} \frac{d}{dx}(e^x) &= \lim_{h \rightarrow 0} \left(\frac{e^h - 1}{h} \right) \cdot e^x && \text{Eq. (1) with } a = e \\ &= 1 \cdot e^x = e^x. && \text{Eq. (2)} \end{aligned}$$

Therefore the natural exponential function is its own derivative.

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

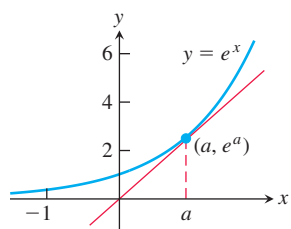


FIGURE 3.14 The line through the origin is tangent to the graph of $y = e^x$ when $a = 1$ (Example 5).

EXAMPLE 5 Find an equation for a line that is tangent to the graph of $y = e^x$ and goes through the origin.

Solution Since the line passes through the origin, its equation is of the form $y = mx$, where m is the slope. If it is tangent to the graph at the point (a, e^a) , the slope is $m = (e^a - 0)/(a - 0)$. The slope of the natural exponential at $x = a$ is e^a . Because these slopes are the same, we then have that $e^a = e^a/a$. It follows that $a = 1$ and $m = e$, so the equation of the tangent line is $y = ex$. See Figure 3.14. ■

We might ask if there are functions *other* than the natural exponential function that are their own derivatives. The answer is that the only functions that satisfy the property that $f'(x) = f(x)$ are functions that are constant multiples of the natural exponential function, $f(x) = c \cdot e^x$, c any constant. We prove this fact in Section 7.2. Note from the Constant Multiple Rule that indeed

$$\frac{d}{dx}(c \cdot e^x) = c \cdot \frac{d}{dx}(e^x) = c \cdot e^x.$$

Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is *not* the product of their derivatives. For instance,

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x, \quad \text{while} \quad \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1.$$

The derivative of a product of two functions is the sum of *two* products, as we now explain.

Derivative Product Rule

If u and v are differentiable at x , then so is their product uv , and

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + \frac{du}{dx} v.$$

The derivative of the product uv is u times the derivative of v plus the derivative of u times v . In prime notation, $(uv)' = uv' + u'v$. In function notation,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x), \quad \text{or} \quad (fg)' = fg' + f'g. \quad (3)$$

EXAMPLE 6 Find the derivative of (a) $y = \frac{1}{x}(x^2 + e^x)$, (b) $y = e^{2x}$.

Solution

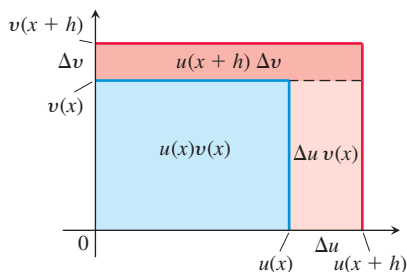
(a) We apply the Product Rule with $u = 1/x$ and $v = x^2 + e^x$:

$$\begin{aligned} \frac{d}{dx}\left[\frac{1}{x}(x^2 + e^x)\right] &= \frac{1}{x}(2x + e^x) + \left(-\frac{1}{x^2}\right)(x^2 + e^x) && \frac{d}{dx}(uv) = u \frac{dv}{dx} + \frac{du}{dx} v \text{ and} \\ &= 2 + \frac{e^x}{x} - 1 - \frac{e^x}{x^2} && \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2} \\ &= 1 + (x - 1) \frac{e^x}{x^2}. \end{aligned}$$

$$(b) \frac{d}{dx}(e^{2x}) = \frac{d}{dx}(e^x \cdot e^x) = e^x \cdot \frac{d}{dx}(e^x) + \frac{d}{dx}(e^x) \cdot e^x = 2e^x \cdot e^x = 2e^{2x} \quad \blacksquare$$

Picturing the Product Rule

Suppose $u(x)$ and $v(x)$ are positive and increase when x increases, and $h > 0$.



Then the change in the product uv is the difference in areas of the larger and smaller “boxes,” which is the sum of the areas of the upper and right-hand reddish-shaded rectangles. That is,

$$\begin{aligned} \Delta(uv) &= u(x+h)v(x+h) - u(x)v(x) \\ &= u(x+h)\Delta v + \Delta u v(x). \end{aligned}$$

Division by h gives

$$\frac{\Delta(uv)}{h} = u(x+h) \frac{\Delta v}{h} + \frac{\Delta u}{h} v(x).$$

The limit as $h \rightarrow 0^+$ yields the Product Rule.

Proof of the Derivative Product Rule

$$\frac{d}{dx}(uv) = \lim_{h \rightarrow 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

To change this fraction into an equivalent one that contains difference quotients for the derivatives of u and v , we subtract and add $u(x+h)v(x)$ in the numerator:

$$\begin{aligned} \frac{d}{dx}(uv) &= \lim_{h \rightarrow 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \\ &= \lim_{h \rightarrow 0} \left[u(x+h) \frac{v(x+h) - v(x)}{h} + \frac{u(x+h) - u(x)}{h} v(x) \right] \\ &= \lim_{h \rightarrow 0} u(x+h) \cdot \lim_{h \rightarrow 0} \frac{v(x+h) - v(x)}{h} + \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} \cdot v(x). \end{aligned}$$

As h approaches zero, $u(x+h)$ approaches $u(x)$ because u , being differentiable at x , is continuous at x . The two fractions approach the values of dv/dx at x and du/dx at x . Therefore,

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + \frac{du}{dx} v. \quad \blacksquare$$

The derivative of the quotient of two functions is given by the Quotient Rule.

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x , and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}.$$

In function notation,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

EXAMPLE 7 Find the derivative of (a) $y = \frac{t^2 - 1}{t^3 + 1}$, (b) $y = e^{-x}$.

Solution

(a) We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^3 + 1$:

$$\begin{aligned} \frac{dy}{dt} &= \frac{(t^3 + 1) \cdot 2t - (t^2 - 1) \cdot 3t^2}{(t^3 + 1)^2} & \frac{d}{dt}\left(\frac{u}{v}\right) &= \frac{v(du/dt) - u(dv/dt)}{v^2} \\ &= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2} \\ &= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}. \end{aligned}$$

$$(b) \frac{d}{dx}(e^{-x}) = \frac{d}{dx}\left(\frac{1}{e^x}\right) = \frac{e^x \cdot 0 - 1 \cdot e^x}{(e^x)^2} = \frac{-1}{e^x} = -e^{-x}$$

Proof of the Derivative Quotient Rule

$$\begin{aligned} \frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h \rightarrow 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{h v(x+h)v(x)} \end{aligned}$$

To change the last fraction into an equivalent one that contains the difference quotients for the derivatives of u and v , we subtract and add $v(x)u(x)$ in the numerator. We then get

$$\begin{aligned} \frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h \rightarrow 0} \frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{h v(x+h)v(x)} \\ &= \lim_{h \rightarrow 0} \frac{v(x) \frac{u(x+h) - u(x)}{h} - u(x) \frac{v(x+h) - v(x)}{h}}{v(x+h)v(x)}. \end{aligned}$$

Taking the limits in the numerator and denominator now gives the Quotient Rule. Exercise 76 outlines another proof. ■

The choice of which rules to use in solving a differentiation problem can make a difference in how much work you have to do. Here is an example.

HISTORICAL BIOGRAPHY

Maria Gaetana Agnesi
(1718–1799)

Agnesi was a well-published scientist by age 20 and an honorary faculty member of the University of Bologna by age 30. Today, Agnesi is remembered chiefly for a bell-shaped curve called the “Witch of Agnesi.”

To know more, visit the companion Website.

EXAMPLE 8 Find the derivative of

$$y = \frac{(x-1)(x^2-2x)}{x^4}.$$

Solution Using the Quotient Rule here will result in a complicated expression with many terms. Instead, use some algebra to simplify the expression. First expand the numerator and divide by x^4 :

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3-3x^2+2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the Sum, Constant Multiple, and Power Rules:

$$\begin{aligned}\frac{dy}{dx} &= -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4} \\ &= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.\end{aligned}$$

Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f'' . So $f'' = (f')'$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

The symbol D^2 means that the operation of differentiation is performed twice.

If $y = x^6$, then $y' = 6x^5$ and we have

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}(6x^5) = 30x^4.$$

Thus $D^2(x^6) = 30x^4$.

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$, is the **third derivative** of y with respect to x . The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^n y$$

denoting the **n th derivative** of y with respect to x for any positive integer n .

We can interpret the second derivative as the rate of change of the slope of the tangent line to the graph of $y = f(x)$ at each point. You will see in the next chapter that the second derivative reveals whether the graph bends upward or downward from the tangent line as we move off the point of tangency. In the next section, we interpret both the second and third derivatives in terms of motion along a straight line.

EXAMPLE 9 The first four derivatives of $y = x^3 - 3x^2 + 2$ are

$$\begin{aligned}\text{First derivative:} & \quad y' = 3x^2 - 6x \\ \text{Second derivative:} & \quad y'' = 6x - 6 \\ \text{Third derivative:} & \quad y''' = 6 \\ \text{Fourth derivative:} & \quad y^{(4)} = 0.\end{aligned}$$

All polynomial functions have derivatives of all orders. In this example, the fifth and later derivatives are all zero.

How to Read the Symbols for Derivatives

y' “y prime”

y'' “y double prime”

$\frac{d^2y}{dx^2}$ “d squared y by dx squared”

y''' “y triple prime”

$y^{(n)}$ “y super n”

$\frac{d^ny}{dx^n}$ “d to the n of y by dx to the n”

D^n “d to the n”

EXERCISES 3.3

Derivative Calculations

In Exercises 1–12, find the first and second derivatives.

1. $y = -x^2 + 3$
2. $y = x^2 + x + 8$
3. $s = 5t^3 - 3t^5$
4. $w = 3z^7 - 7z^3 + 21z^2$
5. $y = \frac{4x^3}{3} - x + 2e^x$
6. $y = \frac{x^3}{3} + \frac{x^2}{2} + e^{-x}$
7. $w = 3z^{-2} - \frac{1}{z}$
8. $s = -2t^{-1} + \frac{4}{t^2}$
9. $y = 6x^2 - 10x - 5x^{-2}$
10. $y = 4 - 2x - x^{-3}$
11. $r = \frac{1}{3s^2} - \frac{5}{2s}$
12. $r = \frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$

In Exercises 13–16, find y' (a) by applying the Product Rule and (b) by multiplying the factors to produce a sum of simpler terms to differentiate.

13. $y = (3 - x^2)(x^3 - x + 1)$
14. $y = (2x + 3)(5x^2 - 4x)$
15. $y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right)$
16. $y = (1 + x^2)(x^{3/4} - x^{-3})$

Find the derivatives of the functions in Exercises 17–40.

17. $y = \frac{2x + 5}{3x - 2}$
18. $z = \frac{4 - 3x}{3x^2 + x}$
19. $g(x) = \frac{x^2 - 4}{x + 0.5}$
20. $f(t) = \frac{t^2 - 1}{t^2 + t - 2}$
21. $v = (1 - t)(1 + t^2)^{-1}$
22. $w = (2x - 7)^{-1}(x + 5)$
23. $f(s) = \frac{\sqrt{s} - 1}{\sqrt{s} + 1}$
24. $u = \frac{5x + 1}{2\sqrt{x}}$
25. $v = \frac{1 + x - 4\sqrt{x}}{x}$
26. $r = 2\left(\frac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$
27. $y = \frac{1}{(x^2 - 1)(x^2 + x + 1)}$
28. $y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$
29. $y = 2e^{-x} + e^{3x}$
30. $y = \frac{x^2 + 3e^x}{2e^x - x}$
31. $y = x^3e^x$
32. $w = re^{-r}$
33. $y = x^{9/4} + e^{-2x}$
34. $y = x^{-3/5} + \pi^{3/2}$
35. $s = 2t^{3/2} + 3e^2$
36. $w = \frac{1}{z^{1.4}} + \frac{\pi}{\sqrt{z}}$
37. $y = \sqrt[3]{x^2} - x^e$
38. $y = \sqrt[3]{x^{9.6}} + 2e^{1.3}$
39. $r = \frac{e^s}{s}$
40. $r = e^{\theta}\left(\frac{1}{\theta^2} + \theta^{-\pi/2}\right)$

Find the derivatives of all orders of the functions in Exercises 41–44.

41. $y = \frac{x^4}{2} - \frac{3}{2}x^2 - x$
42. $y = \frac{x^5}{120}$
43. $y = (x - 1)(x + 2)(x + 3)$
44. $y = (4x^2 + 3)(2 - x)x$

Find the first and second derivatives of the functions in Exercises 45–52.

45. $y = \frac{x^3 + 7}{x}$
46. $s = \frac{t^2 + 5t - 1}{t^2}$
47. $r = \frac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$
48. $u = \frac{(x^2 + x)(x^2 - x + 1)}{x^4}$

$$49. w = \left(\frac{1 + 3z}{3z}\right)(3 - z) \quad 50. p = \frac{q^2 + 3}{(q - 1)^3 + (q + 1)^3}$$

$$51. w = 3z^2e^{2z} \quad 52. w = e^z(z - 1)(z^2 + 1)$$

53. Suppose u and v are functions of x that are differentiable at $x = 0$ and that

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

Find the values of the following derivatives at $x = 0$.

$$\text{a. } \frac{d}{dx}(uv) \quad \text{b. } \frac{d}{dx}\left(\frac{u}{v}\right) \quad \text{c. } \frac{d}{dx}\left(\frac{v}{u}\right) \quad \text{d. } \frac{d}{dx}(7v - 2u)$$

54. Suppose u and v are differentiable functions of x and that

$$u(1) = 2, \quad u'(1) = 0, \quad v(1) = 5, \quad v'(1) = -1.$$

Find the values of the following derivatives at $x = 1$.

$$\text{a. } \frac{d}{dx}(uv) \quad \text{b. } \frac{d}{dx}\left(\frac{u}{v}\right) \quad \text{c. } \frac{d}{dx}\left(\frac{v}{u}\right) \quad \text{d. } \frac{d}{dx}(7v - 2u)$$

Slopes and Tangent Lines

55. **a. Normal line to a curve** Find an equation for the line perpendicular to the tangent line to the curve $y = x^3 - 4x + 1$ at the point $(2, 1)$.

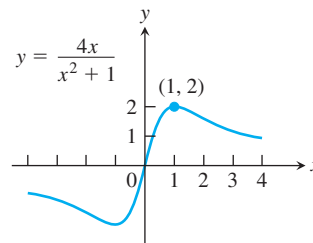
b. Smallest slope What is the smallest slope on the curve? At what point on the curve does the curve have this slope?

c. Tangent lines having specified slope Find equations for the tangent lines to the curve at the points where the slope of the curve is 8.

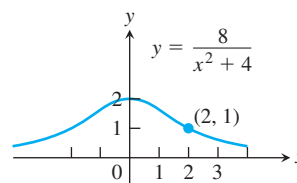
56. **a. Horizontal tangent lines** Find equations for the horizontal tangent lines to the curve $y = x^3 - 3x - 2$. Also find equations for the lines that are perpendicular to these tangent lines at the points of tangency.

b. Smallest slope What is the smallest slope on the curve? At what point on the curve does the curve have this slope? Find an equation for the line that is perpendicular to the curve's tangent line at this point.

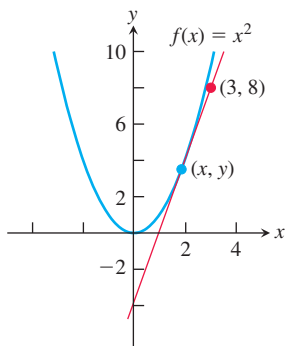
57. Find the tangent lines to *Newton's serpentine* (graphed here) at the origin and the point $(1, 2)$.



58. Find the tangent line to the *Witch of Agnesi* (graphed here) at the point $(2, 1)$.



- 59. Quadratic tangent to identity function** The curve $y = ax^2 + bx + c$ passes through the point $(1, 2)$ and is tangent to the line $y = x$ at the origin. Find a , b , and c .
- 60. Quadratics having a common tangent line** The curves $y = x^2 + ax + b$ and $y = cx - x^2$ have a common tangent line at the point $(1, 0)$. Find a , b , and c .
- 61.** Find all points (x, y) on the graph of $f(x) = 3x^2 - 4x$ with tangent lines parallel to the line $y = 8x + 5$.
- 62.** Find all points (x, y) on the graph of $g(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 1$ with tangent lines parallel to the line $8x - 2y = 1$.
- 63.** Find all points (x, y) on the graph of $y = x/(x - 2)$ with tangent lines perpendicular to the line $y = 2x + 3$.
- 64.** Find all points (x, y) on the graph of $f(x) = x^2$ with tangent lines passing through the point $(3, 8)$.



- 65.** Assume that functions f and g are differentiable with $f(1) = 2$, $f'(1) = -3$, $g(1) = 4$, and $g'(1) = -2$. Find the equation of the line tangent to the graph of $F(x) = f(x)g(x)$ at $x = 1$.
- 66.** Assume that functions f and g are differentiable with $f(2) = 3$, $f'(2) = -1$, $g(2) = -4$, and $g'(2) = 1$. Find an equation of the line perpendicular to the line tangent to the graph of $F(x) = \frac{f(x) + 3}{x - g(x)}$ at $x = 2$.
- 67. a.** Find an equation for the line that is tangent to the curve $y = x^3 - x$ at the point $(-1, 0)$.
- T b.** Graph the curve and tangent line together. The tangent line intersects the curve at another point. Use Zoom and Trace to estimate the point's coordinates.
- T c.** Confirm your estimates of the coordinates of the second intersection point by solving the equations for the curve and tangent line simultaneously.
- 68. a.** Find an equation for the line that is tangent to the curve $y = x^3 - 6x^2 + 5x$ at the origin.
- T b.** Graph the curve and tangent line together. The tangent line intersects the curve at another point. Use Zoom and Trace to estimate the point's coordinates.
- T c.** Confirm your estimates of the coordinates of the second intersection point by solving the equations for the curve and tangent line simultaneously.

Theory and Examples

For Exercises 69 and 70, evaluate each limit by first converting each to a derivative at a particular x -value.

69. $\lim_{x \rightarrow 1} \frac{x^{50} - 1}{x - 1}$

70. $\lim_{x \rightarrow -1} \frac{x^{2/9} - 1}{x + 1}$

- 71.** Find the value of a that makes the following function differentiable for all x -values.

$$g(x) = \begin{cases} ax, & \text{if } x < 0 \\ x^2 - 3x, & \text{if } x \geq 0 \end{cases}$$

- 72.** Find the values of a and b that make the following function differentiable for all x -values.

$$f(x) = \begin{cases} ax + b, & x > -1 \\ bx^2 - 3, & x \leq -1 \end{cases}$$

- 73.** The general polynomial of degree n has the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $a_n \neq 0$. Find $P'(x)$.

- 74. The body's reaction to medicine** The reaction of the body to a dose of medicine can sometimes be represented by an equation of the form

$$R = M^2 \left(\frac{C}{2} - \frac{M}{3} \right),$$

where C is a positive constant and M is the amount of medicine absorbed in the blood. If the reaction is a change in blood pressure, R is measured in millimeters of mercury. If the reaction is a change in temperature, R is measured in degrees, and so on.

Find dR/dM . This derivative, as a function of M , is called the sensitivity of the body to the medicine. In Section 4.6, we will see how to find the amount of medicine to which the body is most sensitive.

- 75.** Suppose that the function v in the Derivative Product Rule has a constant value c . What does the Derivative Product Rule then say? What does this say about the Derivative Constant Multiple Rule?

76. The Reciprocal Rule

- a.** The *Reciprocal Rule* says that at any point where the function $v(x)$ is differentiable and different from zero,

$$\frac{d}{dx} \left(\frac{1}{v} \right) = -\frac{1}{v^2} \frac{dv}{dx}.$$

Show that the Reciprocal Rule is a special case of the Derivative Quotient Rule.

- b.** Show that the Reciprocal Rule and the Derivative Product Rule together imply the Derivative Quotient Rule.

- 77. Generalizing the Product Rule** The Derivative Product Rule gives the formula

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + \frac{du}{dx} v$$

for the derivative of the product uv of two differentiable functions of x .

- a.** What is the analogous formula for the derivative of the product uvw of *three* differentiable functions of x ?
- b.** What is the formula for the derivative of the product $u_1 u_2 u_3 u_4$ of *four* differentiable functions of x ?
- c.** What is the formula for the derivative of a product $u_1 u_2 u_3 \cdots u_n$ of a finite number n of differentiable functions of x ?

- 78. Power Rule for negative integers** Use the Derivative Quotient Rule to prove the Power Rule for negative integers, that is,

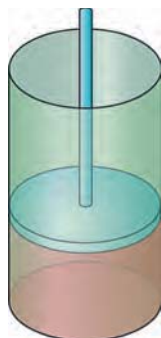
$$\frac{d}{dx}(x^{-m}) = -mx^{-m-1}$$

where m is a positive integer.

- 79. Cylinder pressure** If gas in a cylinder is maintained at a constant temperature T , the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$

in which a , b , n , and R are constants. Find dP/dV . (See accompanying figure.)



- 80. The best quantity to order** One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, TVs, brooms, or whatever the item might be); k is the cost of placing an order (the same, no matter how often you order); c is the cost of one item (a constant); m is the number of items sold each week (a constant); and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security). Find dA/dq and d^2A/dq^2 .

3.4 The Derivative as a Rate of Change

In this section we study applications where derivatives model the rates at which things change. It is natural to think of a quantity changing with respect to time, but other variables can be treated in the same way. For example, an economist may want to study how the cost of producing steel varies with the number of tons produced, or an engineer may want to know how the power output of a generator varies with its temperature.

Instantaneous Rates of Change

If we interpret the difference quotient $(f(x+h) - f(x))/h$ as the average rate of change in f over the interval from x to $x+h$, we can interpret its limit as $h \rightarrow 0$ as the instantaneous rate at which f is changing at the point x . This gives an important interpretation of the derivative.

DEFINITION The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

Thus, instantaneous rates are limits of average rates.

It is conventional to use the word *instantaneous* even when x does not represent time. The word is, however, frequently omitted. When we say *rate of change*, we mean *instantaneous rate of change*.

EXAMPLE 1 The area A of a circle is related to its diameter by the equation

$$A = \frac{\pi}{4}D^2.$$

How fast does the area change with respect to the diameter when the diameter is 10 m?

Solution The rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4} \cdot 2D = \frac{\pi D}{2}.$$

When $D = 10$ m, the area is changing with respect to the diameter at the rate of $(\pi/2)10 = 5\pi \text{ m}^2/\text{m} \approx 15.71 \text{ m}^2/\text{m}$. ■

Motion Along a Line: Displacement, Velocity, Speed, Acceleration, and Jerk

Suppose that an object (or body, considered as a whole mass) is moving along a coordinate line (an s -axis), usually horizontal or vertical, so that we know its position s on that line as a function of time t :

$$s = f(t).$$

The **displacement** of the object over the time interval from t to $t + \Delta t$ (Figure 3.15) is

$$\Delta s = f(t + \Delta t) - f(t),$$

and the **average velocity** of the object over that time interval is

$$v_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

To find the body's velocity at the exact instant t , we take the limit of the average velocity over the interval from t to $t + \Delta t$ as Δt approaches zero. This limit is the derivative of f with respect to t .

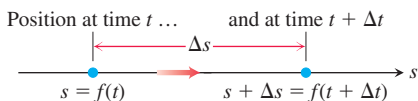
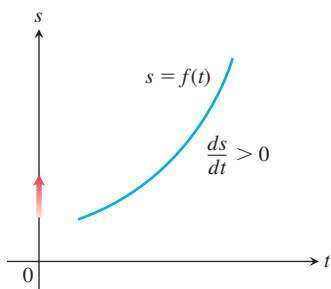
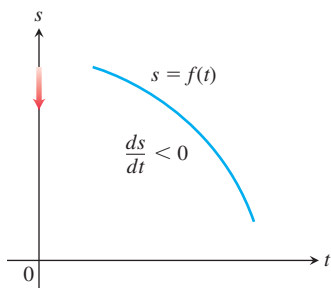


FIGURE 3.15 The positions of a body moving along a coordinate line at time t and shortly later at time $t + \Delta t$. Here the coordinate line is horizontal.



(a) s increasing:
positive slope so
moving upward



(b) s decreasing:
negative slope so
moving downward

FIGURE 3.16 For motion $s = f(t)$ along a straight line (the vertical axis), $v = ds/dt$ is (a) positive when s increases and (b) negative when s decreases.

DEFINITION Velocity (instantaneous velocity) is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

Besides telling how fast an object is moving along the horizontal line in Figure 3.15, its velocity tells the direction of motion. When the object is moving forward (s increasing), the velocity is positive; when the object is moving backward (s decreasing), the velocity is negative. If the coordinate line is vertical, the object moves upward for positive velocity and downward for negative velocity. The blue curves in Figure 3.16 represent position along the line over time; they do not portray the path of motion, which lies along the vertical s -axis.

If we drive to a friend's house and back at 50 km/h, say, the speedometer will show 50 on the way over but it will not show -50 on the way back, even though our distance from home is decreasing. The speedometer always shows *speed*, which is the absolute value of velocity. Speed measures the rate of progress regardless of direction.

DEFINITION Speed is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left| \frac{ds}{dt} \right|$$

EXAMPLE 2 Figure 3.17 shows the graph of the velocity $v = f'(t)$ of a particle moving along a horizontal line as in Figure 3.15 (as opposed to the graph of a position function $s = f(t)$, such as in Figure 3.16). In the graph of the velocity function, it's not the

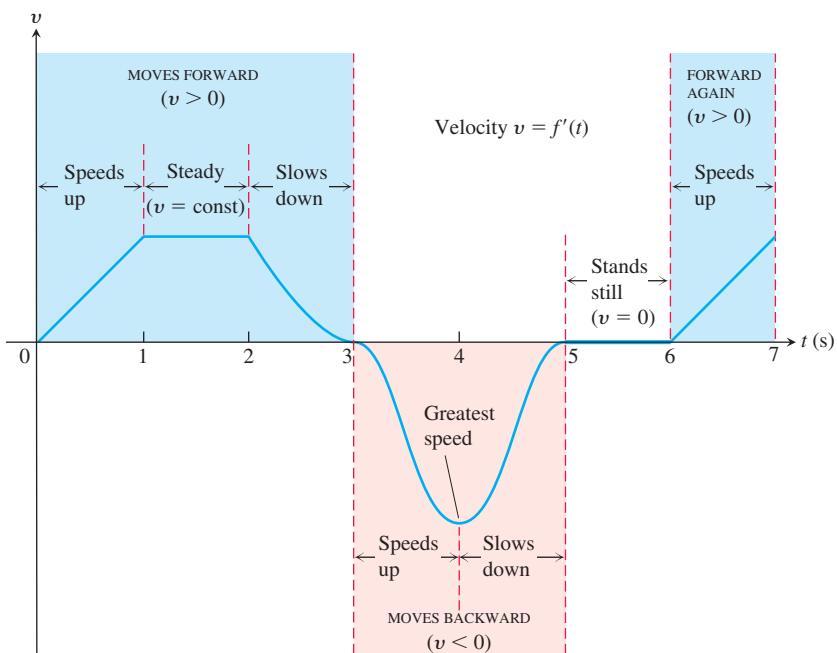


FIGURE 3.17 The velocity graph of a particle moving along a horizontal line, discussed in Example 2.

slope of the curve that tells us whether the particle is moving forward or backward along the line (which is not shown in the figure), but rather the sign of the velocity. Figure 3.17 shows that the particle moves forward for the first 3 s (when the velocity is positive), moves backward for the next 2 s (the velocity is negative), stands motionless for a full second, and then moves forward again. The particle is speeding up when its positive velocity increases during the first second, moves at a steady speed during the next second, and then slows down as the velocity decreases to zero during the third second. It stops for an instant at $t = 3$ s (when the velocity is zero) and reverses direction as the velocity starts to become negative. The particle is now moving backward and gaining in speed until $t = 4$ s, at which time it achieves its greatest speed during its backward motion. Continuing its backward motion at time $t = 4$, the particle starts to slow down again until it finally stops at time $t = 5$ (when the velocity is once again zero). The particle now remains motionless for one full second, and then moves forward again at $t = 6$ s, speeding up during the final second of the forward motion indicated in the velocity graph. ■

HISTORICAL BIOGRAPHY

Bernard Bolzano
(1781–1848)

Bolzano was born in Prague, Czechoslovakia. He studied at the University of Prague, where he took courses in philosophy, physics, and mathematics.

To know more, visit the companion Website.

The rate at which a body's velocity changes is the body's *acceleration*. The acceleration measures how quickly the body picks up or loses speed. In Chapter 12 we will study motion in the plane and in space, where acceleration of an object may also lead to a change in direction.

A sudden change in acceleration is called a *jerk*. When a ride in a car or a bus is jerky, it is not that the accelerations involved are necessarily large but that the changes in acceleration are abrupt.

DEFINITIONS **Acceleration** is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Near the surface of Earth, all bodies fall with the same constant acceleration. Galileo's experiments with free fall (see Section 2.1) lead to the equation

$$s = \frac{1}{2}gt^2,$$

where s is the distance fallen and g is the acceleration due to Earth's gravity. This equation holds in a vacuum, where there is no air resistance, and closely models the fall of dense, heavy objects, such as rocks or steel tools, for the first few seconds of their fall, before the effects of air resistance are significant.

The value of g in the equation $s = (1/2)gt^2$ depends on the units used to measure t and s . With t in seconds (the usual unit), the value of g determined by measurement at sea level is approximately 9.8 m/s^2 (meters per second squared) in metric units. (This gravitational constant depends on the distance from Earth's center of mass, and is slightly lower on top of Mt. Everest, for example.)

The jerk associated with the constant acceleration of gravity ($g = 9.8 \text{ m/s}^2$) is zero:

$$j = \frac{d}{dt}(g) = 0.$$

An object does not exhibit jerkiness during free fall.

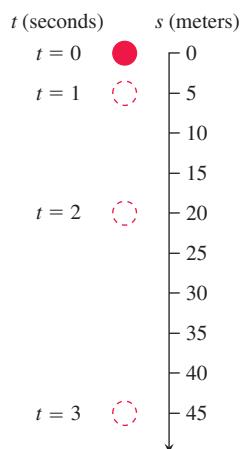


FIGURE 3.18 A ball bearing falling from rest (Example 3).

EXAMPLE 3 Figure 3.18 shows the free fall of a heavy ball bearing released from rest at time $t = 0 \text{ s}$.

- (a) How many meters does the ball fall in the first 3 s?
- (b) What are its velocity, speed, and acceleration when $t = 3$?

Solution

- (a) The metric free-fall equation is $s = 4.9t^2$. During the first 3 s, the ball falls

$$s(3) = 4.9(3)^2 = 44.1 \text{ m}.$$

- (b) At any time t , *velocity* is the derivative of position:

$$v(t) = s'(t) = \frac{d}{dt}(4.9t^2) = 9.8t.$$

At $t = 3$, the velocity is

$$v(3) = 29.4 \text{ m/s}$$

in the downward (increasing s) direction. The *speed* at $t = 3$ is

$$\text{speed} = |v(3)| = 29.4 \text{ m/s}.$$

The *acceleration* at any time t is

$$a(t) = v'(t) = s''(t) = 9.8 \text{ m/s}^2.$$

At $t = 3$, the acceleration is 9.8 m/s^2 . ■

EXAMPLE 4 A dynamite blast blows a heavy rock straight up with a launch velocity of 49 m/s (176.4 km/h) (Figure 3.19a). It reaches a height of $s = 49t - 4.9t^2 \text{ m}$ after t seconds.

- (a) How high does the rock go?
- (b) What are the velocity and speed of the rock when it is 78.4 m above the ground on the way up? On the way down?
- (c) What is the acceleration of the rock at any time t during its flight (after the blast)?
- (d) When does the rock hit the ground again?

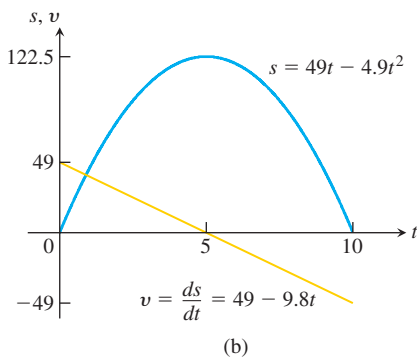
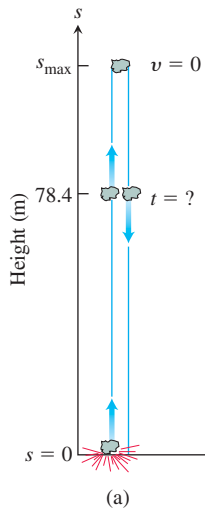


FIGURE 3.19 (a) The rock in Example 4. (b) The graphs of s and v as functions of time; s is largest when $v = ds/dt = 0$. The graph of s is *not* the path of the rock: It is a plot of height versus time. The slope of the plot is the rock's velocity, graphed here as a straight line.

Solution

- (a) In the coordinate system we have chosen, s measures height from the ground up, so the velocity is positive on the way up and negative on the way down. The instant the rock is at its highest point is the one instant during the flight when the velocity is 0. To find the maximum height, all we need to do is to find when $v = 0$ and evaluate s at this time.

At any time t during the rock's motion, its velocity is

$$v = \frac{ds}{dt} = \frac{d}{dt}(49t - 4.9t^2) = 49 - 9.8t \text{ m/s.}$$

The velocity is zero when

$$49 - 9.8t = 0 \quad \text{or} \quad t = 5 \text{ s.}$$

The rock's height at $t = 5$ s is

$$s_{\max} = s(5) = 49(5) - 4.9(5)^2 = 245 - 122.5 = 122.5 \text{ m.}$$

See Figure 3.19b.

- (b) To find the rock's velocity at 78.4 m on the way up and again on the way down, we first find the two values of t for which

$$s(t) = 49t - 4.9t^2 = 78.4.$$

To solve this equation, we write

$$4.9t^2 - 49t + 78.4 = 0$$

$$4.9(t^2 - 10t + 16) = 0$$

$$(t - 2)(t - 8) = 0$$

$$t = 2 \text{ s, } t = 8 \text{ s.}$$

The rock is 78.4 m above the ground 2 s after the explosion and again 8 s after the explosion. The rock's velocities at these times are

$$v(2) = 49 - 9.8(2) = 49 - 19.6 = 29.4 \text{ m/s.}$$

$$v(8) = 49 - 9.8(8) = 49 - 78.4 = -29.4 \text{ m/s.}$$

At both instants, the rock's speed is 29.4 m/s. Since $v(2) > 0$, the rock is moving upward (s is increasing) at $t = 2$ s; it is moving downward (s is decreasing) at $t = 8$ s because $v(8) < 0$.

- (c) At any time during its flight following the explosion, the rock's acceleration is a constant

$$a = \frac{dv}{dt} = \frac{d}{dt}(49 - 9.8t) = -9.8 \text{ m/s}^2.$$

The acceleration is always downward and is the effect of gravity on the rock. As the rock rises, it slows down; as it falls, it speeds up.

- (d) The rock hits the ground at the positive time t for which $s = 0$. The equation $49t - 4.9t^2 = 0$ factors to give $4.9t(10 - t) = 0$, so it has solutions $t = 0$ and $t = 10$. At $t = 0$, the blast occurred and the rock was thrown upward. It returns to the ground 10 s later. ■

Derivatives in Economics and Biology

Economists have a specialized vocabulary for rates of change and derivatives. They call them *marginals*. In a manufacturing operation, the *cost of production* $c(x)$ is a function of x , the number of units produced. The **marginal cost of production** is the rate of change of cost with respect to level of production, so it is dc/dx .

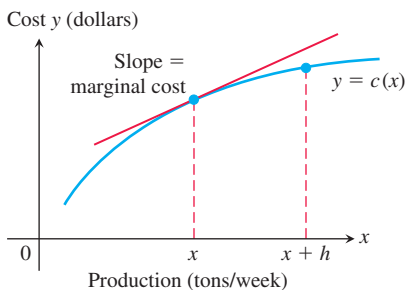


FIGURE 3.20 Weekly steel production: $c(x)$ is the cost of producing x tons per week. The cost of producing an additional h tons is $c(x+h) - c(x)$.

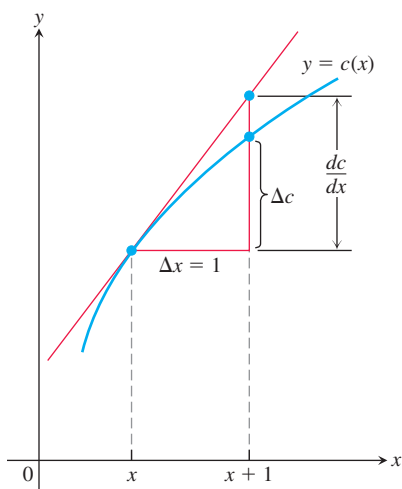


FIGURE 3.21 The marginal cost dc/dx is approximately the extra cost Δc of producing $\Delta x = 1$ more unit.

Suppose that $c(x)$ represents the dollars needed to produce x tons of steel in one week. It costs more to produce $x+h$ tons per week, and the cost difference, divided by h , is the average cost of producing each additional ton:

$$\frac{c(x+h) - c(x)}{h} = \text{average cost of each of the additional } h \text{ tons of steel produced.}$$

The limit of this ratio as $h \rightarrow 0$ is the *marginal cost* of producing more steel per week when the current weekly production is x tons (Figure 3.20):

$$\frac{dc}{dx} = \lim_{h \rightarrow 0} \frac{c(x+h) - c(x)}{h} = \text{marginal cost of production.}$$

Sometimes the marginal cost of production is loosely defined to be the extra cost of producing one additional unit:

$$\frac{\Delta c}{\Delta x} = \frac{c(x+1) - c(x)}{1},$$

which is approximated by the value of dc/dx at x . This approximation is acceptable if the slope of the graph of c does not change quickly near x . Then the difference quotient will be close to its limit dc/dx , which is the rise in the tangent line if $\Delta x = 1$ (Figure 3.21). The approximation often works well for large values of x .

Economists often represent a total cost function by a cubic polynomial

$$c(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta,$$

where δ represents *fixed costs*, such as rent, heat, equipment capitalization, and management costs. The other terms represent *variable costs*, such as the costs of raw materials, taxes, and labor. Fixed costs are independent of the number of units produced, whereas variable costs depend on the quantity produced. A cubic polynomial is usually adequate to capture the cost behavior on a realistic quantity interval.

EXAMPLE 5 Suppose that it costs

$$c(x) = x^3 - 6x^2 + 15x$$

dollars to produce x radiators when 8 to 30 radiators are produced and that

$$r(x) = x^3 - 3x^2 + 12x$$

gives the dollar revenue from selling x radiators. Your shop currently produces 10 radiators a day. About how much extra will it cost to produce one more radiator a day, and what is your estimated increase in revenue and increase in profit for selling 11 radiators a day?

Solution The cost of producing one more radiator a day when 10 are produced is about $c'(10)$:

$$\begin{aligned} c'(x) &= \frac{d}{dx}(x^3 - 6x^2 + 15x) = 3x^2 - 12x + 15 \\ c'(10) &= 3(100) - 12(10) + 15 = 195. \end{aligned}$$

The additional cost will be about \$195. The marginal revenue is

$$r'(x) = \frac{d}{dx}(x^3 - 3x^2 + 12x) = 3x^2 - 6x + 12.$$

The marginal revenue function estimates the increase in revenue that will result from selling one additional unit. If you currently sell 10 radiators a day, you can expect your revenue to increase by about

$$r'(10) = 3(100) - 6(10) + 12 = \$252$$

if you increase sales to 11 radiators a day. The estimated increase in profit is obtained by subtracting the increased cost of \$195 from the increased revenue, leading to an estimated profit increase of $\$252 - \$195 = \$57$. ■

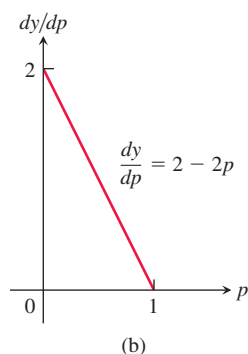
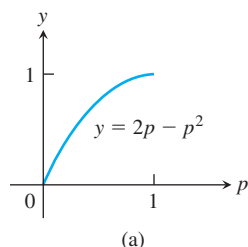


FIGURE 3.22 (a) The graph of $y = 2p - p^2$, describing the proportion of smooth-skin peas in the next generation. (b) The graph of dy/dp (Example 7).

EXAMPLE 6 Marginal rates frequently arise in discussions of tax rates. If your marginal income tax rate is 28% and your income increases by \$1000, you can expect to pay an extra \$280 in taxes. This does not mean that you pay 28% of your entire income in taxes. It just means that at your current income level I , the rate of increase of taxes T with respect to income is $dT/dI = 0.28$. You will pay \$0.28 in taxes out of every extra dollar you earn. As your income increases, you may land in a higher tax bracket, and your marginal rate will increase. ■

Sensitivity to Change

When a small change in x produces a large change in the value of a function $f(x)$, we say that the function is sensitive to changes in x . The derivative $f'(x)$ is a measure of this sensitivity. The function is more sensitive when $|f'(x)|$ is larger (when the slope of the graph of f is steeper).

EXAMPLE 7 Genetic Data and Sensitivity to Change

The Austrian monk Gregor Johann Mendel (1822–1884), working with garden peas and other plants, provided the first scientific explanation of hybridization.

His careful records showed that if p (a number between 0 and 1) is the frequency of the gene for smooth skin in peas (dominant) and $(1 - p)$ is the frequency of the gene for wrinkled skin in peas, then the proportion of smooth-skin peas in the next generation will be

$$y = 2p(1 - p) + p^2 = 2p - p^2.$$

The graph of y versus p in Figure 3.22a suggests that the value of y is more sensitive to a change in p when p is small than when p is large. Indeed, this fact is borne out by the derivative graph in Figure 3.22b, which shows that dy/dp is close to 2 when p is near 0 and close to 0 when p is near 1.

The implication for genetics is that introducing a few more smooth skin genes into a population where the frequency of wrinkled-skin peas is large will have a more dramatic effect on later generations than will a similar increase when the population has a large proportion of smooth-skin peas. ■

EXERCISES 3.4

Motion Along a Coordinate Line

Exercises 1–6 give the positions $s = f(t)$ of a body moving on a coordinate line, with s in meters and t in seconds.

- Find the body's displacement and average velocity for the given time interval.
- Find the body's speed and acceleration at the endpoints of the interval.
- When, if ever, during the interval does the body change direction?

1. $s = t^2 - 3t + 2$, $0 \leq t \leq 2$

2. $s = 6t - t^2$, $0 \leq t \leq 6$

3. $s = -t^3 + 3t^2 - 3t$, $0 \leq t \leq 3$

4. $s = (t^4/4) - t^3 + t^2$, $0 \leq t \leq 3$

5. $s = \frac{25}{t^2} - \frac{5}{t}$, $1 \leq t \leq 5$ 6. $s = \frac{25}{t+5}$, $-4 \leq t \leq 0$

7. **Particle motion** At time t , the position of a body moving along the s -axis is $s = t^3 - 6t^2 + 9t$ m.

- Find the body's acceleration each time the velocity is zero.
- Find the body's speed each time the acceleration is zero.
- Find the total distance traveled by the body from $t = 0$ to $t = 2$.

8. **Particle motion** At time $t \geq 0$, the velocity of a body moving along the horizontal s -axis is $v = t^2 - 4t + 3$.

- Find the body's acceleration each time the velocity is zero.
- When is the body moving forward? Backward?
- When is the body's velocity increasing? Decreasing?

Free-Fall Applications

9. **Free fall on Mars and Jupiter** The equations for free fall at the surfaces of Mars and Jupiter (s in meters, t in seconds) are $s = 1.86t^2$ on Mars and $s = 11.44t^2$ on Jupiter. How long does it take a rock falling from rest to reach a velocity of 27.8 m/s (about 100 km/h) on each planet?

10. **Lunar projectile motion** A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/s (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ m in t s.

- Find the rock's velocity and acceleration at time t . (The acceleration in this case is the acceleration of gravity on the moon.)
- How long does it take the rock to reach its highest point?
- How high does the rock go?
- How long does it take the rock to reach half its maximum height?
- How long is the rock aloft?

11. Finding g on a small airless planet Explorers on a small airless planet used a spring gun to launch a ball bearing vertically upward from the surface at a launch velocity of 15 m/s. Because the acceleration of gravity at the planet's surface was g_s m/s², the explorers expected the ball bearing to reach a height of $s = 15t - (1/2)g_s t^2$ m t s later. The ball bearing reached its maximum height 20 s after being launched. What was the value of g_s ?

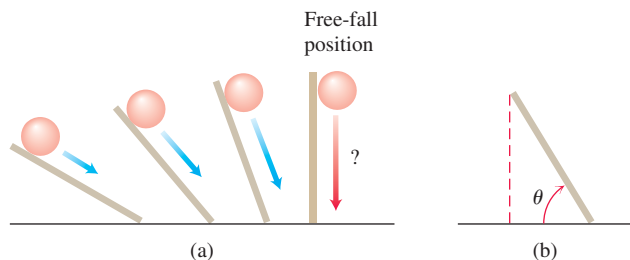
12. Speeding bullet A 45-caliber bullet shot straight up from the surface of the moon would reach a height of $s = 250t - 0.8t^2$ m after t seconds. On Earth, in the absence of air, its height would be $s = 250t - 4.9t^2$ m after t seconds. How long will the bullet be aloft in each case? How high will the bullet go?

13. Free fall from the Tower of Pisa Had Galileo dropped a cannonball from the Tower of Pisa, 56 m above the ground, the ball's height above the ground t seconds into the fall would have been $s = 56 - 4.9t^2$.

- What would have been the ball's velocity, speed, and acceleration at time t ?
- About how long would it have taken the ball to hit the ground?
- What would have been the ball's velocity at the moment of impact?

14. Galileo's free-fall formula Galileo developed a formula for a body's velocity during free fall by rolling balls from rest down increasingly steep inclined planks and looking for a limiting formula that would predict a ball's behavior when the plank was vertical and the ball fell freely; see part (a) of the accompanying figure. He found that, for any given angle of the plank, the ball's velocity t seconds into motion was a constant multiple of t . That is, the velocity was given by a formula of the form $v = kt$. The value of the constant k depended on the inclination of the plank.

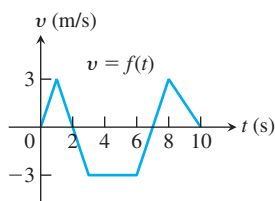
In modern notation—part (b) of the figure—with distance in meters and time in seconds, what Galileo determined by experiment was that, for any given angle θ , the ball's velocity t s into the roll was $v = 9.8(\sin \theta)t$ m/s.



- What is the equation for the ball's velocity during free fall?
- Building on your work in part (a), what constant acceleration does a freely falling body experience near the surface of Earth?

Understanding Motion from Graphs

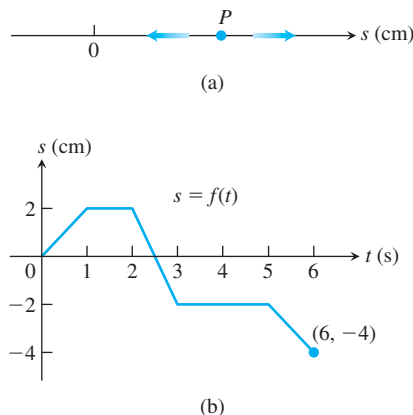
15. The accompanying figure shows the velocity $v = ds/dt = f(t)$ (m/s) of a body moving along a coordinate line.



- When does the body reverse direction?

- When (approximately) is the body moving at a constant speed?
- Graph the body's speed for $0 \leq t \leq 10$.
- Graph the acceleration, where defined.

16. A particle P moves on the number line shown in part (a) of the accompanying figure. Part (b) shows the position of P as a function of time t .

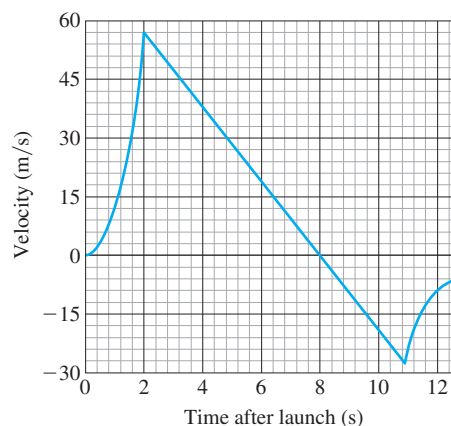


- When is P moving to the left? Moving to the right? Standing still?
- Graph the particle's velocity and speed (where defined).

17. Launching a rocket When a model rocket is launched, the propellant burns for a few seconds, accelerating the rocket upward. After burnout, the rocket coasts upward for a while and then begins to fall. A small explosive charge pops out a parachute shortly after the rocket starts down. The parachute slows the rocket to keep it from breaking when it lands.

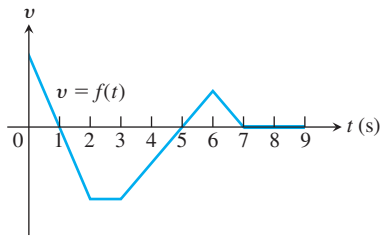
The figure here shows velocity data from the flight of the model rocket. Use the data to answer the following.

- How fast was the rocket climbing when the engine stopped?
- For how many seconds did the engine burn?

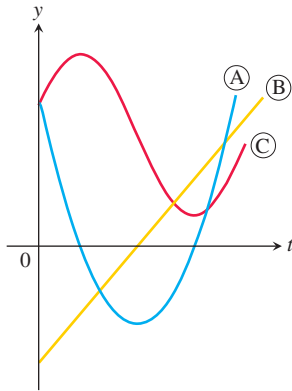


- When did the rocket reach its highest point? What was its velocity then?
- When did the parachute pop out? How fast was the rocket falling then?
- How long did the rocket fall before the parachute opened?
- When was the rocket's acceleration greatest?
- When was the acceleration constant? What was its value then (to the nearest integer)?

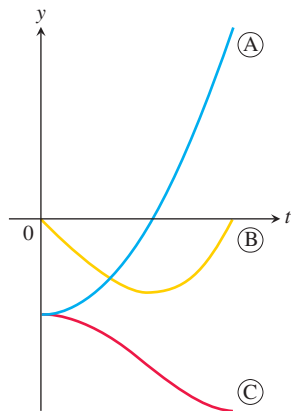
18. The accompanying figure shows the velocity $v = f(t)$ of a particle moving on a horizontal coordinate line.



- When does the particle move forward? Move backward? Speed up? Slow down?
 - When is the particle's acceleration positive? Negative? Zero?
 - When does the particle move at its greatest speed?
 - When does the particle stand still for more than an instant?
19. The graphs in the accompanying figure show the position s , velocity $v = ds/dt$, and acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line as functions of time t . Which graph is which? Give reasons for your answers.



20. The graphs in the accompanying figure show the position s , the velocity $v = ds/dt$, and the acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line as functions of time t . Which graph is which? Give reasons for your answers.



Economics

21. **Marginal cost** Suppose that the dollar cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$.
- Find the average cost per machine of producing the first 100 washing machines.

- Find the marginal cost when 100 washing machines are produced.
- Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

22. **Marginal revenue** Suppose that the revenue from selling x washing machines is

$$r(x) = 20,000 \left(1 - \frac{1}{x} \right)$$

dollars.

- Find the marginal revenue when 100 machines are produced.
- Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing production from 100 machines a week to 101 machines a week.
- Find the limit of $r'(x)$ as $x \rightarrow \infty$. How would you interpret this number?

Additional Applications

23. **Bacterium population** When a bactericide was added to a nutrient broth in which bacteria were growing, the bacterium population continued to grow for a while, but then stopped growing and began to decline. The size of the population at time t (hours) was $b = 10^6 + 10^4 t - 10^3 t^2$. Find the growth rates at

- $t = 0$ hours.
- $t = 5$ hours.
- $t = 10$ hours.

24. **Body surface area** A typical male's body surface area S in square meters is often modeled by the formula $S = \frac{1}{60} \sqrt{wh}$, where h is the height in centimeters, and w the weight in kilograms, of the person. Find the rate of change of body surface area with respect to weight for males of constant height $h = 180$ cm. Does S increase more rapidly with respect to weight at lower or higher body weights? Explain.

- T** 25. **Draining a tank** It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth y of fluid in the tank t hours after the valve is opened is given by the formula

$$y = 6 \left(1 - \frac{t}{12} \right)^2 \text{ m.}$$

- Find the rate dy/dt (m/h) at which the tank is draining at time t .
- When is the fluid level in the tank falling fastest? Slowest? What are the values of dy/dt at these times?
- Graph y and dy/dt together and discuss the behavior of y in relation to the signs and values of dy/dt .

26. **Draining a tank** The number of liters of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30 - t)^2$. How fast is the water running out at the end of 10 min? What is the average rate at which the water flows out during the first 10 min?

27. **Vehicular stopping distance** Based on data from the U.S. Bureau of Public Roads, a model for the total stopping distance of a moving car in terms of its speed is

$$s = 0.21v + 0.0063v^2,$$

where s is measured in meters and v is measured in km/h. The linear term $0.21v$ models the distance the car travels during the time the

driver perceives a need to stop until the brakes are applied, and the quadratic term $0.00636v^2$ models the additional braking distance once they are applied. Find ds/dv at $v = 50$ and $v = 100$ km/h, and interpret the meaning of the derivative.

28. Inflating a balloon The volume $V = (4/3)\pi r^3$ of a spherical balloon changes with the radius.

- At what rate (m^3/m) does the volume change with respect to the radius when $r = 2$ m?
- By approximately how much does the volume increase when the radius changes from 2 to 2.2 m?

29. Airplane takeoff Suppose that the distance an aircraft travels along a runway before takeoff is given by $D = (10/9)t^2$, where D is measured in meters from the starting point and t is measured in seconds from the time the brakes are released. The aircraft will become airborne when its speed reaches 200 km/h. How long will it take to become airborne, and what distance will it travel in that time?

30. Volcanic lava fountains Although the November 1959 Kilauea Iki eruption on the island of Hawaii began with a line of fountains along the wall of the crater, activity was later confined to a single vent in the crater's floor, which at one point shot lava 580 m straight into the air (a Hawaiian record). What was the lava's exit velocity in meters per second? In kilometers per hour? (Hint: If v_0

is the exit velocity of a particle of lava, its height t seconds later will be $s = v_0t - 4.9t^2$ m. Begin by finding the time at which $ds/dt = 0$. Neglect air resistance.)

Analyzing Motion Using Graphs

T Exercises 31–34 give the position function $s = f(t)$ of an object moving along the s -axis as a function of time t . Graph f together with the velocity function $v(t) = ds/dt = f'(t)$ and the acceleration function $a(t) = d^2s/dt^2 = f''(t)$. Comment on the object's behavior in relation to the signs and values of v and a . Include in your commentary such topics as the following:

- When is the object momentarily at rest?
- When does it move to the left (down) or to the right (up)?
- When does it change direction?
- When does it speed up and slow down?
- When is it moving fastest (highest speed)? Slowest?
- When is it farthest from the axis origin?

31. $s = 60t - 4.9t^2$, $0 \leq t \leq 12.5$ (a heavy object fired straight up from Earth's surface at 60 m/s)

32. $s = t^2 - 3t + 2$, $0 \leq t \leq 5$

33. $s = t^3 - 6t^2 + 7t$, $0 \leq t \leq 4$

34. $s = 4 - 7t + 6t^2 - t^3$, $0 \leq t \leq 4$

3.5 Derivatives of Trigonometric Functions

Many phenomena of nature are approximately periodic (electromagnetic fields, heart rhythms, tides, weather). The derivatives of sines and cosines play a key role in describing periodic changes. This section shows how to differentiate the six basic trigonometric functions.

Derivative of the Sine Function

To calculate the derivative of $f(x) = \sin x$, for x measured in radians, we combine the limits in Example 5a and Theorem 6 in Section 2.4 with the angle sum identity for the sine function (see Figure 1.46):

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = \sin x$, then

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} && \text{Derivative definition} \\ &= \lim_{h \rightarrow 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} && \text{Identity for } \sin(x+h) \\ &= \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \left(\sin x \cdot \frac{\cos h - 1}{h} \right) + \lim_{h \rightarrow 0} \left(\cos x \cdot \frac{\sin h}{h} \right) \\ &= \sin x \cdot \underbrace{\lim_{h \rightarrow 0} \frac{\cos h - 1}{h}}_{\text{limit 0}} + \cos x \cdot \underbrace{\lim_{h \rightarrow 0} \frac{\sin h}{h}}_{\text{limit 1}} && \text{Example 5a and Theorem 6, Section 2.4} \\ &= \sin x \cdot 0 + \cos x \cdot 1 = \cos x. \end{aligned}$$

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

EXAMPLE 1 We find derivatives of a difference, a product, and a quotient, each of which involves the sine function.

$$\begin{aligned} \text{(a)} \quad y = x^2 - \sin x: \quad \frac{dy}{dx} &= 2x - \frac{d}{dx}(\sin x) && \text{Difference Rule} \\ &= 2x - \cos x \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad y = e^x \sin x: \quad \frac{dy}{dx} &= e^x \frac{d}{dx}(\sin x) + \left(\frac{d}{dx}e^x\right)\sin x && \text{Product Rule} \\ &= e^x \cos x + e^x \sin x \\ &= e^x(\cos x + \sin x) \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad y = \frac{\sin x}{x}: \quad \frac{dy}{dx} &= \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} && \text{Quotient Rule} \\ &= \frac{x \cos x - \sin x}{x^2} \end{aligned}$$

Derivative of the Cosine Function

With the help of the angle sum formula for the cosine function (see Figure 1.46),

$$\cos(x + h) = \cos x \cos h - \sin x \sin h,$$

we can compute the limit of the difference quotient:

$$\begin{aligned} \frac{d}{dx}(\cos x) &= \lim_{h \rightarrow 0} \frac{\cos(x + h) - \cos x}{h} && \text{Derivative definition} \\ &= \lim_{h \rightarrow 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} && \text{Cosine angle sum identity} \\ &= \lim_{h \rightarrow 0} \frac{\cos x (\cos h - 1) - \sin x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \left(\cos x \cdot \frac{\cos h - 1}{h} \right) - \lim_{h \rightarrow 0} \left(\sin x \cdot \frac{\sin h}{h} \right) \\ &= \cos x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} && \text{Example 5a and Theorem 6, Section 2.4} \\ &= \cos x \cdot 0 - \sin x \cdot 1 \\ &= -\sin x. \end{aligned}$$

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x.$$

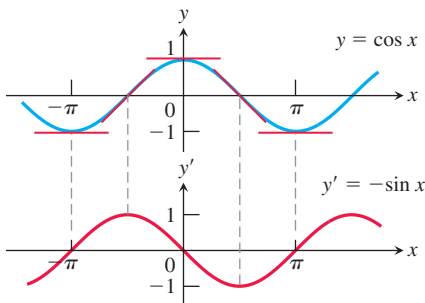


FIGURE 3.23 The curve $y' = -\sin x$ as the graph of the slopes of the tangent lines to the curve $y = \cos x$.

Figure 3.23 shows a way to visualize this result by graphing the slopes of the tangent lines to the curve $y = \cos x$.

EXAMPLE 2 We find derivatives of the cosine function in combinations with other functions.

$$\text{(a)} \quad y = 5e^x + \cos x:$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(5e^x) + \frac{d}{dx}(\cos x) && \text{Sum Rule} \\ &= 5e^x - \sin x \end{aligned}$$

(b) $y = \sin x \cos x$:

$$\begin{aligned}\frac{dy}{dx} &= \sin x \frac{d}{dx}(\cos x) + \left(\frac{d}{dx}(\sin x)\right) \cos x && \text{Product Rule} \\ &= (\sin x)(-\sin x) + (\cos x)(\cos x) \\ &= \cos^2 x - \sin^2 x\end{aligned}$$

(c) $y = \frac{\cos x}{1 - \sin x}$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1 - \sin x) \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} && \text{Quotient Rule} \\ &= \frac{(1 - \sin x)(-\sin x) - (\cos x)(0 - \cos x)}{(1 - \sin x)^2} \\ &= \frac{1 - \sin x}{(1 - \sin x)^2} && \sin^2 x + \cos^2 x = 1 \\ &= \frac{1}{1 - \sin x}\end{aligned}$$

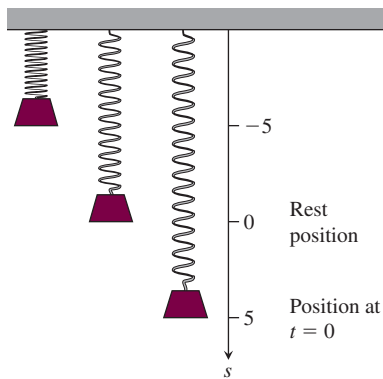


FIGURE 3.24 A weight hanging from a vertical spring and then displaced oscillates above and below its rest position (Example 3).

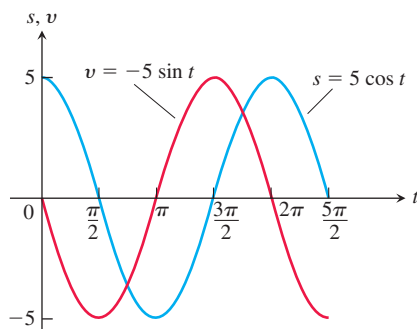


FIGURE 3.25 The graphs of the position and velocity of the weight in Example 3.

Simple Harmonic Motion

Simple harmonic motion models the motion of an object or weight bobbing freely up and down on the end of a spring, with no resistance. The motion is periodic and repeats indefinitely, so we represent it using trigonometric functions. The next example models motion with no opposing forces (such as friction).

EXAMPLE 3 A weight hanging from a spring (Figure 3.24) is stretched down 5 units beyond its rest position and released at time $t = 0$ to bob up and down. Its position at any later time t is

$$s = 5 \cos t.$$

What are its velocity and acceleration at time t ?

Solution We have

Position: $s = 5 \cos t$

Velocity: $v = \frac{ds}{dt} = \frac{d}{dt}(5 \cos t) = -5 \sin t$

Acceleration: $a = \frac{dv}{dt} = \frac{d}{dt}(-5 \sin t) = -5 \cos t.$

Notice how much we can learn from these equations:

1. As time passes, the weight moves down and up between $s = -5$ and $s = 5$ on the s -axis. The amplitude of the motion is 5. The period of the motion is 2π , the period of the cosine function.
2. The velocity $v = -5 \sin t$ attains its greatest magnitude, 5, when $\cos t = 0$, as the graphs in Figure 3.25 show. Hence, the speed of the weight, $|v| = 5|\sin t|$, is greatest when $\cos t = 0$, that is, when $s = 0$ (the rest position). The speed of the weight is zero when $\sin t = 0$. This occurs when $s = 5 \cos t = \pm 5$, at the endpoints of the interval of motion. At these points the weight reverses direction.
3. The weight is acted on by the spring and by gravity. When the weight is below the rest position, the combined forces pull it up, and when it is above the rest position, they pull it down. The weight's acceleration is always proportional to the negative of its displacement. This property of springs is called *Hooke's Law*, and is studied further in Section 6.5.

4. The acceleration, $a = -5 \cos t$, is zero only at the rest position, where $\cos t = 0$ and the force of gravity and the force from the spring balance each other. When the weight is anywhere else, the two forces are unequal, and acceleration is nonzero. The acceleration is greatest in magnitude at the points farthest from the rest position, where $\cos t = \pm 1$. ■

EXAMPLE 4 The jerk associated with the simple harmonic motion in Example 3 is

$$j = \frac{da}{dt} = \frac{d}{dt}(-5 \cos t) = 5 \sin t.$$

It has its greatest magnitude when $\sin t = \pm 1$, not at the extremes of the displacement but at the rest position, where the acceleration changes direction and sign. ■

Derivatives of the Other Basic Trigonometric Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x , it follows from the Quotient Rule that the related functions

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{and} \quad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined. Their derivatives are given by the following formulas. Notice the negative signs in the derivative formulas for the cofunctions.

The derivatives of the other trigonometric functions:

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \sec^2 x & \frac{d}{dx}(\cot x) &= -\csc^2 x \\ \frac{d}{dx}(\sec x) &= \sec x \tan x & \frac{d}{dx}(\csc x) &= -\csc x \cot x \end{aligned}$$

To show a typical calculation, we find the derivative of the tangent function. The other derivations are left for Exercise 54.

EXAMPLE 5 Find $d(\tan x)/dx$.

Solution We use the Derivative Quotient Rule to calculate the derivative:

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} && \text{Quotient Rule} \\ &= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x. \end{aligned}$$

EXAMPLE 6 Find y'' if $y = \sec x$.**Solution** Finding the second derivative involves a combination of trigonometric derivatives.

$$y = \sec x$$

$$y' = \sec x \tan x$$

Derivative rule for secant function

$$y'' = \frac{d}{dx}(\sec x \tan x)$$

$$= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x)$$

Derivative Product Rule

$$= (\sec x)(\sec^2 x) + (\tan x)(\sec x \tan x)$$

Derivative rules

$$= \sec^3 x + \sec x \tan^2 x$$

EXERCISES 3.5**Derivatives**In Exercises 1–18, find dy/dx .

1. $y = -10x + 3 \cos x$

2. $y = \frac{3}{x} + 5 \sin x$

3. $y = x^2 \cos x$

4. $y = \sqrt{x} \sec x + 3$

5. $y = \csc x - 4\sqrt{x} + \frac{7}{e^x}$

6. $y = x^2 \cot x - \frac{1}{x^2}$

7. $f(x) = \sin x \tan x$

8. $g(x) = \frac{\cos x}{\sin^2 x}$

9. $y = xe^{-x} \sec x$

10. $y = (\sin x + \cos x) \sec x$

11. $y = \frac{\cot x}{1 + \cot x}$

12. $y = \frac{\cos x}{1 + \sin x}$

13. $y = \frac{4}{\cos x} + \frac{1}{\tan x}$

14. $y = \frac{\cos x}{x} + \frac{x}{\cos x}$

15. $y = (\sec x + \tan x)(\sec x - \tan x)$

16. $y = x^2 \cos x - 2x \sin x - 2 \cos x$

17. $f(x) = x^3 \sin x \cos x$

18. $g(x) = (2 - x) \tan^2 x$

In Exercises 19–22, find ds/dt .

19. $s = \tan t - e^{-t}$

20. $s = t^2 - \sec t + 5e^t$

21. $s = \frac{1 + \csc t}{1 - \csc t}$

22. $s = \frac{\sin t}{1 - \cos t}$

In Exercises 23–26, find $dr/d\theta$.

23. $r = 4 - \theta^2 \sin \theta$

24. $r = \theta \sin \theta + \cos \theta$

25. $r = \sec \theta \csc \theta$

26. $r = (1 + \sec \theta) \sin \theta$

In Exercises 27–32, find dp/dq .

27. $p = 5 + \frac{1}{\cot q}$

28. $p = (1 + \csc q) \cos q$

29. $p = \frac{\sin q + \cos q}{\cos q}$

30. $p = \frac{\tan q}{1 + \tan q}$

31. $p = \frac{q \sin q}{q^2 - 1}$

32. $p = \frac{3q + \tan q}{q \sec q}$

33. Find y'' if

a. $y = \csc x$.

b. $y = \sec x$.

34. Find $y^{(4)} = d^4 y/dx^4$ if

a. $y = -2 \sin x$.

b. $y = 9 \cos x$.

Tangent LinesIn Exercises 35–38, graph the curves over the given intervals, together with their tangent lines at the given values of x . Label each curve and tangent line with its equation.

35. $y = \sin x, -3\pi/2 \leq x \leq 2\pi$
 $x = -\pi, 0, 3\pi/2$

36. $y = \tan x, -\pi/2 < x < \pi/2$
 $x = -\pi/3, 0, \pi/3$

37. $y = \sec x, -\pi/2 < x < \pi/2$
 $x = -\pi/3, \pi/4$

38. $y = 1 + \cos x, -3\pi/2 \leq x \leq 2\pi$
 $x = -\pi/3, 3\pi/2$

T Do the graphs of the functions in Exercises 39–44 have any horizontal tangent lines in the interval $0 \leq x \leq 2\pi$? If so, where? If not, why not? Visualize your findings by graphing the functions with a grapher.

39. $y = x + \sin x$

40. $y = 2x + \sin x$

41. $y = x - \cot x$

42. $y = x + 2 \cos x$

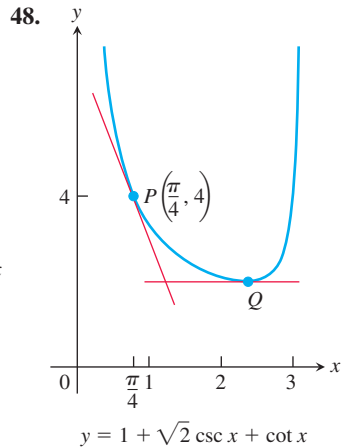
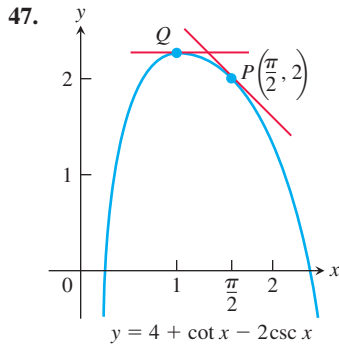
43. $y = \frac{\sec x}{3 + \sec x}$

44. $y = \frac{\cos x}{3 - 4 \sin x}$

45. Find all points on the curve $y = \tan x, -\pi/2 < x < \pi/2$, where the tangent line is parallel to the line $y = 2x$. Sketch the curve and tangent lines together, labeling each with its equation.

46. Find all points on the curve $y = \cot x$, $0 < x < \pi$, where the tangent line is parallel to the line $y = -x$. Sketch the curve and tangent lines together, labeling each with its equation.

In Exercises 47 and 48, find an equation for (a) the tangent line to the curve at P and (b) the horizontal tangent line to the curve at Q .



Theory and Examples

The equations in Exercises 49 and 50 give the position $s = f(t)$ of a body moving on a coordinate line (s in meters, t in seconds). Find the body's velocity, speed, acceleration, and jerk at time $t = \pi/4$ s.

49. $s = 2 - 2 \sin t$

50. $s = \sin t + \cos t$

51. Is there a value of c that will make

$$f(x) = \begin{cases} \frac{\sin^2 3x}{x^2}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$? Give reasons for your answer.

52. Is there a value of b that will make

$$g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \geq 0 \end{cases}$$

continuous at $x = 0$? Differentiable at $x = 0$? Give reasons for your answers.

53. By computing the first few derivatives and looking for a pattern, find the following derivatives.

a. $\frac{d^{999}}{dx^{999}}(\cos x)$

b. $\frac{d^{110}}{dx^{110}}(\sin x - 3 \cos x)$

c. $\frac{d^{73}}{dx^{73}}(x \sin x)$

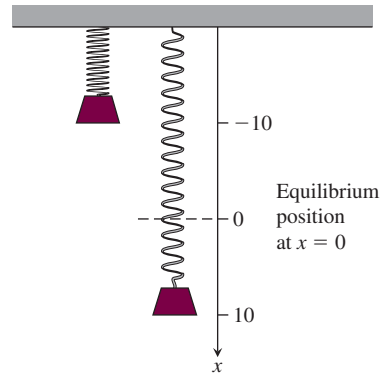
54. Derive the formula for the derivative with respect to x of

a. $\sec x$. b. $\csc x$. c. $\cot x$.

55. A weight is attached to a spring and reaches its equilibrium position ($x = 0$). It is then set in motion resulting in a displacement of

$$x = 10 \cos t,$$

where x is measured in centimeters and t is measured in seconds. See the accompanying figure.



- a. Find the spring's displacement when $t = 0$, $t = \pi/3$, and $t = 3\pi/4$.
b. Find the spring's velocity when $t = 0$, $t = \pi/3$, and $t = 3\pi/4$.

56. Assume that a particle's position on the x -axis is given by

$$x = 3 \cos t + 4 \sin t,$$

where x is measured in meters and t is measured in seconds.

- a. Find the particle's position when $t = 0$, $t = \pi/2$, and $t = \pi$.
b. Find the particle's velocity when $t = 0$, $t = \pi/2$, and $t = \pi$.

- T** 57. Graph $y = \cos x$ for $-\pi \leq x \leq 2\pi$. On the same screen, graph

$$y = \frac{\sin(x+h) - \sin x}{h}$$

for $h = 1, 0.5, 0.3$, and 0.1 . Then, in a new window, try $h = -1, -0.5$, and -0.3 . What happens as $h \rightarrow 0^+$? As $h \rightarrow 0^-$? What phenomenon is being illustrated here?

- T** 58. Graph $y = -\sin x$ for $-\pi \leq x \leq 2\pi$. On the same screen, graph

$$y = \frac{\cos(x+h) - \cos x}{h}$$

for $h = 1, 0.5, 0.3$, and 0.1 . Then, in a new window, try $h = -1, -0.5$, and -0.3 . What happens as $h \rightarrow 0^+$? As $h \rightarrow 0^-$? What phenomenon is being illustrated here?

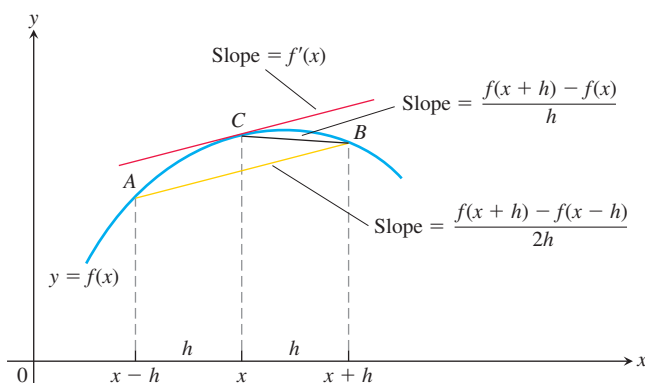
- T** 59. **Centered difference quotients** The centered difference quotient

$$\frac{f(x+h) - f(x-h)}{2h}$$

is used to approximate $f'(x)$ in numerical work because (1) its limit as $h \rightarrow 0$ equals $f'(x)$ when $f'(x)$ exists, and (2) it usually gives a better approximation of $f'(x)$ for a given value of h than the difference quotient

$$\frac{f(x+h) - f(x)}{h}.$$

See the accompanying figure.



- a. To see how rapidly the centered difference quotient for $f(x) = \sin x$ converges to $f'(x) = \cos x$, graph $y = \cos x$ together with

$$y = \frac{\sin(x+h) - \sin(x-h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3 . Compare the results with those obtained in Exercise 57 for the same values of h .

- b. To see how rapidly the centered difference quotient for $f(x) = \cos x$ converges to $f'(x) = -\sin x$, graph $y = -\sin x$ together with

$$y = \frac{\cos(x+h) - \cos(x-h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3 . Compare the results with those obtained in Exercise 58 for the same values of h .

60. **A caution about centered difference quotients** (Continuation of Exercise 59.) The quotient

$$\frac{f(x+h) - f(x-h)}{2h}$$

may have a limit as $h \rightarrow 0$ when f has no derivative at x . As a case in point, take $f(x) = |x|$ and calculate

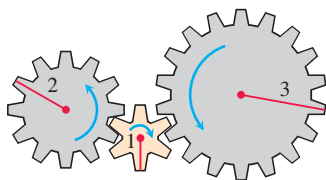
$$\lim_{h \rightarrow 0} \frac{|0+h| - |0-h|}{2h}.$$

As you will see, the limit exists even though $f(x) = |x|$ has no derivative at $x = 0$. *Moral:* Before using a centered difference quotient, be sure the derivative exists.

61. **Slopes on the graph of the tangent function** Graph $y = \tan x$ and its derivative together on $(-\pi/2, \pi/2)$. Does the graph of the tangent function appear to have a smallest slope? A largest slope? Is the slope ever negative? Give reasons for your answers.

62. **Exploring $(\sin kx)/x$** Graph $y = (\sin x)/x$, $y = (\sin 2x)/x$, and $y = (\sin 4x)/x$ together over the interval $-2 \leq x \leq 2$. Where does each graph appear to cross the y -axis? Do the graphs really intersect the axis? What would you expect the graphs of $y = (\sin 5x)/x$ and $y = (\sin(-3x))/x$ to do as $x \rightarrow 0$? Why? What about the graph of $y = (\sin kx)/x$ for other values of k ? Give reasons for your answers.

3.6 The Chain Rule



C: y turns B: u turns A: x turns

FIGURE 3.26 When gear A makes x turns, gear B makes u turns and gear C makes y turns. By comparing circumferences or counting teeth, we see that $y = u/2$ (C turns one-half turn for each B turn) and $u = 3x$ (B turns three times for A's one), so $y = 3x/2$. Thus, $dy/dx = 3/2 = (1/2)(3) = (dy/du)(du/dx)$.

How do we differentiate $F(x) = \sin(x^2 - 4)$? This function is the composition $f \circ g$ of two functions $y = f(u) = \sin u$ and $u = g(x) = x^2 - 4$ that we know how to differentiate. The answer, given by the *Chain Rule*, says that the derivative is the product of the derivatives of f and g . We develop the rule in this section.

Derivative of a Composite Function

The function $y = \frac{3}{2}x = \frac{1}{2}(3x)$ is the composition of the functions $y = \frac{1}{2}u$ and $u = 3x$.

We have

$$\frac{dy}{dx} = \frac{3}{2}, \quad \frac{dy}{du} = \frac{1}{2}, \quad \text{and} \quad \frac{du}{dx} = 3.$$

Since $\frac{3}{2} = \frac{1}{2} \cdot 3$, we see in this case that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

If we think of the derivative as a rate of change, this relationship is intuitively reasonable. If $y = f(u)$ changes half as fast as u , and $u = g(x)$ changes three times as fast as x , then we expect y to change $3/2$ times as fast as x . This effect is much like that of a multiple gear train (Figure 3.26). Let's look at another example.

EXAMPLE 1 The function

$$y = (3x^2 + 1)^2$$

is obtained by composing the functions $y = f(u) = u^2$ and $u = g(x) = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned} \frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1) \cdot 6x && \text{Substitute for } u. \\ &= 36x^3 + 12x. \end{aligned}$$

Calculating the derivative from the expanded formula $(3x^2 + 1)^2 = 9x^4 + 6x^2 + 1$ gives the same result:

$$\frac{dy}{dx} = \frac{d}{dx}(9x^4 + 6x^2 + 1) = 36x^3 + 12x. \quad \blacksquare$$

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x . This is known as the Chain Rule (Figure 3.27).

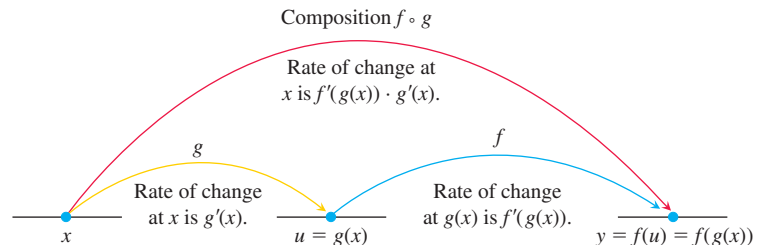


FIGURE 3.27 Rates of change multiply: The derivative of $f \circ g$ at x is the derivative of f at $g(x)$ times the derivative of g at x .

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x , then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x , and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

A Proof of One Case of the Chain Rule: Let Δu be the change in u when x changes by Δx , so that

$$\Delta u = g(x + \Delta x) - g(x).$$

Then the corresponding change in y is

$$\Delta y = f(u + \Delta u) - f(u).$$

If $\Delta u \neq 0$, we can write the fraction $\Delta y/\Delta x$ as the product

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \quad (1)$$

and take the limit as $\Delta x \rightarrow 0$:

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} \quad (\text{Note that } \Delta u \rightarrow 0 \text{ as } \Delta x \rightarrow 0 \\ &\quad \text{since } g \text{ is continuous.}) \\ &= \lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} \\ &= \frac{dy}{du} \cdot \frac{du}{dx}. \end{aligned}$$

The problem with this argument is that if the function $g(x)$ oscillates rapidly near x , then Δu can be zero even when $\Delta x \neq 0$, so the cancelation of Δu in Equation (1) would be invalid. A complete proof requires a different approach that avoids this problem, and we give one such proof in Section 3.11. ■

EXAMPLE 2 An object moves along the x -axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t .

Solution We know that the velocity is dx/dt . In this instance, x is a composition of two functions: $x = \cos(u)$ and $u = t^2 + 1$. We have

$$\begin{aligned} \frac{dx}{du} &= -\sin(u) & x &= \cos(u) \\ \frac{du}{dt} &= 2t. & u &= t^2 + 1 \end{aligned}$$

By the Chain Rule,

$$\begin{aligned} \frac{dx}{dt} &= \frac{dx}{du} \cdot \frac{du}{dt} \\ &= -\sin(u) \cdot 2t \\ &= -\sin(t^2 + 1) \cdot 2t \\ &= -2t \sin(t^2 + 1). \end{aligned}$$

Ways to Write the Chain Rule

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx} f(u) = f'(u) \frac{du}{dx}$$

“Outside-Inside” Rule

A difficulty with the Leibniz notation is that it doesn't state specifically where the derivatives in the Chain Rule are supposed to be evaluated. So it sometimes helps to write the Chain Rule using functional notation. If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In words, differentiate the “outside” function f and evaluate this derivative at the “inside” function $g(x)$ left alone; then multiply by the derivative of the “inside function.”

EXAMPLE 3 Differentiate $\sin(x^2 + e^x)$ with respect to x .

Solution We apply the Chain Rule directly and find

$$\frac{d}{dx} \sin(\underbrace{x^2 + e^x}_{\text{inside}}) = \cos(\underbrace{x^2 + e^x}_{\text{inside left alone}}) \cdot (\underbrace{2x + e^x}_{\text{derivative of the inside}}).$$

EXAMPLE 4 Differentiate $y = e^{\cos x}$.

Solution Here the inside function is $u = g(x) = \cos x$ and the outside function is the exponential function $f(x) = e^x$. Applying the Chain Rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\cos x}) = e^{\cos x} \frac{d}{dx}(\cos x) = e^{\cos x}(-\sin x) = -e^{\cos x} \sin x.$$

Generalizing Example 4, we see that the Chain Rule gives the formula

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}.$$

For example,

$$\frac{d}{dx}(e^{kx}) = e^{kx} \cdot \frac{d}{dx}(kx) = ke^{kx}, \quad \text{for any constant } k$$

and

$$\frac{d}{dx}(e^{x^2}) = e^{x^2} \cdot \frac{d}{dx}(x^2) = 2xe^{x^2}.$$

Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative.

EXAMPLE 5 Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

Solution Notice here that the tangent is a function of $5 - \sin 2t$, whereas the sine is a function of $2t$, which is itself a function of t . Therefore, by the Chain Rule,

$$\begin{aligned} g'(t) &= \frac{d}{dt} \tan(5 - \sin 2t) \\ &= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t) && \text{Derivative of } \tan u \text{ with } u = 5 - \sin 2t \\ &= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right) && \text{Derivative of } -\sin u \text{ with } u = 2t \\ &= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2 \\ &= -2(\cos 2t) \sec^2(5 - \sin 2t). \end{aligned}$$

HISTORICAL BIOGRAPHY

Johann Bernoulli
(1667–1748)

Johann Bernoulli was born in Switzerland and attended the University of Basel. His doctoral dissertation was in mathematics despite its medical title, which was used to hide his mathematical work from his father who wanted Johann to become a doctor.

To know more, visit the companion Website.

The Chain Rule with Powers of a Function

If n is any real number and f is a power function, $f(u) = u^n$, the Power Rule tells us that $f'(u) = nu^{n-1}$. If u is a differentiable function of x , then we can use the Chain Rule to extend this to the **Power Chain Rule**:

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}. \quad \frac{d}{du}(u^n) = nu^{n-1}$$

EXAMPLE 6 The Power Chain Rule simplifies computing the derivative of a power of an expression.

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx}(5x^3 - x^4)^7 &= 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4) && \text{Power Chain Rule with } u = 5x^3 - x^4, n = 7 \\ &= 7(5x^3 - x^4)^6(15x^2 - 4x^3) \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{d}{dx}\left(\frac{1}{3x-2}\right) &= \frac{d}{dx}(3x-2)^{-1} \\ &= -1(3x-2)^{-2} \frac{d}{dx}(3x-2) && \text{Power Chain Rule with } u = 3x-2, n = -1 \\ &= -1(3x-2)^{-2}(3) \\ &= -\frac{3}{(3x-2)^2} \end{aligned}$$

In part (b) we could also find the derivative with the Quotient Rule.

$$\begin{aligned} \text{(c)} \quad \frac{d}{dx}(\sin^5 x) &= 5 \sin^4 x \cdot \frac{d}{dx} \sin x && \text{Power Chain Rule with } u = \sin x, n = 5, \\ &= 5 \sin^4 x \cos x && \text{because } \sin^n x \text{ means } (\sin x)^n, n \neq -1 \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \frac{d}{dx}(e^{\sqrt{3x+1}}) &= e^{\sqrt{3x+1}} \cdot \frac{d}{dx}(\sqrt{3x+1}) \\ &= e^{\sqrt{3x+1}} \cdot \frac{1}{2}(3x+1)^{-1/2} \cdot 3 && \text{Power Chain Rule with } u = 3x+1, n = 1/2 \\ &= \frac{3}{2\sqrt{3x+1}} e^{\sqrt{3x+1}} \end{aligned}$$

EXAMPLE 7 In Example 4 of Section 3.2 we saw that the absolute value function $y = |x|$ is not differentiable at $x = 0$. However, the function is differentiable at all other real numbers, as we now show. Since $|x| = \sqrt{x^2}$, we can derive the following formula, which gives an alternative to the more direct analysis seen before.

Derivative of the Absolute Value Function

$$\begin{aligned} \frac{d}{dx}(|x|) &= \frac{x}{|x|}, \quad x \neq 0 \\ &= \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}(|x|) &= \frac{d}{dx} \sqrt{x^2} \\ &= \frac{1}{2\sqrt{x^2}} \cdot \frac{d}{dx}(x^2) && \text{Power Chain Rule with } u = x^2, n = 1/2, x \neq 0 \\ &= \frac{1}{2|x|} \cdot 2x && \sqrt{x^2} = |x| \\ &= \frac{x}{|x|}, \quad x \neq 0. \end{aligned}$$

EXAMPLE 8 Show that the slope of every line tangent to the curve $y = 1/(1-2x)^3$ is positive.

Solution We find the derivative:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(1-2x)^{-3} \\ &= -3(1-2x)^{-4} \cdot \frac{d}{dx}(1-2x) && \text{Power Chain Rule with } u = (1-2x), n = -3 \\ &= -3(1-2x)^{-4} \cdot (-2) \\ &= \frac{6}{(1-2x)^4}. \end{aligned}$$

At any point (x, y) on the curve, the denominator is nonzero, and the slope of the tangent line is

$$\frac{dy}{dx} = \frac{6}{(1-2x)^4},$$

which is the quotient of two positive numbers.

EXAMPLE 9 The formulas for the derivatives of both $\sin x$ and $\cos x$ were obtained under the assumption that x is measured in radians, *not* degrees. The Chain Rule gives us new insight into the difference between the two. Since $180^\circ = \pi$ radians, $x^\circ = \pi x/180$ radians, where x° is the size of the angle measured in degrees.

By the Chain Rule,

$$\frac{d}{dx} \sin(x^\circ) = \frac{d}{dx} \sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180} \cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180} \cos(x^\circ).$$

See Figure 3.28. Similarly, the derivative of $\cos(x^\circ)$ is $-(\pi/180) \sin(x^\circ)$.

The factor $\pi/180$ would propagate with repeated differentiation, showing an advantage for the use of radian measure in computations. ■

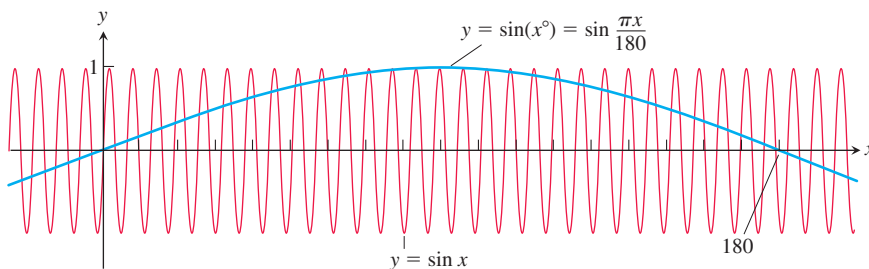


FIGURE 3.28 The function $\sin(x^\circ)$ oscillates only $\pi/180$ times as often as $\sin x$ oscillates. Its maximum slope is $\pi/180$ at $x = 0$ (Example 9).

EXERCISES 3.6

Derivative Calculations

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9$, $u = (1/2)x^4$
2. $y = 2u^3$, $u = 8x - 1$
3. $y = \sin u$, $u = 3x + 1$
4. $y = \cos u$, $u = e^{-x}$
5. $y = \sqrt{u}$, $u = \sin x$
6. $y = \sin u$, $u = x - \cos x$
7. $y = \tan u$, $u = \pi x^2$
8. $y = -\sec u$, $u = \frac{1}{x} + 7x$

In Exercises 9–22, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x .

9. $y = (2x + 1)^5$
10. $y = (4 - 3x)^9$
11. $y = \left(1 - \frac{x}{7}\right)^{-7}$
12. $y = \left(\frac{\sqrt{x}}{2} - 1\right)^{-10}$
13. $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$
14. $y = \sqrt{3x^2 - 4x + 6}$
15. $y = \sec(\tan x)$
16. $y = \cot\left(\pi - \frac{1}{x}\right)$
17. $y = \tan^3 x$
18. $y = 5 \cos^{-4} x$
19. $y = e^{-5x}$
20. $y = e^{2x/3}$
21. $y = e^{5-7x}$
22. $y = e^{(4\sqrt{x}+x^2)}$

Find the derivatives of the functions in Exercises 23–50.

23. $p = \sqrt{3-t}$
24. $q = \sqrt[3]{2r-r^2}$
25. $s = \frac{4}{3\pi} \sin 3t + \frac{4}{5\pi} \cos 5t$
26. $s = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right)$
27. $r = (\csc \theta + \cot \theta)^{-1}$
28. $r = 6(\sec \theta - \tan \theta)^{3/2}$
29. $y = x^2 \sin^4 x + x \cos^{-2} x$
30. $y = \frac{1}{x} \sin^{-5} x - \frac{x}{3} \cos^3 x$
31. $y = \frac{1}{18}(3x-2)^6 + \left(4 - \frac{1}{2x^2}\right)^{-1}$

32. $y = (5-2x)^{-3} + \frac{1}{8}\left(\frac{2}{x} + 1\right)^4$
33. $y = (4x+3)^4(x+1)^{-3}$
34. $y = (2x-5)^{-1}(x^2-5x)^6$
35. $y = xe^{-x} + e^{x^3}$
36. $y = (1+2x)e^{-2x}$
37. $y = (x^2-2x+2)e^{5x/2}$
38. $y = (9x^2-6x+2)e^{x^3}$
39. $h(x) = x \tan(2\sqrt{x}) + 7$
40. $k(x) = x^2 \sec\left(\frac{1}{x}\right)$
41. $f(x) = \sqrt{7+x \sec x}$
42. $g(x) = \frac{\tan 3x}{(x+7)^4}$
43. $f(\theta) = \left(\frac{\sin \theta}{1+\cos \theta}\right)^2$
44. $g(t) = \left(\frac{1+\sin 3t}{3-2t}\right)^{-1}$
45. $r = \sin(\theta^2) \cos(2\theta)$
46. $r = \sec \sqrt{\theta} \tan\left(\frac{1}{\theta}\right)$
47. $q = \sin\left(\frac{t}{\sqrt{t+1}}\right)$
48. $q = \cot\left(\frac{\sin t}{t}\right)$
49. $y = \cos(e^{-\theta^2})$
50. $y = \theta^3 e^{-2\theta} \cos 5\theta$

In Exercises 51–70, find dy/dt .

51. $y = \sin^2(\pi t - 2)$
52. $y = \sec^2 \pi t$
53. $y = (1 + \cos 2t)^{-4}$
54. $y = (1 + \cot(t/2))^{-2}$
55. $y = (t \tan t)^{10}$
56. $y = (t^{-3/4} \sin t)^{4/3}$
57. $y = e^{\cos^2(\pi t - 1)}$
58. $y = (e^{\sin(t/2)})^3$
59. $y = \left(\frac{t^2}{t^3 - 4t}\right)^3$
60. $y = \left(\frac{3t-4}{5t+2}\right)^{-5}$
61. $y = \sin(\cos(2t-5))$
62. $y = \cos\left(5 \sin\left(\frac{t}{3}\right)\right)$
63. $y = \left(1 + \tan^4\left(\frac{t}{12}\right)\right)^3$
64. $y = \frac{1}{6}(1 + \cos^2(7t))^3$

65. $y = \sqrt{1 + \cos(t^2)}$ 66. $y = 4 \sin(\sqrt{1 + \sqrt{t}})$
 67. $y = \tan^2(\sin^3 t)$ 68. $y = \cos^4(\sec^2 3t)$
 69. $y = 3t(2t^2 - 5)^4$ 70. $y = \sqrt{3t + \sqrt{2 + \sqrt{1 - t}}}$

Second DerivativesFind y'' in Exercises 71–78.

71. $y = \left(1 + \frac{1}{x}\right)^3$ 72. $y = (1 - \sqrt{x})^{-1}$
 73. $y = \frac{1}{9} \cot(3x - 1)$ 74. $y = 9 \tan\left(\frac{x}{3}\right)$
 75. $y = x(2x + 1)^4$ 76. $y = x^2(x^3 - 1)^5$
 77. $y = e^{x^2} + 5x$ 78. $y = \sin(x^2 e^x)$

For each of the following functions, solve both $f'(x) = 0$ and $f''(x) = 0$ for x .

79. $f(x) = x(x - 4)^3$
 80. $f(x) = \sec^2 x - 2 \tan x$ for $0 \leq x \leq 2\pi$

Finding Derivative ValuesIn Exercises 81–86, find the value of $(f \circ g)'$ at the given value of x .

81. $f(u) = u^5 + 1$, $u = g(x) = \sqrt{x}$, $x = 1$
 82. $f(u) = 1 - \frac{1}{u}$, $u = g(x) = \frac{1}{1 - x}$, $x = -1$
 83. $f(u) = \cot \frac{\pi u}{10}$, $u = g(x) = 5\sqrt{x}$, $x = 1$
 84. $f(u) = u + \frac{1}{\cos^2 u}$, $u = g(x) = \pi x$, $x = 1/4$
 85. $f(u) = \frac{2u}{u^2 + 1}$, $u = g(x) = 10x^2 + x + 1$, $x = 0$
 86. $f(u) = \left(\frac{u - 1}{u + 1}\right)^2$, $u = g(x) = \frac{1}{x^2} - 1$, $x = -1$
 87. Assume that $f'(3) = -1$, $g'(2) = 5$, $g(2) = 3$, and $y = f(g(x))$. What is y' at $x = 2$?
 88. If $r = \sin(f(t))$, $f(0) = \pi/3$, and $f'(0) = 4$, then what is dr/dt at $t = 0$?
 89. Suppose that functions f and g and their derivatives with respect to x have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	$1/3$	-3
3	3	-4	2π	5

Find the derivatives with respect to x of the following combinations at the given value of x .

- a. $2f(x)$, $x = 2$ b. $f(x) + g(x)$, $x = 3$
 c. $f(x) \cdot g(x)$, $x = 3$ d. $f(x)/g(x)$, $x = 2$
 e. $f(g(x))$, $x = 2$ f. $\sqrt{f(x)}$, $x = 2$
 g. $1/g^2(x)$, $x = 3$ h. $\sqrt{f^2(x) + g^2(x)}$, $x = 2$
 90. Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	$1/3$
1	3	-4	$-1/3$	$-8/3$

Find the derivatives with respect to x of the following combinations at the given value of x .

- a. $5f(x) - g(x)$, $x = 1$ b. $f(x)g^3(x)$, $x = 0$
 c. $\frac{f(x)}{g(x) + 1}$, $x = 1$ d. $f(g(x))$, $x = 0$
 e. $g(f(x))$, $x = 0$ f. $(x^{11} + f(x))^{-2}$, $x = 1$
 g. $f(x + g(x))$, $x = 0$
 91. Find ds/dt when $\theta = 3\pi/2$ if $s = \cos \theta$ and $d\theta/dt = 5$.
 92. Find dy/dt when $x = 1$ if $y = x^2 + 7x - 5$ and $dx/dt = 1/3$.

Theory and Examples

What happens if you can write a function as a composition in different ways? Do you get the same derivative each time? The Chain Rule says you should. Try it with the functions in Exercises 93 and 94.

93. Find dy/dx if $y = x$ by using the Chain Rule with y as a composition of
 a. $y = (u/5) + 7$ and $u = 5x - 35$
 b. $y = 1 + (1/u)$ and $u = 1/(x - 1)$.
 94. Find dy/dx if $y = x^{3/2}$ by using the Chain Rule with y as a composition of
 a. $y = u^3$ and $u = \sqrt{x}$
 b. $y = \sqrt{u}$ and $u = x^3$.
 95. Find the tangent line to $y = ((x - 1)/(x + 1))^2$ at $x = 0$.
 96. Find the tangent line to $y = \sqrt{x^2 - x + 7}$ at $x = 2$.
 97. a. Find the tangent line to the curve $y = 2 \tan(\pi x/4)$ at $x = 1$.
 b. **Slopes on a tangent curve** What is the smallest value the slope of the curve can ever have on the interval $-2 < x < 2$? Give reasons for your answer.
 98. **Slopes on sine curves**
 a. Find equations for the tangent lines to the curves $y = \sin 2x$ and $y = -\sin(x/2)$ at the origin. Is there anything special about how the tangent lines are related? Give reasons for your answer.
 b. Can anything be said about the tangent lines to the curves $y = \sin mx$ and $y = -\sin(x/m)$ at the origin (m a constant $\neq 0$)? Give reasons for your answer.
 c. For a given m , what are the largest values the slopes of the curves $y = \sin mx$ and $y = -\sin(x/m)$ can ever have? Give reasons for your answer.
 d. The function $y = \sin x$ completes one period on the interval $[0, 2\pi]$, the function $y = \sin 2x$ completes two periods, the function $y = \sin(x/2)$ completes half a period, and so on. Is there any relation between the number of periods $y = \sin mx$ completes on $[0, 2\pi]$ and the slope of the curve $y = \sin mx$ at the origin? Give reasons for your answer.

99. **Running machinery too fast** Suppose that a piston is moving straight up and down and that its position at time t seconds is

$$s = A \cos(2\pi bt),$$

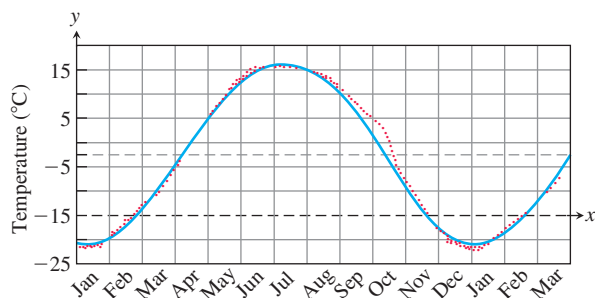
with A and b positive. The value of A is the amplitude of the motion, and b is the frequency (number of times the piston moves up and down each second). What effect does doubling the frequency have on the piston's velocity, acceleration, and jerk? (Once you find out, you will know why some machinery breaks when you run it too fast.)

- 100. Temperatures in Fairbanks, Alaska** The graph in the accompanying figure shows the average Celsius temperature in Fairbanks, Alaska, during a typical 365-day year. The equation that approximates the temperature on day x is

$$y = 20 \sin \left[\frac{2\pi}{365} (x - 101) \right] - 4$$

and is graphed in the accompanying figure.

- On what day is the temperature increasing the fastest?
- About how many degrees per day is the temperature increasing when it is increasing at its fastest?



- 101. Particle motion** The position of a particle moving along a coordinate line is $s = \sqrt{1 + 4t}$, with s in meters and t in seconds. Find the particle's velocity and acceleration at $t = 6$ s.
- 102. Constant acceleration** Suppose that the velocity of a falling body is $v = k\sqrt{s}$ m/s (k a constant) at the instant the body has fallen s meters from its starting point. Show that the body's acceleration is constant.
- 103. Falling meteorite** The velocity of a heavy meteorite entering Earth's atmosphere is inversely proportional to $\sqrt{s}$ when it is s km from Earth's center. Show that the meteorite's acceleration is inversely proportional to s^2 .
- 104. Particle acceleration** A particle moves along the x -axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.
- 105. Temperature and the period of a pendulum** For oscillations of small amplitude (short swings), we may safely model the relationship between the period T and the length L of a simple pendulum with the equation

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where g is the constant acceleration of gravity at the pendulum's location. If we measure g in centimeters per second squared, we measure L in centimeters and T in seconds. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to L . In symbols, with u being temperature and k the proportionality constant,

$$\frac{dL}{du} = kL.$$

Assuming this to be the case, show that the rate at which the period changes with respect to temperature is $kT/2$.

- 106. Chain Rule** Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the compositions

$$(f \circ g)(x) = |x|^2 = x^2 \quad \text{and} \quad (g \circ f)(x) = |x^2| = x^2$$

are both differentiable at $x = 0$ even though g itself is not differentiable at $x = 0$. Does this contradict the Chain Rule? Explain.

- T 107. The derivative of $\sin 2x$** Graph the function $y = 2 \cos 2x$ for $-2 \leq x \leq 3.5$. Then, on the same screen, graph

$$y = \frac{\sin 2(x + h) - \sin 2x}{h}$$

for $h = 1.0, 0.5$, and 0.2 . Experiment with other values of h , including negative values. What do you see happening as $h \rightarrow 0$? Explain this behavior.

- T 108. The derivative of $\cos(x^2)$** Graph $y = -2x \sin(x^2)$ for $-2 \leq x \leq 3$. Then, on the same screen, graph

$$y = \frac{\cos((x + h)^2) - \cos(x^2)}{h}$$

for $h = 1.0, 0.7$, and 0.3 . Experiment with other values of h . What do you see happening as $h \rightarrow 0$? Explain this behavior.

Using the Chain Rule, show that the Power Rule $(d/dx)x^n = nx^{n-1}$ holds for the functions x^n in Exercises 109 and 110.

109. $x^{1/4} = \sqrt[4]{x}$

110. $x^{3/4} = \sqrt[4]{x^3}$

- 111.** Consider the function

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x > 0 \\ 0, & x \leq 0 \end{cases}$$

- Show that f is continuous at $x = 0$.
- Determine f' for $x \neq 0$.
- Show that f is not differentiable at $x = 0$.

- 112.** Consider the function

$$f(x) = \begin{cases} x^2 \cos\left(\frac{2}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

- Show that f is continuous at $x = 0$.
- Determine f' for $x \neq 0$.
- Show that f is not differentiable at $x = 0$.
- Show that f' is not continuous at $x = 0$.

- 113.** Verify each of the following statements.

- If f is even, then f' is odd.
- If f is odd, then f' is even.

COMPUTER EXPLORATIONS

Trigonometric Polynomials

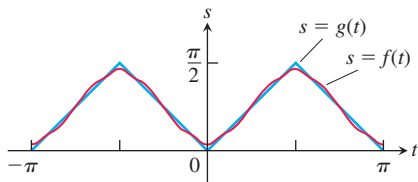
- 114.** As the accompanying figure shows, the trigonometric "polynomial"

$$s = f(t) = 0.78540 - 0.63662 \cos 2t - 0.07074 \cos 6t - 0.02546 \cos 10t - 0.01299 \cos 14t$$

gives a good approximation of the sawtooth function $s = g(t)$ on the interval $[-\pi, \pi]$. How well does the derivative of f approximate the derivative of g at the points where dg/dt is defined? To find out, carry out the following steps.

- Graph dg/dt (where defined) over $[-\pi, \pi]$.
- Find df/dt .

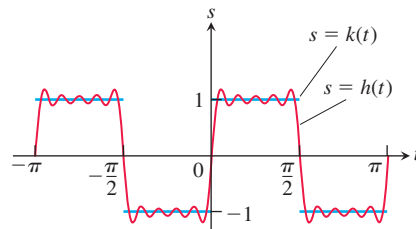
- c. Graph df/dt . Where does the approximation of dg/dt by df/dt seem to be best? Least good? Approximations by trigonometric polynomials are important in the theories of heat and oscillation, but we must not expect too much of them, as we see in the next exercise.



115. (Continuation of Exercise 114.) In Exercise 114, the trigonometric polynomial $f(t)$ that approximated the sawtooth function $g(t)$ on $[-\pi, \pi]$ had a derivative that approximated the derivative of the sawtooth function. It is possible, however, for a trigonometric polynomial to approximate a function in a reasonable way without its derivative approximating the function's derivative at all well. As a case in point, the trigonometric "polynomial"

$$s = h(t) = 1.2732 \sin 2t + 0.4244 \sin 6t + 0.25465 \sin 10t \\ + 0.18189 \sin 14t + 0.14147 \sin 18t$$

graphed in the accompanying figure approximates the step function $s = k(t)$ shown there. Yet the derivative of h is nothing like the derivative of k .



- Graph dk/dt (where defined) over $[-\pi, \pi]$.
- Find dh/dt .
- Graph dh/dt to see how badly the graph fits the graph of dk/dt . Comment on what you see.

3.7 Implicit Differentiation

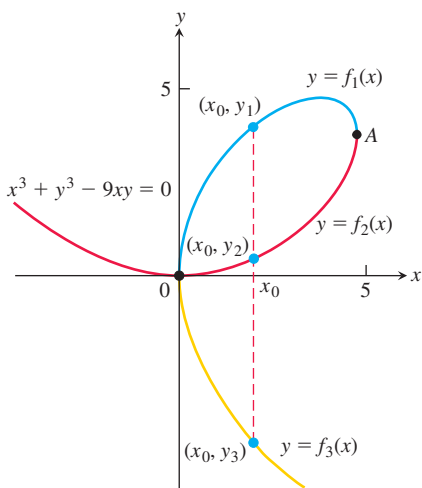


FIGURE 3.29 The curve $x^3 + y^3 - 9xy = 0$ is not the graph of any one function of x . The curve can, however, be divided into separate arcs that are the graphs of functions of x . This particular curve, called a *folium*, dates to Descartes in 1638.

Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses y explicitly in terms of the variable x . We have learned rules for differentiating functions defined in this way. A different situation occurs when we encounter equations like

$$x^3 + y^3 - 9xy = 0, \quad y^2 - x = 0, \quad \text{or} \quad x^2 + y^2 - 25 = 0.$$

(See Figures 3.29, 3.30, and 3.31.) Each of these equations defines an *implicit* relation between the variables x and y , meaning that a value of x may determine more than one value of y , even though we do not have a simple formula for the y -values. In some cases we may be able to solve such an equation for y as an explicit function (or even several functions) of x . When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find dy/dx by *implicit differentiation*. This section describes the technique.

Implicitly Defined Functions

We begin with examples involving familiar equations that we can solve for y as a function of x and then calculate dy/dx in the usual way. Then we differentiate the equations implicitly, and find the derivative. We will see that the two methods give the same answer. Following the examples, we summarize the steps involved in the new method. In the examples and exercises, it is always assumed that the given equation determines y implicitly as a differentiable function of x so that dy/dx exists.

EXAMPLE 1 Find dy/dx if $y^2 = x$.

Solution The equation $y^2 = x$ defines two differentiable functions of x that we can actually find, namely $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$ (Figure 3.30). We know how to calculate the derivative of each of these for $x > 0$:

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}.$$

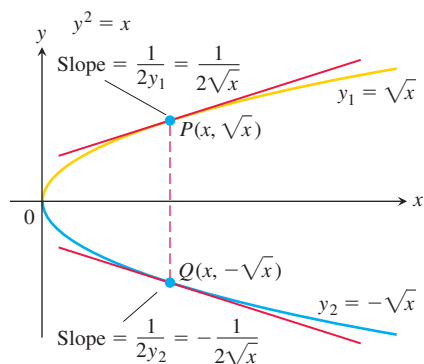


FIGURE 3.30 The equation $y^2 - x = 0$, or $y^2 = x$ as it is usually written, defines two differentiable functions of x on the interval $x > 0$. Example 1 shows how to find the derivatives of these functions without solving the equation $y^2 = x$ for y .

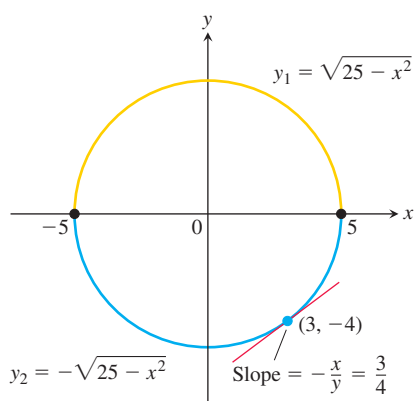


FIGURE 3.31 The circle combines the graphs of two functions. The graph of y_2 is the lower semicircle and passes through $(3, -4)$.

But suppose that we knew only that the equation $y^2 = x$ defined y as one or more differentiable functions of x for $x > 0$ without knowing exactly what these functions were. Can we still find dy/dx ?

The answer is yes. To find dy/dx , we simply differentiate both sides of the equation $y^2 = x$ with respect to x , treating $y = f(x)$ as a differentiable function of x :

$$\begin{aligned} y^2 &= x \\ 2y \frac{dy}{dx} &= 1 && \text{The Chain Rule gives } \frac{d}{dx}(y^2) = \frac{d}{dx}[f(x)]^2 = 2f(x)f'(x) = 2y \frac{dy}{dx}. \\ \frac{dy}{dx} &= \frac{1}{2y}. \end{aligned}$$

This one formula gives the derivatives we calculated for *both* explicit solutions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$:

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = \frac{1}{2y_2} = \frac{1}{2(-\sqrt{x})} = -\frac{1}{2\sqrt{x}}.$$

EXAMPLE 2 Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution The circle is not the graph of a single function of x . Rather, it is the combined graphs of two differentiable functions, $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$ (Figure 3.31). The point $(3, -4)$ lies on the graph of y_2 , so we can find the slope by calculating the derivative directly, using the Power Chain Rule:

$$\left. \frac{dy_2}{dx} \right|_{x=3} = \left. -\frac{2x}{2\sqrt{25 - x^2}} \right|_{x=3} = \left. -\frac{6}{2\sqrt{25 - 9}} \right|_{x=3} = \frac{3}{4}. \quad \frac{d}{dx}(-(25 - x^2)^{1/2}) = -\frac{1}{2}(25 - x^2)^{-1/2}(-2x)$$

We can solve this problem more easily by differentiating the given equation of the circle implicitly with respect to x :

$$\begin{aligned} \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(25) \\ 2x + 2y \frac{dy}{dx} &= 0 && \text{See Example 1.} \\ \frac{dy}{dx} &= -\frac{x}{y}. \end{aligned}$$

$$\text{The slope at } (3, -4) \text{ is } \left. -\frac{x}{y} \right|_{(3, -4)} = -\frac{3}{-4} = \frac{3}{4}.$$

Notice that unlike the slope formula for dy_2/dx , which applies only to points below the x -axis, the formula $dy/dx = -x/y$ applies everywhere the circle has a slope—that is, at all circle points (x, y) where $y \neq 0$. Notice also that the derivative involves *both* variables x and y , not just the independent variable x .

To calculate the derivatives of other implicitly defined functions, we proceed as in Examples 1 and 2: We treat y as a differentiable implicit function of x and apply the usual rules to differentiate both sides of the defining equation.

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x , treating y as a differentiable function of x .
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx .

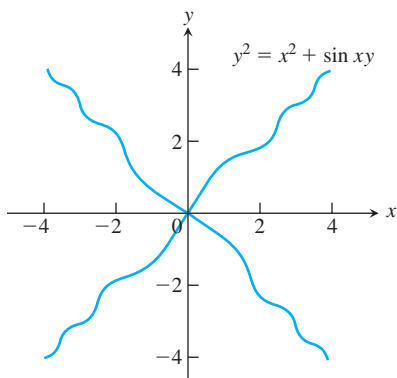


FIGURE 3.32 The graph of the equation in Example 3.

EXAMPLE 3 Find dy/dx if $y^2 = x^2 + \sin xy$ (Figure 3.32).

Solution We differentiate the equation implicitly.

$$\begin{aligned}
 y^2 &= x^2 + \sin xy \\
 \frac{d}{dx}(y^2) &= \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy) && \text{Differentiate both sides with respect to } x \dots \\
 2y \frac{dy}{dx} &= 2x + (\cos xy) \frac{d}{dx}(xy) && \dots \text{treating } y \text{ as a function of } x \text{ and using the Chain Rule.} \\
 2y \frac{dy}{dx} &= 2x + (\cos xy) \left(y + x \frac{dy}{dx} \right) && \text{Treat } xy \text{ as a product.} \\
 2y \frac{dy}{dx} - (\cos xy) \left(x \frac{dy}{dx} \right) &= 2x + (\cos xy)y && \text{Collect terms with } dy/dx. \\
 (2y - x \cos xy) \frac{dy}{dx} &= 2x + y \cos xy \\
 \frac{dy}{dx} &= \frac{2x + y \cos xy}{2y - x \cos xy} && \text{Solve for } dy/dx.
 \end{aligned}$$

Notice that the formula for dy/dx applies everywhere that the implicitly defined curve has a slope. Notice again that the derivative involves *both* variables x and y , not just the independent variable x . ■

Derivatives of Higher Order

Implicit differentiation can also be used to find higher derivatives.

EXAMPLE 4 Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Solution To start, we differentiate both sides of the equation with respect to x in order to find $y' = dy/dx$.

$$\begin{aligned}
 \frac{d}{dx}(2x^3 - 3y^2) &= \frac{d}{dx}(8) && \text{Treat } y \text{ as a function of } x. \\
 6x^2 - 6yy' &= 0 \\
 y' &= \frac{x^2}{y}, \quad \text{when } y \neq 0 && \text{Solve for } y'.
 \end{aligned}$$

We now apply the Quotient Rule to find y'' .

$$y'' = \frac{d}{dx} \left(\frac{x^2}{y} \right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y .

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2} \left(\frac{x^2}{y} \right) = \frac{2x}{y} - \frac{x^4}{y^3}, \quad \text{when } y \neq 0$$

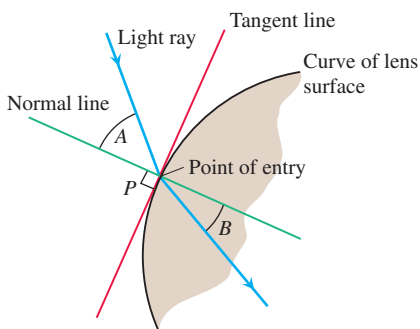


FIGURE 3.33 The profile of a lens, showing the bending (refraction) of a ray of light as it passes through the lens surface.

Lenses, Tangent Lines, and Normal Lines

In the law that describes how light changes direction as it enters a lens, the important angles are the angles the light makes with the line perpendicular to the surface of the lens at the point of entry (angles A and B in Figure 3.33). This line is called the *normal line* to the surface at the point of entry. In a profile view of a lens like the one in Figure 3.33, the **normal line** is the line perpendicular (also said to be *orthogonal*) to the tangent line of the profile curve at the point of entry.

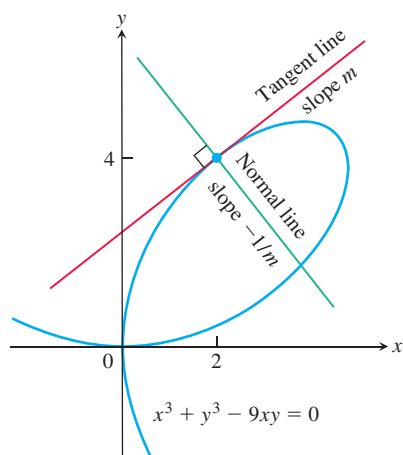


FIGURE 3.34 Example 5 shows how to find equations for the tangent line and normal line to the folium of Descartes at $(2, 4)$.

EXAMPLE 5 Show that the point $(2, 4)$ lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent line and normal line to the curve there (Figure 3.34).

Solution The point $(2, 4)$ lies on the curve because its coordinates satisfy the equation given for the curve: $2^3 + 4^3 - 9(2)(4) = 8 + 64 - 72 = 0$.

To find the slope of the curve at $(2, 4)$, we first use implicit differentiation to find a formula for dy/dx :

$$\begin{aligned}
 x^3 + y^3 - 9xy &= 0 \\
 \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) &= \frac{d}{dx}(0) && \text{Differentiate both sides} \\
 &&& \text{with respect to } x. \\
 3x^2 + 3y^2 \frac{dy}{dx} - 9\left(x \frac{dy}{dx} + \frac{dx}{dx}y\right) &= 0 && \text{Treat } xy \text{ as a product} \\
 &&& \text{and } y \text{ as a function of } x. \\
 (3y^2 - 9x) \frac{dy}{dx} + 3x^2 - 9y &= 0 \\
 3(y^2 - 3x) \frac{dy}{dx} &= 9y - 3x^2 \\
 \frac{dy}{dx} &= \frac{3y - x^2}{y^2 - 3x}. && \text{Solve for } dy/dx.
 \end{aligned}$$

We then evaluate the derivative at $(x, y) = (2, 4)$:

$$\left. \frac{dy}{dx} \right|_{(2,4)} = \left. \frac{3y - x^2}{y^2 - 3x} \right|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}.$$

The tangent line at $(2, 4)$ is the line through $(2, 4)$ with slope $4/5$:

$$\begin{aligned}
 y &= 4 + \frac{4}{5}(x - 2) \\
 y &= \frac{4}{5}x + \frac{12}{5}.
 \end{aligned}$$

The normal line to the curve at $(2, 4)$ is the line perpendicular to the tangent line there, the line through $(2, 4)$ with slope $-5/4$:

$$\begin{aligned}
 y &= 4 - \frac{5}{4}(x - 2) \\
 y &= -\frac{5}{4}x + \frac{13}{2}.
 \end{aligned}$$

Slopes of two nonvertical perpendicular lines are negative reciprocals of each other (see Appendix A.4). ■

EXERCISES 3.7

Differentiating Implicitly

Use implicit differentiation to find dy/dx in Exercises 1–16.

1. $x^2y + xy^2 = 6$
2. $x^3 + y^3 = 18xy$
3. $2xy + y^2 = x + y$
4. $x^3 - xy + y^3 = 1$
5. $x^2(x - y)^2 = x^2 - y^2$
6. $(3xy + 7)^2 = 6y$
7. $y^2 = \frac{x-1}{x+1}$
8. $x^3 = \frac{2x-y}{x+3y}$
9. $x = \sec y$
10. $xy = \cot(xy)$
11. $x + \tan(xy) = 0$
12. $x^4 + \sin y = x^3y^2$
13. $y \sin\left(\frac{1}{y}\right) = 1 - xy$
14. $x \cos(2x + 3y) = y \sin x$
15. $e^{2x} = \sin(x + 3y)$
16. $e^{x^2y} = 2x + 2y$

Find $dr/d\theta$ in Exercises 17–20.

17. $\theta^{1/2} + r^{1/2} = 1$
18. $r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4}$
19. $\sin(r\theta) = \frac{1}{2}$
20. $\cos r + \cot \theta = e^{r\theta}$

Second Derivatives

In Exercises 21–28, use implicit differentiation to find dy/dx and then d^2y/dx^2 . Write the solutions in terms of x and y only.

21. $x^2 + y^2 = 1$
22. $x^{2/3} + y^{2/3} = 1$
23. $y^2 = e^{x^2} + 2x$
24. $y^2 - 2x = 1 - 2y$
25. $2\sqrt{y} = x - y$
26. $xy + y^2 = 1$
27. $3 + \sin y = y - x^3$
28. $\sin y = x \cos y - 2$

29. If $x^3 + y^3 = 16$, find the value of d^2y/dx^2 at the point $(2, 2)$.
 30. If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point $(0, -1)$.

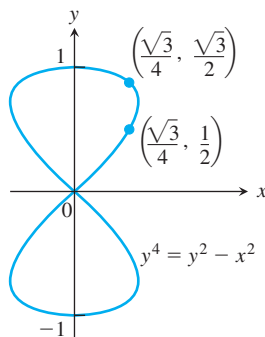
In Exercises 31 and 32, find the slope of the curve at the given points.

31. $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1)$
 32. $(x^2 + y^2)^2 = (x - y)^2$ at $(1, 0)$ and $(1, -1)$

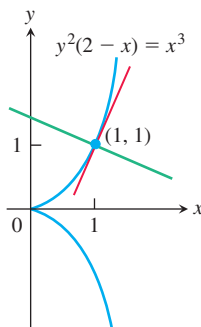
Slopes, Tangent Lines, and Normal Lines

In Exercises 33–42, verify that the given point is on the curve and find the lines that are (a) tangent and (b) normal to the curve at the given point.

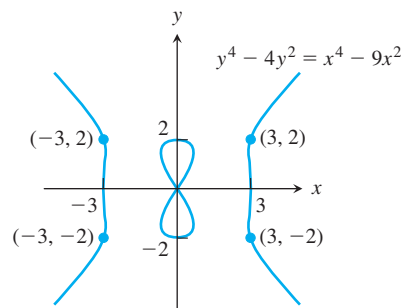
33. $x^2 + xy - y^2 = 1$, $(2, 3)$
 34. $x^2 + y^2 = 25$, $(3, -4)$
 35. $x^2y^2 = 9$, $(-1, 3)$
 36. $y^2 - 2x - 4y - 1 = 0$, $(-2, 1)$
 37. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$, $(-1, 0)$
 38. $x^2 - \sqrt{3}xy + 2y^2 = 5$, $(\sqrt{3}, 2)$
 39. $2xy + \pi \sin y = 2\pi$, $(1, \pi/2)$
 40. $x \sin 2y = y \cos 2x$, $(\pi/4, \pi/2)$
 41. $y = 2 \sin(\pi x - y)$, $(1, 0)$
 42. $x^2 \cos^2 y - \sin y = 0$, $(0, \pi)$
 43. **Parallel tangent lines** Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x -axis, and show that the tangent lines to the curve at these points are parallel. What is the common slope of these tangent lines?
 44. **Normal lines parallel to a line** Find the normal lines to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.
 45. **The eight curve** Find the slopes of the curve $y^4 = y^2 - x^2$ at the two points shown here.



46. **The cissoid of Diocles (from about 200 B.C.)** Find equations for the tangent line and normal line to the cissoid of Diocles $y^2(2 - x) = x^3$ at $(1, 1)$.



47. **The devil's curve (Gabriel Cramer, 1750)** Find the slopes of the devil's curve $y^4 - 4y^2 = x^4 - 9x^2$ at the four indicated points.



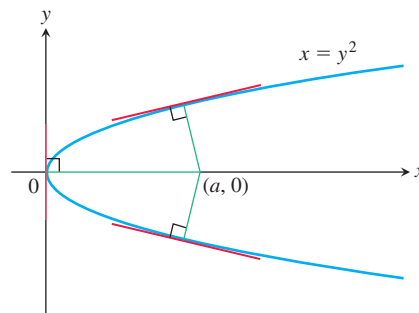
48. **The folium of Descartes** (See Figure 3.29)
 a. Find the slope of the folium of Descartes $x^3 + y^3 - 9xy = 0$ at the points $(4, 2)$ and $(2, 4)$.
 b. At what point other than the origin does the folium have a horizontal tangent line?
 c. Find the coordinates of the point A in Figure 3.29 where the folium has a vertical tangent line.

Theory and Examples

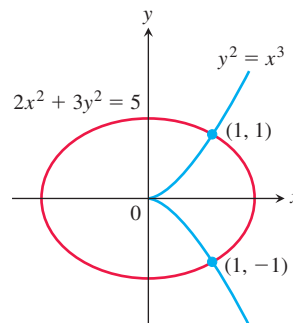
49. **Intersecting normal line** The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at $(1, 1)$ intersects the curve at what other point?
 50. **Power rule for rational exponents** Let p and q be integers with $q > 0$. If $y = x^{p/q}$, differentiate the equivalent equation $y^q = x^p$ implicitly and show that, for $y \neq 0$,

$$\frac{d}{dx} x^{p/q} = \frac{p}{q} x^{(p/q)-1}.$$

51. **Normal lines to a parabola** Show that if it is possible to draw three normal lines from the point $(a, 0)$ to the parabola $x = y^2$ shown in the accompanying diagram, then a must be greater than $1/2$. One of the normal lines is the x -axis. For what value of a are the other two normal lines perpendicular?



52. Is there anything special about the tangent lines to the curves $y^2 = x^3$ and $2x^2 + 3y^2 = 5$ at the points $(1, \pm 1)$? Give reasons for your answer.

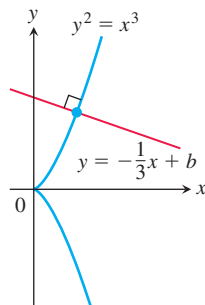


53. Verify that the following pairs of curves meet orthogonally.

a. $x^2 + y^2 = 4$, $x^2 = 3y^2$

b. $x = 1 - y^2$, $x = \frac{1}{3}y^2$

54. The graph of $y^2 = x^3$ is called a **semicubical parabola** and is shown in the accompanying figure. Determine the constant b so that the line $y = -\frac{1}{3}x + b$ meets this graph orthogonally.



In Exercises 55 and 56, find both dy/dx (treating y as a differentiable function of x) and dx/dy (treating x as a differentiable function of y). How do dy/dx and dx/dy seem to be related?

55. $xy^3 + x^2y = 6$

56. $x^3 + y^2 = \sin^2 y$

57. **Derivative of arcsine** Assume that $y = \sin^{-1}x$ is a differentiable function of x . By differentiating the equation $x = \sin y$ implicitly, show that $dy/dx = 1/\sqrt{1-x^2}$.

58. Use the formula in Exercise 57 to find dy/dx if

a. $y = (\sin^{-1}x)^2$

b. $y = \sin^{-1}\left(\frac{1}{x}\right)$.

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps in Exercises 59–66.

a. Plot the equation with the implicit plotter of a CAS. Check to see that the given point P satisfies the equation.

b. Using implicit differentiation, find a formula for the derivative dy/dx and evaluate it at the given point P .

c. Use the slope found in part (b) to find an equation for the tangent line to the curve at P . Then plot the implicit curve and tangent line together on a single graph.

59. $x^3 - xy + y^3 = 7$, $P(2, 1)$

60. $x^5 + y^3x + yx^2 + y^4 = 4$, $P(1, 1)$

61. $y^2 + y = \frac{2+x}{1-x}$, $P(0, 1)$

62. $y^3 + \cos xy = x^2$, $P(1, 0)$

63. $x + \tan\left(\frac{y}{x}\right) = 2$, $P\left(1, \frac{\pi}{4}\right)$

64. $xy^3 + \tan(x+y) = 1$, $P\left(\frac{\pi}{4}, 0\right)$

65. $2y^2 + (xy)^{1/3} = x^2 + 2$, $P(1, 1)$

66. $x\sqrt{1+2y} + y = x^2$, $P(1, 0)$

3.8 Derivatives of Inverse Functions and Logarithms

In Section 1.5 we saw how the inverse of a function undoes, or inverts, the effect of that function. We defined there the natural logarithm function $f^{-1}(x) = \ln x$ as the inverse of the natural exponential function $f(x) = e^x$. This is one of the most important function-inverse pairs in mathematics and science. We learned how to differentiate the exponential function in Section 3.3. Here we develop a rule for differentiating the inverse of a differentiable function, and we apply the rule to find the derivative of the natural logarithm function.

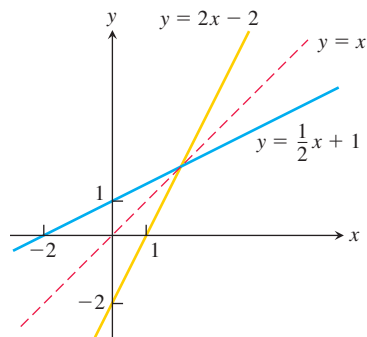


FIGURE 3.35 Graphing a line and its inverse together shows the graphs' symmetry with respect to the line $y = x$. The slopes are reciprocals of each other.

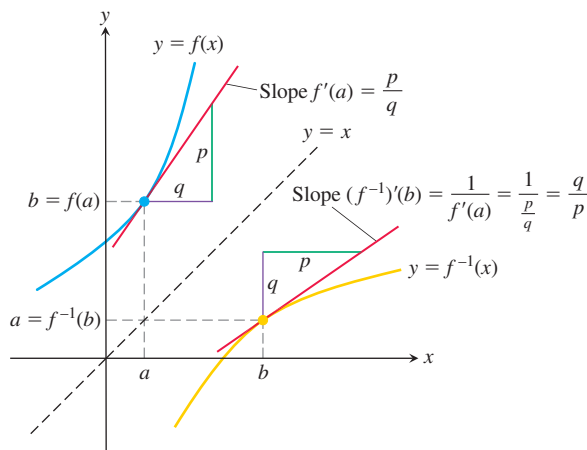
Derivatives of Inverses of Differentiable Functions

We calculated the inverse of the function $f(x) = (1/2)x + 1$ to be $f^{-1}(x) = 2x - 2$ in Example 3 of Section 1.5. Figure 3.35 shows the graphs of both functions. If we calculate their derivatives, we see that

$$\begin{aligned}\frac{d}{dx}f(x) &= \frac{d}{dx}\left(\frac{1}{2}x + 1\right) = \frac{1}{2} \\ \frac{d}{dx}f^{-1}(x) &= \frac{d}{dx}(2x - 2) = 2.\end{aligned}$$

The derivatives are reciprocals of one another, so the slope of one line is the reciprocal of the slope of its inverse line. (See Figure 3.35.)

This is not a special case. Reflecting any nonhorizontal or nonvertical line across the line $y = x$ always inverts the line's slope. If the original line has slope $m \neq 0$, the reflected line has slope $1/m$.



The slopes are reciprocals: $(f^{-1})'(b) = \frac{1}{f'(a)}$ or $(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$

FIGURE 3.36 The graphs of inverse functions have reciprocal slopes at corresponding points.

The reciprocal relationship between the slopes of f and f^{-1} holds for other functions as well, but we must be careful to compare slopes at corresponding points. If the slope of $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$ and $f'(a) \neq 0$, then the slope of $y = f^{-1}(x)$ at the point $(f(a), a)$ is the reciprocal $1/f'(a)$ (Figure 3.36). If we set $b = f(a)$, then

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

If $y = f(x)$ has a horizontal tangent line at $(a, f(a))$, then the inverse function f^{-1} has a vertical tangent line at $(f(a), a)$, so the slope is undefined and f^{-1} is not differentiable at $f(a)$. Theorem 3 gives the conditions under which f^{-1} is differentiable in its domain (which is the same as the range of f).

THEOREM 3—The Derivative Rule for Inverses

If f has an interval I as domain and $f'(x)$ exists and is never zero on I , then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \quad (1)$$

or

$$\left. \frac{df^{-1}}{dx} \right|_{x=b} = \frac{1}{\left. \frac{df}{dx} \right|_{x=f^{-1}(b)}}.$$

Theorem 3 makes two assertions. The first of these has to do with the conditions under which f^{-1} is differentiable; the second assertion is a formula for the derivative of f^{-1}

when it exists. While we omit the proof of the first assertion, the second one is proved in the following way:

$$\begin{aligned}
 f(f^{-1}(x)) &= x && \text{Inverse function relationship} \\
 \frac{d}{dx} f(f^{-1}(x)) &= 1 && \text{Differentiate both sides.} \\
 f'(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) &= 1 && \text{Chain Rule} \\
 \frac{d}{dx} f^{-1}(x) &= \frac{1}{f'(f^{-1}(x))}. && \text{Solve for the derivative.}
 \end{aligned}$$

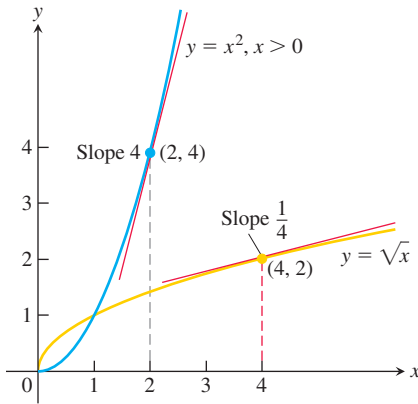


FIGURE 3.37 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point $(4, 2)$ is the reciprocal of the derivative of $f(x) = x^2$ at $(2, 4)$ (Example 1).

EXAMPLE 1 The function $f(x) = x^2, x > 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = 1/(2\sqrt{x})$.

Let's verify that Theorem 3 gives the same formula for the derivative of $f^{-1}(x)$:

$$\begin{aligned}
 (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\
 &= \frac{1}{2(f^{-1}(x))} && f'(x) = 2x \text{ with } x \text{ replaced by } f^{-1}(x) \\
 &= \frac{1}{2(\sqrt{x})}. && f^{-1}(x) = \sqrt{x}
 \end{aligned}$$

Theorem 3 gives a derivative that agrees with the known derivative of the square root function.

Let's examine Theorem 3 at a specific point. We pick $x = 2$ (the number a) and $f(2) = 4$ (the value b). Theorem 3 says that the derivative of f at 2, which is $f'(2) = 4$, and the derivative of f^{-1} at $f(2)$, which is $(f^{-1})'(4)$, are reciprocals. It states that

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \frac{1}{2x}\bigg|_{x=2} = \frac{1}{4}.$$

See Figure 3.37. ■

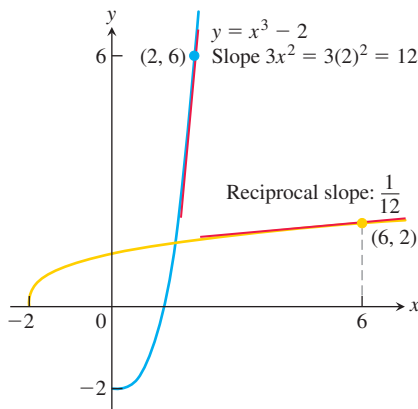


FIGURE 3.38 The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of f^{-1} at $x = 6$ (Example 2).

We will use the procedure illustrated in Example 1 to calculate formulas for the derivatives of many inverse functions throughout this chapter. Equation (1) sometimes enables us to find specific values of df^{-1}/dx without knowing a formula for f^{-1} .

EXAMPLE 2 Let $f(x) = x^3 - 2, x > 0$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$. See Figure 3.38.

Solution We apply Theorem 3 to obtain the value of the derivative of f^{-1} at $x = 6$:

$$\begin{aligned}
 \frac{df}{dx}\bigg|_{x=2} &= 3x^2\bigg|_{x=2} = 12 \\
 \frac{df^{-1}}{dx}\bigg|_{x=f(2)} &= \frac{1}{\frac{df}{dx}\bigg|_{x=2}} = \frac{1}{12}. && \text{Eq. (1)}
 \end{aligned}$$
■

Derivative of the Natural Logarithm Function

Since we know that the exponential function $f(x) = e^x$ is differentiable everywhere, we can apply Theorem 3 to find the derivative of its inverse $f^{-1}(x) = \ln x$:

$$\begin{aligned}
 (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3} \\
 &= \frac{1}{e^{f^{-1}(x)}} && f'(u) = e^u \\
 &= \frac{1}{e^{\ln x}} && x > 0 \\
 &= \frac{1}{x}. && \text{Inverse function relationship}
 \end{aligned}$$

Alternative Derivation Instead of applying Theorem 3 directly, we can find the derivative of $y = \ln x$ using implicit differentiation, as follows:

$$\begin{aligned}
 y &= \ln x && x > 0 \\
 e^y &= x && \text{Inverse function relationship} \\
 \frac{d}{dx}(e^y) &= \frac{d}{dx}(x) && \text{Differentiate implicitly.} \\
 e^y \frac{dy}{dx} &= 1 && \text{Chain Rule} \\
 \frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{x}. && e^y = x
 \end{aligned}$$

No matter which derivation we use, the derivative of $y = \ln x$ with respect to x is

$$\frac{d}{dx} \ln x = \frac{1}{x}, \quad x > 0. \quad (2)$$

The Chain Rule extends this formula to positive differentiable functions $u(x)$:

$$\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}, \quad u > 0. \quad (3)$$

EXAMPLE 3 We use Equations (2) and (3) to find derivatives.

(a) $\frac{d}{dx} \ln 2x = \frac{1}{2x} \frac{d}{dx}(2x) = \frac{1}{2x} \cdot 2 = \frac{1}{x}, \quad x > 0$

(b) Equation (3) with $u = x^2 + 3$ gives

$$\frac{d}{dx} \ln(x^2 + 3) = \frac{1}{x^2 + 3} \cdot \frac{d}{dx}(x^2 + 3) = \frac{1}{x^2 + 3} \cdot 2x = \frac{2x}{x^2 + 3}.$$

(c) Using the Chain Rule and Equation (2), we find

$$\frac{d}{dx} (\ln x)^4 = 4(\ln x)^3 \cdot \frac{d}{dx} (\ln x) = 4(\ln x)^3 \cdot \frac{1}{x} = \frac{4(\ln x)^3}{x}, \quad x > 0.$$

(d) Equation (3) with $u = |x|$ gives an important derivative:

$$\begin{aligned}\frac{d}{dx} \ln|x| &= \frac{d}{du} \ln u \cdot \frac{du}{dx} & u &= |x|, x \neq 0 \\ &= \frac{1}{u} \cdot \frac{x}{|x|} & \frac{d}{dx}(|x|) &= \frac{x}{|x|} \text{ (Example 7, Section 3.6)} \\ &= \frac{1}{|x|} \cdot \frac{x}{|x|} & \text{Substitute for } u. \\ &= \frac{x}{x^2} \\ &= \frac{1}{x}.\end{aligned}$$

Derivative of $\ln|x|$

$$\frac{d}{dx} \ln|x| = \frac{1}{x}, \quad x \neq 0$$

So $1/x$ is the derivative of $\ln x$ on the domain $x > 0$, and the derivative of $\ln(-x)$ on the domain $x < 0$. ■

Notice from Example 3a that the function $y = \ln 2x$ has the same derivative as the function $y = \ln x$. This is true of $y = \ln bx$ for any constant b , provided that $bx > 0$:

$$\frac{d}{dx} \ln bx = \frac{1}{x}, \quad bx > 0$$

$$\frac{d}{dx} \ln bx = \frac{1}{bx} \cdot \frac{d}{dx}(bx) = \frac{1}{bx}(b) = \frac{1}{x}. \quad (4)$$

EXAMPLE 4 A line with slope m passes through the origin and is tangent to the graph of $y = \ln x$. What is the value of m ?

Solution Suppose the point of tangency occurs at the unknown point $x = a > 0$. Then we know that the point $(a, \ln a)$ lies on the graph and that the tangent line at that point has slope $m = 1/a$ (Figure 3.39). Since the tangent line passes through the origin, its slope is

$$m = \frac{\ln a - 0}{a - 0} = \frac{\ln a}{a}.$$

Setting these two formulas for m equal to each other, we have

$$\begin{aligned}\frac{\ln a}{a} &= \frac{1}{a} \\ \ln a &= 1 \\ e^{\ln a} &= e^1 \\ a &= e \\ m &= \frac{1}{e}.\end{aligned}$$

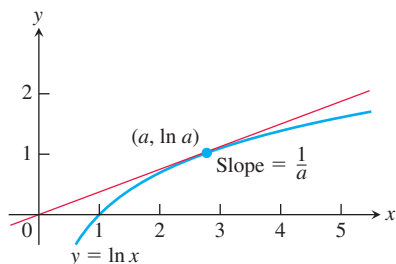


FIGURE 3.39 The tangent line meets the curve at some point $(a, \ln a)$, where the slope of the curve is $1/a$ (Example 4).

The Derivatives of a^x and $\log_a x$

We start with the equation $a^x = e^{\ln(a^x)} = e^{x \ln a}$, $a > 0$, which was seen in Section 1.5, where it was used to define the function a^x :

$$\begin{aligned}\frac{d}{dx} a^x &= \frac{d}{dx} e^{x \ln a} \\ &= e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) & \frac{d}{dx} e^u &= e^u \frac{du}{dx} \\ &= a^x \ln a. & \ln a &\text{ is a constant.}\end{aligned}$$

That is, if $a > 0$, then a^x is differentiable and

$$\frac{d}{dx} a^x = a^x \ln a. \quad (5)$$

This equation shows why e^x is the preferred exponential function in calculus. If $a = e$, then $\ln a = 1$ and the derivative of a^x simplifies to

$$\frac{d}{dx}e^x = e^x \ln e = e^x. \quad \ln e = 1$$

If $a > 0$ and u is a differentiable function of x , then by the Chain Rule, a^u is a differentiable function of x and

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}. \quad (6)$$

EXAMPLE 5 Here are some derivatives of general exponential functions.

$$(a) \frac{d}{dx}3^x = 3^x \ln 3 \quad \text{Eq. (6) with } a = 3, u = x$$

$$(b) \frac{d}{dx}3^{-x} = 3^{-x}(\ln 3) \frac{d}{dx}(-x) = -3^{-x} \ln 3 \quad \text{Eq. (6) with } a = 3, u = -x$$

$$(c) \frac{d}{dx}3^{\sin x} = 3^{\sin x}(\ln 3) \frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3)\cos x \quad \text{Eq. (6) with } u = \sin x$$

$$(d) \frac{d}{dx}\sin(3^x) = \cos(3^x) \frac{d}{dx}3^x = \cos(3^x) \cdot 3^x \ln 3 \quad \text{Chain Rule and Eq. (5)} \quad \blacksquare$$

In Section 3.3 we looked at the derivative $f'(0)$ for the exponential functions $f(x) = a^x$ at various values of the base a . The number $f'(0)$ is the limit, $\lim_{h \rightarrow 0} (a^h - 1)/h$, and gives the slope of the graph of a^x when it crosses the y -axis at the point $(0, 1)$. We now see from Equation (5) that the value of this slope is

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = \ln a. \quad (7)$$

In particular, when $a = e$ we obtain

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = \ln e = 1.$$

However, we have not fully justified that these limits actually exist. While all of the arguments given in deriving the derivatives of the exponential and logarithmic functions are correct, they do assume the existence of these limits. In Chapter 7 we will give another development of the theory of logarithmic and exponential functions which fully justifies that both limits do in fact exist and have the values derived above.

To find the derivative of $\log_a x$ for an arbitrary base ($a > 0$, $a \neq 1$), we use the change-of-base formula for logarithms (reviewed in Section 1.5) to express $\log_a x$ in terms of natural logarithms:

$$\log_a x = \frac{\ln x}{\ln a}.$$

Then we take derivatives

$$\begin{aligned} \frac{d}{dx} \log_a x &= \frac{d}{dx} \left(\frac{\ln x}{\ln a} \right) && \text{Differentiate both sides.} \\ &= \frac{1}{\ln a} \cdot \frac{d}{dx} \ln x && \ln a \text{ is a constant.} \\ &= \frac{1}{\ln a} \cdot \frac{1}{x}, \end{aligned}$$

which yields

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a} \quad a > 0, a \neq 1. \quad (8)$$

If u is a differentiable function of x and $u > 0$, the Chain Rule gives a more general formula:

$$\frac{d}{dx} \log_a u = \frac{1}{u \ln a} \frac{du}{dx} \quad a > 0, a \neq 1. \quad (9)$$

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called **logarithmic differentiation**, is illustrated in the next example.

EXAMPLE 6 Find dy/dx if

$$y = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}, \quad x > 1.$$

Solution We take the natural logarithm of both sides and simplify the result with the algebraic properties of logarithms from Theorem 1 in Section 1.5:

$$\begin{aligned} \ln y &= \ln \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1} \\ &= \ln((x^2 + 1)(x + 3)^{1/2}) - \ln(x - 1) && \text{Rule 2} \\ &= \ln(x^2 + 1) + \ln(x + 3)^{1/2} - \ln(x - 1) && \text{Rule 1} \\ &= \ln(x^2 + 1) + \frac{1}{2} \ln(x + 3) - \ln(x - 1). && \text{Rule 4} \end{aligned}$$

We then take derivatives of both sides with respect to x , using Equation (3):

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x^2 + 1} \cdot 2x + \frac{1}{2} \cdot \frac{1}{x + 3} - \frac{1}{x - 1}.$$

Next we solve for dy/dx :

$$\frac{dy}{dx} = y \left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1} \right).$$

Finally, we substitute for y :

$$\frac{dy}{dx} = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1} \left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1} \right). \quad \blacksquare$$

The computation in Example 6 would be much longer if we used the product, quotient, and power rules.

Irrational Exponents and the Power Rule (General Version)

The natural logarithm and the exponential function will be defined precisely in Chapter 7. We can use the exponential function to define the general exponential function, which enables us to raise any positive number to any real power n , rational or irrational. That is, we can define the power function $y = x^n$ for any exponent n .

DEFINITION For any $x > 0$ and for any real number n ,

$$x^n = e^{n \ln x}.$$

Because the logarithm and exponential functions are inverses of each other, the definition gives

$$\ln x^n = n \ln x, \quad \text{for all real numbers } n.$$

That is, the rule for taking the natural logarithm of a power holds for *all* real exponents n , not just for rational exponents.

The definition of the power function also enables us to establish the derivative Power Rule for any real power n , as stated in Section 3.3.

General Power Rule for Derivatives

For $x > 0$ and any real number n ,

$$\frac{d}{dx} x^n = nx^{n-1}.$$

If $x \leq 0$, then the formula holds whenever the derivative, x^n , and x^{n-1} all exist.

Proof Differentiating x^n with respect to x gives

$$\begin{aligned} \frac{d}{dx} x^n &= \frac{d}{dx} e^{n \ln x} && \text{Definition of } x^n, x > 0 \\ &= e^{n \ln x} \cdot \frac{d}{dx} (n \ln x) && \text{Chain Rule for } e^u \\ &= x^n \cdot \frac{n}{x} && \text{Definition and derivative of } \ln x \\ &= nx^{n-1}. && x^n \cdot \frac{1}{x} = x^{n-1} \end{aligned}$$

In short, whenever $x > 0$,

$$\frac{d}{dx} x^n = nx^{n-1}.$$

For $x < 0$, if $y = x^n$, y' , and x^{n-1} all exist, then

$$\ln |y| = \ln |x|^n = n \ln |x|.$$

Using implicit differentiation (which *assumes* the existence of the derivative y') and Example 3(c), we have

$$\frac{y'}{y} = \frac{n}{x}.$$

Solving for the derivative, we find that

$$y' = n \frac{y}{x} = n \frac{x^n}{x} = nx^{n-1}. \quad y = x^n$$

It can be shown directly from the definition of the derivative that the derivative equals 0 when $x = 0$ and $n > 1$ (see Exercise 107). This completes the proof of the general version of the Power Rule for all values of x . ■

EXAMPLE 7 Differentiate $f(x) = x^x, x > 0$.

Solution The Power Rule tells us how to differentiate a function of the form x^a , where a is a fixed real number. However, the exponent in x^x is not a fixed constant, so we cannot use the Power Rule to differentiate x^x . Equation (5) tells us how to differentiate a^x when the base a is constant. We cannot use that equation either, because the base x in x^x is not constant. Instead, to find the derivative of this function, we note that $f(x) = x^x = e^{x \ln x}$, so differentiation gives

$$\begin{aligned} f'(x) &= \frac{d}{dx}(e^{x \ln x}) && \text{The base } e \text{ is a constant.} \\ &= e^{x \ln x} \frac{d}{dx}(x \ln x) && \frac{d}{dx}e^u, u = x \ln x \\ &= e^{x \ln x} \left(\ln x + x \cdot \frac{1}{x} \right) && \text{Product Rule} \\ &= x^x (\ln x + 1). && x > 0 \end{aligned}$$

We can also find the derivative of $y = x^x$ using logarithmic differentiation, assuming y' exists. ■

The Number e Expressed as a Limit

In Section 1.4 we defined the number e as the base value for which the exponential function $y = a^x$ has slope 1 when it crosses the y -axis at $(0, 1)$. Thus e is the constant that satisfies the equation

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = \ln e = 1. \quad \text{Slope equals } \ln e \text{ from Eq. (7).}$$

We now prove that e can be calculated as a certain limit.

THEOREM 4—The Number e as a Limit The number e can be calculated as the limit

$$e = \lim_{x \rightarrow 0} (1 + x)^{1/x}.$$

Proof If $f(x) = \ln x$, then $f'(x) = 1/x$, so $f'(1) = 1$. But, by the definition of derivative,

$$\begin{aligned} f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \rightarrow 0} \frac{f(1+x) - f(1)}{x} && \ln 1 = 0 \\ &= \lim_{x \rightarrow 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \rightarrow 0} \frac{1}{x} \ln(1+x) \\ &= \lim_{x \rightarrow 0} \ln(1+x)^{1/x} = \ln \left[\lim_{x \rightarrow 0} (1+x)^{1/x} \right]. && \ln \text{ is continuous, Theorem 9 in Chapter 2} \end{aligned}$$

Because $f'(1) = 1$, we have

$$\ln \left[\lim_{x \rightarrow 0} (1+x)^{1/x} \right] = 1.$$

Therefore, exponentiating both sides, we get

$$\lim_{x \rightarrow 0} (1+x)^{1/x} = e.$$

See Figure 3.40. ■

Approximating the limit in Theorem 4 by taking x very small gives approximations to e . Its value is $e \approx 2.718281828459045$ to 15 decimal places.

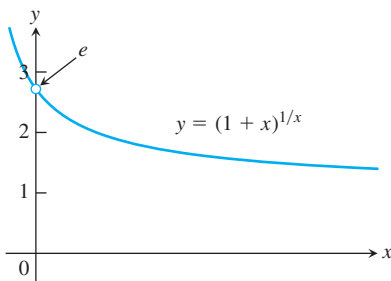


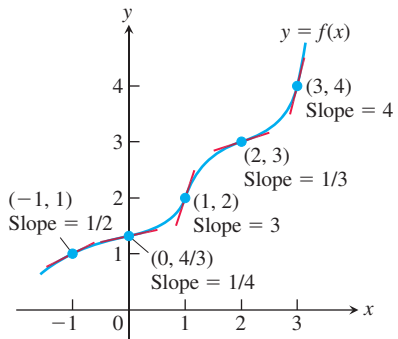
FIGURE 3.40 The number e is the limit of the function graphed here as $x \rightarrow 0$.

EXERCISES 3.8

Derivatives of Inverse Functions

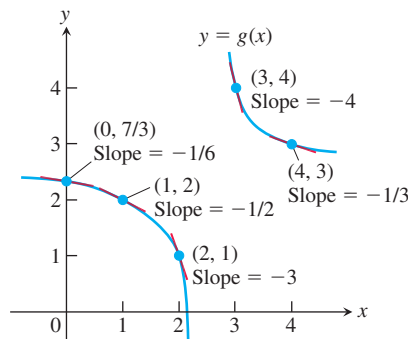
In Exercises 1–4:

- Find $f^{-1}(x)$.
 - Graph f and f^{-1} together.
 - Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$ to show that $(df^{-1}/dx)|_{x=f(a)} = 1/(df/dx)|_{x=a}$.
- $f(x) = 2x + 3$, $a = -1$
 - $f(x) = \frac{x+2}{1-x}$, $a = \frac{1}{2}$
 - $f(x) = 5 - 4x$, $a = 1/2$
 - $f(x) = 2x^2$, $x \geq 0$, $a = 5$
 - Show that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of one another.
 - Graph f and g over an x -interval large enough to show the graphs intersecting at $(1, 1)$ and $(-1, -1)$. Be sure the picture shows the required symmetry about the line $y = x$.
 - Find the slopes of the tangent lines to the graphs of f and g at $(1, 1)$ and $(-1, -1)$ (four tangent lines in all).
 - What lines are tangent to the curves at the origin?
 - Show that $h(x) = x^{3/4}$ and $k(x) = (4x)^{1/3}$ are inverses of one another.
 - Graph h and k over an x -interval large enough to show the graphs intersecting at $(2, 2)$ and $(-2, -2)$. Be sure the picture shows the required symmetry about the line $y = x$.
 - Find the slopes of the tangent lines to the graphs at h and k at $(2, 2)$ and $(-2, -2)$.
 - What lines are tangent to the curves at the origin?
 - Let $f(x) = x^3 - 3x^2 - 1$, $x \geq 2$. Find the value of df^{-1}/dx at the point $x = -1 = f(3)$.
 - Let $f(x) = x^2 - 4x - 5$, $x > 2$. Find the value of df^{-1}/dx at the point $x = 0 = f(5)$.
 - Suppose that the differentiable function $y = f(x)$ has an inverse and that the graph of f passes through the point $(2, 4)$ and has a slope of $1/3$ there. Find the value of df^{-1}/dx at $x = 4$.
 - Suppose that the differentiable function $y = g(x)$ has an inverse and that the graph of g passes through the origin with slope 2. Find the slope of the graph of g^{-1} at the origin.
 - The accompanying figure shows the graph of the function f .

Assuming the inverse function f^{-1} is differentiable, find the slope of $f^{-1}(x)$ at

- $x = 1$
- $x = 2$
- $x = 3$

12. The accompanying figure shows the graph of the function
- g
- .

Assuming the inverse function g^{-1} is differentiable, find the slope of $g^{-1}(x)$ at

- $x = 1$
- $x = 2$
- $x = 3$

13. Suppose that the function
- f
- and its derivative with respect to
- x
- have the following values at
- $x = 0, 1, 2, 3$
- , and 4.

x	0	1	2	3	4
$f(x)$	3	6	0	1	2
$f'(x)$	$4/3$	5	4	$1/2$	$1/7$

Assuming the inverse function f^{-1} is differentiable, find the slope of $f^{-1}(x)$ at

- $x = 1$
- $x = 2$
- $x = 3$

14. Suppose that the function
- g
- and its derivative with respect to
- x
- have the following values at
- $x = 0, 1, 2, 3$
- , and 4.

x	0	1	2	3	4
$g(x)$	-4	-1	1	2	3
$g'(x)$	3	2	$5/4$	$2/3$	$1/5$

Assuming the inverse function g^{-1} is differentiable, find the slope of $g^{-1}(x)$ at

- $x = 1$
- $x = 2$
- $x = 3$

Derivatives of Logarithms

In Exercises 15–44, find the derivative of y with respect to x , t , or θ , as appropriate.

- $y = \ln 3x + x$
- $y = \ln(t^2)$
- $y = \ln \frac{3}{x}$
- $y = \ln(\theta + 1) - e^\theta$
- $y = \ln x^3$
- $y = t(\ln t)^2$
- $y = \frac{1}{\ln 3x}$
- $y = \ln(t^{3/2}) + \sqrt{t}$
- $y = \ln(\sin x)$
- $y = (\cos \theta) \ln(2\theta + 2)$
- $y = (\ln x)^3$
- $y = t \ln \sqrt{t}$

27. $y = \frac{x^4}{4} \ln x - \frac{x^4}{16}$

29. $y = \frac{\ln t}{t}$

31. $y = \frac{\ln x}{1 + \ln x}$

33. $y = \ln(\ln x)$

35. $y = \theta(\sin(\ln \theta) + \cos(\ln \theta))$

36. $y = \ln(\sec \theta + \tan \theta)$

37. $y = \ln \frac{1}{x\sqrt{x+1}}$

39. $y = \frac{1 + \ln t}{1 - \ln t}$

41. $y = \ln(\sec(\ln \theta))$

43. $y = \ln \left(\frac{(x^2 + 1)^5}{\sqrt{1 - x}} \right)$

28. $y = (x^2 \ln x)^4$

30. $y = \frac{t}{\sqrt{\ln t}}$

32. $y = \frac{x \ln x}{1 + \ln x}$

34. $y = \ln(\ln(\ln x))$

38. $y = \frac{1}{2} \ln \frac{1+x}{1-x}$

40. $y = \sqrt{\ln \sqrt{t}}$

42. $y = \ln \left(\frac{\sqrt{\sin \theta \cos \theta}}{1 + 2 \ln \theta} \right)$

44. $y = \ln \sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$

Logarithmic Differentiation

In Exercises 45–58, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

45. $y = \sqrt{x(x+1)}$

46. $y = \sqrt{(x^2 + 1)(x - 1)^2}$

47. $y = \sqrt{\frac{t}{t+1}}$

48. $y = \sqrt{\frac{1}{t(t+1)}}$

49. $y = (\sin \theta) \sqrt{\theta + 3}$

50. $y = (\tan \theta) \sqrt{2\theta + 1}$

51. $y = t(t+1)(t+2)$

52. $y = \frac{1}{t(t+1)(t+2)}$

53. $y = \frac{\theta + 5}{\theta \cos \theta}$

54. $y = \frac{\theta \sin \theta}{\sqrt{\sec \theta}}$

55. $y = \frac{x\sqrt{x^2 + 1}}{(x+1)^{2/3}}$

56. $y = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}$

57. $y = \sqrt[3]{\frac{x(x-2)}{x^2 + 1}}$

58. $y = \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2 + 1)(2x + 3)}}$

Finding Derivatives

In Exercises 59–70, find the derivative of y with respect to x , t , or θ , as appropriate.

59. $y = \ln(\cos^2 \theta)$

60. $y = \ln(3\theta e^{-\theta})$

61. $y = \ln(3te^{-t})$

62. $y = \ln(2e^{-t} \sin t)$

63. $y = \ln \left(\frac{e^\theta}{1 + e^\theta} \right)$

64. $y = \ln \left(\frac{\sqrt{\theta}}{1 + \sqrt{\theta}} \right)$

65. $y = e^{(\cos t + \ln t)}$

66. $y = e^{\sin t} (\ln t^2 + 1)$

In Exercises 67–70, find dy/dx .

67. $\ln y = e^y \sin x$

68. $\ln xy = e^{x+y}$

69. $x^y = y^x$

70. $\tan y = e^x + \ln x$

In Exercises 71–92, find the derivative of y with respect to the given independent variable.

71. $y = 2^x$

72. $y = 3^{-x}$

73. $y = 5^{\sqrt{s}}$

74. $y = 2^{(s^2)}$

75. $y = x^\pi$

76. $y = t^{1-e}$

77. $y = \log_2 5\theta$

78. $y = \log_3(1 + \theta \ln 3)$

79. $y = \log_4 x + \log_4 x^2$

81. $y = \log_2 r \cdot \log_4 r$

83. $y = \log_3 \left(\left(\frac{x+1}{x-1} \right)^{\ln 3} \right)$

85. $y = \theta \sin(\log_7 \theta)$

87. $y = \log_5 e^x$

89. $y = 3^{\log_2 t}$

91. $y = \log_2(8t^{\ln 2})$

80. $y = \log_{25} e^x - \log_5 \sqrt{x}$

82. $y = \log_3 r \cdot \log_9 r$

84. $y = \log_5 \sqrt{\left(\frac{7x}{3x+2} \right)^{\ln 5}}$

86. $y = \log_7 \left(\frac{\sin \theta \cos \theta}{e^\theta 2^\theta} \right)$

88. $y = \log_2 \left(\frac{x^2 e^2}{2\sqrt{x+1}} \right)$

90. $y = 3 \log_8(\log_2 t)$

92. $y = t \log_3(e^{(\sin t)(\ln 3)})$

Powers with Variable Bases and Exponents

In Exercises 93–104, use logarithmic differentiation or the method in Example 7 to find the derivative of y with respect to the given independent variable.

93. $y = (x+1)^x$

94. $y = x^{(x+1)}$

95. $y = (\sqrt{t})^t$

96. $y = t^{\sqrt{t}}$

97. $y = (\sin x)^x$

98. $y = x^{\sin x}$

99. $y = x^{\ln x}$

100. $y = (\ln x)^{\ln x}$

101. $y^x = x^3 y$

102. $x^{\sin y} = \ln y$

103. $x = y^{xy}$

104. $e^y = y^{\ln x}$

Theory and Applications

105. If we write $g(x)$ for $f^{-1}(x)$, Equation (1) can be written as

$$g'(f(a)) = \frac{1}{f'(a)}, \text{ or } g'(f(a)) \cdot f'(a) = 1.$$

If we then write x for a , we get

$$g'(f(x)) \cdot f'(x) = 1.$$

The latter equation may remind you of the Chain Rule, and indeed there is a connection.

Assume that f and g are differentiable functions that are inverses of one another, so that $(g \circ f)(x) = x$. Differentiate both sides of this equation with respect to x , using the Chain Rule to express $(g \circ f)'(x)$ as a product of derivatives of g and f . What do you find? (This is not a proof of Theorem 3 because we assume here the theorem's conclusion that $g = f^{-1}$ is differentiable.)

106. Show that $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n} \right)^n = e^x$ for any $x > 0$.

107. If $f(x) = x^n$, $n > 1$, show from the definition of the derivative that $f'(0) = 0$.

108. Using mathematical induction, show that for $n > 1$,

$$\frac{d^n}{dx^n} \ln x = (-1)^{n-1} \frac{(n-1)!}{x^n}.$$

COMPUTER EXPLORATIONS

In Exercises 109–116, you will explore some functions and their inverses together with their derivatives and tangent line approximations at specified points. Perform the following steps using your CAS:

- Plot the function $y = f(x)$ together with its derivative over the given interval. Explain why you know that f is one-to-one over the interval.
- Solve the equation $y = f(x)$ for x as a function of y , and name the resulting inverse function g .

- c. Find an equation for the tangent line to f at the specified point $(x_0, f(x_0))$.
- d. Find an equation for the tangent line to g at the point $(f(x_0), x_0)$ located symmetrically across the 45° line $y = x$ (which is the graph of the identity function). Use Theorem 3 to find the slope of this tangent line.
- e. Plot the functions f and g , the identity, the two tangent lines, and the line segment joining the points $(x_0, f(x_0))$ and $(f(x_0), x_0)$. Discuss the symmetries you see across the main diagonal (the line $y = x$).
109. $y = \sqrt{3x - 2}$, $\frac{2}{3} \leq x \leq 4$, $x_0 = 3$
110. $y = \frac{3x + 2}{2x - 11}$, $-2 \leq x \leq 2$, $x_0 = 1/2$
111. $y = \frac{4x}{x^2 + 1}$, $-1 \leq x \leq 1$, $x_0 = 1/2$

112. $y = \frac{x^3}{x^2 + 1}$, $-1 \leq x \leq 1$, $x_0 = 1/2$

113. $y = x^3 - 3x^2 - 1$, $2 \leq x \leq 5$, $x_0 = \frac{27}{10}$

114. $y = 2 - x - x^3$, $-2 \leq x \leq 2$, $x_0 = \frac{3}{2}$

115. $y = e^x$, $-3 \leq x \leq 5$, $x_0 = 1$

116. $y = \sin x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $x_0 = 1$

In Exercises 117 and 118, repeat the steps above to solve for the functions $y = f(x)$ and $x = f^{-1}(y)$ defined implicitly by the given equations over the interval.

117. $y^{1/3} - 1 = (x + 2)^3$, $-5 \leq x \leq 5$, $x_0 = -3/2$

118. $\cos y = x^{1/5}$, $0 \leq x \leq 1$, $x_0 = 1/2$

3.9 Inverse Trigonometric Functions

We introduced the six basic inverse trigonometric functions in Section 1.5 but focused there on the arcsine and arccosine functions. Here we complete the study of how all six basic inverse trigonometric functions are defined, graphed, and evaluated, and how their derivatives are computed.

Inverses of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

The graphs of these four basic inverse trigonometric functions are shown in Figure 3.41. We obtain these graphs by reflecting the graphs of the restricted trigonometric functions (as discussed in Section 1.5) through the line $y = x$. Let's take a closer look at the arctangent, arccotangent, arcsecant, and arccosecant functions.

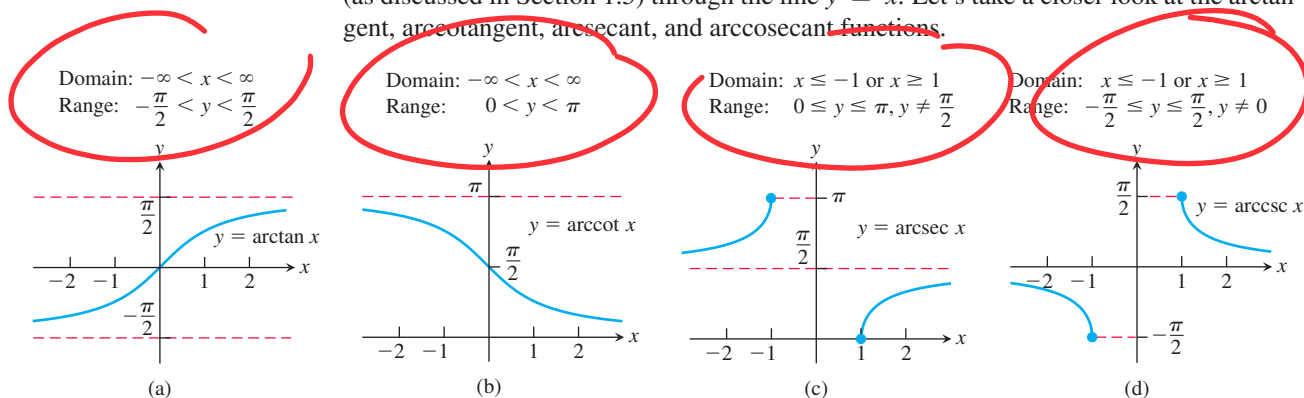


FIGURE 3.41 Graphs of the arctangent, arccotangent, arcsecant, and arccosecant functions.

The arctangent of x is a radian angle whose tangent is x . The arccotangent of x is an angle whose cotangent is x , and so forth. The angles belong to the restricted domains of the tangent, cotangent, secant, and cosecant functions.

DEFINITIONS

$y = \arctan x$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$y = \operatorname{arccot} x$ is the number in $(0, \pi)$ for which $\cot y = x$.

$y = \operatorname{arcsec} x$ is the number in $[0, \pi/2) \cup (\pi/2, \pi]$ for which $\sec y = x$.

$y = \operatorname{arccsc} x$ is the number in $[-\pi/2, 0) \cup (0, \pi/2]$ for which $\csc y = x$.

We use open or half-open intervals to avoid values for which the tangent, cotangent, secant, and cosecant functions are undefined. (See Figure 3.41.)

As we discussed in Section 1.5, the arcsine and arccosine functions are often written as $\sin^{-1}x$ and $\cos^{-1}x$ instead of $\arcsin x$ and $\arccos x$. Likewise, we often denote the other inverse trigonometric functions by $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, and $\csc^{-1}x$.

The graph of $y = \arctan x$ is symmetric about the origin because it is a branch of the graph $x = \tan y$ that is symmetric about the origin (Figure 3.41a). Algebraically this means that

$$\arctan(-x) = -\arctan x;$$

the arctangent is an odd function. The graph of $y = \operatorname{arccot} x$ has no such symmetry (Figure 3.41b). Notice from Figure 3.41a that the graph of the arctangent function has two horizontal asymptotes: one at $y = \pi/2$ and the other at $y = -\pi/2$.

The inverses of the restricted forms of $\sec x$ and $\csc x$ are chosen to be the functions graphed in Figures 3.41c and 3.41d.

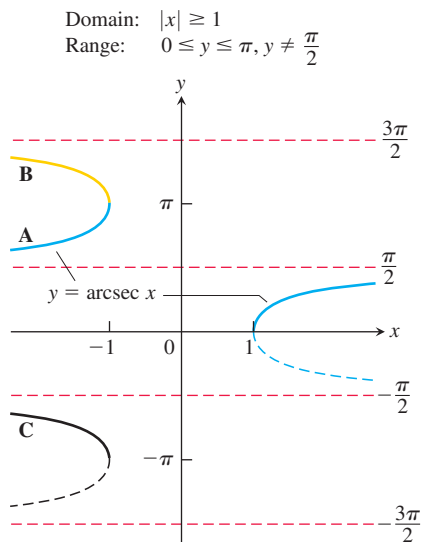


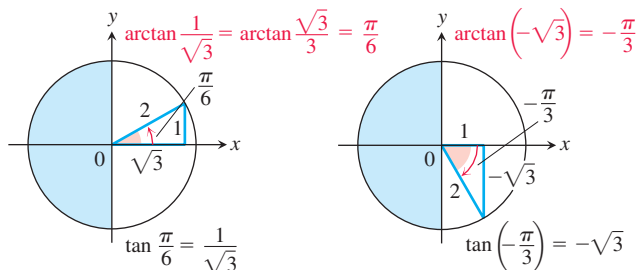
FIGURE 3.42 There are several logical choices for the left-hand branch of $y = \operatorname{arcsec} x$. With choice **A**, $\operatorname{arcsec} x = \arccos(1/x)$, a useful identity employed by many calculators.

Caution There is no general agreement about how to define $\operatorname{arcsec} x$ for negative values of x . We chose angles in the second quadrant between $\pi/2$ and π . This choice makes $\operatorname{arcsec} x = \arccos(1/x)$. It also makes $\operatorname{arcsec} x$ an increasing function on each interval of its domain. Some tables choose $\operatorname{arcsec} x$ to lie in $[-\pi, -\pi/2)$ for $x < 0$, and some texts choose it to lie in $[\pi, 3\pi/2)$ (Figure 3.42). These choices simplify the formula for the derivative (our formula needs absolute value signs) but fail to satisfy the computational equation $\operatorname{arcsec} x = \arccos(1/x)$. From this, we can derive the identity

$$\operatorname{arcsec} x = \arccos\left(\frac{1}{x}\right) = \frac{\pi}{2} - \arcsin\left(\frac{1}{x}\right) \quad (1)$$

by applying Equation (5) in Section 1.5.

EXAMPLE 1 The accompanying figures show two values of $\arctan x$.



x	$\arctan x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
0	0
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

The angles come from the first and fourth quadrants because the range of $\arctan x$ is $(-\pi/2, \pi/2)$.

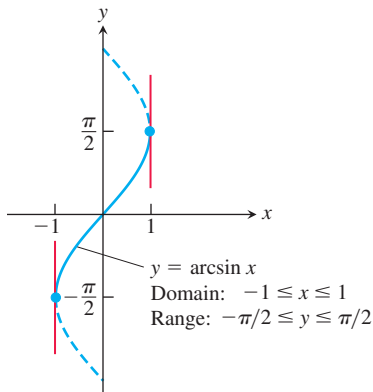


FIGURE 3.43 The graph of $y = \arcsin x$ has vertical tangent lines at $x = -1$ and $x = 1$.

The Derivative of $y = \arcsin u$

We know that the function $x = \sin y$ is differentiable in the interval $-\pi/2 < y < \pi/2$ and that its derivative, the cosine, is positive there. Theorem 3 in Section 3.8 therefore assures us that the inverse function $y = \arcsin x$ is differentiable throughout the interval $-1 < x < 1$. We cannot expect it to be differentiable at $x = 1$ or $x = -1$ because the tangent lines to the graph are vertical at these points (see Figure 3.43).

We find the derivative of $y = \arcsin x$ by applying Theorem 3 with $f(x) = \sin x$ and $f^{-1}(x) = \arcsin x$:

$$\begin{aligned}
 (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3} \\
 &= \frac{1}{\cos(\arcsin x)} && f'(y) = \cos y \\
 &= \frac{1}{\sqrt{1 - \sin^2(\arcsin x)}} && \cos y = \sqrt{1 - \sin^2 y} \\
 &= \frac{1}{\sqrt{1 - x^2}}. && \sin(\arcsin x) = x
 \end{aligned}$$

For $|x| < 1$,

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}.$$

If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get the general formula

$$\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1.$$

EXAMPLE 2 Using the Chain Rule, we calculate the derivative

$$\frac{d}{dx}(\arcsin x^2) = \frac{1}{\sqrt{1 - (x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1 - x^4}}. \quad \blacksquare$$

The Derivative of $y = \arctan u$

We find the derivative of $y = \arctan x$ by applying Theorem 3 with $f(x) = \tan x$ and $f^{-1}(x) = \arctan x$. Theorem 3 can be applied because the derivative of $\tan x$ is positive for $-\pi/2 < x < \pi/2$:

$$\begin{aligned}
 (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3} \\
 &= \frac{1}{\sec^2(\arctan x)} && f'(u) = \sec^2 u \\
 &= \frac{1}{1 + \tan^2(\arctan x)} && \sec^2 u = 1 + \tan^2 u \\
 &= \frac{1}{1 + x^2}. && \tan(\arctan x) = x
 \end{aligned}$$

The derivative is defined for all real numbers:

$$\frac{d}{dx}(\arctan x) = \frac{1}{1 + x^2}.$$

The derivative is defined for all real numbers. If u is a differentiable function of x , we get the Chain Rule form:

$$\frac{d}{dx}(\arctan u) = \frac{1}{1 + u^2} \frac{du}{dx}.$$

The Chain Rule can also be combined with the arctangent function in other ways, as illustrated by the following example.

EXAMPLE 3

$$\begin{aligned}
 \frac{d}{dx} \left(\frac{1}{\arctan x} \right) &= \frac{d}{dx} (\arctan x)^{-1} && \text{Derivative of the reciprocal} \\
 &= (-1)(\arctan x)^{-2} \frac{d}{dx} (\arctan x) && \text{(not the inverse) of arctangent} \\
 &= \frac{-1}{(\arctan x)^2} \cdot \frac{1}{1+x^2} && \text{Apply the Chain Rule.}
 \end{aligned}$$

The Derivative of $y = \operatorname{arcsec} u$

Theorem 3 does not apply to the function $\sec x$ directly, since its domain is not connected. However, we can apply Theorem 3 to each of the two intervals in its domain to see that the inverse of the one-to-one function $\sec x$ is indeed differentiable. The formula for the derivative of $\operatorname{arcsec} x$ on its domain $|x| > 1$ can then be found by using implicit differentiation and the Chain Rule as follows:

$$\begin{aligned}
 y &= \operatorname{arcsec} x \\
 \sec y &= x && \text{Inverse function relationship} \\
 \frac{d}{dx}(\sec y) &= \frac{d}{dx} x && \text{Differentiate both sides.} \\
 (\sec y \tan y) \frac{dy}{dx} &= 1 && \text{Chain Rule} \\
 \frac{dy}{dx} &= \frac{1}{\sec y \tan y}. && \text{Since } |x| > 1, y \text{ lies in } (0, \pi/2) \cup (\pi/2, \pi) \\
 &&& \text{and } \sec y \tan y \neq 0.
 \end{aligned}$$

To express the result in terms of x , we use the relationships

$$\sec y = x \quad \text{and} \quad \tan y = \pm \sqrt{\sec^2 y - 1} = \pm \sqrt{x^2 - 1}$$

to get

$$\frac{dy}{dx} = \pm \frac{1}{x\sqrt{x^2 - 1}}.$$

Can we do anything about the $\pm$ sign? A glance at Figure 3.44 shows that the slope of the graph $y = \operatorname{arcsec} x$ is always positive. Thus,

$$\frac{d}{dx} \operatorname{arcsec} x = \begin{cases} +\frac{1}{x\sqrt{x^2 - 1}} & \text{if } x > 1 \\ -\frac{1}{x\sqrt{x^2 - 1}} & \text{if } x < -1. \end{cases}$$

With the absolute value symbol, we can write a single expression that eliminates the “ $\pm$ ” ambiguity:

$$\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2 - 1}}, \quad |x| > 1.$$

If u is a differentiable function of x with $|u| > 1$, we have the formula

$$\frac{d}{dx}(\operatorname{arcsec} u) = \frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \quad |u| > 1.$$

EXAMPLE 4 Using the Chain Rule and derivative of the arcsecant function, we find

$$\begin{aligned}
 \frac{d}{dx} \operatorname{arcsec}(5x^4) &= \frac{1}{|5x^4|\sqrt{(5x^4)^2 - 1}} \frac{d}{dx}(5x^4) \\
 &= \frac{1}{5x^4\sqrt{25x^8 - 1}} (20x^3) && 5x^4 > 1 \\
 &= \frac{4}{x\sqrt{25x^8 - 1}}.
 \end{aligned}$$

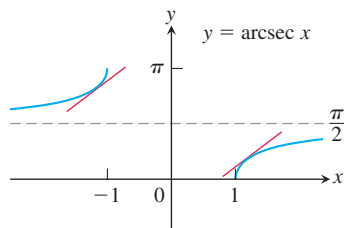


FIGURE 3.44 The slope of the curve $y = \operatorname{arcsec} x$ is positive for both $x < -1$ and $x > 1$.

Derivatives of the Other Three Inverse Trigonometric Functions

We could use the same techniques to find the derivatives of the other three inverse trigonometric functions—arccosine, arccotangent, and arccosecant—but there is an easier way, thanks to the following identities.

Inverse Function–Inverse Cofunction Identities

$$\arccos x = \pi/2 - \arcsin x$$

$$\operatorname{arccot} x = \pi/2 - \arctan x$$

$$\operatorname{arccsc} x = \pi/2 - \operatorname{arcsec} x$$

We saw the first of these identities in Equation (5) of Section 1.5. The others are derived in a similar way. It follows easily that the derivatives of the inverse cofunctions are the negatives of the derivatives of the corresponding inverse functions. For example, the derivative of $\arccos x$ is calculated as follows:

$$\begin{aligned} \frac{d}{dx}(\arccos x) &= \frac{d}{dx}\left(\frac{\pi}{2} - \arcsin x\right) && \text{Identity} \\ &= -\frac{d}{dx}(\arcsin x) \\ &= -\frac{1}{\sqrt{1-x^2}}. && \text{Derivative of arcsine} \end{aligned}$$

$$\sin^{-1} x = \arcsin x$$

The derivatives of the inverse trigonometric functions are summarized in Table 3.1.

TABLE 3.1 Derivatives of the inverse trigonometric functions

1. $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \quad (x < 1)$	4. $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \quad (x < 1)$
2. $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$	5. $\frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{1+x^2}$
3. $\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{ x \sqrt{x^2-1}} \quad (x > 1)$	6. $\frac{d}{dx}(\operatorname{arccsc} x) = -\frac{1}{ x \sqrt{x^2-1}} \quad (x > 1)$

EXERCISES 3.9

Remember that $\arcsin$ and $\sin^{-1}$ represent the same function, and similarly for the other trigonometric functions.

Common Values

Use reference triangles in an appropriate quadrant, as in Example 1, to find the angles in Exercises 1–8.

- a. $\arctan 1$ b. $\arctan(-\sqrt{3})$ c. $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$
- a. $\arctan(-1)$ b. $\tan^{-1} \sqrt{3}$ c. $\arctan\left(\frac{-1}{\sqrt{3}}\right)$
- a. $\arcsin\left(\frac{-1}{2}\right)$ b. $\arcsin\left(\frac{1}{\sqrt{2}}\right)$ c. $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$
- a. $\sin^{-1}\left(\frac{1}{2}\right)$ b. $\arcsin\left(\frac{-1}{\sqrt{2}}\right)$ c. $\arcsin\left(\frac{\sqrt{3}}{2}\right)$
- a. $\arccos\left(\frac{1}{2}\right)$ b. $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ c. $\arccos\left(\frac{\sqrt{3}}{2}\right)$
- a. $\csc^{-1} \sqrt{2}$ b. $\operatorname{arccsc}\left(\frac{-2}{\sqrt{3}}\right)$ c. $\operatorname{arccsc} 2$

- a. $\sec^{-1}(-\sqrt{2})$ b. $\operatorname{arcsec}\left(\frac{2}{\sqrt{3}}\right)$ c. $\operatorname{arcsec}(-2)$
- a. $\operatorname{arccot}(-1)$ b. $\operatorname{arccot}(\sqrt{3})$ c. $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

Evaluations

Find the values in Exercises 9–12.

- $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$
- $\sec\left(\arccos\frac{1}{2}\right)$
- $\tan\left(\arcsin\left(-\frac{1}{2}\right)\right)$
- $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

Limits

Find the limits in Exercises 13–20. (If in doubt, look at the function's graph.)

- $\lim_{x \rightarrow 1^-} \arcsin x$
- $\lim_{x \rightarrow -1^+} \cos^{-1} x$
- $\lim_{x \rightarrow \infty} \tan^{-1} x$
- $\lim_{x \rightarrow -\infty} \arctan x$

17. $\lim_{x \rightarrow \infty} \operatorname{arcsec} x$

19. $\lim_{x \rightarrow \infty} \csc^{-1} x$

18. $\lim_{x \rightarrow -\infty} \sec^{-1} x$

20. $\lim_{x \rightarrow -\infty} \operatorname{arccsc} x$

Finding Derivatives

In Exercises 21–48, find the derivative of y with respect to the appropriate variable.

21. $y = \cos^{-1}(x^2)$

22. $y = \arccos(1/x)$

23. $y = \arcsin \sqrt{2}t$

24. $y = \sin^{-1}(1-t)$

25. $y = \operatorname{arcsec}(2s+1)$

26. $y = \sec^{-1} 5s$

27. $y = \csc^{-1}(x^2+1), \quad x > 0$

28. $y = \operatorname{arccsc} \frac{x}{2}$

29. $y = \sec^{-1} \frac{1}{t}, \quad 0 < t < 1$

30. $y = \arcsin \frac{3}{t^2}$

31. $y = \operatorname{arccot} \sqrt{t}$

32. $y = \cot^{-1} \sqrt{t-1}$

33. $y = \ln(\tan^{-1} x)$

34. $y = \tan^{-1}(\ln x)$

35. $y = \operatorname{arccsc}(e^t)$

36. $y = \arccos(e^{-t})$

37. $y = s\sqrt{1-s^2} + \cos^{-1} s$

38. $y = \sqrt{s^2-1} - \sec^{-1} s$

39. $y = \tan^{-1} \sqrt{x^2-1} + \csc^{-1} x, \quad x > 1$

40. $y = \cot^{-1} \frac{1}{x} - \tan^{-1} x$

41. $y = x \arcsin x + \sqrt{1-x^2}$

42. $y = \ln(x^2+4) - x \arctan \left(\frac{x}{2}\right)$

43. $y = \sqrt{\arcsin x}$

44. $y = e^{\operatorname{arcsec} x}$

45. $y = \cos(x - \arccos x)$

46. $y = \frac{x}{1 + \arctan x}$

47. $y = (\operatorname{arccot}(x^3))^3$

48. $y = \log_2 \operatorname{arccsc} \sqrt{x}$

For problems 49–52 use implicit differentiation to find $\frac{dy}{dx}$ at the given point P .

49. $3 \arctan x + \arcsin y = \frac{\pi}{4}; \quad P(1, -1)$

50. $\arcsin(x+y) + \arccos(x-y) = \frac{5\pi}{6}; \quad P\left(0, \frac{1}{2}\right)$

51. $y \cos^{-1}(xy) = \frac{-3\sqrt{2}}{4}\pi; \quad P\left(\frac{1}{2}, -\sqrt{2}\right)$

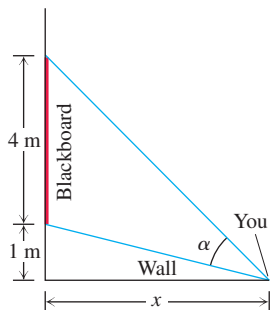
52. $16(\tan^{-1} 3y)^2 + 9(\tan^{-1} 2x)^2 = 2\pi^2; \quad P\left(\frac{\sqrt{3}}{2}, \frac{1}{3}\right)$

Theory and Examples

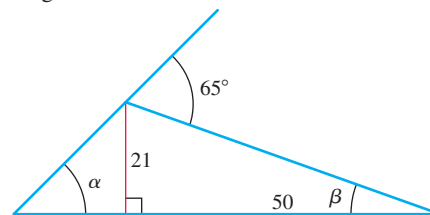
53. You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 4 m long and starts 1 m from the wall you are sitting next to. Show that your viewing angle is

$$\alpha = \operatorname{arccot} \frac{x}{15} - \operatorname{arccot} \frac{x}{3}$$

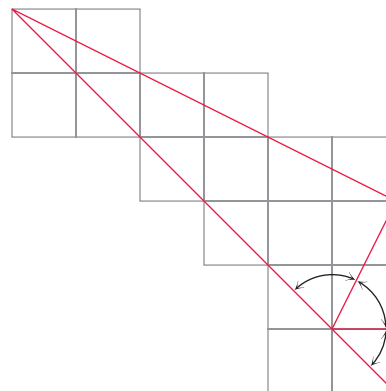
if you are x meters from the front wall.



54. Find the angle α .

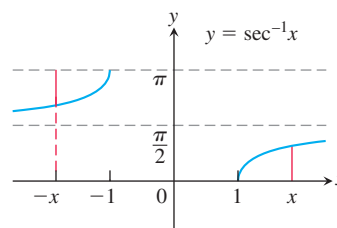


55. Here is an informal proof that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$. Explain what is going on.



56. Two derivations of the identity $\sec^{-1}(-x) = \pi - \sec^{-1} x$

a. (Geometric) Here is a pictorial proof that $\sec^{-1}(-x) = \pi - \sec^{-1} x$. See if you can tell what is going on.



b. (Algebraic) Derive the identity $\sec^{-1}(-x) = \pi - \sec^{-1} x$ by combining the following two equations from the text:

$$\cos^{-1}(-x) = \pi - \cos^{-1} x \quad \text{Eq. (4), Section 1.5}$$

$$\sec^{-1} x = \cos^{-1}(1/x) \quad \text{Eq. (1)}$$

Which of the expressions in Exercises 57–60 are defined, and which are not? Give reasons for your answers.

57. a. $\arctan 2$

b. $\cos^{-1} 2$

58. a. $\operatorname{arccsc}(1/2)$

b. $\csc^{-1} 2$

59. a. $\sec^{-1} 0$

b. $\arcsin \sqrt{2}$

60. a. $\cot^{-1}(-1/2)$

b. $\arccos(-5)$

61. Use the identity

$$\operatorname{arccsc} x = \frac{\pi}{2} - \operatorname{arcsec} x$$

to derive the formula for the derivative of $\operatorname{arccsc} x$ in Table 3.1 from the formula for the derivative of $\operatorname{arcsec} x$.

62. Derive the formula

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

for the derivative of $y = \arctan x$ by differentiating both sides of the equivalent equation $\tan y = x$.

63. Use the Derivative Rule in Section 3.8, Theorem 3, to derive

$$\frac{d}{dx} \operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1.$$

64. Use the identity

$$\operatorname{arccot} x = \frac{\pi}{2} - \arctan x$$

to derive the formula for the derivative of $\operatorname{arccot} x$ in Table 3.1 from the formula for the derivative of $\arctan x$.

65. What is special about the functions

$$f(x) = \sin^{-1} \frac{x-1}{x+1}, x \geq 0, \quad \text{and} \quad g(x) = 2 \tan^{-1} \sqrt{x}?$$

Explain.

66. What is special about the functions

$$f(x) = \sin^{-1} \frac{1}{\sqrt{x^2+1}} \quad \text{and} \quad g(x) = \tan^{-1} \frac{1}{x}?$$

Explain.

T 67. Find the values of

$$\text{a. } \operatorname{arcsec} 1.5 \quad \text{b. } \csc^{-1}(-1.5) \quad \text{c. } \operatorname{arccot} 2$$

T 68. Find the values of

$$\text{a. } \sec^{-1}(-3) \quad \text{b. } \operatorname{arccsc} 1.7 \quad \text{c. } \operatorname{arccot}(-2)$$

T In Exercises 69–71, find the domain and range of each composite function. Then graph the composition of the two functions on separate

screens. Do the graphs make sense in each case? Give reasons for your answers. Comment on any differences you see.

$$69. \text{ a. } y = \arctan(\tan x) \quad \text{b. } y = \tan(\arctan x)$$

$$70. \text{ a. } y = \arcsin(\sin x) \quad \text{b. } y = \sin(\arcsin x)$$

$$71. \text{ a. } y = \arccos(\cos x) \quad \text{b. } y = \cos(\arccos x)$$

T Use your graphing utility for Exercises 72–76.

72. Graph $y = \sec(\sec^{-1} x) = \sec(\cos^{-1}(1/x))$. Explain what you see.

73. **Newton's serpentine** Graph Newton's serpentine, $y = 4x/(x^2 + 1)$. Then graph $y = 2 \sin(2 \tan^{-1} x)$ in the same graphing window. What do you see? Explain.

74. Graph the rational function $y = (2 - x^2)/x^2$. Then graph $y = \cos(2 \sec^{-1} x)$ in the same graphing window. What do you see? Explain.

75. Graph $f(x) = \arcsin x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f'' .

76. Graph $f(x) = \arctan x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f'' .

3.10 Related Rates

In this section we look at questions that arise when two or more related quantities are changing. The problem of determining how the rate of change of one of them affects the rates of change of the others is called a *related rates problem*.

Related Rates Equations

Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If V is the volume and r is the radius of the balloon at an instant of time, then

$$V = \frac{4}{3} \pi r^3.$$

Using the Chain Rule, we differentiate both sides with respect to t to find an equation relating the rates of change of V and r ,

$$\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

So if we know the radius r of the balloon and the rate dV/dt at which the volume is increasing at a given instant of time, then we can solve this last equation for dr/dt to find how fast the radius is increasing at that instant. Note that it is easier to directly measure the rate of increase of the volume (the rate at which air is being pumped into the balloon) than it is to measure the increase in the radius. The related rates equation allows us to calculate dr/dt from dV/dt .

Very often the key to relating the variables in a related rates problem is drawing a picture that shows the geometric relations between them, as illustrated in the following example.

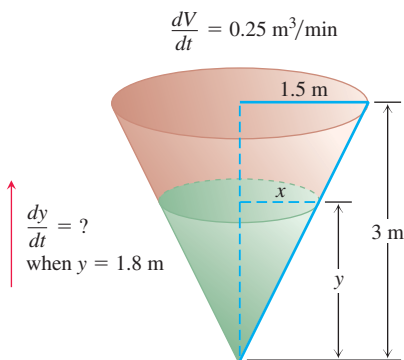


FIGURE 3.45 The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 1).

EXAMPLE 1 Water runs into a conical tank at the rate of $0.25 \text{ m}^3/\text{min}$. The tank stands point down and has a height of 3 m and a base radius of 1.5 m. How fast is the water level rising when the water is 1.8 m deep?

Solution Figure 3.45 shows a partially filled conical tank. The variables in the problem are

V = volume (m^3) of the water in the tank at time t (min)

x = radius (m) of the surface of the water at time t

y = depth (m) of the water in the tank at time t .

We assume that V , x , and y are differentiable functions of t . The constants are the dimensions of the tank. We are asked for dy/dt when

$$y = 1.8 \text{ m} \quad \text{and} \quad \frac{dV}{dt} = 0.25 \text{ m}^3/\text{min}.$$

The water forms a cone with volume

$$V = \frac{1}{3}\pi x^2 y.$$

This equation involves x as well as V and y . Because no information is given about x and dx/dt at the time in question, we need to eliminate x . The similar triangles in Figure 3.45 give us a way to express x in terms y :

$$\frac{x}{y} = \frac{1.5}{3} \quad \text{or} \quad x = \frac{y}{2}.$$

Therefore, we find

$$V = \frac{1}{3}\pi \left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3$$

to give the derivative

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3y^2 \frac{dy}{dt} = \frac{\pi}{4}y^2 \frac{dy}{dt}.$$

Finally, use $y = 1.8$ and $dV/dt = 0.25$ to solve for dy/dt .

$$\begin{aligned} 0.25 &= \frac{\pi}{4}(1.8)^2 \frac{dy}{dt} \\ \frac{dy}{dt} &= \frac{1}{3.24\pi} \approx 0.098 \end{aligned}$$

At the moment in question, the water level is rising at about 0.098 m/min. ■

Related Rates Problem Strategy

1. Let t denote time, and **choose names** for all of the variables that change over time (we will assume that those variables are differentiable functions of t). Identify any quantities that remain constant (these do not need to be given names). In most problems it will be very helpful to **draw a picture** that depicts the setup of the problem.
2. Write an **equation that relates the variables** (and any constants that are present). You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates or values you know.
3. Differentiate with respect to t to obtain a **related rates equation**.
4. Substitute all of the numerical values provided in the problem into the related rates equation. You may need to use the equation(s) relating the variables (which you obtained in Step 2), or use other relationships (such as trigonometric identities), until you reach the point at which *the only remaining unknown quantity is the rate of change that you are asked to find*. Solve for this unknown.

EXAMPLE 2 A hot air balloon rising straight up from a level field is tracked by a range finder 150 m from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

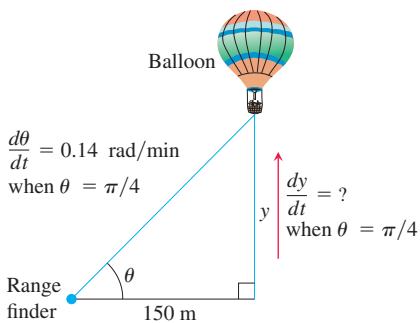


FIGURE 3.46 The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

Solution We draw a picture (Figure 3.46) and name the variables that appear in the problem (which we assume are differentiable functions of t , where time is measured in minutes):

- θ = the angle in radians the range finder makes with the ground, and
- y = the height in meters of the balloon above the ground.

One constant in the picture is the distance from the range finder to the liftoff point (150 m). There is no need to give this distance a special symbol.

Trigonometry (Section 1.3) yields

$$\frac{y}{150} = \tan \theta \quad \text{or} \quad y = 150 \tan \theta. \quad \text{Equation relating the variables}$$

By differentiating with respect to t using the Chain Rule, we obtain

$$\frac{dy}{dt} = 150(\sec^2 \theta) \frac{d\theta}{dt}. \quad \text{Related rates equation}$$

Substituting the known values $\frac{d\theta}{dt} = 0.14$ rad/min and $\theta = \frac{\pi}{4}$ gives

$$\frac{dy}{dt} = 150 \left(\sec^2 \frac{\pi}{4} \right) (0.14) = 150 (\sqrt{2})^2 (0.14) = 42. \quad \sec \frac{\pi}{4} = \sqrt{2}$$

Thus, at the moment in question, the balloon is rising at the rate of 42 m/min. ■

EXAMPLE 3 A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 km north of the intersection and the car is 0.8 km to the east, the police determine with radar that the distance between them and the car is increasing at 30 km/h. If the cruiser is moving at 100 km/h at the instant of measurement, what is the speed of the car?

Solution We picture the car and cruiser in the coordinate plane, using the positive x -axis as the eastbound highway and the positive y -axis as the southbound highway (Figure 3.47). We let t represent time and set

- x = position of car at time t
- y = position of cruiser at time t
- s = distance between car and cruiser at time t

We assume that x , y , and s are differentiable functions of t .

We want to find dx/dt when

$$x = 0.8 \text{ km}, \quad y = 0.6 \text{ km}, \quad \frac{dy}{dt} = -100 \text{ km/h}, \quad \frac{ds}{dt} = 30 \text{ km/h}.$$

Note that dy/dt is negative because y is decreasing.

We differentiate the distance equation between the car and the cruiser,

$$s^2 = x^2 + y^2$$

(we could also use $s = \sqrt{x^2 + y^2}$), and obtain

$$\begin{aligned} 2s \frac{ds}{dt} &= 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \\ \frac{ds}{dt} &= \frac{1}{s} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right) = \frac{1}{\sqrt{x^2 + y^2}} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right). \end{aligned}$$

Finally, we use $x = 0.8$, $y = 0.6$, $dy/dt = -100$, $ds/dt = 30$, and solve for dx/dt .

$$\begin{aligned} 30 &= \frac{1}{\sqrt{(0.8)^2 + (0.6)^2}} \left(0.8 \frac{dx}{dt} + (0.6)(-100) \right) \\ \frac{dx}{dt} &= \frac{30\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(100)}{0.8} = 112.5 \end{aligned}$$

At the moment in question, the car's speed is 112.5 km/h. ■

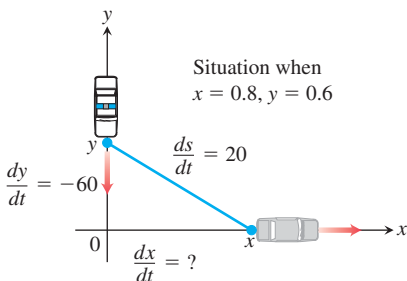


FIGURE 3.47 The speed of the car is related to the speed of the police cruiser and the rate of change of the distance s between them (Example 3).

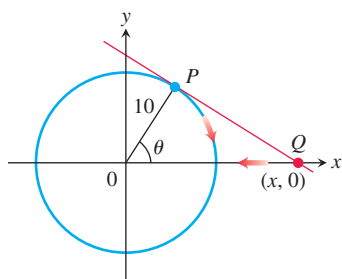


FIGURE 3.48 The particle P travels clockwise along the circle (Example 4).

EXAMPLE 4 A particle P moves clockwise at a constant rate along a circle of radius 10 m centered at the origin. The particle's initial position is $(0, 10)$ on the y -axis, and its final destination is the point $(10, 0)$ on the x -axis. Once the particle is in motion, the tangent line at P intersects the x -axis at a point Q (which moves over time). If it takes the particle 30 s to travel from start to finish, how fast is the point Q moving along the x -axis when it is 20 m from the center of the circle?

Solution We picture the situation in the coordinate plane with the circle centered at the origin (see Figure 3.48). We let t represent time and let θ denote the angle in radians from the x -axis to the radial line joining the origin to P . Since the particle travels from start to finish in 30 s, it is traveling along the circle at a constant rate of $\pi/2$ radians in $1/2$ min, or π rad/min. In other words, $d\theta/dt = -\pi$, with t being measured in minutes. The negative sign appears because θ is decreasing over time.

Setting $x(t)$ to be the distance in meters at time t from the point Q to the origin, we see from Figure 3.48 that

$$x \cos \theta = 10. \quad \text{Equation relating the variables}$$

Differentiation with respect to t gives

$$x(-\sin \theta) \frac{d\theta}{dt} + \frac{dx}{dt} \cos \theta = 0. \quad \text{Related rates equation}$$

We want to find dx/dt given that $x = 20$ and $d\theta/dt = -\pi$. The equation $x \cos \theta = 10$ implies $\cos \theta = 10/20 = 1/2$. Furthermore, for angles θ in the first quadrant, the identity $\sin^2 \theta + \cos^2 \theta = 1$ yields $\sin \theta = \sqrt{1 - (1/2)^2} = \sqrt{3}/2$. Substituting into the related rates equation, we obtain

$$\begin{aligned} (20) \left(-\frac{\sqrt{3}}{2} \right) (-\pi) + \frac{dx}{dt} \cdot \frac{1}{2} &= 0 \\ \frac{dx}{dt} &= -20\sqrt{3}\pi. \end{aligned}$$

Note that x is decreasing because dx/dt is negative. At the moment in question, the point Q is moving toward the origin at the speed of $20\sqrt{3}\pi \approx 109$ m/min. ■

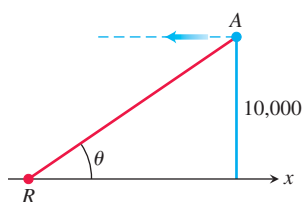


FIGURE 3.49 Jet airliner A traveling at constant altitude toward radar station R (Example 5).

EXAMPLE 5 A jet airliner is flying at a constant altitude of 10,000 m above sea level as it approaches a Pacific island. The aircraft comes within the direct line of sight of a radar station located on the island, and the radar indicates the initial angle between sea level and its line of sight to the aircraft is 30° . How fast (in kilometers per hour) is the aircraft approaching the island when first detected by the radar instrument if it is turning upward (counterclockwise) at the rate of $1/3$ deg/s in order to keep the aircraft within its direct line of sight?

Solution The aircraft A and radar station R are pictured in the coordinate plane, using the positive x -axis as the horizontal distance at sea level from R to A , and the positive y -axis as the vertical altitude above sea level. We let t represent time and observe that $y = 10,000$ is a constant. The general situation and line-of-sight angle θ are depicted in Figure 3.49. We want to find dx/dt when $\theta = \pi/6$ rad and $d\theta/dt = 1/3$ deg/s.

From Figure 3.49, we see that

$$\frac{10,000}{x} = \tan \theta \quad \text{or} \quad x = 10,000 \cot \theta.$$

Using kilometers instead of meters for our distance units, the last equation translates to

$$x = \frac{10,000}{1000} \cot \theta.$$

Differentiation with respect to t gives

$$\frac{dx}{dt} = -10 \csc^2 \theta \frac{d\theta}{dt}.$$

When $\theta = \pi/6$, $\sin^2 \theta = 1/4$, so $\csc^2 \theta = 4$. Converting $d\theta/dt = 1/3 \text{ deg/s}$ to radians per hour, we find

$$\frac{d\theta}{dt} = \frac{1}{3} \left(\frac{\pi}{180} \right) (3600) \text{ rad/h.} \quad 1 \text{ h} = 3600 \text{ s, } 1 \text{ deg} = \pi/180 \text{ rad}$$

Substitution into the equation for dx/dt then gives

$$\frac{dx}{dt} = (-10)(4) \left(\frac{1}{3} \right) \left(\frac{\pi}{180} \right) (3600) \approx -838.$$

The negative sign appears because the distance x is decreasing, so the aircraft is approaching the island at a speed of approximately 838 km/h when first detected by the radar. ■

Note that the solution of Example 5 involved several unit conversions: from seconds to hours and from degrees to radians. When solving related rates problems, we should check that consistent units are used.

EXAMPLE 6 Figure 3.50a shows a rope running through a pulley at P and bearing a weight W at one end. The other end is held 1.5 m above the ground in the hand M of a worker. Suppose the pulley is 7.5 m above ground, the rope is 13.5 m long, and the worker is walking rapidly away from the vertical line PW at the rate of 1.2 m/s. How fast is the weight being raised when the worker's hand is 6.3 m away from PW ?

Solution We let OM be the horizontal line of length x m from a point O directly below the pulley to the worker's hand M at any instant of time (Figure 3.50). Let h be the height of the weight W above O , and let z denote the length of rope from the pulley P to the worker's hand. We want to know dh/dt when $x = 6.3$ given that $dx/dt = 1.2$. Note that the height of P above O is 6 m because O is 1.5 m above the ground. We assume the angle at O is a right angle.

At any instant of time t , we have the following relationships (see Figure 3.50b):

$$6 - h + z = 13.5 \quad \text{Total length of rope is 13.5 m.}$$

$$6^2 + x^2 = z^2. \quad \text{Angle at } O \text{ is a right angle.}$$

If we solve for $z = 7.5 + h$ in the first equation, and substitute into the second equation, we have

$$6^2 + x^2 = (7.5 + h)^2. \quad (1)$$

Differentiating both sides with respect to t gives

$$2x \frac{dx}{dt} = 2(7.5 + h) \frac{dh}{dt},$$

and solving this last equation for dh/dt we find

$$\frac{dh}{dt} = \frac{x}{7.5 + h} \frac{dx}{dt}. \quad (2)$$

Since we know dx/dt , it remains only to find $7.5 + h$ at the instant when $x = 6.3$. From Equation (1),

$$6^2 + 6.3^2 = (7.5 + h)^2$$

so that

$$(7.5 + h)^2 = 75.69, \text{ or } 7.5 + h = 8.7.$$

Equation (2) now gives

$$\frac{dh}{dt} = \frac{6.3}{8.7} \cdot 1.2 = \frac{756}{870} \approx 0.87 \text{ m/s}$$

as the rate at which the weight is being raised when $x = 6.3$ m. ■

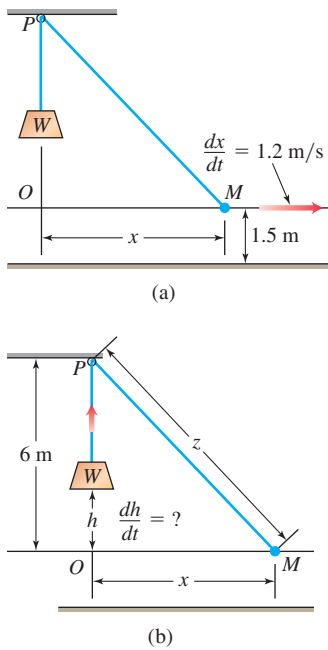


FIGURE 3.50 A worker at M walks to the right, pulling the weight W upward as the rope moves through the pulley P (Example 6).

EXERCISES 3.10

- Area** Suppose that the radius r and area $A = \pi r^2$ of a circle are differentiable functions of t . Write an equation that relates dA/dt to dr/dt .
- Surface area** Suppose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of t . Write an equation that relates dS/dt to dr/dt .
- Assume that $y = 5x$ and $dx/dt = 2$. Find dy/dt .
- Assume that $2x + 3y = 12$ and $dy/dt = -2$. Find dx/dt .
- If $y = x^2$ and $dx/dt = 3$, then what is dy/dt when $x = -1$?
- If $x = y^3 - y$ and $dy/dt = 5$, then what is dx/dt when $y = 2$?
- If $x^2 + y^2 = 25$ and $dx/dt = -2$, then what is dy/dt when $x = 3$ and $y = -4$?
- If $x^2y^3 = 4/27$ and $dy/dt = 1/2$, then what is dx/dt when $x = 2$?
- If $L = \sqrt{x^2 + y^2}$, $dx/dt = -1$, and $dy/dt = 3$, find dL/dt when $x = 5$ and $y = 12$.
- If $r + s^2 + v^3 = 12$, $dr/dt = 4$, and $ds/dt = -3$, find dv/dt when $r = 3$ and $s = 1$.
- If the original 24 m edge length x of a cube decreases at the rate of 5 m/min, when $x = 3$ m at what rate does the cube's
 - surface area change?
 - volume change?
- A cube's surface area increases at the rate of $72 \text{ cm}^2/\text{s}$. At what rate is the cube's volume changing when the edge length is $x = 3$ cm?
- Volume** The radius r and height h of a right circular cylinder are related to the cylinder's volume V by the formula $V = \pi r^2 h$.
 - How is dV/dt related to dh/dt if r is constant?
 - How is dV/dt related to dr/dt if h is constant?
 - How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?
- Volume** The radius r and height h of a right circular cone are related to the cone's volume V by the equation $V = (1/3)\pi r^2 h$.
 - How is dV/dt related to dh/dt if r is constant?
 - How is dV/dt related to dr/dt if h is constant?
 - How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?
- Changing voltage** The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/s while I is decreasing at the rate of $1/3$ amp/s. Let t denote time in seconds.
- Electrical power** The power P (watts) of an electric circuit is related to the circuit's resistance R (ohms) and current I (amperes) by the equation $P = RI^2$.
 - How are dP/dt , dR/dt , and dI/dt related if none of P , R , and I are constant?
 - How is dR/dt related to dI/dt if P is constant?
- Distance** Let x and y be differentiable functions of t , and let $s = \sqrt{x^2 + y^2}$ be the distance between the points $(x, 0)$ and $(0, y)$ in the xy -plane.
 - How is ds/dt related to dx/dt if y is constant?
 - How is ds/dt related to dx/dt and dy/dt if neither x nor y is constant?
 - How is dx/dt related to dy/dt if s is constant?
- Diagonals** If x , y , and z are lengths of the edges of a rectangular box, then the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$.
 - Assuming that x , y , and z are differentiable functions of t , how is ds/dt related to dx/dt , dy/dt , and dz/dt ?
 - How is ds/dt related to dy/dt and dz/dt if x is constant?
 - How are dx/dt , dy/dt , and dz/dt related if s is constant?
- Area** The area A of a triangle with sides of lengths a and b enclosing an angle of measure θ is

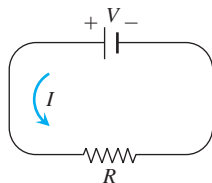
$$A = \frac{1}{2}ab \sin \theta.$$

- How is dA/dt related to $d\theta/dt$ if a and b are constant?
 - How is dA/dt related to $d\theta/dt$ and da/dt if only b is constant?
 - How is dA/dt related to $d\theta/dt$, da/dt , and db/dt if none of a , b , and θ are constant?
- Heating a plate** When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/min . At what rate is the plate's area increasing when the radius is 50 cm?
 - Changing dimensions in a rectangle** The length l of a rectangle is decreasing at the rate of 2 cm/s while the width w is increasing at the rate of 2 cm/s . When $l = 12 \text{ cm}$ and $w = 5 \text{ cm}$, find the rates of change of (a) the area, (b) the perimeter, and (c) the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?
 - Changing dimensions in a rectangular box** Suppose that the edge lengths x , y , and z of a closed rectangular box are changing at the following rates:

$$\frac{dx}{dt} = 1 \text{ m/s}, \quad \frac{dy}{dt} = -2 \text{ m/s}, \quad \frac{dz}{dt} = 1 \text{ m/s}.$$

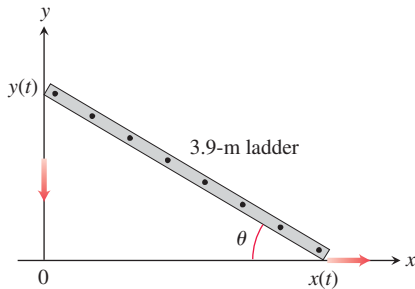
Find the rates at which the box's (a) volume, (b) surface area, and (c) diagonal length $s = \sqrt{x^2 + y^2 + z^2}$ are changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

- A sliding ladder** A 3.9-m ladder is leaning against a house when its base starts to slide away (see accompanying figure). By the time the base is 3.6 m from the house, the base is moving at the rate of 1.5 m/s .
 - How fast is the top of the ladder sliding down the wall then?

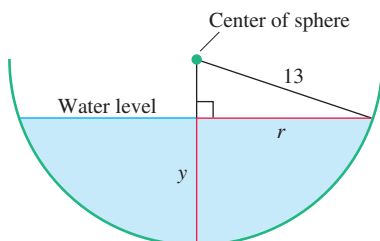


- What is the value of dV/dt ?
- What is the value of dI/dt ?
- What equation relates dR/dt to dV/dt and dI/dt ?
- Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amps. Is R increasing, or decreasing?

- b. At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?
- c. At what rate is the angle θ between the ladder and the ground changing then?

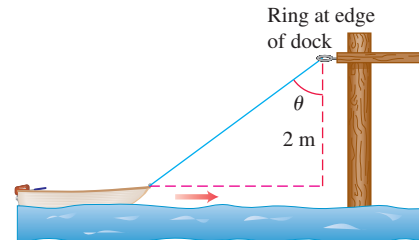


24. **Commercial air traffic** Two commercial airplanes are flying at an altitude of 12,000 m along straight-line courses that intersect at right angles. Plane A is approaching the intersection point at a speed of 442 knots (nautical miles per hour; a nautical mile is 1852 m). Plane B is approaching the intersection at 481 knots. At what rate is the distance between the planes changing when A is 5 nautical miles from the intersection point, and B is 12 nautical miles from the intersection point?
25. **Flying a kite** A girl flies a kite at a height of 90 m, the wind carrying the kite horizontally away from her at a rate of 7.5 m/s. How fast must she let out the string when the kite is 150 m away from her?
26. **Boring a cylinder** The mechanics at Lincoln Automotive are re boring a 15-cm-deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one-thousandth of a centimeter every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 10 cm?
27. **A growing sand pile** Sand falls from a conveyor belt at the rate of $10 \text{ m}^3/\text{min}$ onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the (a) height and (b) radius changing when the pile is 4 m high? Answer in centimeters per minute.
28. **A draining conical reservoir** Water is flowing at the rate of $50 \text{ m}^3/\text{min}$ from a shallow concrete conical reservoir (vertex down) of base radius 45 m and height 6 m.
- a. How fast (in centimeters per minute) is the water level falling when the water is 5 m deep?
- b. How fast is the radius of the water's surface changing then? Answer in centimeters per minute.
29. **A draining hemispherical reservoir** Water is flowing at the rate of $6 \text{ m}^3/\text{min}$ from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions, given that the volume of water in a hemispherical bowl of radius R is $V = (\pi/3)y^2(3R - y)$ when the water is y meters deep.

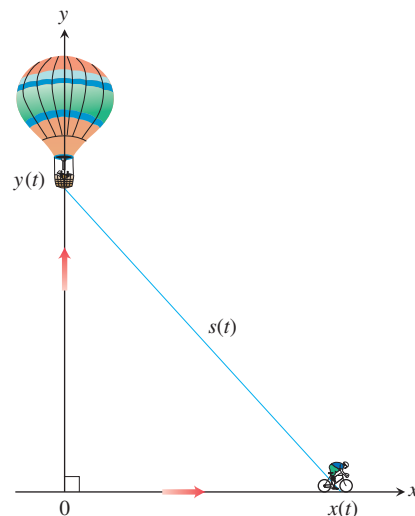


- a. At what rate is the water level changing when the water is 8 m deep?
- b. What is the radius r of the water's surface when the water is y m deep?
- c. At what rate is the radius r changing when the water is 8 m deep?

30. **A growing raindrop** Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop's radius increases at a constant rate.
31. **The radius of an inflating balloon** A spherical balloon is inflated with helium at the rate of $100\pi \text{ m}^3/\text{min}$. How fast is the balloon's radius increasing at the instant the radius is 5 m? How fast is the surface area increasing?
32. **Hauling in a dinghy** A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 2 m above the bow. The rope is hauled in at the rate of 0.5 m/s.
- a. How fast is the boat approaching the dock when 3 m of rope are out?
- b. At what rate is the angle θ changing at this instant (see the figure)?

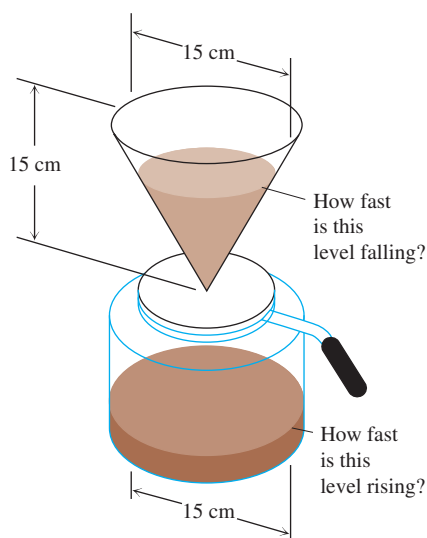


33. **A balloon and a bicycle** A balloon is rising vertically above a level, straight road at a constant rate of 0.3 m/s. Just when the balloon is 20 m above the ground, a bicycle moving at a constant rate of 5 m/s passes under it. How fast is the distance $s(t)$ between the bicycle and balloon increasing 3 s later?



34. **Making coffee** Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of $160 \text{ cm}^3/\text{min}$.
- a. How fast is the level in the pot rising when the coffee in the cone is 12 cm deep?

b. How fast is the level in the cone falling then?



35. Cardiac output In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würzburg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 L/min. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

where Q is the number of milliliters of CO_2 you exhale in a minute and D is the difference between the CO_2 concentration (ml/L) in the blood pumped to the lungs and the CO_2 concentration in the blood returning from the lungs. With $Q = 233$ ml/min and $D = 97 - 56 = 41$ ml/L,

$$y = \frac{233 \text{ ml/min}}{41 \text{ ml/L}} \approx 5.68 \text{ L/min},$$

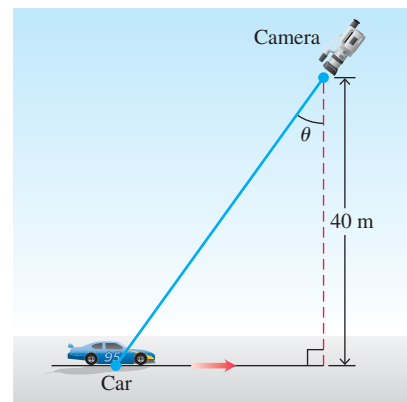
fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

Suppose that when $Q = 233$ and $D = 41$, we also know that D is decreasing at the rate of 2 units a minute but that Q remains unchanged. What is happening to the cardiac output?

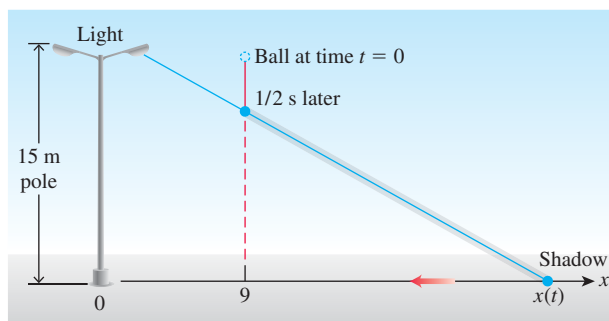
36. Moving along a parabola A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x -coordinate (measured in meters) increases at a steady 10 m/s. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3$ m?

37. Motion in the plane The coordinates of a particle in the metric xy -plane are differentiable functions of time t with $dx/dt = -1$ m/s and $dy/dt = -5$ m/s. How fast is the particle's distance from the origin changing as it passes through the point $(5, 12)$?

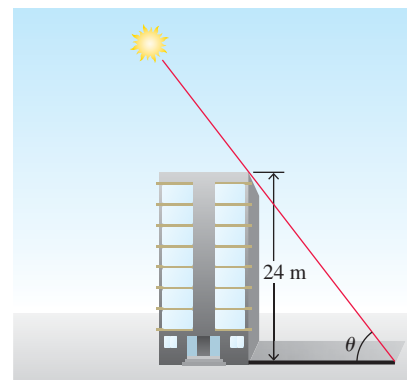
38. Videotaping a moving car You are videotaping a race from a stand 40 m from the track, following a car that is moving at 288 km/h (80 m/s), as shown in the accompanying figure. How fast will your camera angle θ be changing when the car is right in front of you? A half second later?



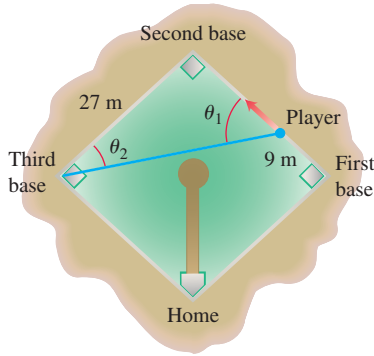
39. A moving shadow A light shines from the top of a pole 15 m high. A ball is dropped from the same height from a point 9 m away from the light. (See accompanying figure.) How fast is the shadow of the ball moving along the ground $1/2$ s later? (Assume the ball falls a distance $s = 4.9t^2$ m in t seconds.)



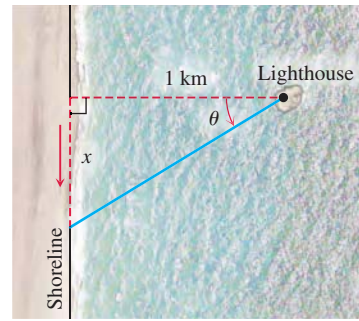
40. A building's shadow On a morning of a day when the sun will pass directly overhead, the shadow of a 24 m building on level ground is 18 m long. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of $0.27^\circ/\text{min}$. At what rate is the shadow decreasing? (Remember to use radians. Express your answer in centimeters per minute, to the nearest tenth.)



- 41. A melting ice layer** A spherical iron ball 8 cm in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of $10 \text{ cm}^3/\text{min}$, how fast is the thickness of the ice decreasing when it is 2 cm thick? How fast is the outer surface area of ice decreasing?
- 42. Highway patrol** A highway patrol plane flies 3 km above a level, straight road at a steady 120 km/h. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 km, the line-of-sight distance is decreasing at the rate of 160 km/h. Find the car's speed along the highway.
- 43. Baseball players** A baseball diamond is a square 27 m on a side. A player runs from first base to second at a rate of 5 m/s.
- At what rate is the player's distance from third base changing when the player is 9 m from first base?
 - At what rates are angles θ_1 and θ_2 (see the figure) changing at that time?
 - The player slides into second base at the rate of 4.5 m/s. At what rates are angles θ_1 and θ_2 changing as the player touches base?



- 44. Ships** Two ships are steaming straight away from a point O along routes that make a 120° angle. Ship A moves at 14 knots (nautical miles per hour; a nautical mile is 1852 m). Ship B moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles?
- 45. Clock's moving hands** At what rate is the angle between a clock's minute and hour hands changing at 4 o'clock in the afternoon?
- 46. Oil spill** An explosion at an oil rig located in gulf waters causes an elliptical oil slick to spread on the surface from the rig. The slick is a constant 20 cm thick. After several days, when the major axis of the slick is 2 km long and the minor axis is $3/4$ km wide, it is determined that its length is increasing at the rate of 9 m/h, and its width is increasing at the rate of 3 m/h. At what rate (in cubic meters per hour) is oil flowing from the site of the rig at that time?
- 47. A lighthouse beam** A lighthouse sits 1 km offshore, and its beam of light rotates counterclockwise at the constant rate of 3 full circles per minute. At what rate is the image of the beam moving down the shoreline when the image is 1 km from the spot on the shoreline nearest the lighthouse?



3.11 Linearization and Differentials

It is often useful to approximate complicated functions with simpler ones that give the accuracy we want for specific applications and, at the same time, are easier to work with than the original functions. The approximating functions discussed in this section are called *linearizations*, and they are based on tangent lines. Other approximating functions, such as polynomials, are discussed in Chapter 9.

We introduce new variables dx and dy , called *differentials*, and define them in a way that makes Leibniz's notation for the derivative dy/dx a true ratio. We use dy to estimate error in measurement, which then provides for a precise proof of the Chain Rule (Section 3.6).

Linearization

As you can see in Figure 3.51, the tangent line to the curve $y = x^2$ lies close to the curve near the point of tangency. For a brief interval to either side, the y -values along the tangent line give good approximations to the y -values on the curve. We observe this phenomenon

by zooming in on the two graphs at the point of tangency, or by looking at tables of values for the difference between $f(x)$ and its tangent line near the x -coordinate of the point of tangency. The phenomenon is true not just for parabolas; every differentiable curve behaves locally like its tangent line.

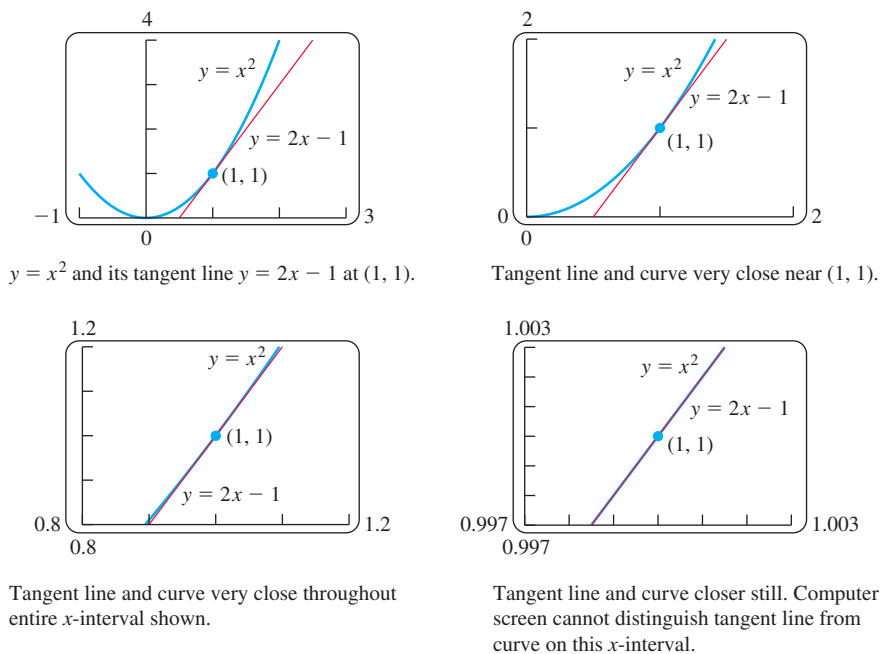


FIGURE 3.51 The more we magnify the graph of a function near a point where the function is differentiable, the flatter the graph becomes and the more it resembles its tangent line.

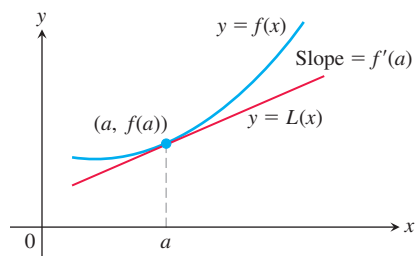


FIGURE 3.52 The tangent line to the curve $y = f(x)$ at $x = a$ is the line $L(x) = f(a) + f'(a)(x - a)$.

In general, the tangent line to $y = f(x)$ at a point $x = a$, where f is differentiable (Figure 3.52), passes through the point $(a, f(a))$, so its point-slope equation is

$$y = f(a) + f'(a)(x - a).$$

Thus, this tangent line is the graph of the linear function

$$L(x) = f(a) + f'(a)(x - a).$$

As long as this line remains close to the graph of f as we move off the point of tangency, $L(x)$ gives a good approximation to $f(x)$.

DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a . The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a . The point $x = a$ is the **center** of the approximation.

EXAMPLE 1 Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$ (Figure 3.53).

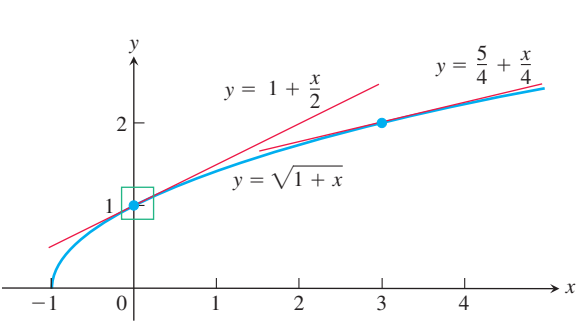


FIGURE 3.53 The graph of $y = \sqrt{1+x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.54 shows a magnified view of the small window about 1 on the y-axis.

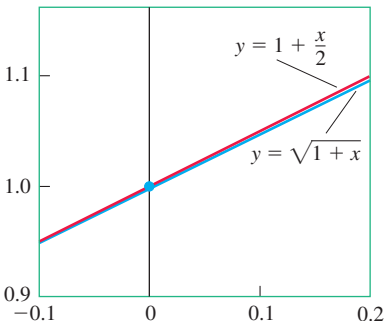


FIGURE 3.54 Magnified view of the window in Figure 3.53.

Solution Since

$$f'(x) = \frac{1}{2}(1+x)^{-1/2},$$

we have $f(0) = 1$ and $f'(0) = 1/2$, giving the linearization

$$L(x) = f(a) + f'(a)(x - a) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}.$$

See Figure 3.54. ■

The following table shows how accurate the approximation $\sqrt{1+x} \approx 1 + (x/2)$ from Example 1 is for some values of x near 0. As we move away from zero, we lose accuracy. For example, for $x = 2$, the linearization gives 2 as the approximation for $\sqrt{3}$, which is not even accurate to one decimal place.

Approximation	True value	True value – approximation
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$0.000003 < 10^{-5}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$0.000305 < 10^{-3}$
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$0.004555 < 10^{-2}$

Do not be misled by the preceding calculations into thinking that whatever we do with a linearization is better done with a calculator. In practice, we would never use a linearization to find a particular square root. The utility of a linearization is its ability to replace a complicated formula by a simpler one over an *entire interval of values*. If we have to work with $\sqrt{1+x}$ for x in an interval close to 0 and can tolerate the small amount of error involved over that interval, we can work with $1 + (x/2)$ instead. Of course, we then need to know how much error there is. We further examine the estimation of error in Chapter 9.

A linear approximation normally loses accuracy away from its center. As Figure 3.53 suggests, the approximation $\sqrt{1+x} \approx 1 + (x/2)$ is too crude to be useful near $x = 3$. There, we need the linearization at $x = 3$.

EXAMPLE 2 Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 3$. (See Figure 3.53.)

Solution We evaluate the equation defining $L(x)$ at $a = 3$. With

$$f(3) = 2, \quad f'(3) = \frac{1}{2}(1+x)^{-1/2} \Big|_{x=3} = \frac{1}{4},$$

we have

$$L(x) = 2 + \frac{1}{4}(x-3) = \frac{5}{4} + \frac{x}{4}.$$

At $x = 3.2$, the linearization in Example 2 gives

$$\sqrt{1+x} = \sqrt{1+3.2} \approx \frac{5}{4} + \frac{3.2}{4} = 1.250 + 0.800 = 2.050,$$

which differs from the true value $\sqrt{4.2} \approx 2.04939$ by less than one one-thousandth. The linearization in Example 1 gives

$$\sqrt{1+x} = \sqrt{1+3.2} \approx 1 + \frac{3.2}{2} = 1 + 1.6 = 2.6,$$

a result that is off by more than 25%. ■

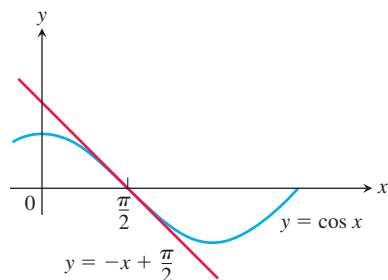


FIGURE 3.55 The graph of $f(x) = \cos x$ and its linearization at $x = \pi/2$.

Near $x = \pi/2$, $\cos x \approx 2x + (\pi/2)$ (Example 3).

EXAMPLE 3 Find the linearization of $f(x) = \cos x$ at $x = \pi/2$ (Figure 3.55).

Solution Since $f(\pi/2) = \cos(\pi/2) = 0$, $f'(x) = -\sin x$, and $f'(\pi/2) = -\sin(\pi/2) = -1$, we find the linearization at $a = \pi/2$ to be

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= 0 + (-1)\left(x - \frac{\pi}{2}\right) \\ &= -x + \frac{\pi}{2}. \end{aligned}$$
■

An important linear approximation for roots and powers is

$$(1+x)^k \approx 1+kx \quad (x \text{ near } 0; \text{ any number } k)$$

(Exercise 15). This approximation, which is good for values of x sufficiently close to zero, has broad application. For example, when x is small,

Approximations Near $x = 0$

$$\sqrt{1+x} \approx 1 + \frac{x}{2}$$

$$\frac{1}{1-x} \approx 1 + x$$

$$\frac{1}{\sqrt{1-x^2}} \approx 1 + \frac{x^2}{2}$$

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x$$

$k = 1/2$

$$\frac{1}{1-x} = (1-x)^{-1} \approx 1 + (-1)(-x) = 1 + x$$

$k = -1$; replace x by $-x$.

$$\sqrt[3]{1+5x^4} = (1+5x^4)^{1/3} \approx 1 + \frac{1}{3}(5x^4) = 1 + \frac{5}{3}x^4$$

$k = 1/3$; replace x by $5x^4$.

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)(-x^2) = 1 + \frac{1}{2}x^2$$

$k = -1/2$; replace x by $-x^2$.

Differentials

We sometimes use the Leibniz notation dy/dx to represent the derivative of y with respect to x . Contrary to its appearance, it is not a ratio. We now introduce two new variables dx and dy with the property that when their ratio exists, it is equal to the derivative.

DEFINITION Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x) dx.$$

Unlike the independent variable dx , the variable dy is always a dependent variable. It depends on both x and dx . If dx is given a specific value and x is a particular number in the domain of the function f , then these values determine the numerical value of dy . Often the variable dx is chosen to be Δx , the change in x .

EXAMPLE 4

- (a) Find dy if $y = x^5 + 37x$.
 (b) Find the value of dy when $x = 1$ and $dx = 0.2$.

Solution

- (a) $dy = (5x^4 + 37) dx$
 (b) Substituting $x = 1$ and $dx = 0.2$ in the expression for dy , we have

$$dy = (5 \cdot 1^4 + 37) 0.2 = 8.4. \quad \blacksquare$$

The geometric meaning of differentials is shown in Figure 3.56. Let $x = a$ and set $dx = \Delta x$. The corresponding change in $y = f(x)$ is

$$\Delta y = f(a + dx) - f(a).$$

The corresponding change in the tangent line L is

$$\begin{aligned} \Delta L &= L(a + dx) - L(a) \\ &= \underbrace{f(a) + f'(a)[(a + dx) - a]}_{L(a + dx)} - \underbrace{f(a)}_{L(a)} \\ &= f'(a) dx. \end{aligned}$$

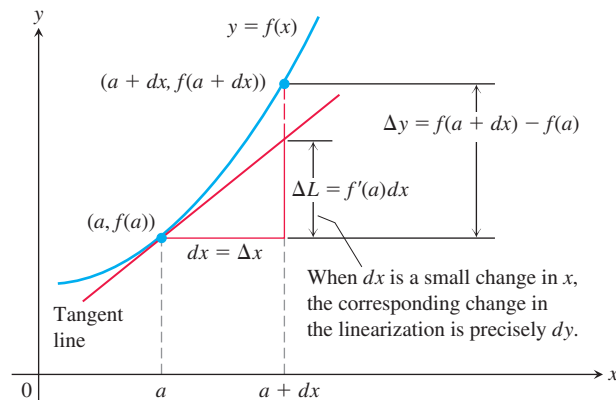


FIGURE 3.56 Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

That is, the change in the linearization of f is precisely the value of the differential dy when $x = a$ and $dx = \Delta x$. Therefore, dy represents the amount the tangent line rises or falls when x changes by an amount $dx = \Delta x$.

If $dx \neq 0$, then the quotient of the differential dy by the differential dx is equal to the derivative $f'(x)$ because

$$dy \div dx = \frac{f'(x) dx}{dx} = f'(x) = \frac{dy}{dx}.$$

We sometimes write

$$df = f'(x) dx$$

in place of $dy = f'(x) dx$, calling df the **differential of f** . For instance, if $f(x) = 3x^2 - 6$, then

$$df = d(3x^2 - 6) = 6x dx.$$

Every differentiation formula like

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{or} \quad \frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}$$

has a corresponding differential form like

$$d(u+v) = du + dv \quad \text{or} \quad d(\sin u) = \cos u du.$$

EXAMPLE 5 We can use the Chain Rule and other differentiation rules to find differentials of functions.

$$(a) \quad d(\tan 2x) = \sec^2(2x) d(2x) = 2 \sec^2 2x dx$$

$$(b) \quad d\left(\frac{x}{x+1}\right) = \frac{(x+1)dx - x d(x+1)}{(x+1)^2} = \frac{x dx + dx - x dx}{(x+1)^2} = \frac{dx}{(x+1)^2} \quad \blacksquare$$

Estimating with Differentials

Suppose we know the value of a differentiable function $f(x)$ at a point a and want to estimate how much this value will change if we move to a nearby point $a + dx$. If $dx = \Delta x$ is small, then we can see from Figure 3.56 that Δy is approximately equal to the differential dy . Since

$$f(a + dx) = f(a) + \Delta y, \quad \Delta x = dx$$

the differential approximation gives

$$f(a + dx) \approx f(a) + dy$$

when $dx = \Delta x$. Thus the approximation $\Delta y \approx dy$ can be used to estimate $f(a + dx)$ when $f(a)$ is known, dx is small, and $dy = f'(a) dx$.

EXAMPLE 6 The radius r of a circle increases from $a = 10$ m to 10.1 m (Figure 3.57). Use dA to estimate the increase in the circle's area A . Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculation.

Solution Since $A = \pi r^2$, the estimated increase is

$$dA = A'(a) dr = 2\pi a dr = 2\pi(10)(0.1) = 2\pi \text{ m}^2.$$

Thus, since $A(r + \Delta r) \approx A(r) + dA$, we have

$$\begin{aligned} A(10 + 0.1) &\approx A(10) + 2\pi \\ &= \pi(10)^2 + 2\pi = 102\pi. \end{aligned}$$

The area of a circle of radius 10.1 m is approximately $102\pi \text{ m}^2$.

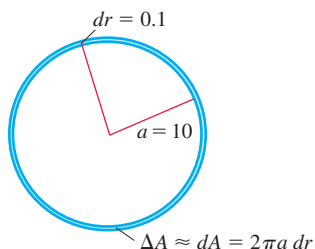


FIGURE 3.57 When dr is small compared with a , the differential dA gives the estimate $A(a + dr) = \pi a^2 + dA$ (Example 6).

The true area is

$$\begin{aligned} A(10.1) &= \pi(10.1)^2 \\ &= 102.01\pi \text{ m}^2. \end{aligned}$$

The error in our estimate is $0.01\pi \text{ m}^2$, which is the difference $\Delta A - dA$. ■

When using differentials to estimate functions, our goal is to choose a nearby point $x = a$ where both $f(a)$ and the derivative $f'(a)$ are easy to evaluate.

EXAMPLE 7 Use differentials to estimate

- (a) $7.97^{1/3}$ (b) $\sin(\pi/6 + 0.01)$.

Solution

- (a) The differential associated with the cube root function $y = x^{1/3}$ is

$$dy = \frac{1}{3x^{2/3}} dx.$$

We set $a = 8$, the closest number near 7.97 where we can easily compute $f(a)$ and $f'(a)$. To arrange that $a + dx = 7.97$, we choose $dx = -0.03$. Approximating with the differential gives

$$\begin{aligned} f(7.97) &= f(a + dx) \approx f(a) + dy \\ &= 8^{1/3} + \frac{1}{3(8)^{2/3}}(-0.03) \\ &= 2 + \frac{1}{12}(-0.03) = 1.9975. \end{aligned}$$

This gives an approximation to the true value of $7.97^{1/3}$, which is 1.997497 to 6 decimal places.

- (b) The differential associated with $y = \sin x$ is

$$dy = \cos x \, dx.$$

To estimate $\sin(\pi/6 + 0.01)$, we set $a = \pi/6$ and $dx = 0.01$. Then

$$\begin{aligned} f(\pi/6 + 0.01) &= f(a + dx) \approx f(a) + dy \\ &= \sin \frac{\pi}{6} + \left(\cos \frac{\pi}{6} \right)(0.01) \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}(0.01) \approx 0.5087. \end{aligned}$$

For comparison, the true value of $\sin(\pi/6 + 0.01)$ to 6 decimal places is 0.508635. ■

The method in part (b) of Example 7 can be used in computer algorithms to give values of trigonometric functions. The algorithms store a large table of sine and cosine values between 0 and $\pi/4$. Values between these stored values are computed using differentials as in Example 7b. Values outside of $[0, \pi/4]$ are computed from values in this interval using trigonometric identities.

Error in Differential Approximation

Let $f(x)$ be differentiable at $x = a$ and suppose that $dx = \Delta x$ is an increment of x . We have two ways to describe the change in f as x changes from a to $a + \Delta x$:

The true change: $\Delta f = f(a + \Delta x) - f(a)$

The differential estimate: $df = f'(a) \Delta x$.

How well does df approximate Δf ?

$$\sin(a + dx) \approx \sin a + (\cos a) dx$$

We measure the approximation error by subtracting df from Δf :

$$\begin{aligned}
 \text{Approximation error} &= \Delta f - df \\
 &= \Delta f - f'(a)\Delta x \\
 &= \underbrace{f(a + \Delta x) - f(a)}_{\Delta f} - f'(a)\Delta x \\
 &= \underbrace{\left(\frac{f(a + \Delta x) - f(a)}{\Delta x} - f'(a) \right)}_{\text{Call this part } \varepsilon} \cdot \Delta x \\
 &= \varepsilon \cdot \Delta x.
 \end{aligned}$$

As $\Delta x \rightarrow 0$, the difference quotient

$$\frac{f(a + \Delta x) - f(a)}{\Delta x}$$

approaches $f'(a)$ (remember the definition of $f'(a)$), so the quantity in parentheses becomes a very small number (which is why we called it ε). In fact, $\varepsilon \rightarrow 0$ as $\Delta x \rightarrow 0$. When Δx is small, the approximation error $\varepsilon \Delta x$ is smaller still.

$$\underbrace{\Delta f}_{\text{true change}} = \underbrace{f'(a)\Delta x}_{\text{estimated change}} + \underbrace{\varepsilon \Delta x}_{\text{error}}$$

Although we do not know the exact size of the error, it is the product $\varepsilon \cdot \Delta x$ of two small quantities that both approach zero as $\Delta x \rightarrow 0$. For many common functions, whenever Δx is small, the error is still smaller.

Change in $y = f(x)$ near $x = a$

If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the change Δy in f is given by

$$\Delta y = f'(a) \Delta x + \varepsilon \Delta x, \quad (1)$$

in which $\varepsilon \rightarrow 0$ as $\Delta x \rightarrow 0$.

In Example 6 we found that

$$\Delta A = \pi(10.1)^2 - \pi(10)^2 = (102.01 - 100)\pi = (\underbrace{2\pi}_{dA} + \underbrace{0.01\pi}_{\text{error}})\text{m}^2$$

so the approximation error is $\Delta A - dA = \varepsilon \Delta r = 0.01\pi$ and $\varepsilon = 0.01\pi/\Delta r = 0.01\pi/0.1 = 0.1\pi$ m.

Proof of the Chain Rule

Equation (1) enables us to give a complete proof of the Chain Rule. Our goal is to show that if $f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x , then the composition $y = f(g(x))$ is a differentiable function of x . Since a function is differentiable if and only if it has a derivative at each point in its domain, we must show that whenever g is differentiable at x_0 and f is differentiable at $g(x_0)$, then the composition is differentiable at x_0 and the derivative of the composition satisfies the equation

$$\left. \frac{dy}{dx} \right|_{x=x_0} = f'(g(x_0)) \cdot g'(x_0).$$

Let Δx be an increment in x and let Δu and Δy be the corresponding increments in u and y . Applying Equation (1), we have

$$\Delta u = g'(x_0) \Delta x + \varepsilon_1 \Delta x = (g'(x_0) + \varepsilon_1) \Delta x,$$

where $\varepsilon_1 \rightarrow 0$ as $\Delta x \rightarrow 0$. Similarly,

$$\Delta y = f'(u_0) \Delta u + \varepsilon_2 \Delta u = (f'(u_0) + \varepsilon_2) \Delta u,$$

where $\varepsilon_2 \rightarrow 0$ as $\Delta u \rightarrow 0$. Notice also that $\Delta u \rightarrow 0$ as $\Delta x \rightarrow 0$. Combining the equations for Δu and Δy gives

$$\Delta y = (f'(u_0) + \varepsilon_2)(g'(x_0) + \varepsilon_1) \Delta x,$$

so

$$\frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) + \varepsilon_2 g'(x_0) + f'(u_0)\varepsilon_1 + \varepsilon_2 \varepsilon_1.$$

Since ε_1 and ε_2 go to zero as Δx goes to zero, the last three terms on the right vanish in the limit, leaving

$$\left. \frac{dy}{dx} \right|_{x=x_0} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) = f'(g(x_0)) \cdot g'(x_0). \quad \blacksquare$$

Sensitivity to Change

The equation $df = f'(x) dx$ tells how *sensitive* the output of f is to a change in input at different values of x . The larger the value of f' at x , the greater the effect of a given change dx . As we move from a to a nearby point $a + dx$, we can describe the change in f in three ways: absolute, relative, and percentage.

	True	Estimated
Absolute change	$\Delta f = f(a + dx) - f(a)$	$df = f'(a) dx$
Relative change	$\frac{\Delta f}{f(a)}$	$\frac{df}{f(a)}$
Percentage change	$\frac{\Delta f}{f(a)} \times 100$	$\frac{df}{f(a)} \times 100$

EXAMPLE 8 You want to calculate the depth of a well from the equation $s = 4.9t^2$ by timing how long it takes a heavy stone you drop to splash into the water below. How sensitive will your calculations be to a 0.1-s error in measuring the time?

Solution The size of ds in the equation

$$ds = 9.8t \, dt$$

depends on how big t is. If $t = 2$ s, the change caused by $dt = 0.1$ is about

$$ds = 9.8(2)(0.1) = 1.96 \text{ m.}$$

Three seconds later at $t = 5$ s, the change caused by the same dt is

$$ds = 9.8(5)(0.1) = 4.9 \text{ m.}$$

For a fixed error in the time measurement, the error in using ds to estimate the depth is larger when it takes a longer time before the stone splashes into the water. That is, the estimate is more sensitive to the effect of the error for larger values of t . ■

EXAMPLE 9 Newton's second law,

$$F = \frac{d}{dt}(mv) = m \frac{dv}{dt} = ma,$$

is stated with the assumption that mass is constant, but we know this is not strictly true because the mass of an object increases with velocity. In Einstein's corrected formula, mass has the value

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

where the "rest mass" m_0 represents the mass of an object that is not moving and c is the speed of light, which is about 300,000 km/s. Use the approximation

$$\frac{1}{\sqrt{1 - x^2}} \approx 1 + \frac{1}{2}x^2 \quad (2)$$

to estimate the increase Δm in mass resulting from the added velocity v .

Solution When v is very small compared with c , v^2/c^2 is close to zero and it is safe to use the approximation

$$\frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right) \quad \text{Eq. (2) with } x = \frac{v}{c}$$

to obtain

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \approx m_0 \left[1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right) \right] = m_0 + \frac{1}{2}m_0 v^2 \left(\frac{1}{c^2}\right),$$

or

$$m \approx m_0 + \frac{1}{2}m_0 v^2 \left(\frac{1}{c^2}\right). \quad (3)$$

Equation (3) expresses the increase in mass that results from the added velocity v . ■

Converting Mass to Energy

Equation (3) derived in Example 9 has an important interpretation. In Newtonian physics, $(1/2)m_0 v^2$ is the kinetic energy (KE) of the object, and if we rewrite Equation (3) in the form

$$(m - m_0)c^2 \approx \frac{1}{2}m_0 v^2,$$

we see that

$$(m - m_0)c^2 \approx \frac{1}{2}m_0 v^2 = \frac{1}{2}m_0 v^2 - \frac{1}{2}m_0 (0)^2 = \Delta(\text{KE}),$$

or

$$(\Delta m)c^2 \approx \Delta(\text{KE}).$$

So the change in kinetic energy $\Delta(\text{KE})$ in going from velocity 0 to velocity v is approximately equal to $(\Delta m)c^2$, the change in mass times the square of the speed of light. Using $c \approx 3 \times 10^8$ m/s, we see that a small change in mass can create a large change in energy.

EXERCISES 3.11

Finding Linearizations

In Exercises 1–5, find the linearization $L(x)$ of $f(x)$ at $x = a$.

1. $f(x) = x^3 - 2x + 3$, $a = 2$

2. $f(x) = \sqrt{x^2 + 9}$, $a = -4$

3. $f(x) = x + \frac{1}{x}$, $a = 1$

4. $f(x) = \sqrt[3]{x}$, $a = -8$

5. $f(x) = \tan x$, $a = \pi$

6. **Common linear approximations at $x = 0$** Find the linearizations of the following functions at $x = 0$.

a. $\sin x$ b. $\cos x$ c. $\tan x$ d. e^x e. $\ln(1 + x)$

Linearization for Approximation

In Exercises 7–14, find a linearization at a suitably chosen integer near a at which the given function and its derivative are easy to evaluate.

7. $f(x) = x^2 + 2x$, $a = 0.1$

8. $f(x) = x^{-1}$, $a = 0.9$

9. $f(x) = 2x^2 + 3x - 3$, $a = -0.9$

10. $f(x) = 1 + x$, $a = 8.1$

11. $f(x) = \sqrt[3]{x}$, $a = 8.5$

12. $f(x) = \frac{x}{x+1}$, $a = 1.3$

13. $f(x) = e^{-x}$, $a = -0.1$

14. $f(x) = \sin^{-1} x$, $a = \pi/12$

15. Show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$.

16. Use the linear approximation $(1 + x)^k \approx 1 + kx$ to find an approximation for the function $f(x)$ for values of x near zero.

a. $f(x) = (1 - x)^6$ b. $f(x) = \frac{2}{1 - x}$

c. $f(x) = \frac{1}{\sqrt{1 + x}}$ d. $f(x) = \sqrt{2 + x^2}$

e. $f(x) = (4 + 3x)^{1/3}$ f. $f(x) = \sqrt[3]{\left(1 - \frac{x}{2 + x}\right)^2}$

17. **Faster than a calculator** Use the approximation $(1 + x)^k \approx 1 + kx$ to estimate the following.

a. $(1.0002)^{50}$ b. $\sqrt[3]{1.009}$

18. Find the linearization of $f(x) = \sqrt{x+1} + \sin x$ at $x = 0$. How is it related to the individual linearizations of $\sqrt{x+1}$ and $\sin x$ at $x = 0$?

Derivatives in Differential Form

In Exercises 19–38, find dy .

19. $y = x^3 - 3\sqrt{x}$

20. $y = x\sqrt{1 - x^2}$

21. $y = \frac{2x}{1 + x^2}$

22. $y = \frac{2\sqrt{x}}{3(1 + \sqrt{x})}$

23. $2y^{3/2} + xy - x = 0$

24. $xy^2 - 4x^{3/2} - y = 0$

25. $y = \sin(5\sqrt{x})$

26. $y = \cos(x^2)$

27. $y = 4 \tan(x^3/3)$

28. $y = \sec(x^2 - 1)$

29. $y = 3 \csc(1 - 2\sqrt{x})$

30. $y = 2 \cot\left(\frac{1}{\sqrt{x}}\right)$

31. $y = e^{\sqrt{x}}$

32. $y = xe^{-x}$

33. $y = \ln(1 + x^2)$

34. $y = \ln\left(\frac{x+1}{\sqrt{x-1}}\right)$

35. $y = \tan^{-1}(e^{x^2})$

36. $y = \cot^{-1}\left(\frac{1}{x^2}\right) + \cos^{-1} 2x$

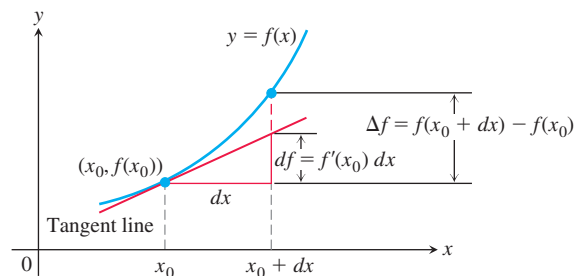
37. $y = \sec^{-1}(e^{-x})$

38. $y = e^{\tan^{-1}\sqrt{x^2+1}}$

Approximation Error

In Exercises 39–44, each function $f(x)$ changes value when x changes from x_0 to $x_0 + dx$. Find

- the change $\Delta f = f(x_0 + dx) - f(x_0)$;
- the value of the estimate $df = f'(x_0) dx$; and
- the approximation error $|\Delta f - df|$.



39. $f(x) = x^2 + 2x$, $x_0 = 1$, $dx = 0.1$

40. $f(x) = 2x^2 + 4x - 3$, $x_0 = -1$, $dx = 0.1$

41. $f(x) = x^3 - x$, $x_0 = 1$, $dx = 0.1$

42. $f(x) = x^4$, $x_0 = 1$, $dx = 0.1$

43. $f(x) = x^{-1}$, $x_0 = 0.5$, $dx = 0.1$

44. $f(x) = x^3 - 2x + 3$, $x_0 = 2$, $dx = 0.1$

Differential Estimates of Change

In Exercises 45–50, write a differential formula that estimates the given change in volume or surface area.

45. The change in the volume $V = (4/3)\pi r^3$ of a sphere when the radius changes from r_0 to $r_0 + dr$

46. The change in the volume $V = x^3$ of a cube when the edge lengths change from x_0 to $x_0 + dx$

47. The change in the surface area $S = 6x^2$ of a cube when the edge lengths change from x_0 to $x_0 + dx$

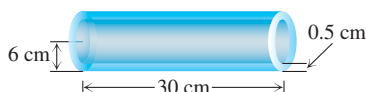
48. The change in the lateral surface area $S = \pi r \sqrt{r^2 + h^2}$ of a right circular cone when the radius changes from r_0 to $r_0 + dr$ and the height does not change

49. The change in the volume $V = \pi r^2 h$ of a right circular cylinder when the radius changes from r_0 to $r_0 + dr$ and the height does not change

50. The change in the lateral surface area $S = 2\pi rh$ of a right circular cylinder when the height changes from h_0 to $h_0 + dh$ and the radius does not change

Applications

51. The radius of a circle is increased from 2.00 to 2.02 m.
- Estimate the resulting change in area.
 - Express the estimate as a percentage of the circle's original area.
52. The diameter of a tree was 25 cm. During the following year, the circumference increased 5 cm. About how much did the tree's diameter increase? What is the tree's cross-sectional area?
53. **Estimating volume** Estimate the volume of material in a cylindrical shell with length 30 cm, radius 6 cm, and shell thickness 0.5 cm.



54. **Estimating height of a building** A surveyor, standing 9 m from the base of a building, measures the angle of elevation to the top of the building to be 75° . How accurately must the angle be measured for the percentage error in estimating the height of the building to be less than 4%?
55. The radius r of a circle is measured with an error of at most 2%. What is the maximum corresponding percentage error in computing the circle's
- circumference?
 - area?
56. The edge x of a cube is measured with an error of at most 0.5%. What is the maximum corresponding percentage error in computing the cube's
- surface area?
 - volume?
57. **Tolerance** The height and radius of a right circular cylinder are equal, so the cylinder's volume is $V = \pi h^3$. The volume is to be calculated with an error of no more than 1% of the true value. Find approximately the greatest error that can be tolerated in the measurement of h , expressed as a percentage of h .
58. **Tolerance**
- About how accurately must the interior diameter of a 10-m-high cylindrical storage tank be measured to calculate the tank's volume to within 1% of its true value?
 - About how accurately must the tank's exterior diameter be measured to calculate the amount of paint it will take to paint the side of the tank to within 5% of the true amount?
59. The diameter of a sphere is measured as 100 ± 1 cm and the volume is calculated from this measurement. Estimate the percentage error in the volume calculation.
60. Estimate the allowable percentage error in measuring the diameter D of a sphere if the volume is to be calculated correctly to within 3%.

- T** 61. **The effect of flight maneuvers on the heart** The amount of work done by the heart's main pumping chamber, the left ventricle, is given by the equation

$$W = PV + \frac{V\delta v^2}{2g},$$

where W is the work per unit time, P is the average blood pressure, V is the volume of blood pumped out during the unit of time, δ ("delta") is the weight density of the blood, v is the average velocity of the exiting blood, and g is the acceleration of gravity.

When P , V , δ , and v remain constant, W becomes a function of g , and the equation takes the simplified form

$$W = a + \frac{b}{g} \quad (a, b \text{ constant}).$$

As a member of NASA's medical team, you want to know how sensitive W is to apparent changes in g caused by flight maneuvers, and this depends on the initial value of g . As part of your investigation, you decide to compare the effect on W of a given change dg on the moon, where $g = 1.6 \text{ m/s}^2$, with the effect the same change dg would have on Earth, where $g = 9.8 \text{ m/s}^2$. Use the simplified equation above to find the ratio of dW_{moon} to dW_{Earth} .

- T** 62. **Drug concentration** The concentration C in milligrams per milliliter (mg/ml) of a certain drug in a person's bloodstream t hours after a pill is swallowed is modeled by

$$C(t) = 1 + \frac{4t}{1+t^3} - e^{-0.06t}.$$

Estimate the change in concentration when t changes from 20 to 30 min.

63. **Unclogging arteries** The formula $V = kr^4$, discovered by the physiologist Jean Poiseuille (1797–1869), allows us to predict how much the radius of a partially clogged artery has to be expanded in order to restore normal blood flow. The formula says that the volume V of blood flowing through the artery in a unit of time at a fixed pressure is a constant k times the radius of the artery to the fourth power. How will a 10% increase in r affect V ?
64. **Measuring acceleration of gravity** When the length L of a clock pendulum is held constant by controlling its temperature, the pendulum's period T depends on the acceleration of gravity g . The period will therefore vary slightly as the clock is moved from place to place on Earth's surface, depending on the change in g . By keeping track of ΔT , we can estimate the variation in g from the equation $T = 2\pi(L/g)^{1/2}$ that relates T , g , and L .
- With L held constant and g as the independent variable, calculate dT and use it to answer parts (b) and (c).
 - If g increases, will T increase or decrease? Will a pendulum clock speed up or slow down? Explain.
- T** c. A clock with a 100-cm pendulum is moved from a location where $g = 980 \text{ cm/s}^2$ to a new location. This increases the period by $dT = 0.001$ s. Find dg and estimate the value of g at the new location.

65. Quadratic approximations

- Let $Q(x) = b_0 + b_1(x - a) + b_2(x - a)^2$ be a quadratic approximation to $f(x)$ at $x = a$ with these properties:
 - $Q(a) = f(a)$
 - $Q'(a) = f'(a)$
 - $Q''(a) = f''(a)$.

Determine the coefficients b_0 , b_1 , and b_2 .

- Find the quadratic approximation to $f(x) = 1/(1 - x)$ at $x = 0$.

- T** c. Graph $f(x) = 1/(1 - x)$ and its quadratic approximation at $x = 0$. Then zoom in on the two graphs at the point $(0, 1)$. Comment on what you see.

- Find the quadratic approximation to $g(x) = 1/x$ at $x = 1$. Graph g and its quadratic approximation together. Comment on what you see.

T e. Find the quadratic approximation to $h(x) = \sqrt{1+x}$ at $x = 0$. Graph h and its quadratic approximation together. Comment on what you see.

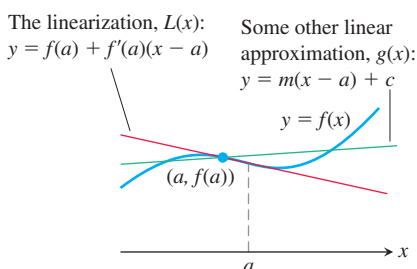
f. What are the linearizations of f , g , and h at the respective points in parts (b), (d), and (e)?

66. The linearization is the best linear approximation Suppose that $y = f(x)$ is differentiable at $x = a$ and that $g(x) = m(x - a) + c$ is a linear function in which m and c are constants. If the error $E(x) = f(x) - g(x)$ were small enough near $x = a$, we might think of using g as a linear approximation of f instead of the linearization $L(x) = f(a) + f'(a)(x - a)$. Show that if we impose on g the conditions

1. $E(a) = 0$ The approximation error is zero at $x = a$.

2. $\lim_{x \rightarrow a} \frac{E(x)}{x - a} = 0$ The error is negligible when compared with $x - a$.

then $g(x) = f(a) + f'(a)(x - a)$. Thus, the linearization $L(x)$ gives the only linear approximation whose error is both zero at $x = a$ and negligible in comparison with $x - a$.



67. The linearization of 2^x

a. Find the linearization of $f(x) = 2^x$ at $x = 0$. Then round its coefficients to two decimal places.

T b. Graph the linearization and function together for $-3 \leq x \leq 3$ and $-1 \leq x \leq 1$.

68. The linearization of $\log_3 x$

a. Find the linearization of $f(x) = \log_3 x$ at $x = 3$. Then round its coefficients to two decimal places.

T b. Graph the linearization and function together in the window $0 \leq x \leq 8$ and $2 \leq x \leq 4$.

COMPUTER EXPLORATIONS

In Exercises 69–74, use a CAS to estimate the magnitude of the error in using the linearization in place of the function over a specified interval I . Perform the following steps:

- Plot the function f over I .
- Find the linearization L of the function at the point a .
- Plot f and L together on a single graph.
- Plot the absolute error $|f(x) - L(x)|$ over I and find its maximum value.
- From your graph in part (d), estimate as large a $\delta > 0$ as you can that satisfies

$$|x - a| < \delta \Rightarrow |f(x) - L(x)| < \varepsilon$$

for $\varepsilon = 0.5, 0.1$, and 0.01 . Then check graphically to see whether your δ -estimate holds true.

69. $f(x) = x^3 + x^2 - 2x$, $[-1, 2]$, $a = 1$

70. $f(x) = \frac{x-1}{4x^2+1}$, $[-\frac{3}{4}, 1]$, $a = \frac{1}{2}$

71. $f(x) = x^{2/3}(x-2)$, $[-2, 3]$, $a = 2$

72. $f(x) = \sqrt{x} - \sin x$, $[0, 2\pi]$, $a = 2$

73. $f(x) = x2^x$, $[0, 2]$, $a = 1$

74. $f(x) = \sqrt{x} \sin^{-1} x$, $[0, 1]$, $a = \frac{1}{2}$

CHAPTER 3 Questions to Guide Your Review

- What is the derivative of a function f ? How is its domain related to the domain of f ? Give examples.
- What role does the derivative play in defining slopes, tangent lines, and rates of change?
- How can you sometimes graph the derivative of a function when all you have is a table of the function's values?
- What does it mean for a function to be differentiable on an open interval? On a closed interval?
- How are derivatives and one-sided derivatives related?
- Describe geometrically when a function typically does *not* have a derivative at a point.
- How is a function's differentiability at a point related to its continuity there, if at all?
- What rules do you know for calculating derivatives? Give some examples.

9. Explain how the three formulas

a. $\frac{d}{dx}(x^n) = nx^{n-1}$ b. $\frac{d}{dx}(cu) = c \frac{du}{dx}$

c. $\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}$

enable us to differentiate any polynomial.

- What formula do we need, in addition to the three listed in Question 9, to differentiate rational functions?
- What is a second derivative? A third derivative? How many derivatives do the functions you know have? Give examples.
- What is the derivative of the exponential function e^x ? How does the domain of the derivative compare with the domain of the function?
- What is the relationship between a function's average and instantaneous rates of change? Give an example.

14. How do derivatives arise in the study of motion? What can you learn about an object's motion along a line by examining the derivatives of the object's position function? Give examples.
15. How can derivatives arise in economics?
16. Give examples of still other applications of derivatives.
17. What do the limits $\lim_{h \rightarrow 0} ((\sin h)/h)$ and $\lim_{h \rightarrow 0} ((\cos h - 1)/h)$ have to do with the derivatives of the sine and cosine functions? What are the derivatives of these functions?
18. Once you know the derivatives of $\sin x$ and $\cos x$, how can you find the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$? What are the derivatives of these functions?
19. What is the rule for calculating the derivative of a composition of two differentiable functions? How is such a derivative evaluated? Give examples.
20. If u is a differentiable function of x , how do you find $(d/dx)(u^n)$ if n is an integer? If n is a real number? Give examples.
21. What is implicit differentiation? When do you need it? Give examples.
22. What is the derivative of the natural logarithm function $\ln x$? How does the domain of the derivative compare with the domain of the function?
23. What is the derivative of the exponential function a^x , $a > 0$ and $a \neq 1$? What is the geometric significance of the limit of $(a^h - 1)/h$ as $h \rightarrow 0$? What is the limit when a is the number e ?
24. What is the derivative of $\log_a x$? Are there any restrictions on a ?
25. What is logarithmic differentiation? Give an example.
26. How can you write any real power of x as a power of e ? Are there any restrictions on x ? How does this lead to the Power Rule for differentiating arbitrary real powers?
27. What is one way of expressing the special number e as a limit? What is an approximate numerical value of e correct to 7 decimal places?
28. What are the derivatives of the inverse trigonometric functions? How do the domains of the derivatives compare with the domains of the functions?
29. How do related rates problems arise? Give examples.
30. Outline a strategy for solving related rates problems. Illustrate with an example.
31. What is the linearization $L(x)$ of a function $f(x)$ at a point $x = a$? What is required of f at a for the linearization to exist? How are linearizations used? Give examples.
32. If x moves from a to a nearby value $a + dx$, how do you estimate the corresponding change in the value of a differentiable function $f(x)$? How do you estimate the relative change? The percentage change? Give an example.

CHAPTER 3 Practice Exercises

Derivatives of Functions

Find the derivatives of the functions in Exercises 1–64.

1. $y = x^5 - 0.125x^2 + 0.25x$
2. $y = 3 - 0.7x^3 + 0.3x^7$
3. $y = x^3 - 3(x^2 + \pi^2)$
4. $y = x^7 + \sqrt{7}x - \frac{1}{\pi + 1}$
5. $y = (x + 1)^2(x^2 + 2x)$
6. $y = (2x - 5)(4 - x)^{-1}$
7. $y = (\theta^2 + \sec \theta + 1)^3$
8. $y = \left(-1 - \frac{\csc \theta}{2} - \frac{\theta^2}{4}\right)^2$
9. $s = \frac{\sqrt{t}}{1 + \sqrt{t}}$
10. $s = \frac{1}{\sqrt{t} - 1}$
11. $y = 2 \tan^2 x - \sec^2 x$
12. $y = \frac{1}{\sin^2 x} - \frac{2}{\sin x}$
13. $s = \cos^4(1 - 2t)$
14. $s = \cot^3\left(\frac{2}{t}\right)$
15. $s = (\sec t + \tan t)^5$
16. $s = \csc^5(1 - t + 3t^2)$
17. $r = \sqrt{2\theta \sin \theta}$
18. $r = 2\theta \sqrt{\cos \theta}$
19. $r = \sin \sqrt{2\theta}$
20. $r = \sin(\theta + \sqrt{\theta + 1})$
21. $y = \frac{1}{2}x^2 \csc \frac{2}{x}$
22. $y = 2\sqrt{x} \sin \sqrt{x}$
23. $y = x^{-1/2} \sec(2x)^2$
24. $y = \sqrt{x} \csc(x + 1)^3$
25. $y = 5 \cot x^2$
26. $y = x^2 \cot 5x$
27. $y = x^2 \sin^2(2x^2)$
28. $y = x^{-2} \sin^2(x^3)$
29. $s = \left(\frac{4t}{t + 1}\right)^{-2}$
30. $s = \frac{-1}{15(15t - 1)^3}$
31. $y = \left(\frac{\sqrt{x}}{1 + x}\right)^2$
32. $y = \left(\frac{2\sqrt{x}}{2\sqrt{x} + 1}\right)^2$
33. $y = \sqrt{\frac{x^2 + x}{x^2}}$
34. $y = 4x\sqrt{x + \sqrt{x}}$
35. $r = \left(\frac{\sin \theta}{\cos \theta - 1}\right)^2$
36. $r = \left(\frac{1 + \sin \theta}{1 - \cos \theta}\right)^2$
37. $y = (2x + 1)\sqrt{2x + 1}$
38. $y = 20(3x - 4)^{1/4}(3x - 4)^{-1/5}$
39. $y = \frac{3}{(5x^2 + \sin 2x)^{3/2}}$
40. $y = (3 + \cos^3 3x)^{-1/3}$
41. $y = 10e^{-x/5}$
42. $y = \sqrt{2}e^{\sqrt{2}x}$
43. $y = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x}$
44. $y = x^2e^{-2/x}$
45. $y = \ln(\sin^2 \theta)$
46. $y = \ln(\sec^2 \theta)$
47. $y = \log_2(x^2/2)$
48. $y = \log_5(3x - 7)$
49. $y = 8^{-t}$
50. $y = 9^{2t}$
51. $y = 5x^{3.6}$
52. $y = \sqrt{2}x^{-\sqrt{2}}$
53. $y = (x + 2)^{x+2}$
54. $y = 2(\ln x)^{x/2}$
55. $y = \arcsin \sqrt{1 - u^2}, \quad 0 < u < 1$
56. $y = \arcsin\left(\frac{1}{\sqrt{v}}\right), \quad v > 1$

57. $y = \ln \arccos x$

58. $y = z \arccos z - \sqrt{1 - z^2}$

59. $y = t \arctan t - \frac{1}{2} \ln t$

60. $y = (1 + t^2) \operatorname{arccot} 2t$

61. $y = z \operatorname{arcsec} z - \sqrt{z^2 - 1}, \quad z > 1$

62. $y = 2\sqrt{x-1} \operatorname{arcsec} \sqrt{x}$

63. $y = \operatorname{arccsc}(\sec \theta), \quad 0 < \theta < \pi/2$

64. $y = (1 + x^2)e^{\arctan x}$

Implicit DifferentiationIn Exercises 65–78, find dy/dx by implicit differentiation.

65. $xy + 2x + 3y = 1$

66. $x^2 + xy + y^2 - 5x = 2$

67. $x^3 + 4xy - 3y^{4/3} = 2x$

68. $5x^{4/5} + 10y^{6/5} = 15$

69. $\sqrt{xy} = 1$

70. $x^2y^2 = 1$

71. $y^2 = \frac{x}{x+1}$

72. $y^2 = \frac{\sqrt{1+x}}{\sqrt{1-x}}$

73. $e^{x+2y} = 1$

74. $y^2 = 2e^{-1/x}$

75. $\ln(x/y) = 1$

76. $x \arcsin y = 1 + x^2$

77. $ye^{\arctan x} = 2$

78. $x^y = \sqrt{2}$

In Exercises 79 and 80, find dp/dq .

79. $p^3 + 4pq - 3q^2 = 2$

80. $q = (5p^2 + 2p)^{-3/2}$

In Exercises 81 and 82, find dr/ds .

81. $r \cos 2s + \sin^2 s = \pi$

82. $2rs - r - s + s^2 = -3$

83. Find d^2y/dx^2 by implicit differentiation:

a. $x^3 + y^3 = 1$

b. $y^2 = 1 - \frac{2}{x}$

84. a. By differentiating $x^2 - y^2 = 1$ implicitly, show that $dy/dx = x/y$.b. Then show that $d^2y/dx^2 = -1/y^3$.**Numerical Values of Derivatives**85. Suppose that functions $f(x)$ and $g(x)$ and their first derivatives have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	-3	1/2
1	3	5	1/2	-4

Find the first derivatives of the following combinations at the given value of x .

a. $6f(x) - g(x), \quad x = 1$

b. $f(x)g^2(x), \quad x = 0$

c. $\frac{f(x)}{g(x)+1}, \quad x = 1$

d. $f(g(x)), \quad x = 0$

e. $g(f(x)), \quad x = 0$

f. $(x + f(x))^{3/2}, \quad x = 1$

g. $f(x + g(x)), \quad x = 0$

86. Suppose that the function $f(x)$ and its first derivative have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$f'(x)$
0	9	-2
1	-3	1/5

Find the first derivatives of the following combinations at the given value of x .

a. $\sqrt{x} f(x), \quad x = 1$

b. $\sqrt{f(x)}, \quad x = 0$

c. $f(\sqrt{x}), \quad x = 1$

d. $f(1 - 5 \tan x), \quad x = 0$

e. $\frac{f(x)}{2 + \cos x}, \quad x = 0$

f. $10 \sin\left(\frac{\pi x}{2}\right) f^2(x), \quad x = 1$

87. Find the value of dy/dt at $t = 0$ if $y = 3 \sin 2x$ and $x = t^2 + \pi$.88. Find the value of ds/du at $u = 2$ if $s = t^2 + 5t$ and $t = (u^2 + 2u)^{1/3}$.89. Find the value of dw/ds at $s = 0$ if $w = \sin(e^{\sqrt{r}})$ and $r = 3 \sin(s + \pi/6)$.90. Find the value of dr/dt at $t = 0$ if $r = (\theta^2 + 7)^{1/3}$ and $\theta^2 t + \theta = 1$.91. If $y^3 + y = 2 \cos x$, find the value of d^2y/dx^2 at the point $(0, 1)$.92. If $x^{1/3} + y^{1/3} = 4$, find d^2y/dx^2 at the point $(8, 8)$.**Applying the Derivative Definition**

In Exercises 93 and 94, find the derivative using the definition.

93. $f(t) = \frac{1}{2t+1}$

94. $g(x) = 2x^2 + 1$

95. a. Graph the function

$$f(x) = \begin{cases} x^2, & -1 \leq x < 0 \\ -x^2, & 0 \leq x \leq 1. \end{cases}$$

b. Is f continuous at $x = 0$?c. Is f differentiable at $x = 0$?

Give reasons for your answers.

96. a. Graph the function

$$f(x) = \begin{cases} x, & -1 \leq x < 0 \\ \tan x, & 0 \leq x \leq \pi/4. \end{cases}$$

b. Is f continuous at $x = 0$?c. Is f differentiable at $x = 0$?

Give reasons for your answers.

97. a. Graph the function

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2. \end{cases}$$

b. Is f continuous at $x = 1$?c. Is f differentiable at $x = 1$?

Give reasons for your answers.

98. For what value or values of the constant m , if any, is

$$f(x) = \begin{cases} \sin 2x, & x \leq 0 \\ mx, & x > 0 \end{cases}$$

- a. continuous at $x = 0$?
b. differentiable at $x = 0$?

Give reasons for your answers.

Slopes, Tangent Lines, and Normal Lines

99. **Tangent lines with specified slope** Are there any points on the curve $y = (x/2) + 1/(2x - 4)$ where the slope is $-3/2$? If so, find them.
100. **Tangent lines with specified slope** Are there any points on the curve $y = x - e^{-x}$ where the slope is 2? If so, find them.
101. **Horizontal tangent lines** Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent line is parallel to the x -axis.
102. **Tangent intercepts** Find the x - and y -intercepts of the line that is tangent to the curve $y = x^3$ at the point $(-2, -8)$.
103. **Tangent lines perpendicular or parallel to lines** Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent line is
a. perpendicular to the line $y = 1 - (x/24)$.
b. parallel to the line $y = \sqrt{2} - 12x$.
104. **Intersecting tangent lines** Show that the tangent lines to the curve $y = (\pi \sin x)/x$ at $x = \pi$ and $x = -\pi$ intersect at right angles.
105. **Normal lines parallel to a line** Find the points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the normal line is parallel to the line $y = -x/2$. Sketch the curve and normal lines together, labeling each with its equation.
106. **Tangent lines and normal lines** Find equations for the tangent and normal lines to the curve $y = 1 + \cos x$ at the point $(\pi/2, 1)$. Sketch the curve, tangent line, and normal line together, labeling each with its equation.
107. **Tangent parabola** The parabola $y = x^2 + C$ is to be tangent to the line $y = x$. Find C .
108. **Slope of a tangent line** Show that the tangent line to the curve $y = x^3$ at any point (a, a^3) meets the curve again at a point where the slope is four times the slope at (a, a^3) .
109. **Tangent curve** For what value of c is the curve $y = c/(x + 1)$ tangent to the line through the points $(0, 3)$ and $(5, -2)$?
110. **Normal lines to a circle** Show that the normal line at any point of the circle $x^2 + y^2 = a^2$ passes through the origin.

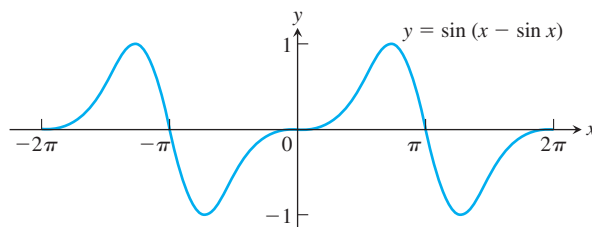
In Exercises 111–116, find equations for the lines that are tangent, and the lines that are normal, to the curve at the given point.

111. $x^2 + 2y^2 = 9$, $(1, 2)$
112. $e^x + y^2 = 2$, $(0, 1)$
113. $xy + 2x - 5y = 2$, $(3, 2)$
114. $(y - x)^2 = 2x + 4$, $(6, 2)$
115. $x + \sqrt{xy} = 6$, $(4, 1)$

116. $x^{3/2} + 2y^{3/2} = 17$, $(1, 4)$

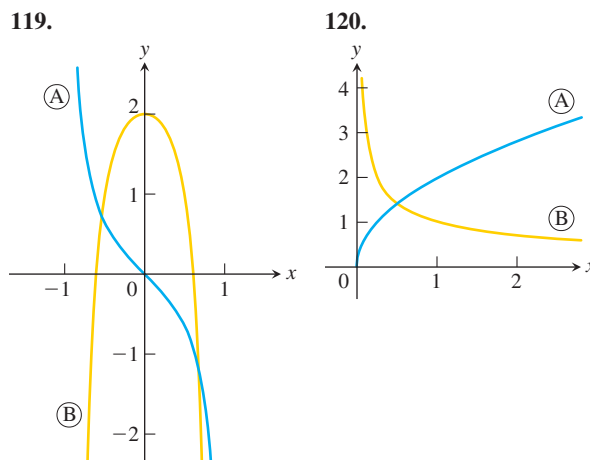
117. Find the slope of the curve $x^3y^3 + y^2 = x + y$ at the points $(1, 1)$ and $(1, -1)$.

118. The graph shown suggests that the curve $y = \sin(x - \sin x)$ might have horizontal tangent lines at the x -axis. Does it? Give reasons for your answer.

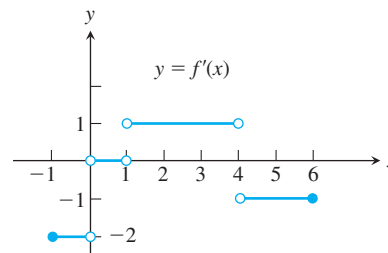


Analyzing Graphs

Each of the figures in Exercises 119 and 120 shows two graphs, the graph of a function $y = f(x)$ together with the graph of its derivative $f'(x)$. Which graph is which? How do you know?



121. Use the following information to graph the function $y = f(x)$ for $-1 \leq x \leq 6$.
- i) The graph of f is made of line segments joined end to end.
 - ii) The graph starts at the point $(-1, 2)$.
 - iii) The derivative of f , where defined, agrees with the step function shown here.



122. Repeat Exercise 121, supposing that the graph starts at $(-1, 0)$ instead of $(-1, 2)$.

Logarithmic Differentiation

In Exercises 123–128, use logarithmic differentiation to find the derivative of y with respect to the appropriate variable.

$$123. y = \frac{2(x^2 + 1)}{\sqrt{\cos 2x}} \quad 124. y = \sqrt[10]{\frac{3x + 4}{2x - 4}}$$

$$125. y = \left(\frac{(t+1)(t-1)}{(t-2)(t+3)} \right)^5, \quad t > 2$$

$$126. y = \frac{2u 2^u}{\sqrt{u^2 + 1}}$$

$$127. y = (\sin \theta)^{\sqrt{\theta}}$$

$$128. y = (\ln x)^{1/(\ln x)}$$

Related Rates

129. Right circular cylinder The total surface area S of a right circular cylinder is related to the base radius r and height h by the equation $S = 2\pi r^2 + 2\pi rh$.

- How is dS/dt related to dr/dt if h is constant?
- How is dS/dt related to dh/dt if r is constant?
- How is dS/dt related to dr/dt and dh/dt if neither r nor h is constant?
- How is dr/dt related to dh/dt if S is constant?

130. Right circular cone The lateral surface area S of a right circular cone is related to the base radius r and height h by the equation $S = \pi r \sqrt{r^2 + h^2}$.

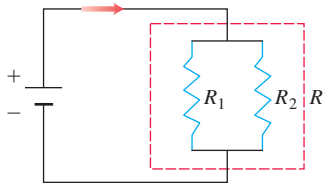
- How is dS/dt related to dr/dt if h is constant?
- How is dS/dt related to dh/dt if r is constant?
- How is dS/dt related to dr/dt and dh/dt if neither r nor h is constant?

131. Circle's changing area The radius r of a circle is changing at the rate of $-2/\pi$ m/s. At what rate is the circle's area changing when $r = 10$ m?

132. Cube's changing edges The volume of a cube is increasing at the rate of $1200 \text{ cm}^3/\text{min}$ at the instant its edges are 20 cm long. At what rate are the lengths of the edges changing at that instant?

133. Resistors connected in parallel If two resistors of R_1 and R_2 ohms are connected in parallel in an electric circuit to make an R -ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$



If R_1 is decreasing at the rate of 1 ohm/s and R_2 is increasing at the rate of 0.5 ohm/s, at what rate is R changing when $R_1 = 75$ ohms and $R_2 = 50$ ohms?

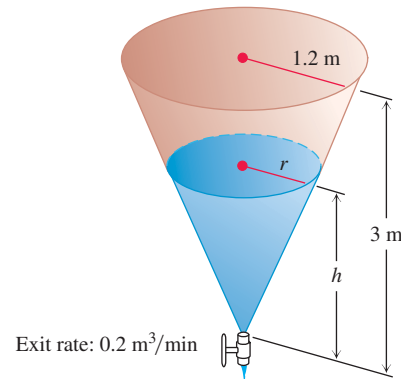
134. Impedance in a series circuit The impedance Z (ohms) in a series circuit is related to the resistance R (ohms) and reactance X (ohms) by the equation $Z = \sqrt{R^2 + X^2}$. If R is increasing at 3 ohms/s and X is decreasing at 2 ohms/s, at what rate is Z changing when $R = 10$ ohms and $X = 20$ ohms?

135. Speed of moving particle The coordinates of a particle moving in the metric xy -plane are differentiable functions of time t with $dx/dt = 10$ m/s and $dy/dt = 5$ m/s. How fast is the particle moving away from the origin as it passes through the point $(3, -4)$?

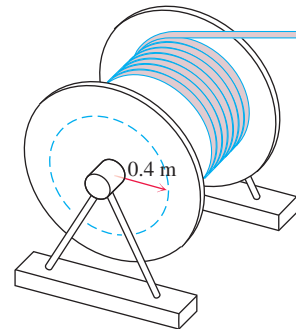
136. Motion of a particle A particle moves along the curve $y = x^{3/2}$ in the first quadrant in such a way that its distance from the origin increases at the rate of 11 units per second. Find dx/dt when $x = 3$.

137. Draining a tank Water drains from the conical tank shown in the accompanying figure at the rate of $0.2 \text{ m}^3/\text{min}$.

- What is the relation between the variables h and r in the figure?
- How fast is the water level dropping when $h = 2$ m?

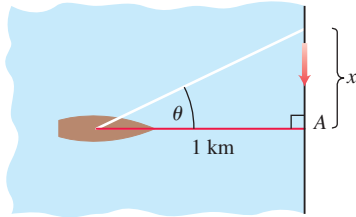


138. Rotating spool As television cable is pulled from a large spool to be strung from the telephone poles along a street, it unwinds from the spool in layers of constant radius (see accompanying figure). If the truck pulling the cable moves at a steady 2 m/s (a touch over 7 km/h), use the equation $s = r\theta$ to find how fast (radians per second) the spool is turning when the layer of radius 0.4 m is being unwound.



139. Moving searchlight beam The figure shows a boat 1 km off-shore, sweeping the shore with a searchlight. The light turns at a constant rate, $d\theta/dt = -0.6$ rad/s.

- How fast is the light moving along the shore when it reaches point A?
- How many revolutions per minute is 0.6 rad/s?



140. Points moving on coordinate axes Points A and B move along the x- and y-axes, respectively, in such a way that the distance r (meters) along the perpendicular from the origin to the line AB remains constant. How fast is OA changing, and is it increasing or decreasing, when $OB = 2r$ and B is moving toward O at the rate of $0.3r$ m/s?

Linearization

141. Find the linearizations of

- $\tan x$ at $x = -\pi/4$
- $\sec x$ at $x = -\pi/4$.

Graph the curves and linearizations together.

142. We can obtain a useful linear approximation of the function $f(x) = 1/(1 + \tan x)$ at $x = 0$ by combining the approximations

$$\frac{1}{1+x} \approx 1-x \quad \text{and} \quad \tan x \approx x$$

to get

$$\frac{1}{1+\tan x} \approx 1-x.$$

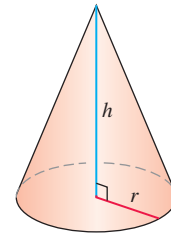
Show that this result is the standard linear approximation of $1/(1 + \tan x)$ at $x = 0$.

143. Find the linearization of $f(x) = \sqrt{1+x} + \sin x - 0.5$ at $x = 0$.

144. Find the linearization of $f(x) = 2/(1-x) + \sqrt{1+x} - 3.1$ at $x = 0$.

Differential Estimates of Change

145. Surface area of a cone Write a formula that estimates the change that occurs in the lateral surface area of a right circular cone when the height changes from h_0 to $h_0 + dh$ and the radius does not change.



$$V = \frac{1}{3} \pi r^2 h$$

$$S = \pi r \sqrt{r^2 + h^2}$$

(Lateral surface area)

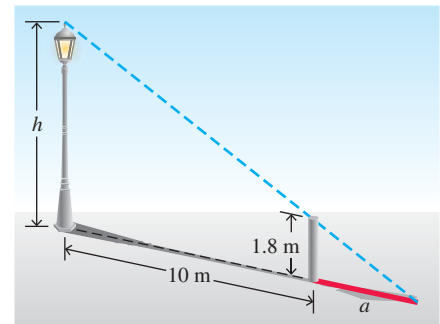
146. Controlling error

- How accurately should you measure the edge of a cube to be reasonably sure of calculating the cube's surface area with an error of no more than 2%?
- Suppose that the edge is measured with the accuracy required in part (a). About how accurately can the cube's volume be calculated from the edge measurement? To find out, estimate the percentage error in the volume calculation that might result from using the edge measurement.

147. Compounding error The circumference of the equator of a sphere is measured as 10 cm with a possible error of 0.4 cm. This measurement is used to calculate the radius. The radius is then used to calculate the surface area and volume of the sphere. Estimate the percentage errors in the calculated values of

- the radius.
- the surface area.
- the volume.

148. Finding height To find the height of a lamppost (see accompanying figure), you stand a 1.8 m pole 10 m from the lamp and measure the length a of its shadow, finding it to be 4.5 m, give or take a centimeter. Calculate the height of the lamppost using the value $a = 4.5$, and estimate the possible error in the result.



CHAPTER 3 Additional and Advanced Exercises

1. An equation like $\sin^2 \theta + \cos^2 \theta = 1$ is called an **identity** because it holds for all values of θ . An equation like $\sin \theta = 0.5$ is not an identity because it holds only for selected values of θ , not all. If you differentiate both sides of a trigonometric identity in θ with respect to θ , the resulting new equation will also be an identity.

Differentiate the following to show that the resulting equations hold for all θ .

- $\sin 2\theta = 2 \sin \theta \cos \theta$
- $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$

2. If the identity $\sin(x + a) = \sin x \cos a + \cos x \sin a$ is differentiated with respect to x (with a assumed to be a constant), is the resulting equation also an identity? Does this principle apply to the equation $x^2 - 2x - 8 = 0$? Explain.

3. a. Find values for the constants a , b , and c that will make

$$f(x) = \cos x \quad \text{and} \quad g(x) = a + bx + cx^2$$

satisfy the conditions

$$f(0) = g(0), \quad f'(0) = g'(0), \quad \text{and} \quad f''(0) = g''(0).$$

- b. Find values for b and c that will make

$$f(x) = \sin(x + a) \quad \text{and} \quad g(x) = b \sin x + c \cos x$$

satisfy the conditions

$$f(0) = g(0) \quad \text{and} \quad f'(0) = g'(0).$$

- c. For the determined values of a , b , and c , what happens for the third and fourth derivatives of f and g in each of parts (a) and (b)?

4. Solutions to differential equations

- a. Show that $y = \sin x$, $y = \cos x$, and $y = a \cos x + b \sin x$ (a and b constants) all satisfy the equation

$$y'' + y = 0.$$

- b. How would you modify the functions in part (a) to satisfy the equation

$$y'' + 4y = 0?$$

Generalize this result.

5. **An osculating circle** Find the values of h , k , and a that make the circle $(x - h)^2 + (y - k)^2 = a^2$ tangent to the parabola $y = x^2 + 1$ at the point $(1, 2)$ and that also make the second derivatives d^2y/dx^2 have the same value on both curves there. Circles like this one that are tangent to a curve and have the same second derivative as the curve at the point of tangency are called *osculating circles* (from the Latin *osculari*, meaning “to kiss”). We will encounter them again in Chapter 12.

6. **Marginal revenue** A bus will hold 60 people. The number x of people per trip who use the bus is related to the fare charged (p dollars) by the law $p = [3 - (x/40)]^2$. Write an expression for the total revenue $r(x)$ per trip received by the bus company. What number of people per trip will make the marginal revenue dr/dx equal to zero? What is the corresponding fare? (This fare is the one that maximizes the revenue.)

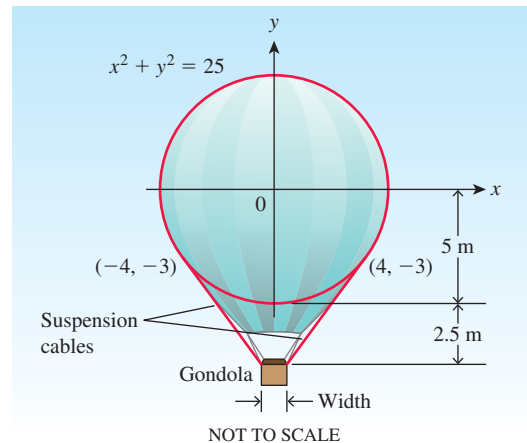
7. Industrial production

- a. Economists often use the expression “rate of growth” in relative rather than absolute terms. For example, let $u = f(t)$ be the number of people in the labor force at time t in a given industry. (We treat this function as though it were differentiable even though it is an integer-valued step function.)

Let $v = g(t)$ be the average production per person in the labor force at time t . The total production is then $y = uv$. If the labor force is growing at the rate of 4% per year ($du/dt = 0.04u$) and the production per worker is growing at the rate of 5% per year ($dv/dt = 0.05v$), find the rate of growth of the total production, y .

- b. Suppose that the labor force in part (a) is decreasing at the rate of 2% per year while the production per person is increasing at the rate of 3% per year. Is the total production increasing, or is it decreasing, and at what rate?

8. **Designing a gondola** The designer of a 10 m-diameter spherical hot air balloon wants to suspend the gondola 2.5 m below the bottom of the balloon with cables tangent to the surface of the balloon, as shown. Two of the cables are shown running from the top edges of the gondola to their points of tangency, $(-4, -3)$ and $(4, -3)$. How wide should the gondola be?



9. **Pisa by parachute** On August 5, 1988, Mike McCarthy of London jumped from the top of the Tower of Pisa. He then opened his parachute in what he said was a world record low-level parachute jump of 54.6 m. Make a rough sketch to show the shape of the graph of his speed during the jump. (Data from: *Boston Globe*, Aug. 6, 1988.)

10. **Motion of a particle** The position at time $t \geq 0$ of a particle moving along a coordinate line is

$$s = 10 \cos(t + \pi/4).$$

- What is the particle's starting position ($t = 0$)?
- What are the points farthest to the left and right of the origin reached by the particle?
- Find the particle's velocity and acceleration at the points in part (b).
- When does the particle first reach the origin? What are its velocity, speed, and acceleration then?

11. **Shooting a paper clip** On Earth, you can easily shoot a paper clip 19.6 m straight up into the air with a rubber band. In t seconds after firing, the paper clip is $s = 19.6t - 4.9t^2$ m above your hand.

- How long does it take the paper clip to reach its maximum height? With what velocity does it leave your hand?
- On the moon, the same acceleration will send the paper clip to a height of $s = 19.6t - 0.8t^2$ m in t s. About how long will it take the paper clip to reach its maximum height, and how high will it go?

12. **Velocities of two particles** At time t seconds, the positions of two particles on a coordinate line are $s_1 = 3t^3 - 12t^2 + 18t + 5$ m and $s_2 = -t^3 + 9t^2 - 12t$ m. When do the particles have the same velocities?

- 13. Velocity of a particle** A particle of constant mass m moves along the x -axis. Its velocity v and position x satisfy the equation

$$\frac{1}{2}m(v^2 - v_0^2) = \frac{1}{2}k(x_0^2 - x^2),$$

where k , v_0 , and x_0 are constants. Show that whenever $v \neq 0$,

$$m \frac{dv}{dt} = -kx.$$

In Exercises 14 and 15, use implicit differentiation to find $\frac{dy}{dx}$.

14. $y^{\ln x} = x^{x^y}$

15. $y^{e^x} = x^y + 1$

16. Average and instantaneous velocity

- a. Show that if the position x of a moving point is given by a quadratic function of t , $x = At^2 + Bt + C$, then the average velocity over any time interval $[t_1, t_2]$ is equal to the instantaneous velocity at the midpoint of the time interval.

- b. What is the geometric significance of the result in part (a)?

- 17.** Find all values of the constants m and b for which the function

$$y = \begin{cases} \sin x, & x < \pi \\ mx + b, & x \geq \pi \end{cases}$$

is

- a. continuous at $x = \pi$.
b. differentiable at $x = \pi$.

- 18.** Does the function

$$f(x) = \begin{cases} \frac{1 - \cos x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a derivative at $x = 0$? Explain.

- 19. a.** For what values of a and b will

$$f(x) = \begin{cases} ax, & x < 2 \\ ax^2 - bx + 3, & x \geq 2 \end{cases}$$

be differentiable for all values of x ?

- b. Discuss the geometry of the resulting graph of f .

- 20. a.** For what values of a and b will

$$g(x) = \begin{cases} ax + b, & x \leq -1 \\ ax^3 + x + 2b, & x > -1 \end{cases}$$

be differentiable for all values of x ?

- b. Discuss the geometry of the resulting graph of g .

- 21. Odd differentiable functions** Is there anything special about the derivative of an odd differentiable function of x ? Give reasons for your answer.

- 22. Even differentiable functions** Is there anything special about the derivative of an even differentiable function of x ? Give reasons for your answer.

- 23.** Suppose that the functions f and g are defined throughout an open interval containing the point x_0 , that f is differentiable at x_0 , that $f(x_0) = 0$, and that g is continuous at x_0 . Show that the product fg is differentiable at x_0 . This process shows, for example, that although $|x|$ is not differentiable at $x = 0$, the product $x|x|$ is differentiable at $x = 0$.

- 24.** (Continuation of Exercise 23.) Use the result of Exercise 23 to show that the following functions are differentiable at $x = 0$.

a. $|x|\sin x$

b. $x^{2/3}\sin x$

c. $\sqrt[3]{x}(1 - \cos x)$

d. $h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

- 25.** Is the derivative of

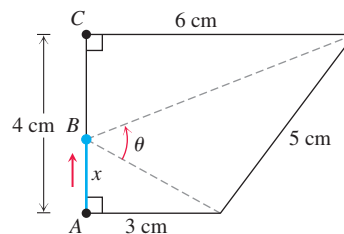
$$h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

continuous at $x = 0$? How about the derivative of $k(x) = xh(x)$? Give reasons for your answers.

- 26.** Let $f(x) = \begin{cases} x^2, & x \text{ is rational} \\ 0, & x \text{ is irrational.} \end{cases}$

Show that f is differentiable at $x = 0$.

- 27.** Point B moves from point A to point C at 2 cm/s in the accompanying diagram. At what rate is θ changing when $x = 4$ cm?



- 28.** Suppose that a function f satisfies the following two conditions for all real values of x and y :

i) $f(x + y) = f(x) \cdot f(y)$.

ii) $f(x) = 1 + xg(x)$, where $\lim_{x \rightarrow 0} g(x) = 1$.

Show that the derivative $f'(x)$ exists at every value of x and that $f'(x) = f(x)$.

- 29. The generalized product rule** Use mathematical induction to prove that if $y = u_1 u_2 \cdots u_n$ is a finite product of differentiable functions, then y is differentiable on their common domain, and

$$\frac{dy}{dx} = \frac{du_1}{dx} u_2 \cdots u_n + u_1 \frac{du_2}{dx} \cdots u_n + \cdots + u_1 u_2 \cdots u_{n-1} \frac{du_n}{dx}.$$

- 30. Leibniz's rule for higher-order derivatives of products** Leibniz's rule for higher-order derivatives of products of differentiable functions says that

a. $\frac{d^2(uv)}{dx^2} = \frac{d^2u}{dx^2}v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}.$

b. $\frac{d^3(uv)}{dx^3} = \frac{d^3u}{dx^3}v + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + u\frac{d^3v}{dx^3}.$

$$\begin{aligned} \text{c. } \frac{d^n(uv)}{dx^n} &= \frac{d^n u}{dx^n} v + n \frac{d^{n-1} u}{dx^{n-1}} \frac{dv}{dx} + \cdots \\ &\quad + \frac{n(n-1) \cdots (n-k+1) d^{n-k} u}{k!} \frac{d^k v}{dx^k} \\ &\quad + \cdots + u \frac{d^n v}{dx^n}. \end{aligned}$$

The equations in parts (a) and (b) are special cases of the equation in part (c). Derive the equation in part (c) by mathematical induction, using

$$\binom{m}{k} + \binom{m}{k+1} = \frac{m!}{k!(m-k)!} + \frac{m!}{(k+1)!(m-k-1)!}.$$

- 31. The period of a clock pendulum** The period T of a clock pendulum (time for one full swing and back) is given by the formula $T^2 = 4\pi^2 L/g$, where T is measured in seconds, $g = 9.8 \text{ m/s}^2$, and L , the length of the pendulum, is measured in meters. Find approximately
- the length of a clock pendulum whose period is $T = 1 \text{ s}$.
 - the change dT in T if the pendulum in part (a) is lengthened 0.01 m .
 - the amount the clock gains or loses in a day as a result of the period's changing by the amount dT found in part (b).

- 32. The melting ice cube** Assume that an ice cube retains its cubical shape as it melts. If we call its edge length s , its volume is $V = s^3$ and its surface area is $6s^2$. We assume that V and s are differentiable functions of time t . We assume also that the cube's volume decreases at a rate that is proportional to its surface area. (This latter assumption seems reasonable enough when we think that the melting takes place at the surface: Changing the amount of surface changes the amount of ice exposed to melt.) In mathematical terms,

$$\frac{dV}{dt} = -k(6s^2), \quad k > 0.$$

The minus sign indicates that the volume is decreasing. We assume that the proportionality factor k is constant. (It probably depends on many things, such as the relative humidity of the surrounding air, the air temperature, and the incidence or absence of sunlight, to name only a few.) Assume a particular set of conditions in which the cube lost $1/4$ of its volume during the first hour, and assume that the volume is V_0 when $t = 0$. How long will it take the ice cube to melt?

CHAPTER 3 Technology Application Projects

Mathematica/Maple Projects

Projects can be found at www.pearsonglobaleditions.com or within MyLab Math.

- **Convergence of Secant Slopes to the Derivative Function**

You will visualize the secant line between successive points on a curve and observe what happens as the distance between them becomes small. The function, sample points, and secant lines are plotted on a single graph, while a second graph compares the slopes of the secant lines with the derivative function.

- **Derivatives, Slopes, Tangent Lines, and Making Movies**

Parts I–III. You will visualize the derivative at a point, the linearization of a function, and the derivative of a function. You will learn how to plot the function and selected tangent lines on the same graph.

Part IV (Plotting Many Tangent Lines)

Part V (Making Movies). Parts IV and V of the module can be used to animate tangent lines as one moves along the graph of a function.

- **Convergence of Secant Slopes to the Derivative Function**

You will visualize right-hand and left-hand derivatives.

- **Motion Along a Straight Line: Position $\rightarrow$ Velocity $\rightarrow$ Acceleration**

Observe dramatic animated visualizations of the derivative relations among the position, velocity, and acceleration functions. Figures in the text can be animated.

4

Applications of Derivatives



OVERVIEW One of the most important applications of the derivative is its use as a tool for finding the optimal (best) solutions to problems. For example, what are the height and diameter of the cylinder of largest volume that can be inscribed in a given sphere? What are the dimensions of the strongest rectangular wooden beam that can be cut from a cylindrical log of given diameter? How many items should a manufacturer produce to maximize profit?

In this chapter we apply derivatives to find extreme values of functions, to determine and analyze the shapes of graphs, and to solve equations numerically. We also investigate how to recover a function from its derivative. The key to many of these applications is the Mean Value Theorem, which connects the derivative and the average change of a function.

4.1 Extreme Values of Functions on Closed Intervals

This section shows how to locate and identify extreme (maximum or minimum) values of a function from its derivative. Once we can do this, we can solve a variety of optimization problems (see Section 4.6). The domains of the functions we consider are intervals or unions of separate intervals.

DEFINITIONS Let f be a function with domain D . Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \quad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \quad \text{for all } x \text{ in } D.$$

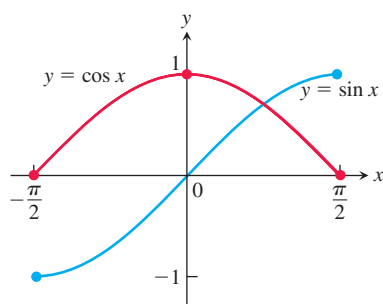


FIGURE 4.1 Absolute extrema for the sine and cosine functions on $[-\pi/2, \pi/2]$. These values can depend on the domain of a function.

Maximum and minimum values are called **extreme values** of the function f . Absolute maxima or minima are also referred to as **global** maxima or minima.

For example, on the closed interval $[-\pi/2, \pi/2]$ the function $f(x) = \cos x$ takes on an absolute maximum value of 1 (once) and an absolute minimum value of 0 (twice). On the same interval, the function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Figure 4.1).

Functions defined by the same equation or formula can have different extrema (maximum or minimum values), depending on the domain. A function might not have a maximum or minimum if the domain is unbounded or fails to contain an endpoint. We see this in the following example.

EXAMPLE 1 The absolute extrema of the following functions on their domains can be seen in Figure 4.2. Each function has the same defining equation, $y = x^2$, but the domains vary.

Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum Absolute minimum of 0 at $x = 0$
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$ Absolute minimum of 0 at $x = 0$
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$ No absolute minimum
(d) $y = x^2$	$(0, 2)$	No absolute extrema

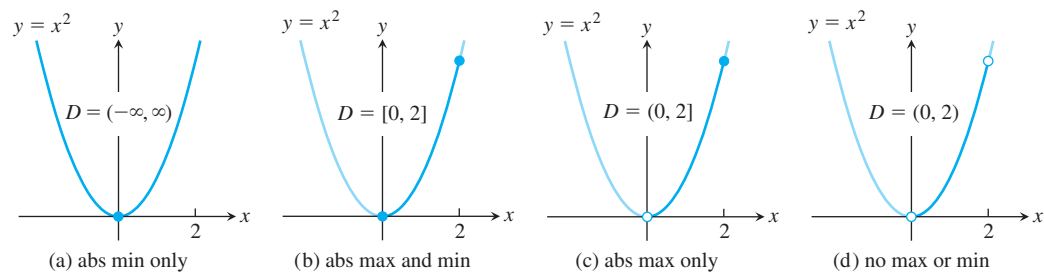


FIGURE 4.2 Graphs for Example 1.

HISTORICAL BIOGRAPHY

Daniel Bernoulli
(1700–1789)
Daniel Bernoulli was the second son of mathematician Johann Bernoulli. In 1724, Bernoulli published his *Exercitationes mathematicae* that attracted considerable attention.

To know more, visit the companion Website.

Some of the functions in Example 1 do not have a maximum or a minimum value. The following theorem asserts that a function which is *continuous* over (or on) a finite *closed* interval $[a, b]$ has an absolute maximum and an absolute minimum value on the interval. We look for these extreme values when we graph a function.

THEOREM 1 – The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for all x in $[a, b]$.

The proof of the Extreme Value Theorem requires a detailed knowledge of the real number system (see Appendix A.9) and we will not give it here. Figure 4.3 illustrates possible locations for the absolute extrema of a continuous function on a closed interval $[a, b]$. As we observed for the function $y = \cos x$, it is possible that an absolute minimum (or absolute maximum) may occur at two or more different points of the interval.

The requirements in Theorem 1 that the interval be closed and finite, and that the function be continuous, are essential. Without them, the conclusion of the theorem need not hold. Example 1 shows that an absolute extreme value may not exist if the interval fails to be both closed and finite. The exponential function $y = e^x$ over $(-\infty, \infty)$ shows that

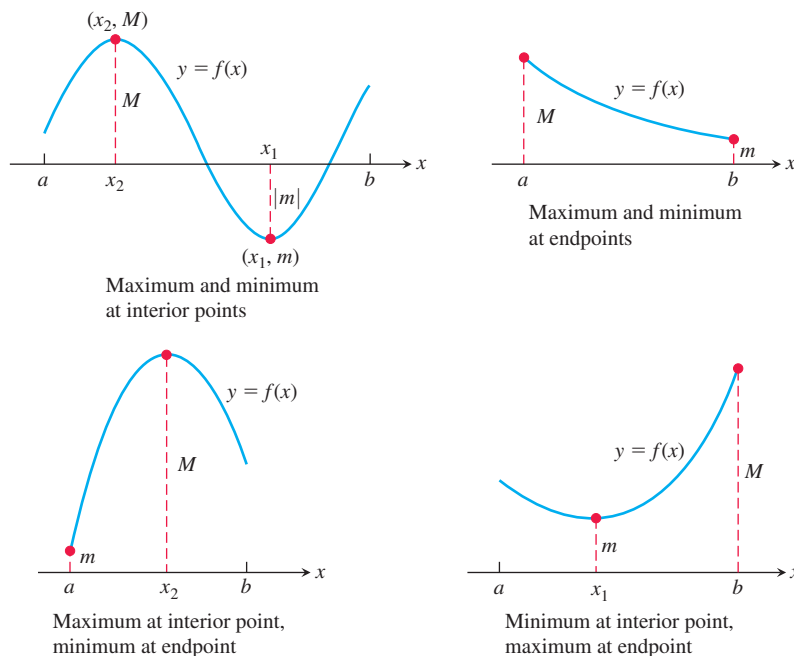


FIGURE 4.3 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

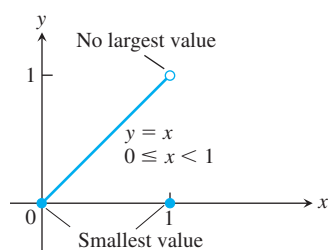


FIGURE 4.4 Even a single point of discontinuity can keep a function from having either a maximum or a minimum value on a closed interval. The function

$$y = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

is continuous at every point of $[0, 1]$ except $x = 1$, yet its graph over $[0, 1]$ does not have a highest point.

neither extreme value need exist on an infinite interval. Figure 4.4 shows that the continuity requirement cannot be omitted.

Local (Relative) Extreme Values

Figure 4.5 shows a graph with five points where a function has extreme values on its domain $[a, b]$. The function's absolute minimum occurs at a even though at e the function's value is smaller than at any other point *nearby*. The curve rises as x approaches c from the left, then falls to the right of c , making $f(c)$ a maximum locally. The function attains its absolute maximum at d . We now define what we mean by local extrema.

DEFINITIONS A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c .

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c .

If the domain of f is the closed interval $[a, b]$, then f has a local maximum at the endpoint $x = a$ if $f(x) \leq f(a)$ for all x in some half-open interval $[a, a + \delta)$, $\delta > 0$. Likewise, f has a local maximum at an interior point $x = c$ if $f(x) \leq f(c)$ for all x in some open interval $(c - \delta, c + \delta)$, $\delta > 0$, and a local maximum at the endpoint $x = b$ if $f(x) \leq f(b)$ for all x in some half-open interval $(b - \delta, b]$, $\delta > 0$. The inequalities are reversed for local minimum values. In Figure 4.5, the function f has local maxima at c and d and local minima at a , e , and b . Local extrema are also called **relative extrema**. Some functions can have infinitely many local extrema, even over a finite interval. One example is the function $f(x) = \sin(1/x)$ on the interval $(0, 1]$. (We graphed this function in Figure 2.41.)

An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood. Hence, *a list of all local maxima will*

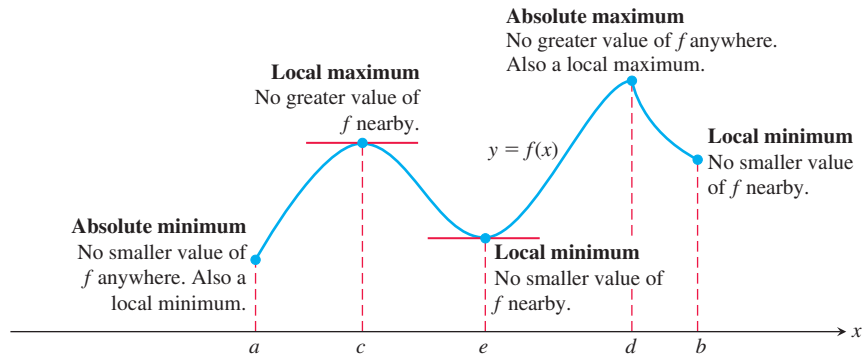


FIGURE 4.5 How to identify types of maxima and minima for a function with domain $a \leq x \leq b$.

automatically include the absolute maximum if there is one. Similarly, a list of all local minima will include the absolute minimum if there is one.

Finding Extrema

The next theorem explains why we usually need to investigate only a few values to find a function's extrema.

THEOREM 2—The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c , then

$$f'(c) = 0.$$

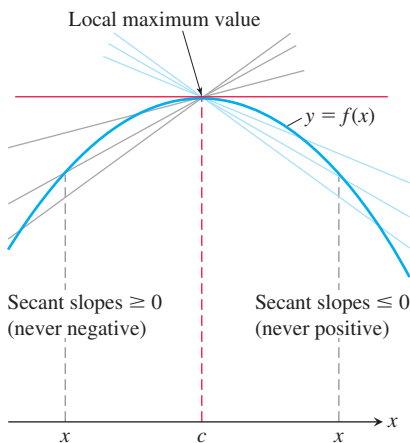


FIGURE 4.6 A curve with a local maximum value. The slope at c , simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.

Proof To prove that $f'(c)$ is zero at a local extremum, we show first that $f'(c)$ cannot be positive and second that $f'(c)$ cannot be negative. The only number that is neither positive nor negative is zero, so that is what $f'(c)$ must be.

To begin, suppose that f has a local maximum value at $x = c$ (Figure 4.6) so that $f(x) - f(c) \leq 0$ for all values of x near enough to c . Since c is an interior point of f 's domain, $f'(c)$ is defined by the two-sided limit

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}.$$

This means that the right-hand and left-hand limits both exist at $x = c$ and equal $f'(c)$. When we examine these limits separately, we find that

$$f'(c) = \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \leq 0. \quad \text{Because } x - c > 0 \text{ and } f(x) \leq f(c) \quad (1)$$

Similarly,

$$f'(c) = \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \geq 0. \quad \text{Because } x - c < 0 \text{ and } f(x) \leq f(c) \quad (2)$$

Together, Equations (1) and (2) imply $f'(c) = 0$.

This proves the theorem for local maximum values. To prove it for local minimum values, we simply use $f(x) \geq f(c)$, which reverses the inequalities in Formulas (1) and (2). ■

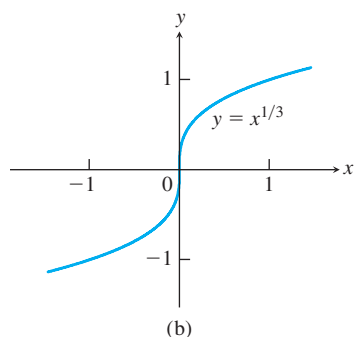
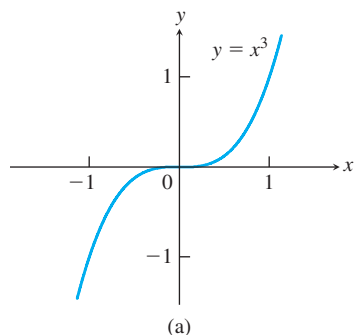


FIGURE 4.7 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

Theorem 2 says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. If we recall that all the domains we consider are intervals or unions of separate intervals, the only places where a function f can possibly have an extreme value (local or global) are

1. interior points where $f' = 0$, At $x = c$ and $x = e$ in Fig. 4.5
2. interior points where f' is undefined, At $x = d$ in Fig. 4.5
3. endpoints of an interval in the domain of f . At $x = a$ and $x = b$ in Fig. 4.5

The following definition helps us to summarize these results.

DEFINITION An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f .

Thus the only domain points where a continuous function on a closed and finite interval can assume extreme values are critical points and endpoints. However, be careful not to misinterpret what is being said here. A function may have a critical point at $x = c$ without having a local extreme value there. For instance, both of the functions $y = x^3$ and $y = x^{1/3}$ have critical points at the origin, but neither function has a local extreme value at the origin. Instead, each function has a *point of inflection* there (see Figure 4.7). We define and explore inflection points in Section 4.4.

If the interval is not closed or not finite (such as $a < x < b$ or $a < x < \infty$), we have seen that absolute extrema need not exist. However, many problems that ask for extreme values call for finding the absolute extrema of a continuous function on a closed and finite interval. Theorem 1 assures us that such values exist; Theorem 2 tells us that they are taken on only at critical points and endpoints. Often we can simply list these points and calculate the corresponding function values to find what the largest and smallest values are, and where they are located.

Finding the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Find all critical points of f on the interval.
2. Evaluate f at all critical points and endpoints.
3. Take the largest and smallest of these values.

EXAMPLE 2 Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution The function is differentiable over its entire domain, so the only critical point occurs where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$:

$$\text{Critical point value: } f(0) = 0$$

$$\text{Endpoint values: } f(-2) = 4$$

$$f(1) = 1.$$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$. ■

EXAMPLE 3 Find the absolute maximum and minimum values of $f(x) = 10x(2 - \ln x)$ on the interval $[1, e^2]$.

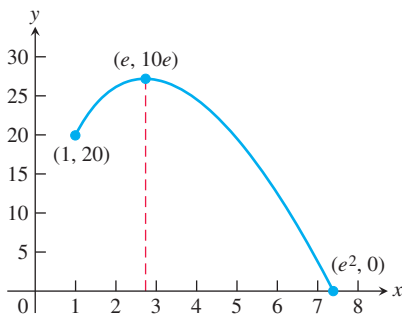


FIGURE 4.8 The extreme values of $f(x) = 10x(2 - \ln x)$ on $[1, e^2]$ occur at $x = e$ and $x = e^2$ (Example 3).

Solution Figure 4.8 suggests that f has its absolute maximum value near $x = 3$ and its absolute minimum value of 0 at $x = e^2$. Let's verify this observation.

We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative is

$$f'(x) = 10(2 - \ln x) - 10x\left(\frac{1}{x}\right) = 10(1 - \ln x).$$

The only critical point in the domain $[1, e^2]$ is the point $x = e$, where $\ln x = 1$. The values of f at this one critical point and at the endpoints are

$$\text{Critical point value: } f(e) = 10e(2 - \ln e) = 10e$$

$$\text{Endpoint values: } f(1) = 10(2 - \ln 1) = 20$$

$$f(e^2) = 10e^2(2 - 2 \ln e) = 0.$$

We can see from this list that the function's absolute maximum value is $10e \approx 27.2$; it occurs at the critical interior point $x = e$. The absolute minimum value is 0 and occurs at the right endpoint $x = e^2$. ■

EXAMPLE 4 Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

Solution We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative,

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}},$$

has no zeros but is undefined at the interior point $x = 0$. The values of f at this one critical point and at the endpoints are

$$\text{Critical point value: } f(0) = 0$$

$$\text{Endpoint values: } f(-2) = (-2)^{2/3} = \sqrt[3]{4}$$

$$f(3) = (3)^{2/3} = \sqrt[3]{9}.$$

We can see from this list that the function's absolute maximum value is $\sqrt[3]{9} \approx 2.08$, and it occurs at the right endpoint $x = 3$. The absolute minimum value is 0, and it occurs at the interior point $x = 0$ where the graph has a cusp (Figure 4.9). ■

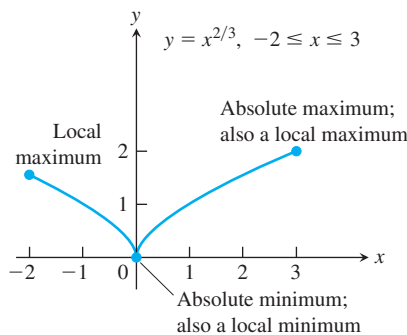


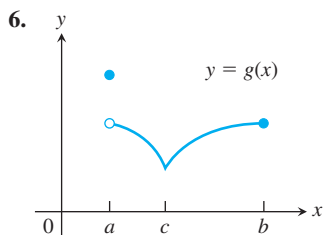
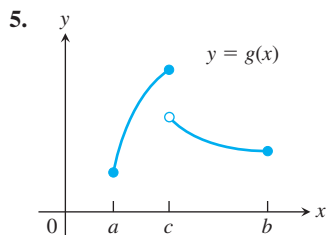
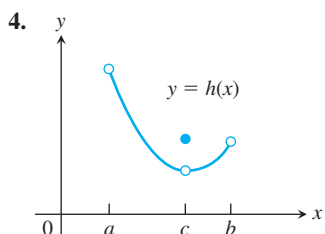
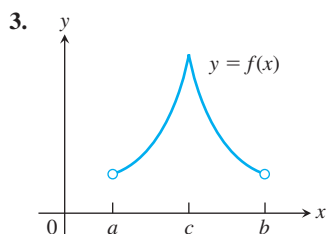
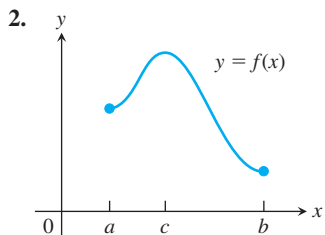
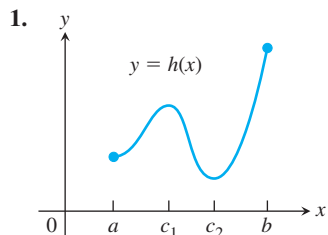
FIGURE 4.9 The extreme values of $f(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 4).

Theorem 1 leads to a method for finding the absolute maxima and absolute minima of a differentiable function on a finite closed interval. On more general domains, such as $(0, 1)$, $[2, 5)$, $[1, \infty)$, and $(-\infty, \infty)$, absolute maxima and minima may or may not exist. To determine if they exist, and to locate them when they do, we will develop methods to sketch the graph of a differentiable function. With knowledge of the asymptotes of the function, as well as the local maxima and minima, we can deduce the locations of the absolute maxima and minima, if any. For now we can find the absolute maxima and the absolute minima of a function on a finite closed interval by comparing the values of the function at its critical points and at the endpoints of the interval. For a differentiable function on a closed and finite interval $[a, b]$, these are the only points where the extrema have the potential to occur.

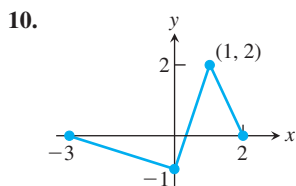
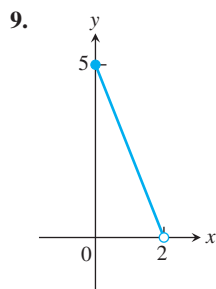
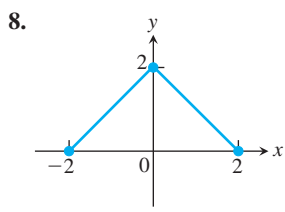
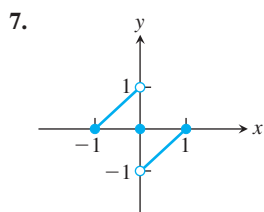
EXERCISES 4.1

Finding Extrema from Graphs

In Exercises 1–6, determine from the graph whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with Theorem 1.



In Exercises 7–10, find the absolute extreme values and where they occur.



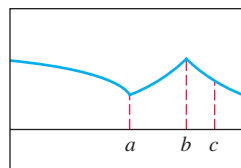
In Exercises 11–14, match the table with a graph.

11. x	$f'(x)$
a	0
b	0
c	5

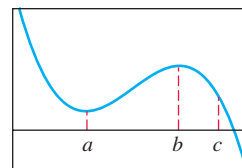
12. x	$f'(x)$
a	0
b	0
c	-5

13. x	$f'(x)$
a	does not exist
b	0
c	-2

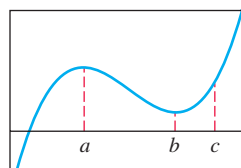
14. x	$f'(x)$
a	does not exist
b	does not exist
c	-1.7



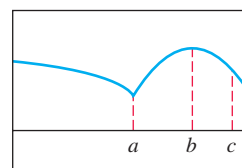
(a)



(b)



(c)



(d)

In Exercises 15–20, sketch the graph of each function and determine whether the function has any absolute extreme values on its domain. Explain how your answer is consistent with Theorem 1.

15. $f(x) = |x|$, $-1 < x < 2$

16. $y = 2 - x^2$, $-1 < x < 1$

17. $g(x) = \begin{cases} -x, & 0 \leq x < 1 \\ x - 1, & 1 \leq x \leq 2 \end{cases}$

18. $h(x) = \begin{cases} \frac{1}{x}, & -1 \leq x < 0 \\ \sqrt{x}, & 0 \leq x \leq 4 \end{cases}$

19. $y = 3 \sin x$, $0 < x < 2\pi$

20. $f(x) = \begin{cases} x + 1, & -1 \leq x < 0 \\ \cos x, & 0 < x \leq \frac{\pi}{2} \end{cases}$

Absolute Extrema on Finite Closed Intervals

In Exercises 21–36, find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

21. $f(x) = \frac{2}{3}x - 5, \quad -2 \leq x \leq 3$

22. $f(x) = -x - 4, \quad -4 \leq x \leq 1$

23. $f(x) = x^2 - 1, \quad -1 \leq x \leq 2$

24. $f(x) = 4 - x^3, \quad -2 \leq x \leq 1$

25. $F(x) = -\frac{1}{x^2}, \quad 0.5 \leq x \leq 2$

26. $F(x) = -\frac{1}{x}, \quad -2 \leq x \leq -1$

27. $h(x) = \sqrt[3]{x}, \quad -1 \leq x \leq 8$

28. $h(x) = -3x^{2/3}, \quad -1 \leq x \leq 1$

29. $g(x) = \sqrt{4 - x^2}, \quad -2 \leq x \leq 1$

30. $g(x) = -\sqrt{5 - x^2}, \quad -\sqrt{5} \leq x \leq 0$

31. $f(\theta) = \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{5\pi}{6}$

32. $f(\theta) = \tan \theta, \quad -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{4}$

33. $g(x) = \csc x, \quad \frac{\pi}{3} \leq x \leq \frac{2\pi}{3}$

34. $g(x) = \sec x, \quad -\frac{\pi}{3} \leq x \leq \frac{\pi}{6}$

35. $f(t) = 2 - |t|, \quad -1 \leq t \leq 3$

36. $f(t) = |t - 5|, \quad 4 \leq t \leq 7$

In Exercises 37–40:

- a. Find the absolute maximum and minimum values of each function on the given interval.

- T** b. Graph the function, identify the points on the graph where the absolute extrema occur, and include their coordinates.

37. $g(x) = xe^{-x}, \quad -1 \leq x \leq 1$

38. $h(x) = \ln(x + 1) - \frac{x}{2}, \quad 0 \leq x \leq 3$

39. $f(x) = \frac{1}{x} + \ln x, \quad 0.5 \leq x \leq 4$

40. $g(x) = e^{-x^2}, \quad -2 \leq x \leq 1$

In Exercises 41–44, find the function's absolute maximum and minimum values and say where they occur.

41. $f(x) = x^{4/3}, \quad -1 \leq x \leq 8$

42. $f(x) = x^{5/3}, \quad -1 \leq x \leq 8$

43. $g(\theta) = \theta^{3/5}, \quad -32 \leq \theta \leq 1$

44. $h(\theta) = 3\theta^{2/3}, \quad -27 \leq \theta \leq 8$

Finding Critical Points

In Exercises 45–56, determine all critical points and all domain endpoints for each function.

45. $y = x^2 - 6x + 7$

46. $f(x) = 6x^2 - x^3$

47. $f(x) = x(4 - x)^3$

48. $g(x) = (x - 1)^2(x - 3)^2$

49. $y = x^2 + \frac{2}{x}$

50. $f(x) = \frac{x^2}{x - 2}$

51. $y = x^2 - 32\sqrt{x}$

52. $g(x) = \sqrt{2x - x^2}$

53. $y = \ln(x + 1) - \tan^{-1}x$

54. $y = 2\sqrt{1 - x^2} + \arcsin x$

55. $y = x^3 + 3x^2 - 24x + 7$

56. $y = x - 3x^{2/3}$

Theory and Examples

In Exercises 57 and 58, give reasons for your answers.

57. Let $f(x) = (x - 2)^{2/3}$.

- Does $f'(2)$ exist?
- Show that the only local extreme value of f occurs at $x = 2$.
- Does the result in part (b) contradict the Extreme Value Theorem?
- Repeat parts (a) and (b) for $f(x) = (x - a)^{2/3}$, replacing 2 by a .

58. Let $f(x) = |x^3 - 9x|$.

- Does $f'(0)$ exist?
- Does $f'(3)$ exist?
- Does $f'(-3)$ exist?
- Determine all extrema of f .

In Exercises 59–62, show that the function has neither an absolute minimum nor an absolute maximum on its natural domain.

59. $y = x^{11} + x^3 + x - 5$

60. $y = 3x + \tan x$

61. $y = \frac{1 - e^x}{e^x + 1}$

62. $y = 2x - \sin 2x$

63. **A minimum with no derivative** The function $f(x) = |x|$ has an absolute minimum value at $x = 0$ even though f is not differentiable at $x = 0$. Is this consistent with Theorem 2? Give reasons for your answer.

64. **Even functions** If an even function $f(x)$ has a local maximum value at $x = c$, can anything be said about the value of f at $x = -c$? Give reasons for your answer.

65. **Odd functions** If an odd function $g(x)$ has a local minimum value at $x = c$, can anything be said about the value of g at $x = -c$? Give reasons for your answer.

66. **No critical points or endpoints exist** We know how to find the extreme values of a continuous function $f(x)$ by investigating its values at critical points and endpoints. But what if there *are* no critical points or endpoints? What happens then? Do such functions really exist? Give reasons for your answers.

67. The function

$$V(x) = x(10 - 2x)(16 - 2x), \quad 0 < x < 5,$$

models the volume of a box.

- Find the extreme values of V .
- Interpret any values found in part (a) in terms of the volume of the box.

68. Cubic functions Consider the cubic function

$$f(x) = ax^3 + bx^2 + cx + d.$$

- Show that f can have 0, 1, or 2 critical points. Give examples and graphs to support your argument.
- How many local extreme values can f have?

69. Maximum height of a vertically moving body The height of a body moving vertically is given by

$$s = -\frac{1}{2}gt^2 + v_0t + s_0, \quad g > 0,$$

with s in meters and t in seconds. Find the body's maximum height.

70. Peak alternating current Suppose that at any given time t (in seconds) the current i (in amperes) in an alternating current circuit is $i = 2 \cos t + 2 \sin t$. What is the peak current for this circuit (largest magnitude)?

T Graph the functions in Exercises 71–74. Then find the extreme values of the function on the interval and say where they occur.

- $f(x) = |x - 2| + |x + 3|$, $-5 \leq x \leq 5$
- $g(x) = |x - 1| - |x - 5|$, $-2 \leq x \leq 7$
- $h(x) = |x + 2| - |x - 3|$, $-\infty < x < \infty$
- $k(x) = |x + 1| + |x - 3|$, $-\infty < x < \infty$

COMPUTER EXPLORATIONS

In Exercises 75–82, you will use a CAS to help find the absolute extrema of the given function over the specified closed interval. Perform the following steps.

- Plot the function over the interval to see its general behavior there.
 - Find the interior points where $f' = 0$. (In some exercises, you may have to use the numerical equation solver to approximate a solution.) You may want to plot f' as well.
 - Find the interior points where f' does not exist.
 - Evaluate the function at all points found in parts (b) and (c) and at the endpoints of the interval.
 - Find the function's absolute extreme values on the interval and identify where they occur.
- $f(x) = x^4 - 8x^2 + 4x + 2$, $[-20/25, 64/25]$
 - $f(x) = -x^4 + 4x^3 - 4x + 1$, $[-3/4, 3]$
 - $f(x) = x^{2/3}(3 - x)$, $[-2, 2]$
 - $f(x) = 2 + 2x - 3x^{2/3}$, $[-1, 10/3]$
 - $f(x) = \sqrt{x} + \cos x$, $[0, 2\pi]$
 - $f(x) = x^{3/4} - \sin x + \frac{1}{2}$, $[0, 2\pi]$
 - $f(x) = \pi x^2 e^{-3x/2}$, $[0, 5]$
 - $f(x) = \ln(2x + x \sin x)$, $[1, 15]$

4.2 The Mean Value Theorem

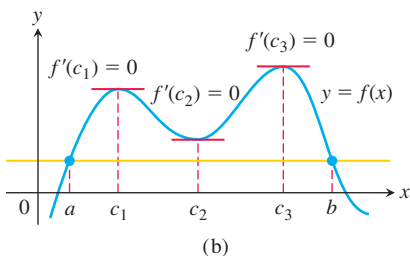
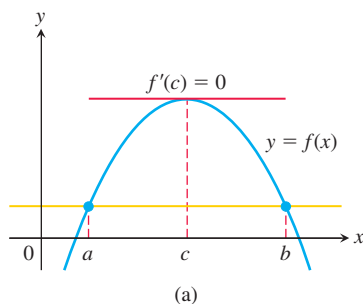


FIGURE 4.10 Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

We know that constant functions have zero derivatives, but could there be a more complicated function whose derivative is always zero? If two functions have identical derivatives over an interval, how are the functions related? We answer these and other questions in this chapter by applying the Mean Value Theorem. First we introduce a special case, known as Rolle's Theorem, which is used to prove the Mean Value Theorem.

Rolle's Theorem

As suggested by its graph, if a differentiable function crosses a horizontal line at two different points, there is at least one point between them where the tangent to the graph is horizontal and the derivative is zero (Figure 4.10). We now state and prove this result.

THEOREM 3—Rolle's Theorem

Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b) . If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

Proof Being continuous, f assumes absolute maximum and minimum values on $[a, b]$ by Theorem 1. These can occur only

- at interior points where f' is zero,
- at interior points where f' does not exist,
- at endpoints of the interval, in this case a and b .

HISTORICAL BIOGRAPHY**Michel Rolle**

(1652–1719)

French mathematician Michel Rolle was largely self-educated in mathematics. He worked as an accountant and studied algebra and the Diophantine equations whenever he found time.

To know more, visit the companion Website.

By hypothesis, f has a derivative at every interior point. That rules out possibility (2), leaving us with interior points where $f' = 0$ and with the two endpoints a and b .

If either the maximum or the minimum occurs at a point c between a and b , then $f'(c) = 0$ by Theorem 2 in Section 4.1, and we have found a point for Rolle's Theorem.

If both the absolute maximum and the absolute minimum occur at the endpoints, then because $f(a) = f(b)$ it must be the case that f is a constant function with $f(x) = f(a) = f(b)$ for every $x \in [a, b]$. Therefore, $f'(x) = 0$ and the point c can be taken anywhere in the interior (a, b) . ■

The hypotheses of Theorem 3 are essential. If they fail at even one point, the graph may not have a horizontal tangent (Figure 4.11).

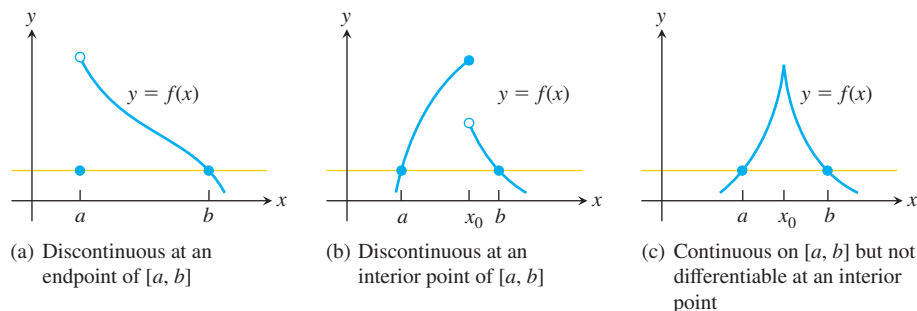


FIGURE 4.11 There may be no horizontal tangent line if the hypotheses of Rolle's Theorem do not hold.

Rolle's Theorem may be combined with the Intermediate Value Theorem to show when there is only one real solution of an equation $f(x) = 0$, as we illustrate in the next example.

EXAMPLE 1 Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

Solution We define the continuous function

$$f(x) = x^3 + 3x + 1.$$

Since $f(-1) = -3$ and $f(0) = 1$, the Intermediate Value Theorem tells us that the graph of f crosses the x -axis somewhere in the open interval $(-1, 0)$. (See Figure 4.12.) Now, if there were even two points $x = a$ and $x = b$ where $f(x)$ was zero, Rolle's Theorem would guarantee the existence of a point $x = c$ between them where f' was zero. However, the derivative

$$f'(x) = 3x^2 + 3$$

is never zero (because it is always positive). Therefore, f has no more than one zero. ■

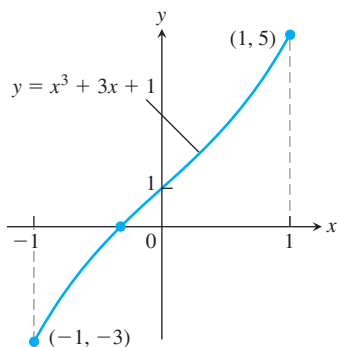


FIGURE 4.12 The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here where the curve crosses the x -axis between -1 and 0 (Example 1).

We will use Rolle's Theorem to prove the Mean Value Theorem.

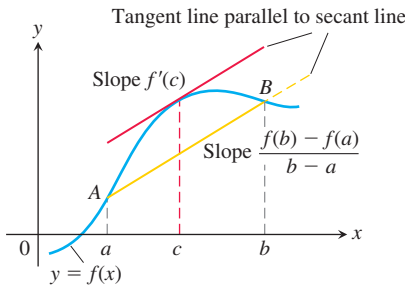


FIGURE 4.13 Geometrically, the Mean Value Theorem says that somewhere between a and b the curve has at least one tangent line parallel to the secant line that joins A and B .

HISTORICAL BIOGRAPHY

Joseph-Louis Lagrange
(1736–1813)

Lagrange was born in Turin, Italy. He enjoyed studying mathematics, despite his father's wish that he study law. Lagrange's mathematical contributions began as early as 1754 with the discovery of the calculus of variations and continued with applications to mechanics in 1756.

To know more, visit the companion Website.

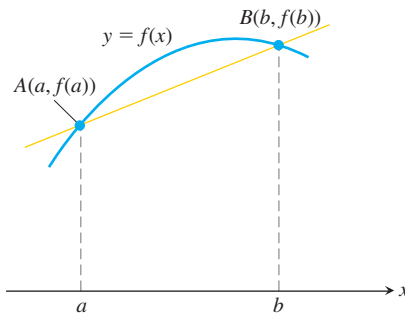


FIGURE 4.14 The graph of f and the secant line AB over the interval $[a, b]$.

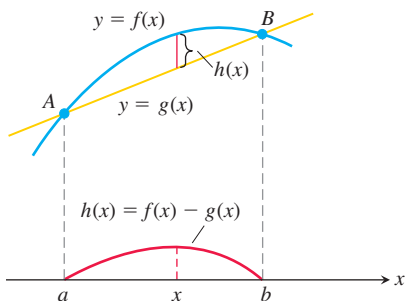


FIGURE 4.15 The secant line AB is the graph of the function $g(x)$. The function $h(x) = f(x) - g(x)$ gives the vertical distance between the graphs of f and g at x .

The Mean Value Theorem

The Mean Value Theorem, which was first stated by Joseph-Louis Lagrange, is a slanted version of Rolle's Theorem (Figure 4.13). The Mean Value Theorem guarantees that there is a point where the tangent line is parallel to the secant line that joins A and B .

THEOREM 4—The Mean Value Theorem

Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b) . Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \quad (1)$$

Proof We picture the graph of f and draw a line through the points $A(a, f(a))$ and $B(b, f(b))$. (See Figure 4.14.) The secant line is the graph of the function

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \quad (2)$$

(point-slope equation). The vertical difference between the graphs of f and g at x is

$$\begin{aligned} h(x) &= f(x) - g(x) \\ &= f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a). \end{aligned} \quad (3)$$

Figure 4.15 shows the graphs of f , g , and h together.

The function h satisfies the hypotheses of Rolle's Theorem on $[a, b]$. It is continuous on $[a, b]$ and differentiable on (a, b) because both f and g are. Also, $h(a) = h(b) = 0$ because the graphs of f and g both pass through A and B . Therefore $h'(c) = 0$ at some point $c \in (a, b)$. This is the point we want for Equation (1) in the theorem.

To verify Equation (1), we differentiate both sides of Equation (3) with respect to x and then set $x = c$:

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \quad \text{Derivative of Eq. (3)}$$

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \quad \text{Evaluated at } x = c$$

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a} \quad h'(c) = 0$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad \text{Rearranged}$$

which is what we set out to prove. ■

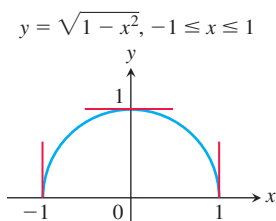


FIGURE 4.16 The function $f(x) = \sqrt{1 - x^2}$ satisfies the hypotheses (and conclusion) of the Mean Value Theorem on $[-1, 1]$ even though f is not differentiable at -1 and 1 .

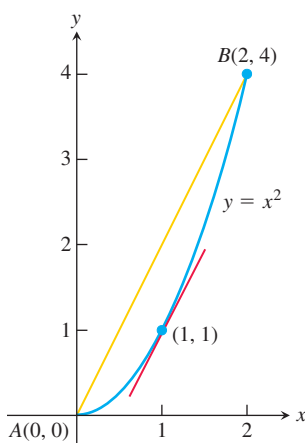


FIGURE 4.17 As we find in Example 1, $c = 1$ is where the tangent line is parallel to the secant line.

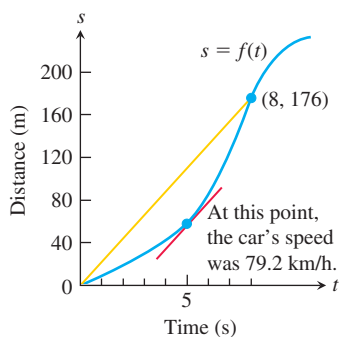


FIGURE 4.18 Distance versus elapsed time for the car in Example 3.

The hypotheses of the Mean Value Theorem do not require f to be differentiable at either a or b . One-sided continuity at a and b is enough (Figure 4.16).

EXAMPLE 2 The function $f(x) = x^2$ (Figure 4.17) is continuous for $0 \leq x \leq 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $(4 - 0)/(2 - 0) = 2$. In this case we can identify c by solving the equation $2c = 2$ to get $c = 1$. However, it is not always easy to find c algebraically, even though we know it always exists. ■

A Physical Interpretation

We can think of the number $(f(b) - f(a))/(b - a)$ as the average change in f over $[a, b]$ and can view $f'(c)$ as an instantaneous change. Then the Mean Value Theorem says that the instantaneous change at some interior point is equal to the average change over the entire interval.

EXAMPLE 3 If a car accelerating from zero takes 8 s to go 176 m, its average velocity for the 8-s interval is $176/8 = 22$ m/s. The Mean Value Theorem says that at some point during the acceleration, the speedometer must read exactly 79.2 km/h (22 m/s) (Figure 4.18). ■

Mathematical Consequences

At the beginning of the section, we asked what kind of function has a zero derivative over an interval. The first corollary of the Mean Value Theorem provides the answer that only constant functions have zero derivatives.

COROLLARY 1 If $f'(x) = 0$ at each point x of an open interval (a, b) , then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

Proof We want to show that f has a constant value on the interval (a, b) . We do so by showing that if x_1 and x_2 are any two points in (a, b) with $x_1 < x_2$, then $f(x_1) = f(x_2)$. Now f satisfies the hypotheses of the Mean Value Theorem on $[x_1, x_2]$: It is differentiable at every point of $[x_1, x_2]$ and hence continuous at every point as well. Therefore,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

at some point c between x_1 and x_2 . Since $f' = 0$ throughout (a, b) , this equation implies successively that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0, \quad f(x_2) - f(x_1) = 0, \quad \text{and} \quad f(x_1) = f(x_2). \quad \blacksquare$$

At the beginning of this section, we also asked about the relationship between two functions that have identical derivatives over an interval. The next corollary tells us that their values on the interval have a constant difference.

COROLLARY 2 If $f'(x) = g'(x)$ at each point x in an open interval (a, b) , then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b) .

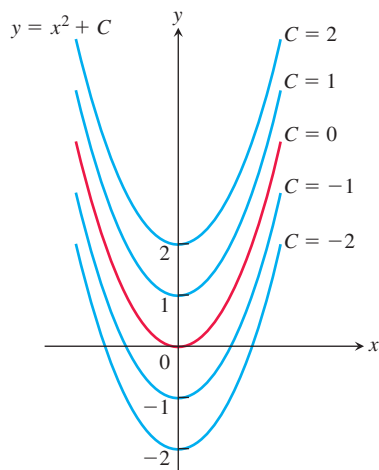


FIGURE 4.19 From a geometric point of view, Corollary 2 of the Mean Value Theorem says that the graphs of functions with identical derivatives on an interval can differ only by a vertical shift. The graphs of the functions with derivative $2x$ are the parabolas $y = x^2 + C$, shown here for several values of C .

Proof At each point $x \in (a, b)$ the derivative of the difference function $h = f - g$ is

$$h'(x) = f'(x) - g'(x) = 0.$$

Thus, $h(x) = C$ on (a, b) by Corollary 1. That is, $f(x) - g(x) = C$ on (a, b) , so $f(x) = g(x) + C$. ■

Corollaries 1 and 2 are also true if the open interval (a, b) fails to be finite. That is, they remain true if the interval is (a, ∞) , $(-\infty, b)$, or $(-\infty, \infty)$.

Corollary 2 will play an important role when we discuss antiderivatives in Section 4.8. It tells us, for instance, that since the derivative of $f(x) = x^2$ on $(-\infty, \infty)$ is $2x$, any other function with derivative $2x$ on $(-\infty, \infty)$ must have the formula $x^2 + C$ for some value of C (Figure 4.19).

EXAMPLE 4 Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

Solution Since the derivative of $g(x) = -\cos x$ is $g'(x) = \sin x$, we see that f and g have the same derivative. Corollary 2 then says that $f(x) = -\cos x + C$ for some constant C . Since the graph of f passes through the point $(0, 2)$, the value of C is determined from the condition that $f(0) = 2$:

$$f(0) = -\cos(0) + C = 2, \quad \text{so} \quad C = 3.$$

The function is $f(x) = -\cos x + 3$. ■

Finding Velocity and Position from Acceleration

We can use Corollary 2 to find the velocity and position functions of an object moving along a vertical line. Assume the object or body is falling freely from rest with acceleration 9.8 m/sec^2 . We assume the position $s(t)$ of the body is measured positive downward from the rest position (so the vertical coordinate line points *downward*, in the direction of the motion, with the rest position at 0).

We know that the velocity $v(t)$ is some function whose derivative is 9.8 . We also know that the derivative of $g(t) = 9.8t$ is 9.8 . By Corollary 2,

$$v(t) = 9.8t + C$$

for some constant C . Since the body falls from rest, $v(0) = 0$. Thus

$$9.8(0) + C = 0, \quad \text{and} \quad C = 0.$$

The velocity function must be $v(t) = 9.8t$. What about the position function $s(t)$?

We know that $s(t)$ is some function whose derivative is $9.8t$. We also know that the derivative of $f(t) = 4.9t^2$ is $9.8t$. By Corollary 2,

$$s(t) = 4.9t^2 + C$$

for some constant C . Since $s(0) = 0$,

$$4.9(0)^2 + C = 0, \quad \text{and} \quad C = 0.$$

The position function is $s(t) = 4.9t^2$ until the body hits the ground.

The ability to find functions from their rates of change is one of the very powerful tools of calculus. As we will see, it lies at the heart of the mathematical developments in Chapter 5.

EXERCISES 4.2

Checking the Mean Value Theorem

Find the value or values of c that satisfy the equation

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals in Exercises 1–8.

1. $f(x) = x^2 + 2x - 1$, $[0, 1]$
2. $f(x) = x^{2/3}$, $[0, 1]$
3. $f(x) = x + \frac{1}{x}$, $[\frac{1}{2}, 2]$
4. $f(x) = \sqrt{x-1}$, $[1, 3]$
5. $f(x) = \arcsin x$, $[-1, 1]$
6. $f(x) = \ln(x-1)$, $[2, 4]$
7. $f(x) = x^3 - x^2$, $[-1, 2]$
8. $g(x) = \begin{cases} x^3, & -2 \leq x \leq 0 \\ x^2, & 0 < x \leq 2 \end{cases}$

Which of the functions in Exercises 9–14 satisfy the hypotheses of the Mean Value Theorem on the given interval, and which do not? Give reasons for your answers.

9. $f(x) = x^{2/3}$, $[-1, 8]$
10. $f(x) = x^{4/5}$, $[0, 1]$
11. $f(x) = \sqrt{x(1-x)}$, $[0, 1]$
12. $f(x) = \begin{cases} \frac{\sin x}{x}, & -\pi \leq x < 0 \\ 0, & x = 0 \end{cases}$
13. $f(x) = \begin{cases} x^2 - x, & -2 \leq x \leq -1 \\ 2x^2 - 3x - 3, & -1 < x \leq 0 \end{cases}$
14. $f(x) = \begin{cases} 2x - 3, & 0 \leq x \leq 2 \\ 6x - x^2 - 7, & 2 < x \leq 3 \end{cases}$
15. The function

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at $x = 0$ and $x = 1$ and differentiable on $(0, 1)$, but its derivative on $(0, 1)$ is never zero. How can this be? Doesn't Rolle's Theorem say the derivative has to be zero somewhere in $(0, 1)$? Give reasons for your answer.

16. For what values of a , m , and b does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \leq x \leq 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

Roots (Zeros)

17. a. Plot the zeros of each polynomial on a line together with the zeros of its first derivative.

- i) $y = x^2 - 4$
- ii) $y = x^2 + 8x + 15$

iii) $y = x^3 - 3x^2 + 4 = (x+1)(x-2)^2$

iv) $y = x^3 - 33x^2 + 216x = x(x-9)(x-24)$

- b. Use Rolle's Theorem to prove that between every two zeros of $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ there lies a zero of $nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$.

18. Suppose that f'' is continuous on $[a, b]$ and that f has three zeros in the interval. Show that f'' has at least one zero in (a, b) . Generalize this result.
19. Show that if $f'' > 0$ throughout an interval $[a, b]$, then f' has at most one zero in $[a, b]$. What if $f'' < 0$ throughout $[a, b]$ instead?
20. Show that a cubic polynomial can have at most three real zeros.

Show that the functions in Exercises 21–28 have exactly one zero in the given interval.

21. $f(x) = x^4 + 3x + 1$, $[-2, -1]$
22. $f(x) = x^3 + \frac{4}{x^2} + 7$, $(-\infty, 0)$
23. $g(t) = \sqrt{t} + \sqrt{1+t} - 4$, $(0, \infty)$
24. $g(t) = \frac{1}{1-t} + \sqrt{1+t} - 3.1$, $(-1, 1)$
25. $r(\theta) = \theta + \sin^2\left(\frac{\theta}{3}\right) - 8$, $(-\infty, \infty)$
26. $r(\theta) = 2\theta - \cos^2\theta + \sqrt{2}$, $(-\infty, \infty)$
27. $r(\theta) = \sec^2\theta - \cos(2\theta) - 1$, $(0, \pi/2)$
28. $r(\theta) = 3\tan\theta - \cot\theta - \theta$, $(0, \pi/2)$

Finding Functions from Derivatives

29. Suppose that $f(-1) = 3$ and that $f'(x) = 0$ for all x . Must $f(x) = 3$ for all x ? Give reasons for your answer.
30. Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all x . Must $f(x) = 2x + 5$ for all x ? Give reasons for your answer.
31. Suppose that $f'(x) = 2x$ for all x . Find $f(2)$ if
 - a. $f(0) = 0$
 - b. $f(1) = 0$
 - c. $f(-2) = 3$.
32. What can be said about functions whose derivatives are constant? Give reasons for your answer.

In Exercises 33–38, find all possible functions with the given derivative.

33. a. $y' = x$ b. $y' = x^2$ c. $y' = x^3$
34. a. $y' = 2x$ b. $y' = 2x - 1$ c. $y' = 3x^2 + 2x - 1$
35. a. $y' = -\frac{1}{x^2}$ b. $y' = 1 - \frac{1}{x^2}$ c. $y' = 5 + \frac{1}{x^2}$
36. a. $y' = \frac{1}{2\sqrt{x}}$ b. $y' = \frac{1}{\sqrt{x}}$ c. $y' = 4x - \frac{1}{\sqrt{x}}$
37. a. $y' = \sin 2t$ b. $y' = \cos \frac{t}{2}$ c. $y' = \sin 2t + \cos \frac{t}{2}$
38. a. $y' = \sec^2\theta$ b. $y' = \sqrt{\theta}$ c. $y' = \sqrt{\theta} - \sec^2\theta$

In Exercises 39–42, find the function with the given derivative whose graph passes through the point P .

39. $f'(x) = 2x - 1$, $P(0, 0)$
40. $g'(x) = \frac{1}{x^2} + 2x$, $P(-1, 1)$

41. $f'(x) = e^{2x}$, $P\left(0, \frac{3}{2}\right)$

42. $r'(t) = \sec t \tan t - 1$, $P(0, 0)$

Finding Position from Velocity or Acceleration

Exercises 43–46 give the velocity $v = ds/dt$ and initial position of an object moving along a coordinate line. Find the object's position at time t .

43. $v = 9.8t + 5$, $s(0) = 10$ 44. $v = 32t - 2$, $s(0.5) = 4$

45. $v = \sin \pi t$, $s(0) = 0$ 46. $v = \frac{2}{\pi} \cos \frac{2t}{\pi}$, $s(\pi^2) = 1$

Exercises 47–50 give the acceleration $a = d^2s/dt^2$, initial velocity, and initial position of an object moving on a coordinate line. Find the object's position at time t .

47. $a = e^t$, $v(0) = 20$, $s(0) = 5$

48. $a = 9.8$, $v(0) = -3$, $s(0) = 0$

49. $a = -4 \sin 2t$, $v(0) = 2$, $s(0) = -3$

50. $a = \frac{9}{\pi^2} \cos \frac{3t}{\pi}$, $v(0) = 0$, $s(0) = -1$

Applications

51. **Temperature change** It took 14 s for a mercury thermometer to rise from -19°C to 100°C when it was taken from a freezer and placed in boiling water. Show that somewhere along the way, the mercury was rising at the rate of 8.5°C/s .
52. A trucker handed in a ticket at a tollbooth showing that in 2 hours she had covered 230 km on a toll road with speed limit 100 km/h. The trucker was cited for speeding. Why?
53. Classical accounts tell us that a 170-oar trireme (ancient Greek or Roman warship) once covered 184 sea miles in 24 hours. Explain why at some point during this feat the trireme's speed exceeded 7.5 knots (sea or nautical miles per hour).
54. A marathoner ran the 42-km New York City Marathon in 2.2 hours. Show that at least twice the marathoner was running at exactly 18 km/h, assuming the initial and final speeds are zero.
55. Show that at some instant during a 2-hour automobile trip the car's speedometer reading will equal the average speed for the trip.
56. **Free fall on the moon** On our moon, the acceleration of gravity is 1.6 m/s^2 . If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 s later?

Theory and Examples

57. **The geometric mean of a and b** The *geometric mean* of two positive numbers a and b is the number $\sqrt{ab}$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = 1/x$ on an interval of positive numbers $[a, b]$ is $c = \sqrt{ab}$.
58. **The arithmetic mean of a and b** The *arithmetic mean* of two numbers a and b is the number $(a + b)/2$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = x^2$ on any interval $[a, b]$ is $c = (a + b)/2$.

T 59. Graph the function

$$f(x) = \sin x \sin(x + 2) - \sin^2(x + 1).$$

What does the graph do? Why does the function behave this way? Give reasons for your answers.

60. Rolle's Theorem

- a. Construct a polynomial $f(x)$ that has zeros at $x = -2, -1, 0, 1$, and 2 .
- b. Graph f and its derivative f' together. How is what you see related to Rolle's Theorem?
- c. Do $g(x) = \sin x$ and its derivative g' illustrate the same phenomenon as f and f' ?
61. **Unique solution** Assume that f is continuous on $[a, b]$ and differentiable on (a, b) . Also assume that $f(a)$ and $f(b)$ have opposite signs and that $f' \neq 0$ between a and b . Show that $f(x) = 0$ exactly once between a and b .
62. **Parallel tangent line** Assume that f and g are differentiable on $[a, b]$ and that $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one point between a and b where the tangent lines to the graphs of f and g are parallel or the same line. Illustrate with a sketch.
63. Suppose that $f'(x) \leq 1$ for $1 \leq x \leq 4$. Show that $f(4) - f(1) \leq 3$.
64. Suppose that $0 < f'(x) < 1/2$ for all x -values. Show that $f(-1) < f(1) < 2 + f(-1)$.
65. Show that $|\cos x - 1| \leq |x|$ for all x -values. (*Hint:* Consider $f(t) = \cos t$ on the closed interval with the endpoints 0 and x .)
66. Show that for any numbers a and b , the sine inequality $|\sin b - \sin a| \leq |b - a|$ is true.
67. If the graphs of two differentiable functions $f(x)$ and $g(x)$ start at the same point in the plane and the functions have the same rate of change at every point, do the graphs have to be identical? Give reasons for your answer.
68. If $|f(w) - f(x)| \leq |w - x|$ for all values w and x and f is a differentiable function, show that $-1 \leq f'(x) \leq 1$ for all x -values.
69. Assume that f is differentiable on $a \leq x \leq b$ and that $f(b) < f(a)$. Show that f' is negative at some point between a and b .
70. Let f be a function defined on an interval $[a, b]$. What conditions could you place on f to guarantee that

$$\min f' \leq \frac{f(b) - f(a)}{b - a} \leq \max f',$$

where $\min f'$ and $\max f'$ refer to the minimum and maximum values of f' on $[a, b]$? Give reasons for your answers.

- T** 71. Use the inequalities in Exercise 70 to estimate $f(0.1)$ if $f'(x) = 1/(1 + x^4 \cos x)$ for $0 \leq x \leq 0.1$ and $f(0) = 1$.
- T** 72. Use the inequalities in Exercise 70 to estimate $f(0.1)$ if $f'(x) = 1/(1 - x^4)$ for $0 \leq x \leq 0.1$ and $f(0) = 2$.
73. Let f be differentiable at every value of x and suppose that $f(1) = 1$, that $f' < 0$ on $(-\infty, 1)$, and that $f' > 0$ on $(1, \infty)$.
 - a. Show that $f(x) \geq 1$ for all x .
 - b. Must $f'(1) = 0$? Explain.
74. Let $f(x) = px^2 + qx + r$ be a quadratic function defined on a closed interval $[a, b]$. Show that there is exactly one point c in (a, b) at which f satisfies the conclusion of the Mean Value Theorem.

4.3 Monotonic Functions and the First Derivative Test

In sketching the graph of a differentiable function, it is useful to know where it increases (rises from left to right) and where it decreases (falls from left to right) over an interval. This section gives a test to determine where it increases and where it decreases. We also show how to test the critical points of a function to identify whether local extreme values are present.

Increasing Functions and Decreasing Functions

As another corollary to the Mean Value Theorem, we show that functions with positive derivatives are increasing functions and functions with negative derivatives are decreasing functions. A function that is either increasing on an interval or decreasing on an interval is said to be **monotonic** on the interval.

COROLLARY 3 Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) .

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Proof Let x_1 and x_2 be any two points in $[a, b]$ with $x_1 < x_2$. The Mean Value Theorem applied to f on $[x_1, x_2]$ says that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c between x_1 and x_2 . The sign of the right-hand side of this equation is the same as the sign of $f'(c)$ because $x_2 - x_1$ is positive. Therefore, $f(x_2) > f(x_1)$ if f' is positive on (a, b) and $f(x_2) < f(x_1)$ if f' is negative on (a, b) . ■

Corollary 3 tells us that $f(x) = \sqrt{x}$ is increasing on the interval $[0, b]$ for any $b > 0$ because $f'(x) = 1/(2\sqrt{x})$ is positive on $(0, b)$. The derivative does not exist at $x = 0$, but Corollary 3 still applies. The corollary is also valid for open and for infinite intervals, so $f(x) = \sqrt{x}$ is also increasing on $(0, 1)$, on $(0, \infty)$, and on $[0, \infty)$.

To find the intervals where a function f is increasing or decreasing, we first find all of the critical points of f . If $a < b$ are two critical points for f , and if the derivative f' is continuous but never zero on the interval (a, b) , then by the Intermediate Value Theorem applied to f' , the derivative must be everywhere positive on (a, b) , or everywhere negative there. One way we can determine the sign of f' on (a, b) is simply by evaluating the derivative at a single point c in (a, b) . If $f'(c) > 0$, then $f'(x) > 0$ for all x in (a, b) so f is increasing on $[a, b]$ by Corollary 3; if $f'(c) < 0$, then f is decreasing on $[a, b]$. It doesn't matter which point c we choose in (a, b) since the sign of $f'(c)$ is the same for all choices. Usually we pick c to be a point where it is easy to evaluate $f'(c)$. The next example illustrates how we use this procedure.

EXAMPLE 1 Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and those on which f is decreasing.

Solution The function f is everywhere continuous and differentiable. The first derivative

$$\begin{aligned} f'(x) &= 3x^2 - 12 = 3(x^2 - 4) \\ &= 3(x + 2)(x - 2) \end{aligned}$$

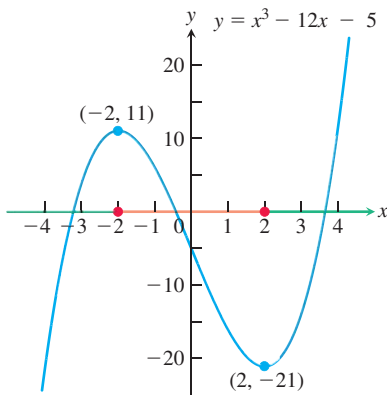


FIGURE 4.20 The function $f(x) = x^3 - 12x - 5$ is monotonic on three separate intervals (Example 1).

is zero at $x = -2$ and $x = 2$. These critical points subdivide the domain of f to create nonoverlapping open intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$ on which f' is either positive or negative. We determine the sign of f' by evaluating f' at a convenient point in each subinterval. We evaluate f' at $x = -3$ in the first interval, $x = 0$ in the second interval, and $x = 3$ in the third since f' is relatively easy to compute at these points. The behavior of f is determined by then applying Corollary 3 to each subinterval. The results are summarized in the following table, and the graph of f is given in Figure 4.20.

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	+	-	+
Behavior of f	increasing	decreasing	increasing

We used “strict” less-than inequalities to identify the intervals in the summary table for Example 1, since open intervals were specified. Corollary 3 says that we could use $\leq$ inequalities as well. That is, the function f in the example is increasing on $-\infty < x \leq -2$, decreasing on $-2 \leq x \leq 2$, and increasing on $2 \leq x < \infty$. We do not talk about whether a function is increasing or decreasing at a single point.

HISTORICAL BIOGRAPHY

Edmund Halley
(1656–1742)

Halley, a British biologist, geologist, sea captain, astronomer, and mathematician, encouraged Newton to write the *Principia*. Despite all of Halley’s accomplishments, he is known today as the man who calculated the orbit of the comet of 1682.

To know more, visit the companion Website.

First Derivative Test for Local Extrema

In Figure 4.21, at the points where f has a minimum value, $f' < 0$ immediately to the left and $f' > 0$ immediately to the right. (If the point is an endpoint, there is only one side to consider.) Thus, the function is decreasing on the left of the minimum value and it is increasing on its right. Similarly, at the points where f has a maximum value, $f' > 0$ immediately to the left and $f' < 0$ immediately to the right. Thus, the function is increasing on the left of the maximum value and decreasing on its right. In summary, at a local extreme point, the sign of $f'(x)$ changes.

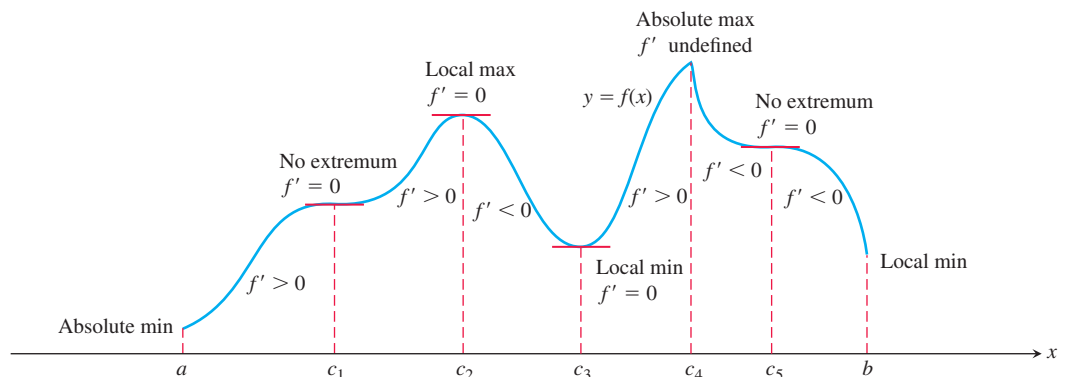


FIGURE 4.21 The critical points of a function locate where it is increasing and where it is decreasing. The first derivative changes sign at a critical point where a local extremum occurs.

These observations lead to a test for the presence and nature of local extreme values of differentiable functions.

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f , and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

- 1. if f' changes from negative to positive at c , then f has a local minimum at c ;
- 2. if f' changes from positive to negative at c , then f has a local maximum at c ;
- 3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c .

The test for local extrema at endpoints is similar, but there is only one side to consider in determining whether f is increasing or decreasing, based on the sign of f' .

Proof of the First Derivative Test Part (1). Since the sign of f' changes from negative to positive at c , there are numbers a and b such that $a < c < b$, $f' < 0$ on (a, c) , and $f' > 0$ on (c, b) . If $x \in (a, c)$, then $f(c) < f(x)$ because $f' < 0$ implies that f is decreasing on $[a, c]$. If $x \in (c, b)$, then $f(c) < f(x)$ because $f' > 0$ implies that f is increasing on $[c, b]$. Therefore, $f(x) \geq f(c)$ for every $x \in (a, b)$. By definition, f has a local minimum at c .

Parts (2) and (3) are proved similarly. ■

EXAMPLE 2 Find the critical points of

$$f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}.$$

Identify the open intervals on which f is increasing and those on which it is decreasing. Find the function's local and absolute extreme values.

Solution The function f is continuous at all x since it is the product of two continuous functions, $x^{1/3}$ and $(x - 4)$. The first derivative,

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^{4/3} - 4x^{1/3}) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} \\ &= \frac{4}{3}x^{-2/3}(x - 1) = \frac{4(x - 1)}{3x^{2/3}}, \end{aligned}$$

is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in the domain, so the critical points $x = 0$ and $x = 1$ are the only places where f might have an extreme value.

The critical points partition the x -axis into open intervals on which f' is either positive or negative. The sign pattern of f' reveals the behavior of f between and at the critical points, as summarized in the following table.

Interval	$x < 0$	$0 < x < 1$	$x > 1$
Sign of f'	−	−	+
Behavior of f	decreasing	decreasing	increasing

Corollary 3 to the Mean Value Theorem implies that f decreases on $(-\infty, 0)$, decreases on $(0, 1)$, and increases on $(1, \infty)$. The First Derivative Test for Local Extrema tells us that f does not have an extreme value at $x = 0$ (f' does not change sign) and that f has a local minimum at $x = 1$ (f' changes from negative to positive).

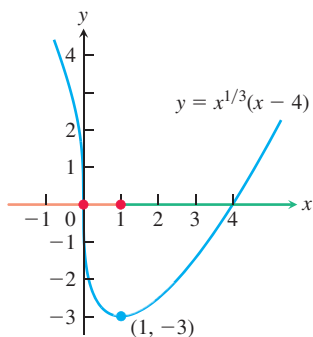


FIGURE 4.22 The function $f(x) = x^{1/3}(x - 4)$ decreases when $x < 1$ and increases when $x > 1$ (Example 2).

The value of the local minimum is $f(1) = 1^{1/3}(1 - 4) = -3$. This is also an absolute minimum since f is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$. Figure 4.22 shows this value in relation to the function's graph.

Note that $\lim_{x \rightarrow 0} f'(x) = -\infty$, so the graph of f has a vertical tangent line at the origin. ■

EXAMPLE 3 Find the critical points of

$$f(x) = (x^2 - 3)e^x.$$

Identify the open intervals on which f is increasing and those on which it is decreasing. Find the function's local and absolute extreme values.

Solution The function f is continuous and differentiable for all real numbers, so the critical points occur only at the zeros of f' .

Using the Derivative Product Rule, we find the derivative

$$\begin{aligned} f'(x) &= (x^2 - 3) \cdot \frac{d}{dx}e^x + \frac{d}{dx}(x^2 - 3) \cdot e^x \\ &= (x^2 - 3)e^x + (2x)e^x \\ &= (x^2 + 2x - 3)e^x. \end{aligned}$$

Since e^x is never zero, the first derivative is zero if and only if

$$\begin{aligned} x^2 + 2x - 3 &= 0 \\ (x + 3)(x - 1) &= 0. \end{aligned}$$

The zeros $x = -3$ and $x = 1$ partition the x -axis into open intervals as follows.

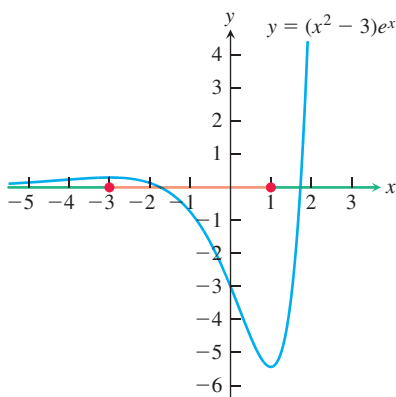


FIGURE 4.23 The graph of $f(x) = (x^2 - 3)e^x$ (Example 3).

Interval	$x < -3$	$-3 < x < 1$	$1 < x$
Sign of f'	+	-	+
Behavior of f	increasing	decreasing	increasing

We can see from the table that there is a local maximum (about 0.299) at $x = -3$ and a local minimum (about -5.437) at $x = 1$. The local minimum value is also an absolute minimum because $f(x) > 0$ for $|x| > \sqrt{3}$. There is no absolute maximum. The function increases on $(-\infty, -3)$ and $(1, \infty)$ and decreases on $(-3, 1)$. Figure 4.23 shows the graph. ■

EXERCISES 4.3

Analyzing Functions from Derivatives

Answer the following questions about the functions whose derivatives are given in Exercises 1–14:

- What are the critical points of f ?
- On what open intervals is f increasing or decreasing?
- At what points, if any, does f assume local maximum or minimum values?

- $f'(x) = x(x - 1)$
- $f'(x) = (x - 1)(x + 2)$
- $f'(x) = (x - 1)^2(x + 2)$
- $f'(x) = (x - 1)^2(x + 2)^2$
- $f'(x) = (x - 1)e^{-x}$
- $f'(x) = (x - 7)(x + 1)(x + 5)$
- $f'(x) = \frac{x^2(x - 1)}{x + 2}, \quad x \neq -2$

$$8. f'(x) = \frac{(x - 2)(x + 4)}{(x + 1)(x - 3)}, \quad x \neq -1, 3$$

$$9. f'(x) = 1 - \frac{4}{x^2}, \quad x \neq 0 \quad 10. f'(x) = 3 - \frac{6}{\sqrt{x}}, \quad x \neq 0$$

$$11. f'(x) = x^{-1/3}(x + 2) \quad 12. f'(x) = x^{-1/2}(x - 3)$$

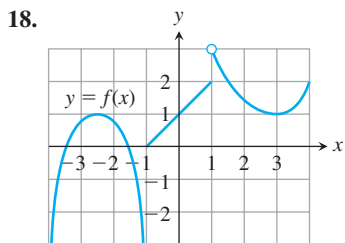
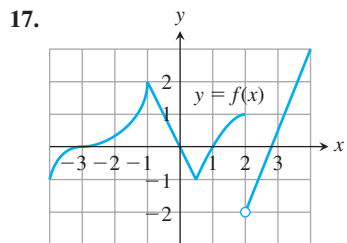
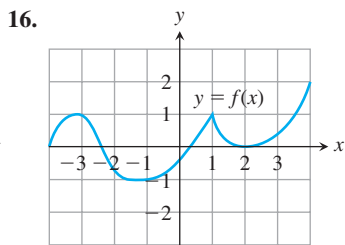
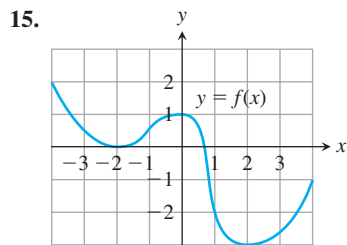
$$13. f'(x) = (\sin x - 1)(2 \cos x + 1), \quad 0 \leq x \leq 2\pi$$

$$14. f'(x) = (\sin x + \cos x)(\sin x - \cos x), \quad 0 \leq x \leq 2\pi$$

Identifying Extrema

In Exercises 15–18:

- Find the open intervals on which the function is increasing and those on which it is decreasing.
- Identify the function's local and absolute extreme values, if any, saying where they occur.



In Exercises 19–46:

- a. Find the open intervals on which the function is increasing and those on which it is decreasing.
- b. Identify the function's local extreme values, if any, saying where they occur.

19. $g(t) = -t^2 - 3t + 5$
20. $g(t) = -3t^2 + 9t + 5$
21. $h(x) = -x^3 + 2x^2$
22. $h(x) = 2x^3 - 18x$
23. $f(\theta) = 3\theta^2 - 4\theta^3$
24. $f(\theta) = 6\theta - \theta^3$
25. $f(r) = 3r^3 + 16r$
26. $h(r) = (r + 7)^3$
27. $f(x) = x^4 - 8x^2 + 16$
28. $g(x) = x^4 - 4x^3 + 4x^2$
29. $H(t) = \frac{3}{2}t^4 - t^6$
30. $K(t) = 15t^3 - t^5$
31. $f(x) = x - 6\sqrt{x-1}$
32. $g(x) = 4\sqrt{x} - x^2 + 3$
33. $g(x) = x\sqrt{8-x^2}$
34. $g(x) = x^2\sqrt{5-x}$
35. $f(x) = \frac{x^2-3}{x-2}, x \neq 2$
36. $f(x) = \frac{x^3}{3x^2+1}$
37. $f(x) = x^{1/3}(x+8)$
38. $g(x) = x^{2/3}(x+5)$
39. $h(x) = x^{1/3}(x^2-4)$
40. $k(x) = x^{2/3}(x^2-4)$
41. $f(x) = e^{2x} + e^{-x}$
42. $f(x) = e^{\sqrt{x}}$
43. $f(x) = x \ln x$
44. $f(x) = x^2 \ln x$
45. $g(x) = x(\ln x)^2$
46. $g(x) = x^2 - 2x - 4 \ln x$

In Exercises 47–58:

- a. Identify the function's local extreme values in the given domain, and say where they occur.
- T** b. Graph the function over the given domain. Which of the extreme values, if any, are absolute?

47. $f(x) = 2x - x^2, -\infty < x \leq 2$
48. $f(x) = (x+1)^2, -\infty < x \leq 0$
49. $g(x) = x^2 - 4x + 4, 1 \leq x < \infty$
50. $g(x) = -x^2 - 6x - 9, -4 \leq x < \infty$
51. $f(t) = 12t - t^3, -3 \leq t < \infty$
52. $f(t) = t^3 - 3t^2, -\infty < t \leq 3$
53. $h(x) = \frac{x^3}{3} - 2x^2 + 4x, 0 \leq x < \infty$
54. $k(x) = x^3 + 3x^2 + 3x + 1, -\infty < x \leq 0$

55. $f(x) = \sqrt{25-x^2}, -5 \leq x \leq 5$

56. $f(x) = \sqrt{x^2-2x-3}, 3 \leq x < \infty$

57. $g(x) = \frac{x-2}{x^2-1}, 0 \leq x < 1$

58. $g(x) = \frac{x^2}{4-x^2}, -2 < x \leq 1$

In Exercises 59–66:

- a. Find the local extrema of each function on the given interval, and say where they occur.

T b. Graph the function and its derivative together. Comment on the behavior of f in relation to the signs and values of f' .

59. $f(x) = \sin 2x, 0 \leq x \leq \pi$

60. $f(x) = \sin x - \cos x, 0 \leq x \leq 2\pi$

61. $f(x) = \sqrt{3} \cos x + \sin x, 0 \leq x \leq 2\pi$

62. $f(x) = -2x + \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

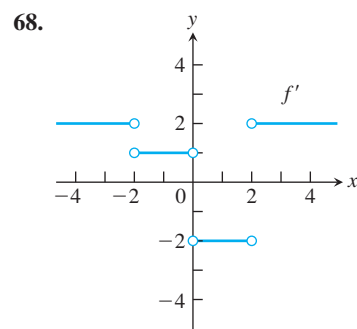
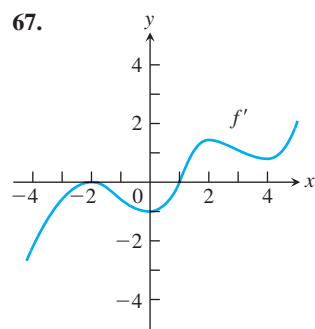
63. $f(x) = \frac{x}{2} - 2 \sin \frac{x}{2}, 0 \leq x \leq 2\pi$

64. $f(x) = -2 \cos x - \cos^2 x, -\pi \leq x \leq \pi$

65. $f(x) = \csc^2 x - 2 \cot x, 0 < x < \pi$

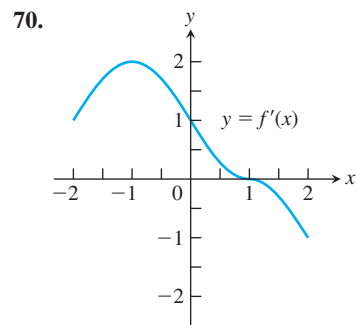
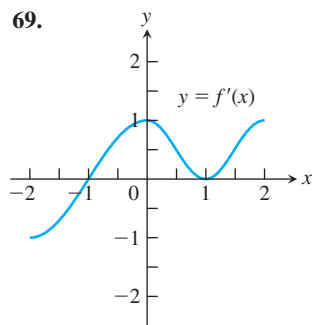
66. $f(x) = \sec^2 x - 2 \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

In Exercises 67 and 68, the graph of f' is given. Assume that f is continuous, and determine the x -values corresponding to local minima and local maxima.



In Exercises 69 and 70, the graph of f' is given. Assume that f has domain $(-2, 2)$.

- a. Either use the graph to determine which intervals f is increasing on and which intervals f is decreasing on, or explain why this information cannot be determined from the graph.
- b. Either use the graph to determine which intervals f is positive on and which intervals f is negative on, or explain why this information cannot be determined from the graph.



Theory and Examples

Show that the functions in Exercises 71 and 72 have local extreme values at the given values of θ , and say which kind of local extreme the function has.

71. $h(\theta) = 3 \cos \frac{\theta}{2}$, $0 \leq \theta \leq 2\pi$, at $\theta = 0$ and $\theta = 2\pi$

72. $h(\theta) = 5 \sin \frac{\theta}{2}$, $0 \leq \theta \leq \pi$, at $\theta = 0$ and $\theta = \pi$

73. Sketch the graph of a differentiable function $y = f(x)$ through the point $(1, 1)$ if $f'(1) = 0$ and

- $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$;
- $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$;
- $f'(x) > 0$ for $x \neq 1$;
- $f'(x) < 0$ for $x \neq 1$.

74. Sketch the graph of a differentiable function $y = f(x)$ that has

- a local minimum at $(1, 1)$ and a local maximum at $(3, 3)$;
- a local maximum at $(1, 1)$ and a local minimum at $(3, 3)$;
- local maxima at $(1, 1)$ and $(3, 3)$;
- local minima at $(1, 1)$ and $(3, 3)$.

75. Sketch the graph of a continuous function $y = g(x)$ such that

- $g(2) = 2$, $0 < g' < 1$ for $x < 2$, $g'(x) \rightarrow 1^-$ as $x \rightarrow 2^-$, $-1 < g' < 0$ for $x > 2$, and $g'(x) \rightarrow -1^+$ as $x \rightarrow 2^+$;
- $g(2) = 2$, $g' < 0$ for $x < 2$, $g'(x) \rightarrow -\infty$ as $x \rightarrow 2^-$, $g' > 0$ for $x > 2$, and $g'(x) \rightarrow \infty$ as $x \rightarrow 2^+$.

76. Sketch the graph of a continuous function $y = h(x)$ such that

- $h(0) = 0$, $-2 \leq h(x) \leq 2$ for all x , $h'(x) \rightarrow \infty$ as $x \rightarrow 0^-$, and $h'(x) \rightarrow \infty$ as $x \rightarrow 0^+$;
- $h(0) = 0$, $-2 \leq h(x) \leq 0$ for all x , $h'(x) \rightarrow \infty$ as $x \rightarrow 0^-$, and $h'(x) \rightarrow -\infty$ as $x \rightarrow 0^+$.

77. Discuss the extreme-value behavior of the function $f(x) = x \sin(1/x)$, $x \neq 0$. How many critical points does this function have? Where are they located on the x -axis? Does f have an absolute minimum? An absolute maximum? (See Exercise 49 in Section 2.3.)

78. Find the open intervals on which the function $f(x) = ax^2 + bx + c$, $a \neq 0$, is increasing and those on which it is decreasing. Describe the reasoning behind your answer.

79. Determine the values of constants a and b so that $f(x) = ax^2 + bx$ has an absolute maximum at the point $(1, 2)$.

80. Determine the values of constants a , b , c , and d so that $f(x) = ax^3 + bx^2 + cx + d$ has a local maximum at the point $(0, 0)$ and a local minimum at the point $(1, -1)$.

81. Locate and identify the absolute extreme values of

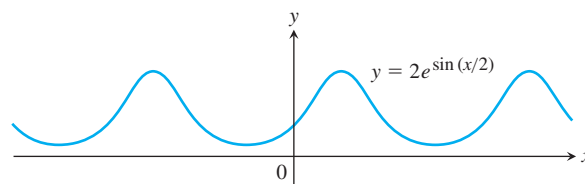
- $\ln(\cos x)$ on $[-\pi/4, \pi/3]$,
- $\cos(\ln x)$ on $[1/2, 2]$.

82. a. Prove that $f(x) = x - \ln x$ is increasing for $x > 1$.

b. Using part (a), show that $\ln x < x$ if $x > 1$.

83. Find the absolute maximum and the absolute minimum values of $f(x) = e^x - 2x$ on $[0, 1]$.

84. Where does the periodic function $f(x) = 2e^{\sin(x/2)}$ take on its extreme values and what are these values?



85. Find the absolute maximum value of $f(x) = x^2 \ln(1/x)$ and say where it occurs.

86. a. Prove that $e^x \geq 1 + x$ if $x \geq 0$.

b. Use the result in part (a) to show that

$$e^x \geq 1 + x + \frac{1}{2}x^2.$$

87. Show that increasing functions and decreasing functions are one-to-one. That is, show that for any x_1 and x_2 in I , $x_2 \neq x_1$ implies $f(x_2) \neq f(x_1)$.

Use the results of Exercise 87 to show that the functions in Exercises 88–92 have inverses over their domains. Find a formula for df^{-1}/dx using Theorem 3, Section 3.8.

88. $f(x) = (1/3)x + (5/6)$

89. $f(x) = 27x^3$

90. $f(x) = 1 - 8x^3$

91. $f(x) = (1 - x)^3$

92. $f(x) = x^{5/3}$

4.4 Concavity and Curve Sketching

We have seen how the first derivative tells us where a function is increasing, where it is decreasing, and whether a local maximum or local minimum occurs at a critical point. In this section we see that the second derivative gives us information about how the graph of a differentiable function bends or turns. With this knowledge about the first and second derivatives, coupled with our previous understanding of symmetry and asymptotic behavior studied in Sections 1.1 and 2.5, we can now draw an accurate graph of a function. By organizing all of these ideas into a coherent procedure, we give a method for sketching graphs and revealing visually the key features of functions. Identifying and knowing the locations of these features is of major importance in mathematics and its applications to science and engineering, especially in the graphical analysis and interpretation of data. When the domain of a function is not a finite closed interval, sketching a graph helps to determine whether absolute maxima or absolute minima exist and, if they do exist, where they are located.

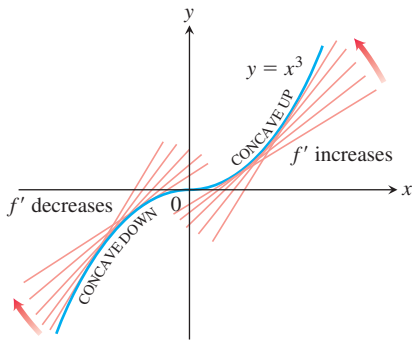


FIGURE 4.24 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

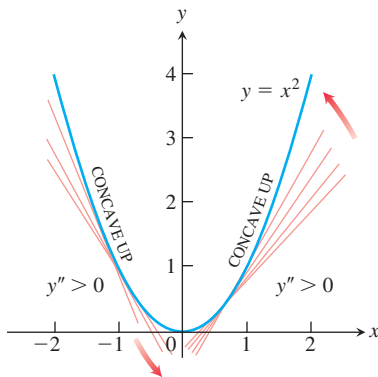


FIGURE 4.25 The graph of $f(x) = x^2$ is concave up on every interval (Example 1b).

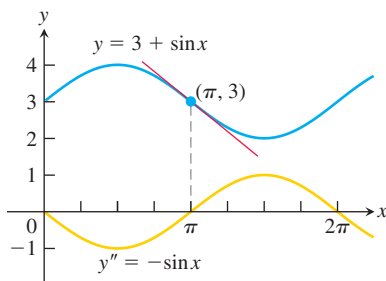


FIGURE 4.26 Using the sign of y'' to determine the concavity of y (Example 2).

Concavity

As you can see in Figure 4.24, the curve $y = x^3$ rises as x increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ turn in different ways. As we approach the origin from the left along the curve, the curve turns to our right and falls below its tangent lines. The slopes of the tangent lines are decreasing on the interval $(-\infty, 0)$. As we move away from the origin along the curve to the right, the curve turns to our left and rises above its tangent lines. The slopes of the tangent lines are increasing on the interval $(0, \infty)$. This turning or bending behavior defines the *concavity* of the curve.

DEFINITION The graph of a differentiable function $y = f(x)$ is

- (a) **concave up** on an open interval I if f' is increasing on I ;
- (b) **concave down** on an open interval I if f' is decreasing on I .

A function whose graph is concave up is also often called **convex**.

If $y = f(x)$ has a second derivative, we can apply Corollary 3 of the Mean Value Theorem to the first derivative function. We conclude that f' increases if $f'' > 0$ on I , and decreases if $f'' < 0$.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I .

1. If $f'' > 0$ on I , the graph of f over I is concave up.
2. If $f'' < 0$ on I , the graph of f over I is concave down.

If $y = f(x)$ is twice-differentiable, we will use the notations f'' and y'' interchangeably when denoting the second derivative.

EXAMPLE 1

- (a) The curve $y = x^3$ (Figure 4.24) is concave down on $(-\infty, 0)$, where $y'' = 6x < 0$, and concave up on $(0, \infty)$, where $y'' = 6x > 0$.
- (b) The curve $y = x^2$ (Figure 4.25) is concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive. ■

EXAMPLE 2 Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Solution The first derivative of $y = 3 + \sin x$ is $y' = \cos x$, and the second derivative is $y'' = -\sin x$. The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive (Figure 4.26). ■

Points of Inflection

The curve $y = 3 + \sin x$ in Example 2 changes concavity at the point $(\pi, 3)$. Since the first derivative $y' = \cos x$ exists for all x , we see that the curve has a tangent line of slope -1 at the point $(\pi, 3)$. This point is called a *point of inflection* of the curve. Notice from Figure 4.26 that the graph crosses its tangent line at this point and that the second derivative $y'' = -\sin x$ has value 0 when $x = \pi$. In general, we have the following definition.

DEFINITION A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

We observed that the second derivative of $f(x) = 3 + \sin x$ is equal to zero at the inflection point $(\pi, 3)$. Generally, if the second derivative exists at a point of inflection $(c, f(c))$, then $f''(c) = 0$. This follows immediately from the Intermediate Value Theorem whenever f'' is continuous over an interval containing $x = c$ because the second derivative changes sign moving across this interval. Even if the continuity assumption is dropped, it is still true that $f''(c) = 0$, provided the second derivative exists (although a more advanced argument is required in this noncontinuous case). Since a tangent line must exist at the point of inflection, either the first derivative $f'(c)$ exists (is finite) or the graph has a vertical tangent line at the point. At a vertical tangent, neither the first nor second derivative exists. In summary, one of two things can happen at a point of inflection.

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

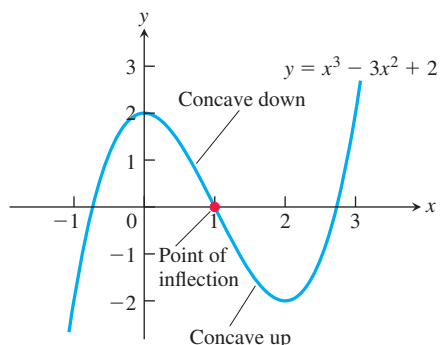


FIGURE 4.27 The concavity of the graph of f changes from concave down to concave up at the inflection point (Example 3).

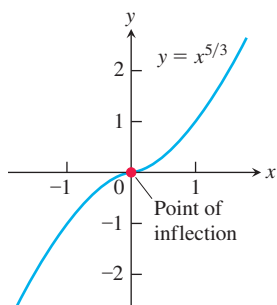


FIGURE 4.28 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin where the concavity changes, although f'' does not exist at $x = 0$ (Example 4).

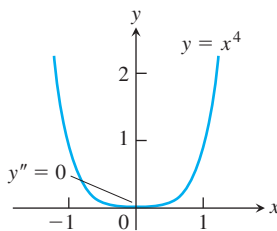


FIGURE 4.29 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there (Example 5).

EXAMPLE 3 Determine the concavity and find the inflection points of the function

$$f(x) = x^3 - 3x^2 + 2.$$

Solution We start by computing the first and second derivatives.

$$f'(x) = 3x^2 - 6x, \quad f''(x) = 6x - 6.$$

To determine concavity, we look at the sign of the second derivative $f''(x) = 6x - 6$. The sign is negative when $x < 1$, is 0 at $x = 1$, and is positive when $x > 1$. It follows that the graph of f is concave down on $(-\infty, 1)$, is concave up on $(1, \infty)$, and has an inflection point at the point $(1, 0)$ where the concavity changes.

The graph of f is shown in Figure 4.27. Notice that we did not need to know the shape of this graph ahead of time in order to determine its concavity. ■

The next example illustrates that a function can have a point of inflection where the first derivative exists but the second derivative fails to exist.

EXAMPLE 4 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f'(x) = (5/3)x^{2/3} = 0$ when $x = 0$. However, the second derivative,

$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{2/3}\right) = \frac{10}{9}x^{-1/3},$$

fails to exist at $x = 0$. Nevertheless, $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$ and there is a point of inflection at the origin. The graph is shown in Figure 4.28. ■

The following example shows that an inflection point need not occur even though both derivatives exist and $f'' = 0$.

EXAMPLE 5 The curve $y = x^4$ has no inflection point at $x = 0$ (Figure 4.29). Even though the second derivative $y'' = 12x^2$ is zero there, it does not change sign. The curve is concave up everywhere. ■

In the next example, a point of inflection occurs at a vertical tangent to the curve where neither the first nor the second derivative exists.

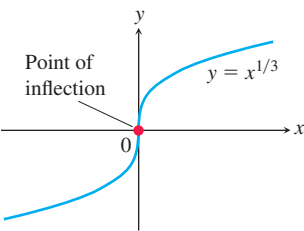


FIGURE 4.30 A point of inflection where y' and y'' fail to exist (Example 6).

EXAMPLE 6 The graph of $y = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$:

$$y'' = \frac{d^2}{dx^2}(x^{1/3}) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}.$$

However, both $y' = x^{-2/3}/3$ and y'' fail to exist at $x = 0$, and there is a vertical tangent there. See Figure 4.30. ■

Caution Example 4 in Section 4.1 (Figure 4.9) shows that the function $f(x) = x^{2/3}$ does not have a second derivative at $x = 0$ and does not have a point of inflection there (there is no change in concavity at $x = 0$). Combined with the behavior of the function in Example 6 above, we see that when the second derivative does not exist at $x = c$, an inflection point may or may not occur there. So we need to be careful about interpreting functional behavior whenever first or second derivatives fail to exist at a point. At such points the graph can have vertical tangent lines, corners, cusps, or various discontinuities. ●

To study the motion of an object moving along a line as a function of time, we often are interested in knowing when the object's acceleration, given by the second derivative, is positive or negative. The points of inflection on the graph of the object's position function reveal where the acceleration changes sign.

EXAMPLE 7 A particle is moving along a horizontal coordinate line (positive to the right) with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \quad t \geq 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

Solution The velocity is

$$v(t) = s'(t) = 6t^2 - 28t + 22 = 2(t - 1)(3t - 11),$$

and the acceleration is

$$a(t) = v'(t) = s''(t) = 12t - 28 = 4(3t - 7).$$

When the function $s(t)$ is increasing, the particle is moving to the right; when $s(t)$ is decreasing, the particle is moving to the left.

Notice that the first derivative ($v = s'$) is zero at the critical points $t = 1$ and $t = 11/3$.

Interval	$0 < t < 1$	$1 < t < 11/3$	$11/3 < t$
Sign of $v = s'$	+	−	+
Behavior of s	increasing	decreasing	increasing
Particle motion	right	left	right

The particle is moving to the right in the time intervals $[0, 1)$ and $(11/3, \infty)$, and moving to the left in $(1, 11/3)$. It is momentarily stationary (at rest) at $t = 1$ and $t = 11/3$.

The acceleration $a(t) = s''(t) = 4(3t - 7)$ is zero when $t = 7/3$.

Interval	$0 < t < 7/3$	$7/3 < t$
Sign of $a = s''$	−	+
Graph of s	concave down	concave up

Under the influence of the leftward acceleration over the time interval $[0, 7/3)$, the particle starts out moving to the right while slowing down, and then at $t = 1$ it reverses and begins moving to the left while speeding up. The acceleration then changes direction at $t = 7/3$, but the particle continues moving leftward, while slowing down under the rightward acceleration. At $t = 11/3$ the particle reverses direction again: moving to the right in the same direction as the acceleration, so it is speeding up. ■

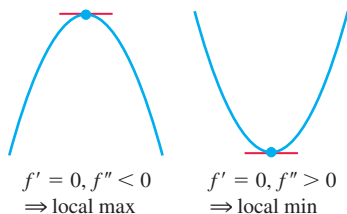
Second Derivative Test for Local Extrema

Instead of looking for sign changes in f' at critical points, we can sometimes use the following test to determine the presence and nature of local extrema.

THEOREM 5—Second Derivative Test for Local Extrema

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.



Proof Part (1). If $f''(c) < 0$, then $f''(x) < 0$ on some open interval I containing the point c , since f'' is continuous. Therefore, f' is decreasing on I . Since $f'(c) = 0$, the sign of f' changes from positive to negative at c , so f has a local maximum at c by the First Derivative Test.

The proof of Part (2) is similar.

For Part (3), consider the three functions $y = x^4$, $y = -x^4$, and $y = x^3$. For each function, the first and second derivatives are zero at $x = 0$. Yet the function $y = x^4$ has a local minimum there, $y = -x^4$ has a local maximum, and $y = x^3$ is increasing in any open interval containing $x = 0$ (having neither a maximum nor a minimum there). Thus the test fails. ■

This test requires us to know f'' only at c itself and not in an interval about c . This makes the test easy to apply. That's the good news. The bad news is that the test is inconclusive if $f'' = 0$ or if f'' does not exist at $x = c$. When this happens, use the First Derivative Test for local extreme values.

Together f' and f'' tell us the shape of the function's graph—that is, where the critical points are located and what happens at a critical point, where the function is increasing and where it is decreasing, and how the curve is turning or bending as indicated by its concavity. We use this information to sketch a graph of the function that captures its key features.

EXAMPLE 8 Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps.

- (a) Identify where the extrema of f occur.
- (b) Find the intervals on which f is increasing and the intervals on which f is decreasing.
- (c) Find where the graph of f is concave up and where it is concave down.
- (d) Sketch the general shape of the graph for f .
- (e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

Solution The function f is continuous since $f'(x) = 4x^3 - 12x^2$ exists. The domain of f is $(-\infty, \infty)$, and the domain of f' is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f' . Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),$$

the first derivative is zero at $x = 0$ and $x = 3$. We use these critical points to define intervals where f is increasing or decreasing.

Interval	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	–	–	+
Behavior of f	decreasing	decreasing	increasing

- (a) Using the First Derivative Test for local extrema and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.
- (b) Using the table above, we see that f is decreasing on $(-\infty, 0]$ and $[0, 3]$, and increasing on $[3, \infty)$.
- (c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$. We use these points to define intervals where the graph of f is concave up or concave down.

Interval	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	+	–	+
Behavior of f	concave up	concave down	concave up

- We see that the graph of f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.
- (d) Summarizing the information in the last two tables, we obtain the following.

$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
decreasing	decreasing	decreasing	increasing
concave up	concave down	concave up	concave up

The general shape of the curve is shown in the accompanying figure.

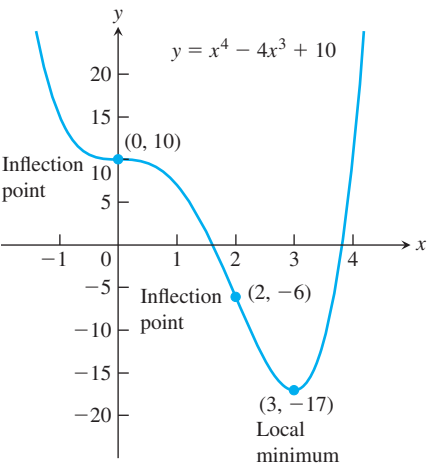
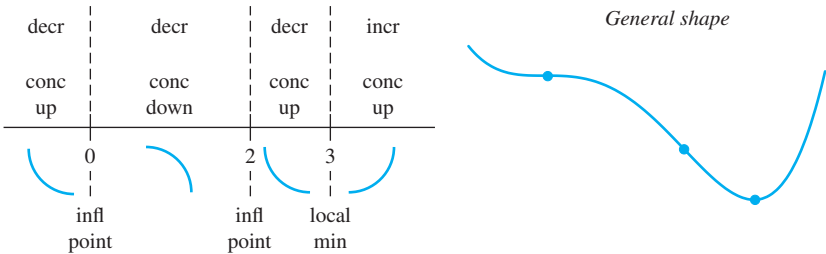


FIGURE 4.31 The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 8).



- (e) Plot the curve's intercepts (if possible) and the points where y' and y'' are zero. Indicate any local extreme values and inflection points. Use the general shape as a guide to sketch the curve. (Plot additional points as needed.) Figure 4.31 shows the graph of f . ■

The steps in Example 8 give a procedure for graphing the key features of a function. Asymptotes were defined and discussed in Section 2.5. We can find them for many classes

of functions (including rational functions), and the methods in the next section give tools to help find them for even more general functions.

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y'' .
3. Find the critical points of f , if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

EXAMPLE 9 Sketch the graph of $f(x) = \frac{(x+1)^2}{1+x^2}$.

Solution

1. The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin (Section 1.1).
2. Find f' and f'' .

$$f(x) = \frac{(x+1)^2}{1+x^2} \quad \begin{array}{l} \text{x-intercept at } x = -1, \\ \text{y-intercept at } y = 1 \end{array}$$

$$\begin{aligned} f'(x) &= \frac{(1+x^2) \cdot 2(x+1) - (x+1)^2 \cdot 2x}{(1+x^2)^2} \\ &= \frac{2(1-x^2)}{(1+x^2)^2} \end{aligned} \quad \text{Critical points: } x = -1, x = 1$$

$$\begin{aligned} f''(x) &= \frac{(1+x^2)^2 \cdot 2(-2x) - 2(1-x^2)[2(1+x^2) \cdot 2x]}{(1+x^2)^4} \\ &= \frac{4x(x^2-3)}{(1+x^2)^3} \end{aligned} \quad \begin{array}{l} \text{After some algebra, including} \\ \text{canceling the common factor} \\ (1+x^2) \end{array}$$

3. *Behavior at critical points.* The critical points occur only at $x = \pm 1$ where $f'(x) = 0$ (Step 2) since f' exists everywhere over the domain of f . At $x = -1$, $f''(-1) = 1 > 0$, yielding a relative minimum by the Second Derivative Test. At $x = 1$, $f''(1) = -1 < 0$, yielding a relative maximum by the Second Derivative test.
4. *Increasing and decreasing.* We see that on the interval $(-\infty, -1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.
5. *Inflection points.* Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f'' is zero when $x = -\sqrt{3}, 0$, and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $(-\infty, -\sqrt{3})$, positive on $(-\sqrt{3}, 0)$, negative on $(0, \sqrt{3})$, and positive again on $(\sqrt{3}, \infty)$. Thus each point is a point of inflection. The curve is concave down on the interval $(-\infty, -\sqrt{3})$, concave up on $(-\sqrt{3}, 0)$, concave down on $(0, \sqrt{3})$, and concave up again on $(\sqrt{3}, \infty)$.

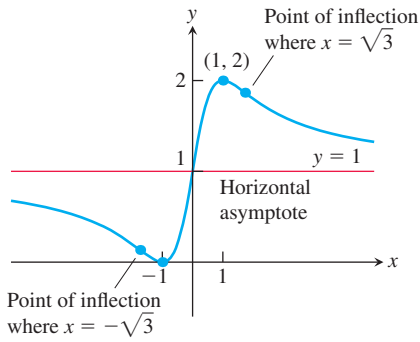


FIGURE 4.32 The graph of $y = \frac{(x+1)^2}{1+x^2}$ (Example 9).

6. *Asymptotes.* Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by x^2 gives

$$\begin{aligned} f(x) &= \frac{(x+1)^2}{1+x^2} = \frac{x^2 + 2x + 1}{1+x^2} && \text{Expanding numerator} \\ &= \frac{1 + (2/x) + (1/x^2)}{(1/x^2) + 1}. && \text{Dividing by } x^2 \end{aligned}$$

We see that $f(x) \rightarrow 1$ as $x \rightarrow \infty$ and that $f(x) \rightarrow 1$ as $x \rightarrow -\infty$. Thus, the line $y = 1$ is a horizontal asymptote. Since the function is continuous on $(-\infty, \infty)$, there are no vertical asymptotes.

7. The graph of f is sketched in Figure 4.32. Notice how the graph is concave down as it approaches the horizontal asymptote $y = 1$ as $x \rightarrow -\infty$, and concave up in its approach to $y = 1$ as $x \rightarrow \infty$. ■

EXAMPLE 10 Sketch the graph of $f(x) = \frac{x^2 + 4}{2x}$.

Solution

- The domain of f is all nonzero real numbers. There are no intercepts because neither x nor $f(x)$ can be zero. Since $f(-x) = -f(x)$, we note that f is an odd function, so the graph of f is symmetric about the origin.
- We calculate the derivatives of the function, but we first rewrite it in order to simplify our computations:

$$f(x) = \frac{x^2 + 4}{2x} = \frac{x}{2} + \frac{2}{x} \quad \text{Function simplified for differentiation}$$

$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2} \quad \text{Combine fractions to solve easily } f'(x) = 0.$$

$$f''(x) = \frac{4}{x^3} \quad \text{Exists throughout the entire domain of } f$$

- The critical points occur at $x = \pm 2$ where $f'(x) = 0$. Since $f''(-2) < 0$ and $f''(2) > 0$, we see from the Second Derivative Test that a relative maximum occurs at $x = -2$ with $f(-2) = -2$, and a relative minimum occurs at $x = 2$ with $f(2) = 2$.
- On the interval $(-\infty, -2)$ the derivative f' is positive because $x^2 - 4 > 0$ so the graph is increasing; on the interval $(-2, 0)$ the derivative is negative and the graph is decreasing. Similarly, the graph is decreasing on the interval $(0, 2)$ and increasing on $(2, \infty)$.
- There are no points of inflection because $f''(x) < 0$ whenever $x < 0$, $f''(x) > 0$ whenever $x > 0$, and f'' exists everywhere and is never zero throughout the domain of f . The graph is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$.
- From the rewritten formula for $f(x)$, we see that

$$\lim_{x \rightarrow 0^+} \left(\frac{x}{2} + \frac{2}{x} \right) = +\infty \quad \text{and} \quad \lim_{x \rightarrow 0^-} \left(\frac{x}{2} + \frac{2}{x} \right) = -\infty,$$

so the y -axis is a vertical asymptote. Also, as $x \rightarrow \infty$ or as $x \rightarrow -\infty$, the graph of $f(x)$ approaches the line $y = x/2$. Thus $y = x/2$ is an oblique asymptote.

7. The graph of f is sketched in Figure 4.33. ■

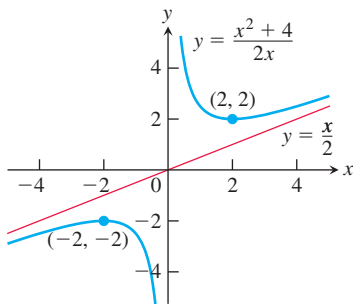


FIGURE 4.33 The graph of $y = \frac{x^2 + 4}{2x}$ (Example 10).

EXAMPLE 11 Sketch the graph of $f(x) = e^{2/x}$.

Solution The domain of f is $(-\infty, 0) \cup (0, \infty)$ and there are no symmetries about either axis or the origin. The derivatives of f are

$$f'(x) = e^{2/x} \left(-\frac{2}{x^2} \right) = -\frac{2e^{2/x}}{x^2}$$

and

$$f''(x) = -\frac{x^2(2e^{2/x})(-2/x^2) - 2e^{2/x}(2x)}{x^4} = \frac{4e^{2/x}(1+x)}{x^4}.$$

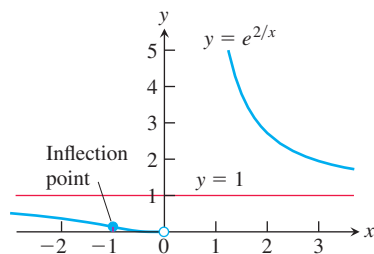


FIGURE 4.34 The graph of $y = e^{2/x}$ has a point of inflection at $(-1, e^{-2})$. The line $y = 1$ is a horizontal asymptote and $x = 0$ is a vertical asymptote (Example 11).

Both derivatives exist everywhere over the domain of f . Moreover, since $e^{2/x}$ and x^2 are both positive for all $x \neq 0$, we see that $f' < 0$ everywhere over the domain and the graph is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$. Examining the second derivative, we see that $f''(x) = 0$ at $x = -1$. Since $e^{2/x} > 0$ and $x^4 > 0$, we have $f'' < 0$ for $x < -1$ and $f'' > 0$ for $x > -1$, $x \neq 0$. Since f'' changes sign, the point $(-1, e^{-2})$ is a point of inflection. The curve is concave down on the interval $(-\infty, -1)$ and concave up over $(-1, 0) \cup (0, \infty)$.

From Example 7, Section 2.5, we see that $\lim_{x \rightarrow 0^+} f(x) = \infty$. As $x \rightarrow 0^+$, we see that $2/x \rightarrow \infty$, so $\lim_{x \rightarrow 0^+} f(x) = \infty$ and the y-axis is a vertical asymptote. Also, as $x \rightarrow -\infty$ or $x \rightarrow \infty$, $2/x \rightarrow 0$ and so $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow \infty} f(x) = e^0 = 1$. Therefore, $y = 1$ is a horizontal asymptote. There are no absolute extrema, since f never takes on the value 0 and has no absolute maximum. The graph of f is sketched in Figure 4.34. ■

EXAMPLE 12 Sketch the graph of $f(x) = \cos x - \frac{\sqrt{2}}{2}x$ over $0 \leq x \leq 2\pi$.

Solution The derivatives of f are

$$f'(x) = -\sin x - \frac{\sqrt{2}}{2} \quad \text{and} \quad f''(x) = -\cos x.$$

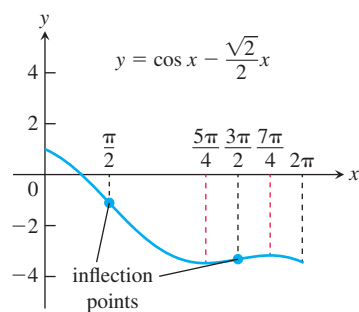


FIGURE 4.35 The graph of the function in Example 12.

Both derivatives exist everywhere over the interval $(0, 2\pi)$. Within that open interval, the first derivative is zero when $\sin x = -\sqrt{2}/2$, so the critical points are $x = 5\pi/4$ and $x = 7\pi/4$. Since $f''(5\pi/4) = -\cos(5\pi/4) = \sqrt{2}/2 > 0$, the function has a local minimum value of $f(5\pi/4) \approx -3.48$ (evaluated with a calculator) by the Second Derivative Test. Also, $f''(7\pi/4) = -\cos(7\pi/4) = -\sqrt{2}/2 < 0$, so the function has a local maximum value of $f(7\pi/4) \approx -3.18$.

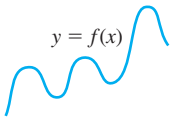
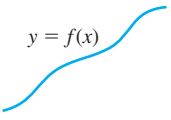
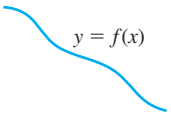
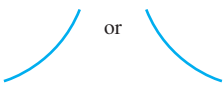
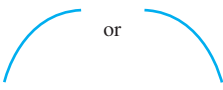
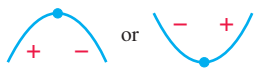
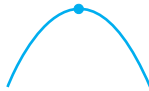
Examining the second derivative, we find that $f'' = 0$ when $x = \pi/2$ or $x = 3\pi/2$. Since f'' changes sign at these two points, we conclude that $(\pi/2, f(\pi/2)) \approx (\pi/2, -1.11)$ and $(3\pi/2, f(3\pi/2)) \approx (3\pi/2, -3.33)$ are points of inflection.

Finally, we evaluate f at the endpoints of the interval to find $f(0) = 1$ and $f(2\pi) \approx -3.44$. Therefore, the values $f(0) = 1$ and $f(5\pi/4) \approx -3.48$ are the absolute maximum and absolute minimum values of f over the closed interval $[0, 2\pi]$. The graph of f is sketched in Figure 4.35. ■

Graphical Behavior of Functions from Derivatives

As we saw in Examples 8–12, we can learn much about a twice-differentiable function $y = f(x)$ by examining its first derivative. We can find where the function's graph rises and falls and where any local extrema are located. We can differentiate y' to learn how the graph bends as it passes over the intervals of rise and fall. Together with information about the function's asymptotes and its value at some key points, such as intercepts, this information about

the derivatives helps us determine the shape of the function's graph. The following figure summarizes how the first derivative and second derivative affect the shape of a graph.

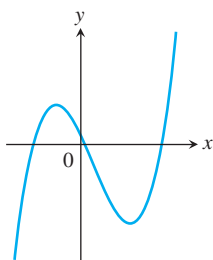
 $y = f(x)$ Differentiable $\Rightarrow$ smooth, connected; graph may rise and fall	 $y = f(x)$ $y' > 0 \Rightarrow$ rises from left to right; may be wavy	 $y = f(x)$ $y' < 0 \Rightarrow$ falls from left to right; may be wavy
 $y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall or both	 $y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall or both	 y'' changes sign at an inflection point
 y' changes sign $\Rightarrow$ graph has local maximum or local minimum	 $y' = 0$ and $y'' < 0$ at a point; graph has local maximum	 $y' = 0$ and $y'' > 0$ at a point; graph has local minimum

EXERCISES 4.4

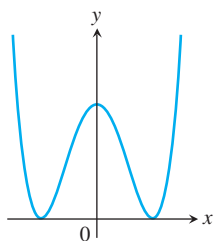
Analyzing Functions from Graphs

Identify the inflection points and local maxima and minima of the functions graphed in Exercises 1–8. Identify the open intervals on which the functions are differentiable and the graphs are concave up and concave down.

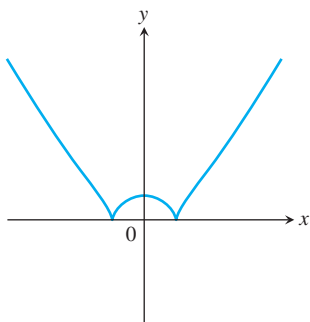
1. $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$



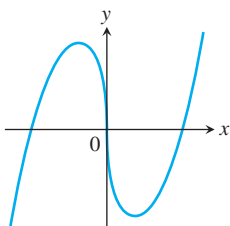
2. $y = \frac{x^4}{4} - 2x^2 + 4$



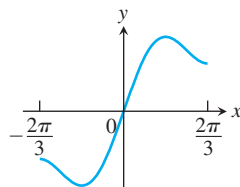
3. $y = \frac{3}{4}(x^2 - 1)^{2/3}$



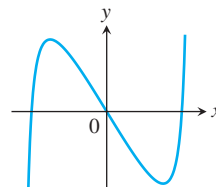
4. $y = \frac{9}{14}x^{1/3}(x^2 - 7)$



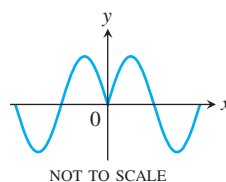
5. $y = x + \sin 2x, -\frac{2\pi}{3} \leq x \leq \frac{2\pi}{3}$



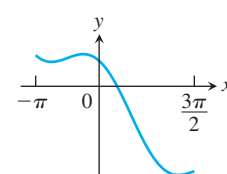
6. $y = \tan x - 4x, -\frac{\pi}{2} < x < \frac{\pi}{2}$



7. $y = \sin|x|, -2\pi \leq x \leq 2\pi$



8. $y = 2 \cos x - \sqrt{2}x, -\pi \leq x \leq \frac{3\pi}{2}$



Graphing Functions

In Exercises 9–70, graph the function using appropriate methods from the graphing procedures presented just before Example 9, identifying the coordinates of any local extreme points and inflection points. Then find coordinates of absolute extreme points, if any.

9. $y = x^2 - 4x + 3$

10. $y = 6 - 2x - x^2$

11. $y = x^3 - 3x + 3$

12. $y = x(6 - 2x)^2$

13. $y = -2x^3 + 6x^2 - 3$

14. $y = 1 - 9x - 6x^2 - x^3$

15. $y = (x - 2)^3 + 1$

16. $y = 1 - (x + 1)^3$

17. $y = x^4 - 2x^2 = x^2(x^2 - 2)$
 18. $y = -x^4 + 6x^2 - 4 = x^2(6 - x^2) - 4$
 19. $y = 4x^3 - x^4 = x^3(4 - x)$ 20. $y = x^4 + 2x^3 = x^3(x + 2)$
 21. $y = x^5 - 5x^4 = x^4(x - 5)$ 22. $y = x\left(\frac{x}{2} - 5\right)^4$
 23. $y = \frac{2x^2 + x - 1}{x^2 - 1}$ 24. $y = \frac{x^2 - 49}{x^2 + 5x - 14}$
 25. $y = \frac{x^4 + 1}{x^2}$ 26. $y = \frac{x^2 - 4}{2x}$
 27. $y = \frac{1}{x^2 - 1}$ 28. $y = \frac{x^2}{x^2 - 1}$
 29. $y = -\frac{x^2 - 2}{x^2 - 1}$ 30. $y = \frac{x^2 - 4}{x^2 - 2}$
 31. $y = \frac{x^2}{x + 1}$ 32. $y = -\frac{x^2 - 4}{x + 1}$
 33. $y = \frac{x^2 - x + 1}{x - 1}$ 34. $y = -\frac{x^2 - x + 1}{x - 1}$
 35. $y = \frac{x^3 - 3x^2 + 3x - 1}{x^2 + x - 2}$ 36. $y = \frac{x^3 + x - 2}{x - x^2}$
 37. $y = \frac{x}{x^2 - 1}$ 38. $y = \frac{4x}{x^2 + 4}$ (Newton's serpentine)
 39. $y = \frac{8}{x^2 + 4}$ (Agnesi's witch) 40. $y = \frac{x}{\sqrt{x^2 + 1}}$
 41. $y = x + \sin x, \quad 0 \leq x \leq 2\pi$
 42. $y = x - \sin x, \quad 0 \leq x \leq 2\pi$
 43. $y = \sqrt{3}x - 2 \cos x, \quad 0 \leq x \leq 2\pi$
 44. $y = \frac{4}{3}x - \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$
 45. $y = \sin x \cos x, \quad 0 \leq x \leq \pi$
 46. $y = \cos x + \sqrt{3} \sin x, \quad 0 \leq x \leq 2\pi$
 47. $y = x^{1/5}$ 48. $y = x^{2/5}$
 49. $y = 2x - 3x^{2/3}$ 50. $y = 5x^{2/5} - 2x$
 51. $y = x^{2/3}\left(\frac{5}{2} - x\right)$ 52. $y = x^{2/3}(x - 5)$
 53. $y = x\sqrt{8 - x^2}$ 54. $y = (2 - x^2)^{3/2}$
 55. $y = \sqrt{16 - x^2}$ 56. $y = x^2 + \frac{2}{x}$
 57. $y = \frac{x^2 - 3}{x - 2}$ 58. $y = \sqrt[3]{x^3 + 1}$
 59. $y = \frac{8x}{x^2 + 4}$ 60. $y = \frac{5}{x^4 + 5}$
 61. $y = |x^2 - 1|$ 62. $y = |x^2 - 2x|$
 63. $y = \sqrt{|x|} = \begin{cases} \sqrt{-x}, & x < 0 \\ \sqrt{x}, & x \geq 0 \end{cases}$
 64. $y = \sqrt{|x - 4|}$
 65. $y = \frac{x}{9 - x^2}$ 66. $y = \frac{x^2}{1 - x}$
 67. $y = \ln(3 - x^2)$ 68. $y = (\ln x)^2$
 69. $y = \ln(\cos x)$ 70. $y = \frac{1}{1 + e^{-x}} = \frac{e^x}{1 + e^x}$

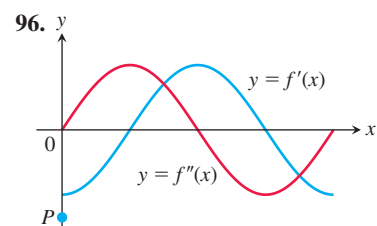
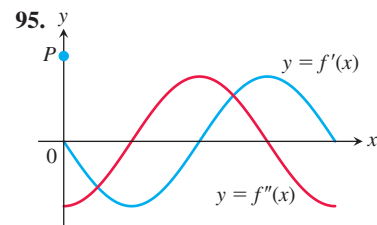
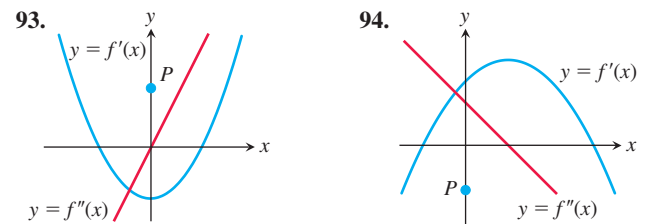
Sketching the General Shape, Knowing y'

Each of Exercises 71–92 gives the first derivative of a continuous function $y = f(x)$. Find y'' and then use Steps 2–4 of the graphing procedure described in this section to sketch the general shape of the graph of f .

71. $y' = 2 + x - x^2$ 72. $y' = x^2 - x - 6$
 73. $y' = x(x - 3)^2$ 74. $y' = x^2(2 - x)$
 75. $y' = x(x^2 - 12)$ 76. $y' = (x - 1)^2(2x + 3)$
 77. $y' = (8x - 5x^2)(4 - x)^2$ 78. $y' = (x^2 - 2x)(x - 5)^2$
 79. $y' = \sec^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$
 80. $y' = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$
 81. $y' = \cot \frac{\theta}{2}, \quad 0 < \theta < 2\pi$ 82. $y' = \csc^2 \frac{\theta}{2}, \quad 0 < \theta < 2\pi$
 83. $y' = \tan^2 \theta - 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$
 84. $y' = 1 - \cot^2 \theta, \quad 0 < \theta < \pi$
 85. $y' = \cos t, \quad 0 \leq t \leq 2\pi$ 86. $y' = \sin t, \quad 0 \leq t \leq 2\pi$
 87. $y' = (x + 1)^{-2/3}$ 88. $y' = (x - 2)^{-1/3}$
 89. $y' = x^{-2/3}(x - 1)$ 90. $y' = x^{-4/5}(x + 1)$
 91. $y' = 2|x| = \begin{cases} -2x, & x \leq 0 \\ 2x, & x > 0 \end{cases}$
 92. $y' = \begin{cases} -x^2, & x \leq 0 \\ x^2, & x > 0 \end{cases}$

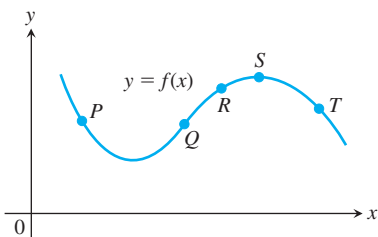
Sketching y from Graphs of y' and y''

Each of Exercises 93–96 shows the graphs of the first and second derivatives of a function $y = f(x)$. Copy the picture and add to it a sketch of the approximate graph of f , given that the graph passes through the point P .



Theory and Examples

97. The accompanying figure shows a portion of the graph of a twice-differentiable function $y = f(x)$. At each of the five labeled points, classify y' and y'' as positive, negative, or zero.



98. Sketch a smooth connected curve $y = f(x)$ with
- $$f(-2) = 8, \quad f'(2) = f'(-2) = 0,$$
- $$f(0) = 4, \quad f'(x) < 0 \quad \text{for } |x| < 2,$$
- $$f(2) = 0, \quad f''(x) < 0 \quad \text{for } x < 0,$$
- $$f'(x) > 0 \quad \text{for } |x| > 2, \quad f''(x) > 0 \quad \text{for } x > 0.$$
99. Sketch the graph of a twice-differentiable function $y = f(x)$ with the following properties. Label coordinates where possible.

x	y	Derivatives
$x < 2$		$y' < 0, \quad y'' > 0$
2	1	$y' = 0, \quad y'' > 0$
$2 < x < 4$		$y' > 0, \quad y'' > 0$
4	4	$y' > 0, \quad y'' = 0$
$4 < x < 6$		$y' > 0, \quad y'' < 0$
6	7	$y' = 0, \quad y'' < 0$
$x > 6$		$y' < 0, \quad y'' < 0$

100. Sketch the graph of a twice-differentiable function $y = f(x)$ that passes through the points $(-2, 2)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, and $(2, 2)$ and whose first two derivatives have the following sign patterns.

$$y': \begin{array}{cccc} + & - & + & - \\ -2 & 0 & 2 & \end{array}$$

$$y'': \begin{array}{ccc} - & + & - \\ -1 & 1 & \end{array}$$

101. Sketch the graph of a twice-differentiable function $y = f(x)$ with the following properties. Label coordinates where possible.

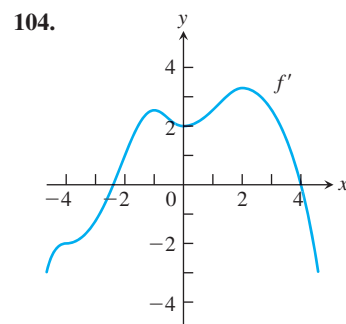
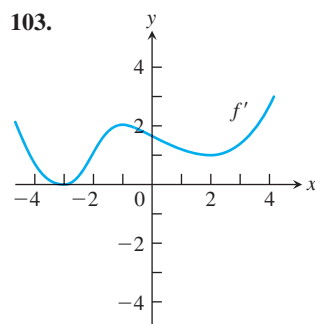
x	y	Derivatives
$x < -2$		$y' > 0, \quad y'' < 0$
-2	-1	$y' = 0, \quad y'' = 0$
$-2 < x < -1$		$y' > 0, \quad y'' > 0$
-1	0	$y' > 0, \quad y'' = 0$
$-1 < x < 0$		$y' > 0, \quad y'' < 0$
0	3	$y' = 0, \quad y'' < 0$
$0 < x < 1$		$y' < 0, \quad y'' < 0$
1	2	$y' < 0, \quad y'' = 0$
$1 < x < 2$		$y' < 0, \quad y'' > 0$
2	0	$y' = 0, \quad y'' > 0$
$x > 2$		$y' > 0, \quad y'' > 0$

102. Sketch the graph of a twice-differentiable function $y = f(x)$ that passes through the points $(-3, -2)$, $(-2, 0)$, $(0, 1)$, $(1, 2)$, and $(2, 3)$ and whose first two derivatives have the following sign patterns.

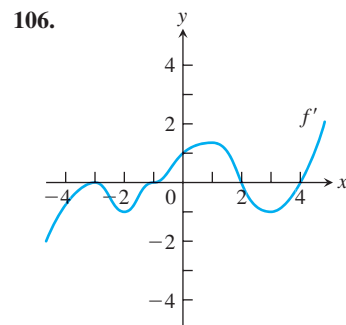
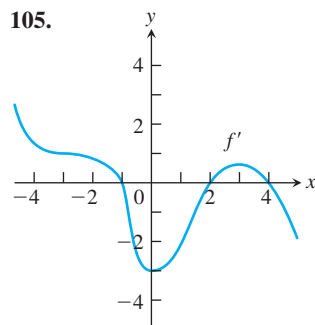
$$y': \begin{array}{cccc} - & + & + & - \\ -3 & 0 & 2 & \end{array}$$

$$y'': \begin{array}{ccc} + & - & - \\ -2 & 0 & 1 \end{array}$$

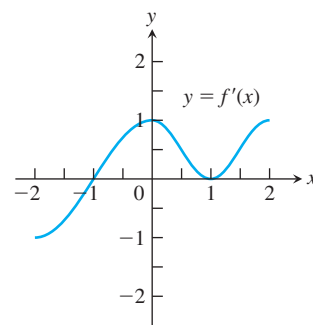
In Exercises 103 and 104, the graph of f' is given. Determine x -values corresponding to inflection points for the graph of f .



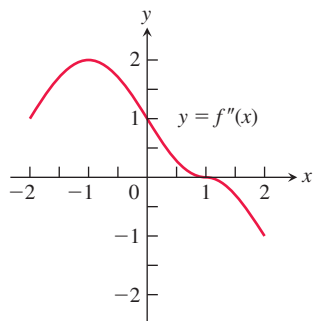
In Exercises 105 and 106, the graph of f' is given. Determine x -values corresponding to local minima, local maxima, and inflection points for the graph of f .



107. A function $f(x)$ has domain $(-2, 2)$. The graph below is a plot of the derivative of f , not a plot of f itself. In other words, this is a graph of $y = f'(x)$. Either use this graph to determine on which intervals the graph of f is concave up and on which intervals the graph of f is concave down, or explain why this information cannot be determined from the graph.

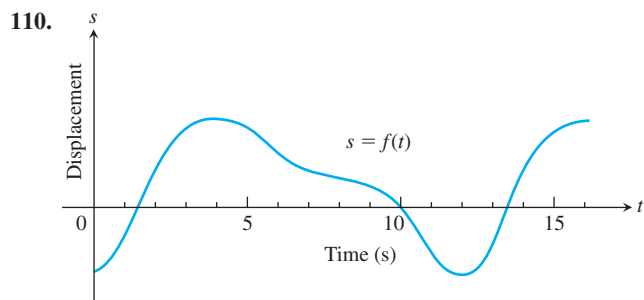
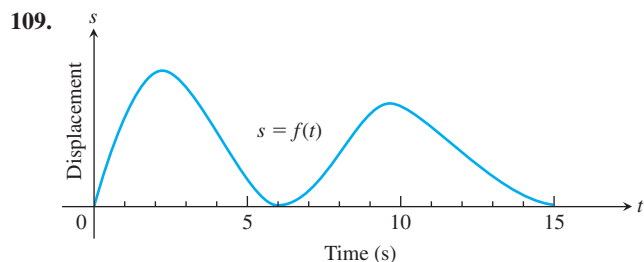


108. A function $f(x)$ has domain $(-2, 2)$. The graph below is a plot of the second derivative of f , not a plot of f itself. In other words, this is a graph of $y = f''(x)$.

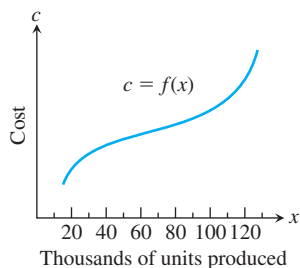


- Either use the graph above to determine on which intervals the graph of f is concave up and on which intervals the graph of f is concave down and the inflection points of f , or explain why this information cannot be determined from the graph.
- Either use the graph above to determine on which intervals f is increasing and on which intervals f is decreasing, or explain why this information cannot be determined from the graph.

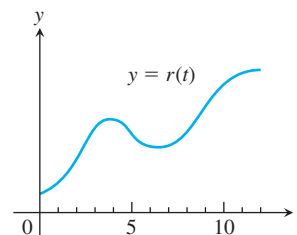
Motion Along a Line The graphs in Exercises 109 and 110 show the position $s = f(t)$ of an object moving up and down on a coordinate line. (a) When is the object moving away from the origin? Toward the origin? At approximately what times is the (b) velocity equal to zero? (c) Acceleration equal to zero? (d) When is the acceleration positive? Negative?



111. **Marginal cost** The accompanying graph shows the hypothetical cost $c = f(x)$ of manufacturing x items. At approximately what production level does the marginal cost change from decreasing to increasing?



112. The accompanying graph shows the monthly revenue of the Widget Corporation for the past 12 years. During approximately what time intervals was the marginal revenue increasing? Decreasing?



113. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection? (Hint: Draw the sign pattern for y' .)

114. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2)(x - 4).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection?

- For $x > 0$, sketch a curve $y = f(x)$ that has $f(1) = 0$ and $f'(x) = 1/x$. Can anything be said about the concavity of such a curve? Give reasons for your answer.
- Can anything be said about the graph of a function $y = f(x)$ that has a continuous second derivative that is never zero? Give reasons for your answer.
- If b , c , and d are constants, for what value of b will the curve $y = x^3 + bx^2 + cx + d$ have a point of inflection at $x = 1$? Give reasons for your answer.

118. Parabolas

- Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c$, $a \neq 0$.
- When is the parabola concave up? Concave down? Give reasons for your answers.

119. **Quadratic curves** What can you say about the inflection points of a quadratic curve $y = ax^2 + bx + c$, $a \neq 0$? Give reasons for your answer.

120. **Cubic curves** What can you say about the inflection points of a cubic curve $y = ax^3 + bx^2 + cx + d$, $a \neq 0$? Give reasons for your answer.

121. Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = (x + 1)(x - 2).$$

For what x -values does the graph of f have an inflection point?

122. Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = x^2(x - 2)^3(x + 3).$$

For what x -values does the graph of f have an inflection point?

- Find the values of constants a , b , and c such that the graph of $y = ax^3 + bx^2 + cx$ has a local maximum at $x = 3$, local minimum at $x = -1$, and inflection point at $(1, 11)$.
- Find the values of constants a , b , and c such that the graph of $y = (x^2 + a)/(bx + c)$ has a local minimum at $x = 3$ and a local maximum at $(-1, -2)$.

COMPUTER EXPLORATIONS

In Exercises 125–128, find the inflection points (if any) on the graph of the function and the coordinates of the points on the graph where the function has a local maximum or local minimum value. Then graph the function in a region large enough to show all these points simultaneously. Add to your picture the graphs of the function's first and second derivatives. How are the values at which these graphs intersect the x -axis related to the graph of the function? In what other ways are the graphs of the derivatives related to the graph of the function?

125. $y = x^5 - 5x^4 - 240$ 126. $y = x^3 - 12x^2$

127. $y = \frac{4}{5}x^5 + 16x^2 - 25$

128. $y = \frac{x^4}{4} - \frac{x^3}{3} - 4x^2 + 12x + 20$

129. Graph $f(x) = 2x^4 - 4x^2 + 1$ and its first two derivatives together. Comment on the behavior of f in relation to the signs and values of f' and f'' .

130. Graph $f(x) = x \cos x$ and its second derivative together for $0 \leq x \leq 2\pi$. Comment on the behavior of the graph of f in relation to the signs and values of f'' .

4.5 Indeterminate Forms and L'Hôpital's Rule

Consider the four limits

$$\lim_{x \rightarrow 0} \frac{x^3}{x} = \lim_{x \rightarrow 0} x^2 = 0,$$

$$\lim_{x \rightarrow 0} \frac{x}{x^3} = \lim_{x \rightarrow 0} \frac{1}{x^2} = \infty,$$

$$\lim_{x \rightarrow 0} \frac{x}{x} = \lim_{x \rightarrow 0} 1 = 1,$$

$$\lim_{x \rightarrow 0} \frac{x}{2x} = \lim_{x \rightarrow 0} \frac{1}{2} = \frac{1}{2}.$$

In each case both the numerator and the denominator approach zero as $x \rightarrow 0$, even though these limits ultimately lead to completely different results: 0, ∞ , 1, and $1/2$. We say the expression

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \quad \text{when} \quad \lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0 \quad (1)$$

involves an *indeterminate form* $0/0$. The expression “ $0/0$ ” has the form of a number, but it is not a meaningful quantity. Stating that both the numerator and the denominator approach zero does not provide sufficient information to obtain the limit of the ratio. We have to examine the behavior of the expression in more detail by performing algebraic manipulation or by applying methods that we will introduce in this section.

Other forms exhibit behavior similar to Equation (1). For instance, if both the numerator and the denominator approach $+\infty$ or $-\infty$, then the limit of the ratio leads to an indeterminate form ∞/∞ . Additional indeterminate forms we consider in this section are $\infty \cdot 0$, $\infty - \infty$, 1^∞ , 0^∞ , and ∞^0 . Their purpose is to summarize the behavior of certain types of limits.

John (Johann) Bernoulli discovered a rule for using derivatives to calculate limits of fractions whose numerators and denominators both approach zero or $\pm\infty$. The rule is known today as **L'Hôpital's Rule**, after Guillaume de l'Hôpital. He was a French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print. Limits involving transcendental functions often require some use of this rule.

Indeterminate Form $0/0$

It is important to understand that the notation “ $0/0$ ” is *not* intended to imply numerically dividing 0 by 0. Instead, the indeterminate form $0/0$ refers to a limit of a ratio of two functions, each of which approaches zero. L'Hôpital's rule can help us evaluate such limits.

THEOREM 6—L'Hôpital's Rule Suppose that $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$, that f and g are differentiable on an open interval I containing a , and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

assuming that the limit on the right side of this equation exists.

$\infty, 0, \infty$
 $1, 0, \infty$

$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0}$
 $= \frac{0}{0}$

HISTORICAL BIOGRAPHY

Guillaume François Antoine de l'Hôpital
(1661–1704)

In the late 1600s, John Bernoulli discovered a rule for calculating limits of fractions whose numerators and denominators both approach zero. Today the rule is known as L'Hôpital's rule.

To know more, visit the companion Website.

Johann Bernoulli
(1667–1748)

Johann Bernoulli was born in Switzerland and attended the University of Basel. His doctoral dissertation was in mathematics despite its medical title, which was used to hide his mathematical work from his father who wanted Johann to become a doctor.

To know more, visit the companion Website.

We give a proof of Theorem 6 at the end of this section. Theorem 6 also applies if $x \rightarrow \pm\infty$ or when $f'(x)/g'(x) \rightarrow \pm\infty$, but we will not prove this.

Caution

To apply l'Hôpital's Rule to f/g , divide the derivative of f by the derivative of g . Do not make the mistake of taking the derivative of f/g . The quotient to use is f'/g' , not $(f/g)'$.

EXAMPLE 1 The following limits involve $0/0$ indeterminate forms, so we apply l'Hôpital's Rule. In some cases, it must be applied repeatedly.

(a) $\lim_{x \rightarrow 0} \frac{3x - \sin x}{x}$ The numerator and the denominator are both approaching 0; apply l'Hôpital's Rule.

$$= \lim_{x \rightarrow 0} \frac{3 - \cos x}{1} \quad \text{Not } 0/0$$

$$= \frac{3 - \cos 0}{1} = 2 \quad \text{Limit is found.}$$

$$(b) \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$$

$$(c) \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2} \quad \frac{0}{0}; \text{ apply l'Hôpital's Rule.}$$

$$= \lim_{x \rightarrow 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x} \quad \text{Still } \frac{0}{0}; \text{ apply l'Hôpital's Rule again.}$$

$$= \lim_{x \rightarrow 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8} \quad \text{Not } \frac{0}{0}; \text{ limit is found.}$$

$$(d) \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} \quad \frac{0}{0}; \text{ apply l'Hôpital's Rule.}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2} \quad \text{Still } \frac{0}{0}; \text{ apply l'Hôpital's Rule again.}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x}{6x} \quad \text{Still } \frac{0}{0}; \text{ apply l'Hôpital's Rule again.}$$

$$= \lim_{x \rightarrow 0} \frac{\cos x}{6} = \frac{1}{6} \quad \text{Not } \frac{0}{0}; \text{ limit is found.}$$

$$(e) \lim_{x \rightarrow \infty} \frac{\ln(1 + 1/x)}{\sin(1/x)} \quad \frac{0}{0}; \text{ apply l'Hôpital's Rule.}$$

$$= \lim_{x \rightarrow \infty} \frac{-(1/x^2) \frac{1}{1+1/x}}{-(1/x^2) \cos(1/x)} \quad \text{Still } \frac{0}{0}; \text{ simplification is easier than applying l'Hôpital's Rule.}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{1+1/x}}{\cos(1/x)} = \frac{1}{\cos 0} = 1 \quad \text{Not } \frac{0}{0}; \text{ limit is found.}$$

Here is a summary of the procedure we followed in Example 1.

Using L'Hôpital's Rule To find

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, we continue to differentiate f and g , so long as we still get the form $0/0$ as $x \rightarrow a$. But as soon as one or the other of these derivatives no longer approaches zero, we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or the denominator has a finite nonzero limit.

EXAMPLE 2 Be careful to apply l'Hôpital's Rule correctly:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1 - \cos x}{x + x^2} & \quad \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{1 + 2x} \quad \text{Not } \frac{0}{0}\end{aligned}$$

It is tempting to try to apply l'Hôpital's Rule again, which would result in

$$\lim_{x \rightarrow 0} \frac{\cos x}{2} = \frac{1}{2},$$

but this is not the correct limit. l'Hôpital's Rule can be applied only to limits that give indeterminate forms, and $\lim_{x \rightarrow 0} (\sin x)/(1 + 2x)$ does not give an indeterminate form.

Instead, this limit is $0/1 = 0$, and the correct answer for the original limit is 0. ■

L'Hôpital's Rule applies to one-sided limits as well.

EXAMPLE 3 In this example the one-sided limits are different.

$$\begin{aligned}\text{(a)} \quad \lim_{x \rightarrow 0^+} \frac{\sin x}{x^2} & \quad \frac{0}{0} \\ &= \lim_{x \rightarrow 0^+} \frac{\cos x}{2x} = \infty \quad \text{Positive for } x > 0\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad \lim_{x \rightarrow 0^-} \frac{\sin x}{x^2} & \quad \frac{0}{0} \\ &= \lim_{x \rightarrow 0^-} \frac{\cos x}{2x} = -\infty \quad \text{Negative for } x < 0\end{aligned}$$

Indeterminate Forms ∞/∞ , $\infty \cdot 0$, $\infty - \infty$

Recall that ∞ and $+\infty$ mean the same thing.

Sometimes when we try to evaluate a limit as $x \rightarrow a$, we get an indeterminate form like ∞/∞ , $\infty \cdot 0$, or $\infty - \infty$, instead of $0/0$. We first consider the form ∞/∞ .

More advanced treatments of calculus prove that l'Hôpital's Rule applies to the indeterminate form ∞/∞ , as well as to $0/0$. If $f(x) \rightarrow \pm\infty$ and $g(x) \rightarrow \pm\infty$ as $x \rightarrow a$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists or approaches ∞ or $-\infty$. In the notation $x \rightarrow a$, a may be either finite or infinite. Moreover, $x \rightarrow a$ may be replaced by the one-sided limits $x \rightarrow a^+$ or $x \rightarrow a^-$.

EXAMPLE 4 Find the limits of these ∞/∞ forms:

$$\text{(a)} \quad \lim_{x \rightarrow \pi/2} \frac{\sec x}{1 + \tan x} \qquad \text{(b)} \quad \lim_{x \rightarrow \infty} \frac{\ln x}{2\sqrt{x}} \qquad \text{(c)} \quad \lim_{x \rightarrow \infty} \frac{e^x}{x^2}.$$

Solution

(a) The numerator and denominator are discontinuous at $x = \pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule, we can choose I to be any open interval with $x = \pi/2$ as an endpoint.

$$\begin{aligned}\lim_{x \rightarrow (\pi/2)^-} \frac{\sec x}{1 + \tan x} & \quad \frac{\infty}{\infty} \text{ from left, apply L'Hôpital's Rule} \\ &= \lim_{x \rightarrow (\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \rightarrow (\pi/2)^-} \sin x = 1\end{aligned}$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.

$$(b) \lim_{x \rightarrow \infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} = 0 \quad \frac{1/x}{1/\sqrt{x}} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$$

$$(c) \lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty$$

Next we turn our attention to the indeterminate forms $\infty \cdot 0$ and $\infty - \infty$. Sometimes these forms can be handled by using algebra to convert them to a $0/0$ or ∞/∞ form. Here again, we do not mean to suggest that $\infty \cdot 0$ or $\infty - \infty$ is a number. They are only notations for functional behaviors when considering limits. Here are examples of how we might work with these indeterminate forms.

EXAMPLE 5 Find the limits of these $\infty \cdot 0$ forms:

$$(a) \lim_{x \rightarrow \infty} \left(x \sin \frac{1}{x} \right)$$

$$(b) \lim_{x \rightarrow 0^+} \sqrt{x} \ln x$$

Solution

$$\begin{aligned}(a) \lim_{x \rightarrow \infty} \left(x \sin \frac{1}{x} \right) &= \lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} && \infty \cdot 0 \text{ converted to } \frac{0}{0} \\ &= \lim_{x \rightarrow \infty} \frac{(\cos(1/x))(-1/x^2)}{-1/x^2} && \text{L'Hôpital's Rule applied} \\ &= \lim_{x \rightarrow \infty} \left(\cos \frac{1}{x} \right) = 1\end{aligned}$$

(See Example 6b in Section 2.5 for an alternative method to solve this problem.)

$$\begin{aligned}(b) \lim_{x \rightarrow 0^+} \sqrt{x} \ln x &= \lim_{x \rightarrow 0^+} \frac{\ln x}{1/\sqrt{x}} && \infty \cdot 0 \text{ converted to } \infty/\infty \\ &= \lim_{x \rightarrow 0^+} \frac{1/x}{-(1/2)x^{3/2}} && \text{L'Hôpital's Rule applied} \\ &= \lim_{x \rightarrow 0^+} (-2\sqrt{x}) = 0\end{aligned}$$

EXAMPLE 6 Find the limit of this $\infty - \infty$ form:

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right).$$

Solution If $x \rightarrow 0^+$, then $\sin x \rightarrow 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \rightarrow \infty - \infty.$$

Similarly, if $x \rightarrow 0^-$, then $\sin x \rightarrow 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \rightarrow -\infty - (-\infty) = -\infty + \infty.$$

Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}. \quad \text{Common denominator is } x \sin x.$$

Then we apply l'Hôpital's Rule to the result:

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) &= \lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} && \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x + x \cos x} && \text{Still } \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0. \end{aligned}$$

Indeterminate Powers

Limits that lead to the indeterminate forms 1^∞ , 0^0 , and ∞^0 can sometimes be handled by first taking the logarithm of the function. We use l'Hôpital's Rule to find the limit of the logarithm expression and then exponentiate the result to find the original function limit. This procedure is justified by the continuity of the exponential function and Theorem 9 in Section 2.6, and it is formulated as follows. (The formula is also valid for one-sided limits.)

If $\lim_{x \rightarrow a} \ln f(x) = L$, then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} e^{\ln f(x)} = e^L.$$

Here a may be either finite or infinite.

EXAMPLE 7 Apply l'Hôpital's Rule to show that $\lim_{x \rightarrow 0^+} (1+x)^{1/x} = e$.

Solution The limit leads to the indeterminate form 1^∞ . We let $f(x) = (1+x)^{1/x}$ and find $\lim_{x \rightarrow 0^+} \ln f(x)$. Since

$$\ln f(x) = \ln(1+x)^{1/x} = \frac{1}{x} \ln(1+x),$$

l'Hôpital's Rule now applies to give

$$\begin{aligned} \lim_{x \rightarrow 0^+} \ln f(x) &= \lim_{x \rightarrow 0^+} \frac{\ln(1+x)}{x} && \frac{0}{0} \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{1+x}}{1} && \text{l'Hôpital's Rule applied} \\ &= \frac{1}{1} = 1. \end{aligned}$$

Therefore, $\lim_{x \rightarrow 0^+} (1+x)^{1/x} = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} e^{\ln f(x)} = e^1 = e$.

EXAMPLE 8 Find $\lim_{x \rightarrow \infty} x^{1/x}$.

Solution The limit leads to the indeterminate form ∞^0 . We let $f(x) = x^{1/x}$ and find $\lim_{x \rightarrow \infty} \ln f(x)$. Since

$$\ln f(x) = \ln x^{1/x} = \frac{\ln x}{x},$$

L'Hôpital's Rule gives

$$\begin{aligned}\lim_{x \rightarrow \infty} \ln f(x) &= \lim_{x \rightarrow \infty} \frac{\ln x}{x} \quad \frac{\infty}{\infty} \\ &= \lim_{x \rightarrow \infty} \frac{1/x}{1} \quad \text{L'Hôpital's Rule applied} \\ &= \frac{0}{1} = 0.\end{aligned}$$

Therefore, $\lim_{x \rightarrow \infty} x^{1/x} = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{\ln f(x)} = e^0 = 1$. ■

Proof of L'Hôpital's Rule

Before we prove L'Hôpital's Rule, we consider a special case to provide some geometric insight for its reasonableness. Consider the two functions $f(x)$ and $g(x)$ having *continuous* derivatives and satisfying $f(a) = g(a) = 0$, $g'(a) \neq 0$. The graphs of $f(x)$ and $g(x)$, together with their linearizations $y = f'(a)(x - a)$ and $y = g'(a)(x - a)$, are shown in Figure 4.36. We know that near $x = a$, the linearizations provide good approximations to the functions. In fact,

$$f(x) = f'(a)(x - a) + \varepsilon_1(x - a) \quad \text{and} \quad g(x) = g'(a)(x - a) + \varepsilon_2(x - a),$$

where $\varepsilon_1 \rightarrow 0$ and $\varepsilon_2 \rightarrow 0$ as $x \rightarrow a$. So, as Figure 4.36 suggests,

$$\begin{aligned}\lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow a} \frac{f'(a)(x - a) + \varepsilon_1(x - a)}{g'(a)(x - a) + \varepsilon_2(x - a)} \\ &= \lim_{x \rightarrow a} \frac{f'(a) + \varepsilon_1}{g'(a) + \varepsilon_2} = \frac{f'(a)}{g'(a)} \quad g'(a) \neq 0 \\ &= \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}, \quad \text{Continuous derivatives}\end{aligned}$$

as asserted by L'Hôpital's Rule. We now proceed to a proof of the rule based on the more general assumptions stated in Theorem 6, which do not require that $g'(a) \neq 0$ or that the two functions have *continuous* derivatives.

The proof of L'Hôpital's Rule is based on Cauchy's Mean Value Theorem, an extension of the Mean Value Theorem that involves two functions instead of one. We prove Cauchy's Theorem first and then show how it leads to L'Hôpital's Rule.

THEOREM 7—Cauchy's Mean Value Theorem

Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b) . Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof We apply the Mean Value Theorem of Section 4.2 twice. First we use it to show that $g(a) \neq g(b)$. For if $g(b)$ did equal $g(a)$, then the Mean Value Theorem would give

$$g'(c) = \frac{g(b) - g(a)}{b - a} = 0$$

for some c between a and b , which cannot happen because $g'(x) \neq 0$ in (a, b) .

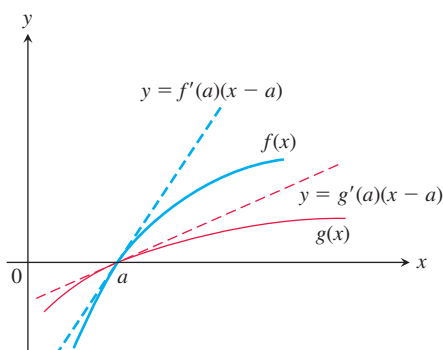


FIGURE 4.36 The two functions in L'Hôpital's Rule, graphed with their linear approximations at $x = a$.

HISTORICAL BIOGRAPHY

Augustin-Louis Cauchy
(1789–1857)

Cauchy was born in Paris the year the French revolution began. He was the first to define fully the ideas of convergence and absolute convergence of infinite series. His classic works *Cours d'analyse* (Course on Analysis, 1821) and *Résumé des leçons ... sur le calcul infinitésimal* (1823) were his greatest contributions to calculus.

To know more, visit the companion Website.

When $g(x) = x$, Theorem 7 is the Mean Value Theorem.

We next apply the Mean Value Theorem to the function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)].$$

This function is continuous and differentiable where f and g are, and $F(b) = F(a) = 0$. Therefore, there is a number c between a and b for which $F'(c) = 0$. When expressed in terms of f and g , this equation becomes

$$F'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}[g'(c)] = 0$$

so that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

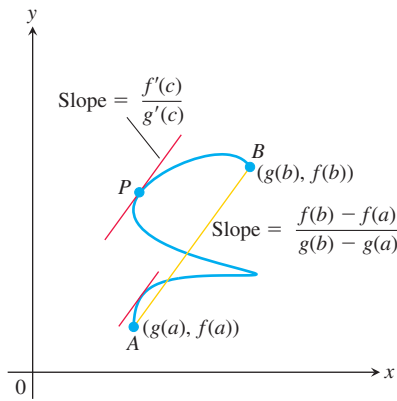


FIGURE 4.37 There is at least one point P on the curve C for which the slope of the tangent line to the curve at P is the same as the slope of the secant line joining the points $A(g(a), f(a))$ and $B(g(b), f(b))$.

Cauchy's Mean Value Theorem has a geometric interpretation for a general winding curve C in the plane joining the two points $A = (g(a), f(a))$ and $B = (g(b), f(b))$. In Chapter 10 you will learn how to describe general curves such as C , along with their tangent lines. There is at least one point P on the curve for which the tangent to the curve at P is parallel to the secant line joining the points A and B . The slope of that tangent line turns out to be the quotient f'/g' evaluated at the number c in the interval (a, b) , which is the left-hand side of the equation in Theorem 7. Because the slope of the secant line joining A and B is

$$\frac{f(b) - f(a)}{g(b) - g(a)},$$

the equation in Cauchy's Mean Value Theorem says that the slope of the tangent line equals the slope of the secant line. This geometric interpretation is shown in Figure 4.37. Notice from the figure that it is possible for more than one point on the curve C to have a tangent line that is parallel to the secant line joining A and B .

Proof of l'Hôpital's Rule Since $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ and both f and g are differentiable at a , they must also be continuous at a , hence $f(a) = g(a) = 0$.

We first establish the limit equation for the case $x \rightarrow a^+$. The method needs almost no change to apply to $x \rightarrow a^-$, and the combination of these two cases establishes the result.

Suppose that x lies in the interval to the right of a . Then $g'(x) \neq 0$, and we can apply Cauchy's Mean Value Theorem to the closed interval from a to x . This step produces a number c between a and x such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}.$$

But $f(a) = g(a) = 0$, so

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}.$$

As x approaches a , c approaches a because it always lies between a and x . Therefore,

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{c \rightarrow a^+} \frac{f'(c)}{g'(c)} = \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)},$$

which establishes l'Hôpital's Rule for the case where x approaches a from above. The case where x approaches a from below is proved by applying Cauchy's Mean Value Theorem to the closed interval $[x, a]$, $x < a$. ■

EXERCISES 4.5

Finding Limits in Two Ways

In Exercises 1–6, use l'Hôpital's Rule to evaluate the limit. Then evaluate the limit using a method studied in Chapter 2.

- $\lim_{x \rightarrow -2} \frac{x+2}{x^2-4}$
- $\lim_{x \rightarrow 0} \frac{\sin 5x}{x}$
- $\lim_{x \rightarrow \infty} \frac{5x^2-3x}{7x^2+1}$
- $\lim_{x \rightarrow 1} \frac{x^3-1}{4x^3-x-3}$
- $\lim_{x \rightarrow 0} \frac{1-\cos x}{x^2}$
- $\lim_{x \rightarrow \infty} \frac{2x^2+3x}{x^3+x+1}$

Applying l'Hôpital's Rule

Use l'Hôpital's rule to find the limits in Exercises 7–52.

- $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4}$
- $\lim_{x \rightarrow -5} \frac{x^2-25}{x+5}$
- $\lim_{t \rightarrow -3} \frac{t^3-4t+15}{t^2-t-12}$
- $\lim_{t \rightarrow -1} \frac{3t^3+3}{4t^3-t+3}$
- $\lim_{x \rightarrow \infty} \frac{5x^3-2x}{7x^3+3}$
- $\lim_{x \rightarrow \infty} \frac{x-8x^2}{12x^2+5x}$
- $\lim_{t \rightarrow 0} \frac{\sin t^2}{t}$
- $\lim_{t \rightarrow 0} \frac{\sin 5t}{2t}$
- $\lim_{x \rightarrow 0} \frac{8x^2}{\cos x - 1}$
- $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}$
- $\lim_{\theta \rightarrow \pi/2} \frac{2\theta - \pi}{\cos(2\pi - \theta)}$
- $\lim_{\theta \rightarrow -\pi/3} \frac{3\theta + \pi}{\sin(\theta + (\pi/3))}$
- $\lim_{\theta \rightarrow \pi/6} \frac{\sin \theta - \frac{1}{2}}{\theta - \frac{\pi}{6}}$
- $\lim_{\theta \rightarrow \pi/4} \frac{\tan \theta - 1}{\theta - \frac{\pi}{4}}$
- $\lim_{\theta \rightarrow \pi/2} \frac{1 - \sin \theta}{1 + \cos 2\theta}$
- $\lim_{x \rightarrow 1} \frac{x-1}{\ln x - \sin \pi x}$
- $\lim_{x \rightarrow 0} \frac{x^2}{\ln(\sec x)}$
- $\lim_{x \rightarrow \pi/2} \frac{\ln(\csc x)}{(x - (\pi/2))^2}$
- $\lim_{t \rightarrow 0} \frac{t(1 - \cos t)}{t - \sin t}$
- $\lim_{t \rightarrow 0} \frac{t \sin t}{1 - \cos t}$
- $\lim_{x \rightarrow (\pi/2)^-} \left(x - \frac{\pi}{2}\right) \sec x$
- $\lim_{x \rightarrow (\pi/2)^-} \left(\frac{\pi}{2} - x\right) \tan x$
- $\lim_{\theta \rightarrow 0} \frac{3 \sin \theta - 1}{\theta}$
- $\lim_{\theta \rightarrow 0} \frac{(1/2)^\theta - 1}{\theta}$
- $\lim_{x \rightarrow 0} \frac{x2^x}{2^x - 1}$
- $\lim_{x \rightarrow 0} \frac{3^x - 1}{2^x - 1}$
- $\lim_{x \rightarrow \infty} \frac{\ln(x+1)}{\log_2 x}$
- $\lim_{x \rightarrow \infty} \frac{\log_2 x}{\log_3(x+3)}$
- $\lim_{x \rightarrow 0^+} \frac{\ln(x^2+2x)}{\ln x}$
- $\lim_{x \rightarrow 0^+} \frac{\ln(e^x-1)}{\ln x}$
- $\lim_{y \rightarrow 0} \frac{\sqrt{5y+25}-5}{y}$
- $\lim_{y \rightarrow 0} \frac{\sqrt{ay+a^2}-a}{y}, \quad a > 0$
- $\lim_{x \rightarrow \infty} (\ln 2x - \ln(x+1))$
- $\lim_{x \rightarrow 0^+} (\ln x - \ln \sin x)$
- $\lim_{x \rightarrow 0^+} \frac{(\ln x)^2}{\ln(\sin x)}$
- $\lim_{x \rightarrow 0^+} \left(\frac{3x+1}{x} - \frac{1}{\sin x} \right)$
- $\lim_{x \rightarrow 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right)$
- $\lim_{x \rightarrow 0^+} (\csc x - \cot x + \cos x)$

$$45. \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{e^\theta - \theta - 1}$$

$$47. \lim_{t \rightarrow \infty} \frac{e^t + t^2}{e^t - t}$$

$$49. \lim_{x \rightarrow 0} \frac{x - \sin x}{x \tan x}$$

$$51. \lim_{\theta \rightarrow 0} \frac{\theta - \sin \theta \cos \theta}{\tan \theta - \theta}$$

$$46. \lim_{h \rightarrow 0} \frac{e^h - (1+h)}{h^2}$$

$$48. \lim_{x \rightarrow \infty} x^2 e^{-x}$$

$$50. \lim_{x \rightarrow 0} \frac{(e^x - 1)^2}{x \sin x}$$

$$52. \lim_{x \rightarrow 0} \frac{\sin 3x - 3x + x^2}{\sin x \sin 2x}$$

Indeterminate Powers and Products

Find the limits in Exercises 53–68.

- $\lim_{x \rightarrow 1^+} x^{1/(1-x)}$
- $\lim_{x \rightarrow 1^+} x^{1/(x-1)}$
- $\lim_{x \rightarrow \infty} (\ln x)^{1/x}$
- $\lim_{x \rightarrow e^+} (\ln x)^{1/(x-e)}$
- $\lim_{x \rightarrow 0^+} x^{-1/\ln x}$
- $\lim_{x \rightarrow \infty} (1+2x)^{1/(2 \ln x)}$
- $\lim_{x \rightarrow 0} (e^x + x)^{1/x}$
- $\lim_{x \rightarrow 0^+} x^x$
- $\lim_{x \rightarrow 0^+} \left(1 + \frac{1}{x}\right)^x$
- $\lim_{x \rightarrow \infty} \left(\frac{x+2}{x-1}\right)^x$
- $\lim_{x \rightarrow \infty} \left(\frac{x^2+1}{x+2}\right)^{1/x}$
- $\lim_{x \rightarrow 0^+} x^2 \ln x$
- $\lim_{x \rightarrow 0^+} x(\ln x)^2$
- $\lim_{x \rightarrow 0^+} x \tan\left(\frac{\pi}{2} - x\right)$
- $\lim_{x \rightarrow 0^+} \sin x \cdot \ln x$

Theory and Applications

L'Hôpital's Rule does not help with the limits in Exercises 69–76. Try it—you just keep on cycling. Find the limits some other way.

- $\lim_{x \rightarrow \infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}}$
- $\lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}}$
- $\lim_{x \rightarrow (\pi/2)^-} \frac{\sec x}{\tan x}$
- $\lim_{x \rightarrow 0^+} \frac{\cot x}{\csc x}$
- $\lim_{x \rightarrow \infty} \frac{2^x - 3^x}{3^x + 4^x}$
- $\lim_{x \rightarrow -\infty} \frac{2^x + 4^x}{5^x - 2^x}$
- $\lim_{x \rightarrow \infty} \frac{e^{x^2}}{xe^x}$
- $\lim_{x \rightarrow 0^+} \frac{x}{e^{-1/x}}$
- Which one is correct, and which one is wrong? Give reasons for your answers.
 a. $\lim_{x \rightarrow 3} \frac{x-3}{x^2-3} = \lim_{x \rightarrow 3} \frac{1}{2x} = \frac{1}{6}$ b. $\lim_{x \rightarrow 3} \frac{x-3}{x^2-3} = \frac{0}{6} = 0$
- Which one is correct, and which one is wrong? Give reasons for your answers.
 a. $\lim_{x \rightarrow 0} \frac{x^2-2x}{x^2-\sin x} = \lim_{x \rightarrow 0} \frac{2x-2}{2x-\cos x} = \lim_{x \rightarrow 0} \frac{2}{2+\sin x} = \frac{2}{2+0} = 1$
 b. $\lim_{x \rightarrow 0} \frac{x^2-2x}{x^2-\sin x} = \lim_{x \rightarrow 0} \frac{2x-2}{2x-\cos x} = \frac{-2}{0-1} = 2$

79. Only one of these calculations is correct. Which one? Why are the others wrong? Give reasons for your answers.

a. $\lim_{x \rightarrow 0^+} x \ln x = 0 \cdot (-\infty) = 0$

b. $\lim_{x \rightarrow 0^+} x \ln x = 0 \cdot (-\infty) = -\infty$

c. $\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{(1/x)} = \frac{-\infty}{\infty} = -1$

d. $\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{(1/x)}$
 $= \lim_{x \rightarrow 0^+} \frac{(1/x)}{(-1/x^2)} = \lim_{x \rightarrow 0^+} (-x) = 0$

80. Find all values of c that satisfy the conclusion of Cauchy's Mean Value Theorem for the given functions and interval.

a. $f(x) = x$, $g(x) = x^2$, $(a, b) = (-2, 0)$

b. $f(x) = x$, $g(x) = x^2$, (a, b) arbitrary

c. $f(x) = x^3/3 - 4x$, $g(x) = x^2$, $(a, b) = (0, 3)$

81. **Continuous extension** Find a value of c that makes the function

$$f(x) = \begin{cases} \frac{9x - 3 \sin 3x}{5x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$. Explain why your value of c works.

82. For what values of a and b is

$$\lim_{x \rightarrow 0} \left(\frac{\tan 2x}{x^3} + \frac{a}{x^2} + \frac{\sin bx}{x} \right) = 0?$$

T 83. $\infty - \infty$ Form

a. Estimate the value of

$$\lim_{x \rightarrow \infty} (x - \sqrt{x^2 + x})$$

by graphing $f(x) = x - \sqrt{x^2 + x}$ over a suitably large interval of x -values.

b. Now confirm your estimate by finding the limit with l'Hôpital's Rule. As the first step, multiply $f(x)$ by the fraction $(x + \sqrt{x^2 + x})/(x + \sqrt{x^2 + x})$ and simplify the new numerator.

84. Find $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - \sqrt{x})$.

T 85. 0/0 Form Estimate the value of

$$\lim_{x \rightarrow 1} \frac{2x^2 - (3x + 1)\sqrt{x} + 2}{x - 1}$$

by graphing. Then confirm your estimate with l'Hôpital's Rule.

86. This exercise explores the difference between

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x^2} \right)^x$$

and

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = e.$$

a. Use l'Hôpital's Rule to show that

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = e.$$

T b. Graph

$$f(x) = \left(1 + \frac{1}{x^2} \right)^x \quad \text{and} \quad g(x) = \left(1 + \frac{1}{x} \right)^x$$

together for $x \geq 0$. How does the behavior of f compare with that of g ? Estimate the value of $\lim_{x \rightarrow \infty} f(x)$.

c. Confirm your estimate of $\lim_{x \rightarrow \infty} f(x)$ by calculating it with l'Hôpital's Rule.

87. Show that

$$\lim_{k \rightarrow \infty} \left(1 + \frac{r}{k} \right)^k = e^r.$$

88. Given that $x > 0$, find the maximum value, if any, of

a. $x^{1/x}$

b. x^{1/x^2}

c. x^{1/x^n} (n a positive integer)

d. Show that $\lim_{x \rightarrow \infty} x^{1/x^n} = 1$ for every positive integer n .

89. Use limits to find horizontal asymptotes for each function.

a. $y = x \tan\left(\frac{1}{x}\right)$

b. $y = \frac{3x + e^{2x}}{2x + e^{3x}}$

90. Find $f'(0)$ for $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

T 91. The continuous extension of $(\sin x)^x$ to $[0, \pi]$

a. Graph $f(x) = (\sin x)^x$ on the interval $0 \leq x \leq \pi$. What value would you assign to f to make it continuous at $x = 0$?

b. Verify your conclusion in part (a) by finding $\lim_{x \rightarrow 0^+} f(x)$ with l'Hôpital's Rule.

c. Returning to the graph, estimate the maximum value of f on $[0, \pi]$. About where is $\max f$ taken on?

d. Sharpen your estimate in part (c) by graphing f' in the same window to see where its graph crosses the x -axis. To simplify your work, you might want to delete the exponential factor from the expression for f' and graph just the factor that has a zero.

T 92. The function $(\sin x)^{\tan x}$ (Continuation of Exercise 91)

a. Graph $f(x) = (\sin x)^{\tan x}$ on the interval $-7 \leq x \leq 7$. How do you account for the gaps in the graph? How wide are the gaps?

b. Now graph f on the interval $0 \leq x \leq \pi$. The function is not defined at $x = \pi/2$, but the graph has no break at this point. What is going on? What value does the graph appear to give for f at $x = \pi/2$? (Hint: Use l'Hôpital's Rule to find $\lim_{x \rightarrow (\pi/2)^-} f$ and $\lim_{x \rightarrow (\pi/2)^+} f$.)

c. Continuing with the graphs in part (b), find $\max f$ and $\min f$ as accurately as you can and estimate the values of x at which they are taken on.

4.6 Applied Optimization

What are the dimensions of a rectangle with fixed perimeter having *maximum area*? What are the dimensions for the *least expensive* cylindrical can of a given volume? How many items should be produced for the *most profitable* production run? Each of these questions asks for the best, or optimal, value of a given function. In this section we use derivatives to solve a variety of optimization problems in mathematics, physics, economics, and business.

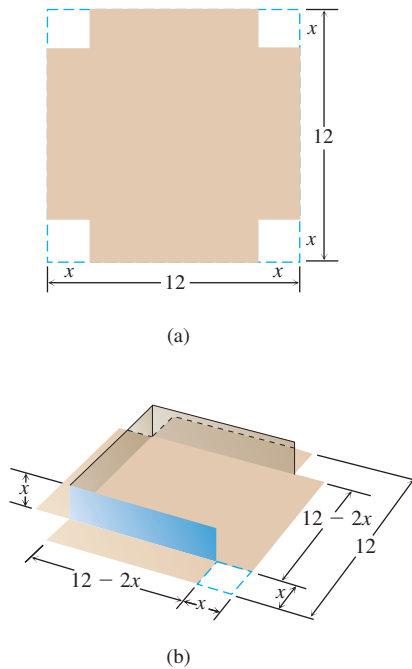


FIGURE 4.38 An open box made by cutting the corners from a square sheet of tin. What size corners maximize the box's volume (Example 1)?

Solving Applied Optimization Problems

1. *Read the problem.* Read the problem until you understand it. What is given? What is the unknown quantity to be optimized (maximized or minimized)?
2. *Introduce variables.* List every relevant relation in the problem as an equation. In most problems it is helpful to draw a picture.
3. *Write an equation for the unknown quantity.* Express the quantity to be optimized as a function of a single variable. This may require considerable manipulation.
4. *Test the critical points and endpoints in the domain of the function found in the previous step.* Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

EXAMPLE 1 An open-top box is to be made by cutting small congruent squares from the corners of a 12-cm-by-12-cm sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Solution We start with a picture (Figure 4.38). In the figure, the corner squares are x cm on a side. The volume of the box is a function of this variable:

$$V(x) = x(12 - 2x)^2 = 144x - 48x^2 + 4x^3. \quad V = hlw$$

Since the sides of the sheet of tin are only 12 cm long, $x \leq 6$ and the domain of V is the interval $0 \leq x \leq 6$.

A graph of V (Figure 4.39) suggests a minimum value of 0 at $x = 0$ and $x = 6$ and a maximum near $x = 2$. To learn more, we examine the first derivative of V with respect to x :

$$\frac{dV}{dx} = 144 - 96x + 12x^2 = 12(12 - 8x + x^2) = 12(2 - x)(6 - x).$$

Of the two zeros, $x = 2$ and $x = 6$, only $x = 2$ lies in the interior of the function's domain and makes the critical-point list. The values of V at this one critical point and two endpoints are

$$\text{Critical-point value: } V(2) = 128$$

$$\text{Endpoint values: } V(0) = 0, \quad V(6) = 0.$$

The maximum volume is 128 cm^3 . The cutout squares should be 2 cm on a side. ■

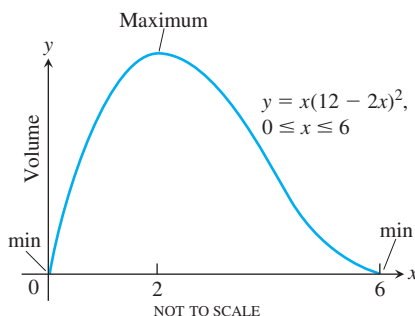


FIGURE 4.39 The volume of the box in Figure 4.38 graphed as a function of x .

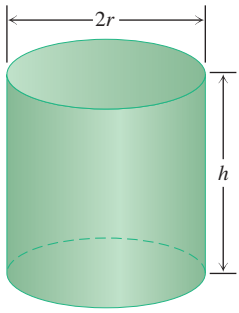


FIGURE 4.40 Example 2 shows that this one-liter can uses the least material when $h = 2r$.

EXAMPLE 2 You have been asked to design a one-liter can shaped like a right circular cylinder (Figure 4.40). What dimensions will use the least material?

Solution *Volume of can:* If r and h are measured in centimeters, then the volume of the can in cubic centimeters is

$$\pi r^2 h = 1000. \quad 1 \text{ liter} = 1000 \text{ cm}^3$$

$$\text{Surface area of can:} \quad A = \underbrace{2\pi r^2}_{\text{circular ends}} + \underbrace{2\pi r h}_{\text{cylindrical wall}}$$

How can we interpret the phrase “least material”? For a first approximation we can ignore the thickness of the material and the waste in manufacturing. Then we ask for dimensions r and h that make the total surface area as small as possible, while satisfying the constraint $\pi r^2 h = 1000 \text{ cm}^3$.

To express the surface area as a function of one variable, we solve for one of the variables in $\pi r^2 h = 1000$ and substitute that expression into the surface area formula. Solving for h is easier:

$$h = \frac{1000}{\pi r^2}.$$

Thus,

$$\begin{aligned} A &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r \left(\frac{1000}{\pi r^2} \right) \\ &= 2\pi r^2 + \frac{2000}{r}. \end{aligned}$$

Our goal is to find a value of $r > 0$ that minimizes the value of A .

Since A is differentiable on $r > 0$, an interval with no endpoints, it can have a minimum value only where its first derivative is zero.

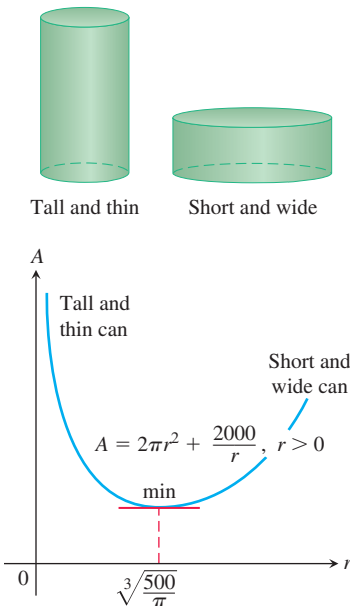


FIGURE 4.41 The graph of $A = 2\pi r^2 + 2000/r$ is concave up.

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2}$$

$$0 = 4\pi r - \frac{2000}{r^2} \quad \text{Set } dA/dr = 0.$$

$$4\pi r^3 = 2000 \quad \text{Multiply by } r^2.$$

$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \quad \text{Solve for } r.$$

What happens at $r = \sqrt[3]{500/\pi}$?

The second derivative

$$\frac{d^2A}{dr^2} = 4\pi + \frac{4000}{r^3}$$

is positive throughout the domain of A . The graph is therefore everywhere concave up, and the value of A at $r = \sqrt[3]{500/\pi}$ is an absolute minimum. See Figure 4.41.

The corresponding value of h (after a little algebra) is

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

The one-liter can that uses the least material has height equal to twice the radius, here with $r \approx 5.42 \text{ cm}$ and $h \approx 10.84 \text{ cm}$. ■

Examples from Mathematics and Physics

EXAMPLE 3 A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

Solution Let $(x, \sqrt{4 - x^2})$ be the coordinates of the upper right corner of the rectangle obtained by placing the circle and rectangle in the coordinate plane (Figure 4.42). The length, height, and area of the rectangle can then be expressed in terms of the position x of the lower right-hand corner:

$$\text{Length: } 2x, \quad \text{Height: } \sqrt{4 - x^2}, \quad \text{Area: } 2x\sqrt{4 - x^2}.$$

Notice that the values of x are to be found in the interval $0 \leq x \leq 2$, where the selected corner of the rectangle lies.

Our goal is to find the absolute maximum value of the function

$$A(x) = 2x\sqrt{4 - x^2}$$

on the domain $[0, 2]$.

The derivative

$$\frac{dA}{dx} = \frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2}$$

is not defined when $x = 2$ and is equal to zero when

$$\begin{aligned} \frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2} &= 0 \\ -2x^2 + 2(4 - x^2) &= 0 \\ 8 - 4x^2 &= 0 \\ x^2 &= 2 \\ x &= \pm\sqrt{2}. \end{aligned}$$

Of the two zeros, $x = \sqrt{2}$ and $x = -\sqrt{2}$, only $x = \sqrt{2}$ lies in the interior of A 's domain and makes the critical-point list. The values of A at the endpoints and at this one critical point are

$$\text{Critical-point value: } A(\sqrt{2}) = 2\sqrt{2}\sqrt{4 - 2} = 4$$

$$\text{Endpoint values: } A(0) = 0, \quad A(2) = 0.$$

The area has a maximum value of 4 when the rectangle is $\sqrt{4 - x^2} = \sqrt{2}$ units high and $2x = 2\sqrt{2}$ units long. ■

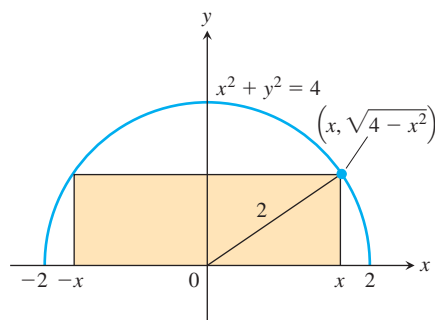


FIGURE 4.42 The rectangle inscribed in the semicircle in Example 3.

HISTORICAL BIOGRAPHY

Willebrord Snell van Royen
(1580–1626)

Snell was born in Leiden, Holland. Snell developed an important result involving the measure of light refraction as it travels into different media. While he never published the result, Descartes did so ten years after Snell's death, and today it is known as Snell's law.

To know more, visit the companion Website.

EXAMPLE 4 The speed of light depends on the medium through which it travels, and is generally slower in denser media.

Fermat's principle in optics states that light travels from one point to another along a path for which the time of travel is a minimum. Describe the path that a ray of light will follow in going from a point A in a medium where the speed of light is c_1 to a point B in a second medium where the speed of light is c_2 .

Solution Since light traveling from A to B follows the quickest route, we look for a path that will minimize the travel time. We assume that A and B lie in the xy -plane and that the line separating the two media is the x -axis (Figure 4.43). We place A at coordinates $(0, a)$ and B at coordinates $(d, -b)$ in the xy -plane.

In a uniform medium, where the speed of light remains constant, "shortest time" means "shortest path," and the ray of light will follow a straight line. Thus the path from A

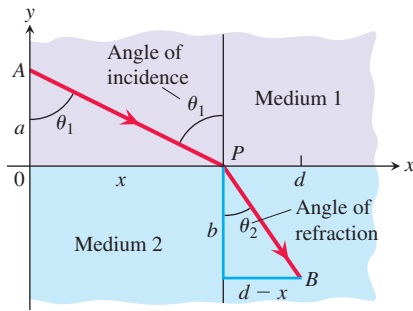


FIGURE 4.43 A light ray refracted (deflected from its path) as it passes from one medium to a denser medium (Example 4).

to B will consist of a line segment from A to a boundary point P , followed by another line segment from P to B . Distance traveled equals rate times time, so

$$\text{Time} = \frac{\text{distance}}{\text{rate}}.$$

From Figure 4.43, the time required for light to travel from A to P is

$$t_1 = \frac{AP}{c_1} = \frac{\sqrt{a^2 + x^2}}{c_1}.$$

From P to B , the time is

$$t_2 = \frac{PB}{c_2} = \frac{\sqrt{b^2 + (d-x)^2}}{c_2}.$$

The time from A to B is the sum of these:

$$t = t_1 + t_2 = \frac{\sqrt{a^2 + x^2}}{c_1} + \frac{\sqrt{b^2 + (d-x)^2}}{c_2}.$$

This equation expresses t as a differentiable function of x whose domain is $[0, d]$. We want to find the absolute minimum value of t on this closed interval. We find the derivative

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2 + x^2}} - \frac{d-x}{c_2\sqrt{b^2 + (d-x)^2}}$$

and observe that it is continuous. In terms of the angles θ_1 and θ_2 in Figure 4.43,

$$\frac{dt}{dx} = \frac{\sin \theta_1}{c_1} - \frac{\sin \theta_2}{c_2}.$$

The function t has a negative derivative at $x = 0$ and a positive derivative at $x = d$. Since dt/dx is continuous over the interval $[0, d]$, by the Intermediate Value Theorem for continuous functions (Section 2.6), there is a point $x_0 \in [0, d]$ where $dt/dx = 0$ (Figure 4.44). There is only one such point because dt/dx is an increasing function of x (Exercise 70). At this unique point we then have

$$\frac{\sin \theta_1}{c_1} = \frac{\sin \theta_2}{c_2}.$$

This equation is **Snell's Law** or the **Law of Refraction**, and it is an important principle in the theory of optics. It describes the path the ray of light follows. ■

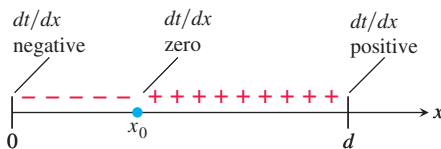


FIGURE 4.44 The sign pattern of dt/dx in Example 4.

Examples from Economics

Suppose that

$r(x)$ = the revenue from selling x items

$c(x)$ = the cost of producing the x items

$p(x) = r(x) - c(x)$ = the profit from producing and selling x items.

Although x is usually an integer in many applications, we can learn about the behavior of these functions by defining them for all nonzero real numbers and by assuming they are differentiable functions. Economists use the terms **marginal revenue**, **marginal cost**, and **marginal profit** to name the derivatives $r'(x)$, $c'(x)$, and $p'(x)$ of the revenue, cost, and profit functions. Let's consider the relationship of the profit p to these derivatives.

If $r(x)$ and $c(x)$ are differentiable for x in some interval of production possibilities, and if $p(x) = r(x) - c(x)$ has a maximum value there, it occurs at a critical point of $p(x)$ or at an endpoint of the interval. If it occurs at a critical point, then $p'(x) = r'(x) - c'(x) = 0$ and we see that $r'(x) = c'(x)$. In economic terms, this last equation means that

At a production level yielding maximum profit, marginal revenue equals marginal cost (Figure 4.45).

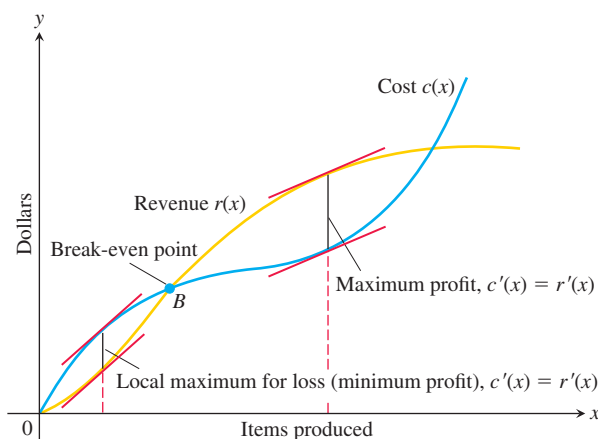


FIGURE 4.45 The graph of a typical cost function starts concave down and later turns concave up. It crosses the revenue curve at the break-even point B . To the left of B , the company operates at a loss. To the right, the company operates at a profit, with the maximum profit occurring where $c'(x) = r'(x)$. Farther to the right, cost exceeds revenue (perhaps because of a combination of rising labor and material costs, and market saturation) and production levels become unprofitable again.

EXAMPLE 5 Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$ are the revenue and the cost functions, given in millions of dollars, where x represents millions of MP3 players produced. Is there a production level that maximizes profit? If so, what is it?

Solution Notice that $r'(x) = 9$ and $c'(x) = 3x^2 - 12x + 15$.

$$3x^2 - 12x + 15 = 9 \quad \text{Set } c'(x) = r'(x).$$

$$3x^2 - 12x + 6 = 0$$

The two solutions of the quadratic equation are

$$x_1 = \frac{12 - \sqrt{72}}{6} = 2 - \sqrt{2} \approx 0.586 \quad \text{and}$$

$$x_2 = \frac{12 + \sqrt{72}}{6} = 2 + \sqrt{2} \approx 3.414.$$

The possible production levels for maximum profit are $x \approx 0.586$ million MP3 players or $x \approx 3.414$ million. The second derivative of $p(x) = r(x) - c(x)$ is $p''(x) = -c''(x)$ since $r''(x)$ is everywhere zero. Thus, $p''(x) = 6(2 - x)$, which is negative at $x = 2 + \sqrt{2}$ and positive at $x = 2 - \sqrt{2}$. By the Second Derivative Test, a maximum profit occurs at about $x = 3.414$ (where revenue exceeds costs) and maximum loss occurs at about $x = 0.586$. The graphs of $r(x)$ and $c(x)$ are shown in Figure 4.46. ■

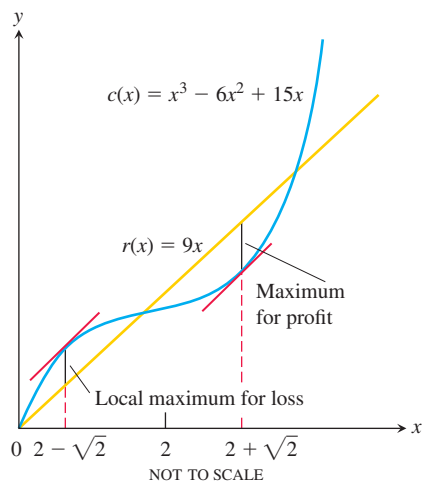


FIGURE 4.46 The cost and revenue curves for Example 5.

EXAMPLE 6 A cabinetmaker uses cherry wood to produce 5 desks each day. Each delivery of one container of wood is \$5000, whereas the storage of that material is \$10 per day per unit stored, where a unit is the amount of material needed by her to produce 1 desk. How much material should be ordered each time, and how often should the material be delivered, to minimize her average daily cost in the production cycle between deliveries?

Solution If she asks for a delivery every x days, then she must order $5x$ units to have enough material for that delivery cycle. The *average* amount in storage is approximately

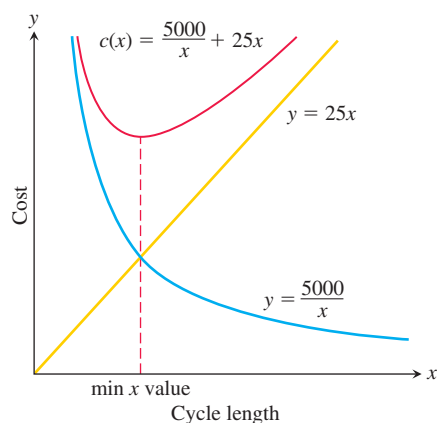


FIGURE 4.47 The average daily cost $c(x)$ is the sum of a hyperbola and a linear function (Example 6).

one-half of the delivery amount, or $5x/2$. Thus, the cost of delivery and storage for each cycle is approximately

Cost per cycle = delivery costs + storage costs

$$\text{Cost per cycle} = \underbrace{5000}_{\text{delivery cost}} + \underbrace{\left(\frac{5x}{2}\right)}_{\text{average amount stored}} \cdot \underbrace{x}_{\text{number of days stored}} \cdot \underbrace{10}_{\text{storage cost per day}}$$

We compute the *average daily cost* $c(x)$ by dividing the cost per cycle by the number of days x in the cycle (see Figure 4.47).

$$c(x) = \frac{5000}{x} + 25x, \quad x > 0.$$

As $x \rightarrow 0$ and as $x \rightarrow \infty$, the average daily cost becomes large. So we expect a minimum to exist, but where? Our goal is to determine the number of days x between deliveries that provides the absolute minimum cost.

We find the critical points by determining where the derivative is equal to zero:

$$\begin{aligned} c'(x) &= -\frac{5000}{x^2} + 25 = 0 \\ x &= \pm\sqrt{200} \approx \pm 14.14. \end{aligned}$$

Of the two critical points, only $\sqrt{200}$ lies in the domain of $c(x)$. The critical point value of the average daily cost is

$$c(\sqrt{200}) = \frac{5000}{\sqrt{200}} + 25\sqrt{200} = 500\sqrt{2} \approx \$707.11.$$

We note that $c(x)$ is defined over the open interval $(0, \infty)$ with $c''(x) = 10000/x^3 > 0$. Thus, an absolute minimum exists at $x = \sqrt{200} \approx 14.14$ days.

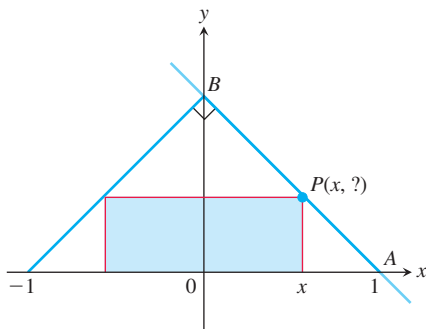
The cabinetmaker should schedule a delivery of $5(14) = 70$ units of wood every 14 days. ■

EXERCISES 4.6

Mathematical Applications

Whenever you are maximizing or minimizing a function of a single variable, we urge you to graph it over the domain that is appropriate to the problem you are solving. The graph will provide insight before you calculate and will furnish a visual context for understanding your answer.

- 1. Minimizing perimeter** What is the smallest perimeter possible for a rectangle whose area is 16 cm^2 , and what are its dimensions?
- Show that among all rectangles with an 8-m perimeter, the one with largest area is a square.
- The figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.



- Express the y-coordinate of P in terms of x . (*Hint:* Write an equation for the line AB .)
 - Express the area of the rectangle in terms of x .
 - What is the largest area the rectangle can have, and what are its dimensions?
- A rectangle has its base on the x -axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?
 - You are planning to make an open rectangular box from a 24-cm-by-45-cm piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way, and what is its volume?
 - You are planning to close off a corner of the first quadrant with a line segment 20 units long running from $(a, 0)$ to $(0, b)$. Show that the area of the triangle enclosed by the segment is largest when $a = b$.
 - The best fencing plan** A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?
 - The shortest fence** A 216 m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?

- 9. Designing a tank** Your iron works has contracted to design and build a 4 m^3 , square-based, open-top, rectangular steel holding tank for a paper company. The tank is to be made by welding thin stainless steel plates together along their edges. As the production engineer, your job is to find dimensions for the base and height that will make the tank weigh as little as possible.

- What dimensions do you tell the shop to use?
- Briefly describe how you took weight into account.

- 10. Catching rainwater** A 20 m^3 open-top rectangular tank with a square base $x \text{ m}$ on a side and $y \text{ m}$ deep is to be built with its top flush with the ground to catch runoff water. The costs associated with the tank involve not only the material from which the tank is made but also an excavation charge proportional to the product xy .

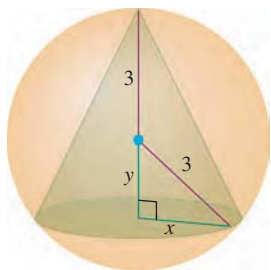
- If the total cost is

$$c = 5(x^2 + 4xy) + 10xy,$$

what values of x and y will minimize it?

- Give a possible scenario for the cost function in part (a).

- 11. Designing a poster** You are designing a rectangular poster to contain 312.5 cm^2 of printing with a 10-cm margin at the top and bottom and a 5-cm margin at each side. What overall dimensions will minimize the amount of paper used?
- 12.** Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.



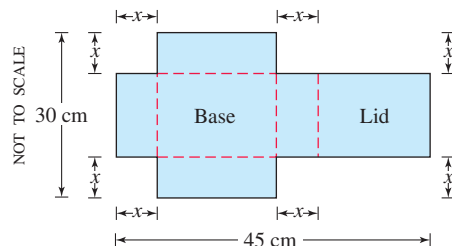
- 13.** Two sides of a triangle have lengths a and b , and the angle between them is θ . What value of θ will maximize the triangle's area? (Hint: $A = (1/2)ab \sin \theta$.)
- 14. Designing a can** What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cm^3 ? Compare the result here with the result in Example 2.
- 15. Designing a can** You are designing a 1000 cm^3 right circular cylindrical can whose manufacture will take waste into account. There is no waste in cutting the aluminum for the side, but the top and bottom of radius r will be cut from squares that measure $2r$ units on a side. The total amount of aluminum used up by the can will therefore be

$$A = 8r^2 + 2\pi rh$$

rather than the $A = 2\pi r^2 + 2\pi rh$ in Example 2. In Example 2, the ratio of h to r for the most economical can was 2 to 1. What is the ratio for the most economical can now?

- T 16. Designing a box with a lid** A piece of cardboard measures 30 cm by 45 cm. Two equal squares are removed from the corners of a 30-cm side as shown in the figure. Two equal rectangles are

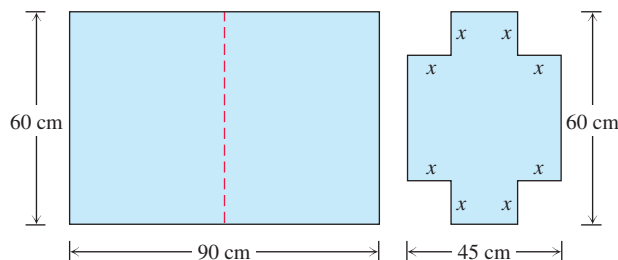
removed from the other corners so that the tabs can be folded to form a rectangular box with lid.



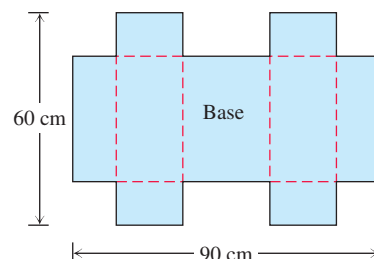
- Write a formula $V(x)$ for the volume of the box.
- Find the domain of V for the problem situation, and graph V over this domain.
- Use a graphical method to find the maximum volume and the value of x that gives it.
- Confirm your result in part (c) analytically.

- T 17. Designing a suitcase** A 60-cm-by-90-cm sheet of cardboard is folded in half to form a 60-cm-by-45-cm rectangle as shown in the accompanying figure. Then four congruent squares of side length x are cut from the corners of the folded rectangle. The sheet is unfolded, and the six tabs are folded up to form a box with sides and a lid.

- Write a formula $V(x)$ for the volume of the box.
- Find the domain of V for the problem situation and graph V over this domain.
- Use a graphical method to find the maximum volume and the value of x that gives it.
- Confirm your result in part (c) analytically.
- Find a value of x that yields a volume of $17,500 \text{ cm}^3$.
- Write a paragraph describing the issues that arise in part (b).

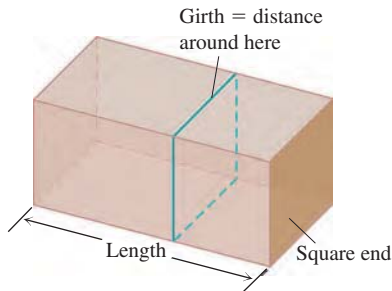


The sheet is then unfolded.



- T 18.** A rectangle is to be inscribed under the arch of the curve $y = 4 \cos(0.5x)$ from $x = -\pi$ to $x = \pi$. What are the dimensions of the rectangle with largest area, and what is the largest area?

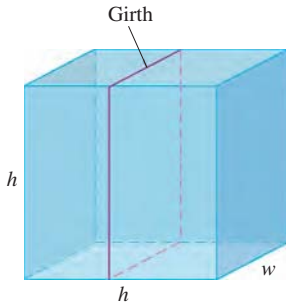
19. Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?
20. a. A certain Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 276 cm. What dimensions will give a box with a square end the largest possible volume?



- T** b. Graph the volume of a 276-cm box (length plus girth equals 276 cm) as a function of its length, and compare what you see with your answer in part (a).

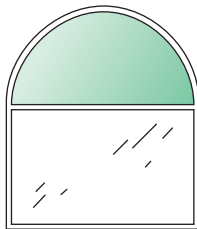
21. (Continuation of Exercise 20)

- a. Suppose that instead of having a box with square ends, you have a box with square sides so that its dimensions are h by h by w and the girth is $2h + 2w$. What dimensions will give the box its largest volume now?



- T** b. Graph the volume as a function of h and compare what you see with your answer in part (a).

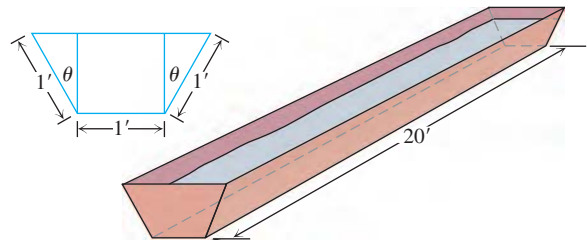
22. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.



23. A silo (base not included) is to be constructed in the form of a cylinder surmounted by a hemisphere. The cost of construction per square unit of surface area is twice as great for the hemisphere as it is for the cylindrical sidewall. Determine the dimensions to

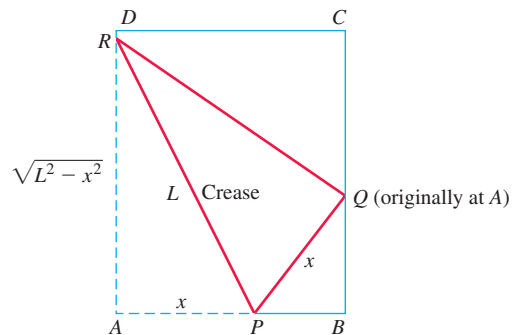
be used if the volume is fixed and the cost of construction is to be kept to a minimum. Neglect the thickness of the silo and waste in construction.

24. The trough in the figure is to be made to the dimensions shown. Only the angle θ can be varied. What value of θ will maximize the trough's volume?



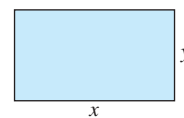
25. **Paper folding** A rectangular sheet of 21.6-cm-by-28-cm paper is placed on a flat surface. One of the corners is placed on the opposite longer edge, as shown in the figure, and held there as the paper is smoothed flat. The problem is to make the length of the crease as small as possible. Call the length L . Try it with paper.

- a. Show that $L^2 = 2x^3/(2x - 21.6)$.
- b. What value of x minimizes L^2 ?
- c. What is the minimum value of L ?

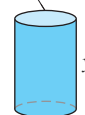


26. **Constructing cylinders** Compare the answers to the following two construction problems.

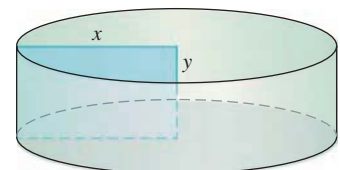
- a. A rectangular sheet of perimeter 36 cm and dimensions x cm by y cm is to be rolled into a cylinder as shown in part (a) of the figure. What values of x and y give the largest volume?
- b. The same sheet is to be revolved about one of the sides of length y to sweep out the cylinder as shown in part (b) of the figure. What values of x and y give the largest volume?



Circumference = x

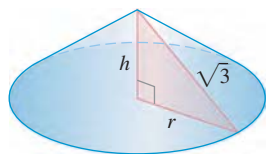


(a)

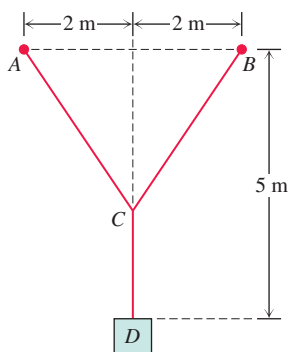


(b)

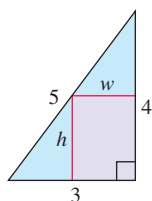
- 27. Constructing cones** A right triangle whose hypotenuse is $\sqrt{3}$ m long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.



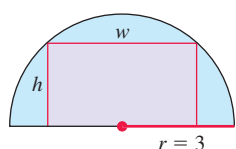
- 28.** Find the point on the line $\frac{x}{a} + \frac{y}{b} = 1$ that is closest to the origin.
- 29.** Find a positive number for which the sum of it and its reciprocal is the smallest (least) possible.
- 30.** Find a positive number for which the sum of its reciprocal and four times its square is the smallest possible.
- 31.** A wire b m long is cut into two pieces. One piece is bent into an equilateral triangle and the other is bent into a circle. If the sum of the areas enclosed by each part is a minimum, what is the length of each part?
- 32.** Answer Exercise 31 if one piece is bent into a square and the other into a circle.
- 33.** Suppose a weight D is to be held 5 m below a horizontal line AB by a wire in the shape of a Y. If the points A and B are 4 m apart, what is the minimum total length of wire that can be used?



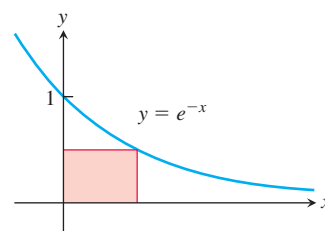
- 34.** Suppose two different gauges of wire must be used to support the weight in Exercise 33: the vertical portion of the wire (the segment CD) costs \$1 per meter, while the remaining wire (the segments AC and CB) must be sturdier and cost \$2 per meter. What is the minimum total cost of the wire that can be used?
- 35.** Determine the dimensions of the rectangle of largest area that can be inscribed in the right triangle shown in the accompanying figure.



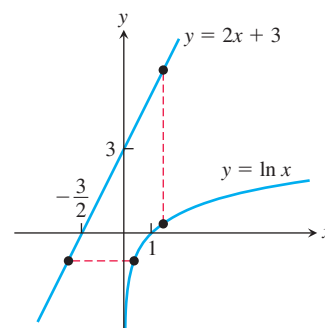
- 36.** Determine the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius 3. (See the accompanying figure.)



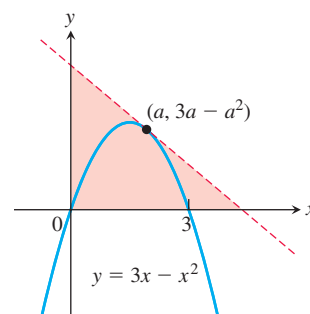
- 37.** What value of a makes $f(x) = x^2 + (a/x)$ have
- a local minimum at $x = 2$?
 - a point of inflection at $x = 1$?
- 38.** What values of a and b make $f(x) = x^3 + ax^2 + bx$ have
- a local maximum at $x = -1$ and a local minimum at $x = 3$?
 - a local minimum at $x = 4$ and a point of inflection at $x = 1$?
- 39.** A right circular cone is circumscribed by a sphere of radius 1. Determine the height h and radius r of the cone of maximum volume.
- 40.** Determine the dimensions of the inscribed rectangle of maximum area.



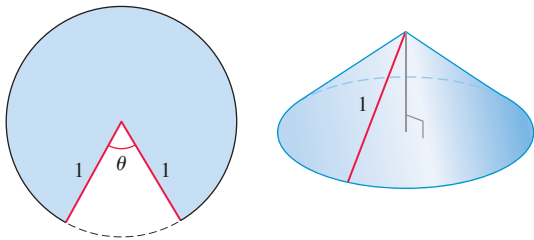
- 41.** Consider the accompanying graphs of $y = 2x + 3$ and $y = \ln x$. Determine the
- minimum vertical distance;
 - minimum horizontal distance between these graphs.



- 42.** Find the point on the graph of $y = 20x^3 + 60x - 3x^5 - 5x^4$ with the largest slope.
- 43.** Among all triangles in the first quadrant formed by the x -axis, the y -axis, and tangent lines to the graph of $y = 3x - x^2$, what is the smallest possible area?



44. A cone is formed from a circular piece of material of radius 1 meter by removing a section of angle θ and then joining the two straight edges. Determine the largest possible volume for the cone.



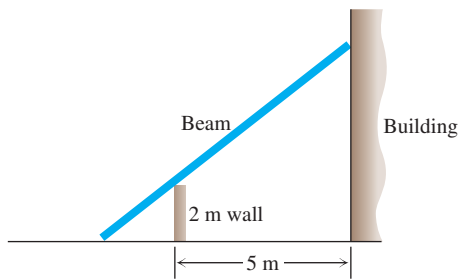
Physical Applications

45. **Vertical motion** The height above ground of an object moving vertically is given by

$$s = -4.9t^2 + 29.4t + 34.3,$$

with s in meters and t in seconds. Find

- the object's velocity when $t = 0$;
 - its maximum height and when it occurs;
 - its velocity when $s = 0$.
46. **Quickest route** Jane is 2 km offshore in a boat and wishes to reach a coastal village 6 km down a straight shoreline from the point nearest the boat. She can row 2 km/h and can walk 5 km/h. Where should she land her boat to reach the village in the least amount of time?
47. **Shortest beam** The 2-m wall shown here stands 5 m from the building. Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.



48. **Motion on a line** The positions of two particles on the s -axis are $s_1 = \sin t$ and $s_2 = \sin(t + \pi/3)$, with s_1 and s_2 in meters and t in seconds.
- At what time(s) in the interval $0 \leq t \leq 2\pi$ do the particles meet?
 - What is the farthest apart that the particles ever get?
 - When in the interval $0 \leq t \leq 2\pi$ is the distance between the particles changing the fastest?
49. The intensity of illumination at any point from a light source is proportional to the square of the reciprocal of the distance between the point and the light source. Two lights, one having an intensity eight times that of the other, are 6 m apart. How far from the stronger light is the total illumination least?
50. **Projectile motion** The range R of a projectile fired from the origin over horizontal ground is the distance from the origin to the

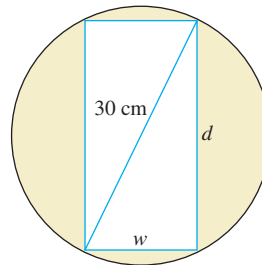
point of impact. If the projectile is fired with an initial velocity v_0 at an angle α with the horizontal, then in Chapter 12 we find that

$$R = \frac{v_0^2}{g} \sin 2\alpha,$$

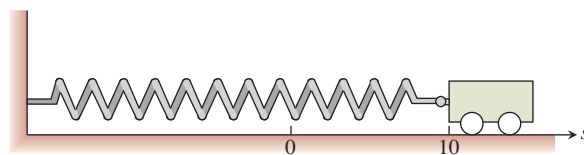
where g is the downward acceleration due to gravity. Find the angle α for which the range R is the largest possible.

- T 51. Strength of a beam** The strength S of a rectangular wooden beam is proportional to its width times the square of its depth. (See the accompanying figure.)

- Find the dimensions of the strongest beam that can be cut from a 30-cm diameter cylindrical log.
- Graph S as a function of the beam's width w , assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).
- On the same screen, graph S as a function of the beam's depth d , again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of k ? Try it.

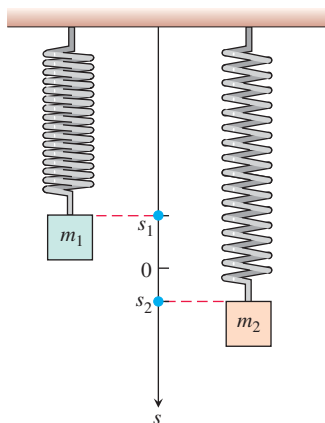


- T 52. Stiffness of a beam** The stiffness S of a rectangular beam is proportional to its width times the cube of its depth.
- Find the dimensions of the stiffest beam that can be cut from a 30-cm-diameter cylindrical log.
 - Graph S as a function of the beam's width w , assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).
 - On the same screen, graph S as a function of the beam's depth d , again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of k ? Try it.
53. **Frictionless cart** A small frictionless cart, attached to the wall by a spring, is pulled 10 cm from its rest position and released at time $t = 0$ to roll back and forth for 4 s. Its position at time t is $s = 10 \cos \pi t$.
- What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?
 - Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?



54. Two masses hanging side by side from springs have positions $s_1 = 2 \sin t$ and $s_2 = \sin 2t$, respectively.

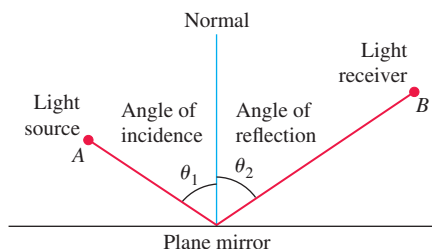
- At what times in the interval $0 < t$ do the masses pass each other? (Hint: $\sin 2t = 2 \sin t \cos t$.)
- When in the interval $0 \leq t \leq 2\pi$ is the vertical distance between the masses the greatest? What is this distance? (Hint: $\cos 2t = 2 \cos^2 t - 1$.)



55. **Distance between two ships** At noon, ship A was 12 nautical miles due north of ship B. Ship A was sailing south at 12 knots (nautical miles per hour; a nautical mile is 1852 m) and continued to do so all day. Ship B was sailing east at 8 knots and continued to do so all day.

- Start counting time with $t = 0$ at noon and express the distance s between the ships as a function of t .
- How rapidly was the distance between the ships changing at noon? One hour later?
- The visibility that day was 5 nautical miles. Did the ships ever sight each other?
- T** Graph s and ds/dt together as functions of t for $-1 \leq t \leq 3$, using different colors if possible. Compare the graphs and reconcile what you see with your answers in parts (b) and (c).
- The graph of ds/dt looks as if it might have a horizontal asymptote in the first quadrant. This in turn suggests that ds/dt approaches a limiting value as $t \rightarrow \infty$. What is this value? What is its relation to the ships' individual speeds?

56. **Fermat's principle in optics** Light from a source A is reflected by a plane mirror to a receiver at point B, as shown in the accompanying figure. Show that for the light to obey Fermat's principle, the angle of incidence must equal the angle of reflection, both measured from the line normal to the reflecting surface. (This result can also be derived without calculus. There is a purely geometric argument, which you may prefer.)



57. **Tin pest** When metallic tin is kept below 13.2°C , it slowly becomes brittle and crumbles to a gray powder. Tin objects eventually crumble to this gray powder spontaneously if kept in a cold climate for years. The Europeans who saw tin organ pipes in their churches crumble away years ago called the change *tin pest* because it seemed to be contagious, and indeed it was, for the gray powder is a catalyst for its own formation.

A *catalyst* for a chemical reaction is a substance that controls the rate of reaction without undergoing any permanent change in itself. An *autocatalytic reaction* is one whose product is a catalyst for its own formation. Such a reaction may proceed slowly at first if the amount of catalyst present is small and slowly again at the end, when most of the original substance is used up. But in between, when both the substance and its catalyst product are abundant, the reaction proceeds at a faster pace.

In some cases, it is reasonable to assume that the rate $v = dx/dt$ of the reaction is proportional both to the amount of the original substance present and to the amount of product. That is, v may be considered to be a function of x alone, and

$$v = kx(a - x) = kax - kx^2,$$

where

x = the amount of product,

a = the amount of substance at the beginning, and

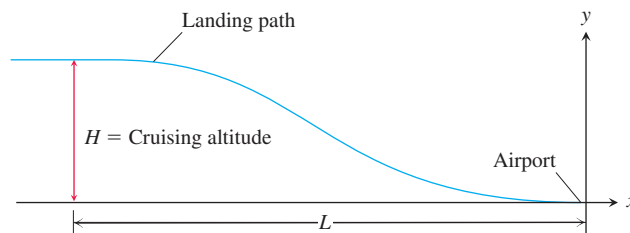
k = a positive constant.

At what value of x does the rate v have a maximum? What is the maximum value of v ?

58. **Airplane landing path** An airplane is flying at altitude H when it begins its descent to an airport runway that is at horizontal ground distance L from the airplane, as shown in the accompanying figure. Assume that the landing path of the airplane is the graph of a cubic polynomial function $y = ax^3 + bx^2 + cx + d$, where $y(-L) = H$ and $y(0) = 0$.

- What is dy/dx at $x = 0$?
- What is dy/dx at $x = -L$?
- Use the values for dy/dx at $x = 0$ and $x = -L$ together with $y(0) = 0$ and $y(-L) = H$ to show that

$$y(x) = H \left[2 \left(\frac{x}{L} \right)^3 + 3 \left(\frac{x}{L} \right)^2 \right].$$



Business and Economics

59. It costs you c dollars each to manufacture and distribute backpacks. If the backpacks sell at x dollars each, the number sold is given by

$$n = \frac{a}{x - c} + b(100 - x),$$

where a and b are positive constants. What selling price will bring a maximum profit?

60. You operate a tour service that offers the following rates:

\$200 per person if 50 people (the minimum number to book the tour) go on the tour.

For each additional person, up to a maximum of 80 people total, the rate per person is reduced by \$2.

It costs \$6000 (a fixed cost) plus \$32 per person to conduct the tour. How many people does it take to maximize your profit?

61. **Wilson lot size formula** One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be), k is the cost of placing an order (the same, no matter how often you order), c is the cost of one item (a constant), m is the number of items sold each week (a constant), and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security).

- a. Your job, as the inventory manager for your store, is to find the quantity that will minimize $A(q)$. What is it? (The formula you get for the answer is called the *Wilson lot size formula*.)
- b. Shipping costs sometimes depend on order size. When they do, it is more realistic to replace k by $k + bq$, the sum of k and a constant multiple of q . What is the most economical quantity to order now?
62. **Production level** Prove that the production level (if any) at which average cost is smallest is a level at which the average cost equals marginal cost.
63. Show that if $r(x) = 6x$ and $c(x) = x^3 - 6x^2 + 15x$ are your revenue and cost functions, then the best you can do is break even (have revenue equal cost).
64. **Production level** Suppose that $c(x) = x^3 - 20x^2 + 20,000x$ is the cost of manufacturing x items. Find a production level that will minimize the average cost of making x items.
65. You are to construct an open rectangular box with a square base and a volume of 6 m^3 . If material for the bottom costs $\$60/\text{m}^2$ and material for the sides costs $\$40/\text{m}^2$, what dimensions will result in the least expensive box? What is the minimum cost?
66. The 800-room Mega Motel chain is filled to capacity when the room charge is $\$50$ per night. For each $\$10$ increase in room charge, 40 fewer rooms are filled each night. What charge per room will result in the maximum revenue per night?

Biology

67. **Sensitivity to medicine** (Continuation of Exercise 74, Section 3.3) Find the amount of medicine to which the body is most sensitive by finding the value of M that maximizes the derivative dR/dM , where

$$R = M^2 \left(\frac{C}{2} - \frac{M}{3} \right)$$

and C is a constant.

68. How we cough

- a. When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the questions of how much it should contract to maximize the velocity and whether it really contracts that much when we cough.

Under reasonable assumptions about the elasticity of the tracheal wall and about how the air near the wall is slowed by friction, the average flow velocity v can be modeled by the equation

$$v = c(r_0 - r)r^2 \text{ cm/s}, \quad \frac{r_0}{2} \leq r \leq r_0,$$

where r_0 is the rest radius of the trachea in centimeters and c is a positive constant whose value depends in part on the length of the trachea.

Show that v is greatest when $r = (2/3)r_0$, that is, when the trachea is about 33% contracted. The remarkable fact is that X-ray photographs confirm that the trachea contracts about this much during a cough.

- T**b. Take r_0 to be 0.5 and c to be 1, and graph v over the interval $0 \leq r \leq 0.5$. Compare what you see with the claim that v is at a maximum when $r = (2/3)r_0$.

Theory and Examples

69. **An inequality for positive integers** Show that if a, b, c , and d are positive integers, then

$$\frac{(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1)}{abcd} \geq 16.$$

70. The derivative dt/dx in Example 4

- a. Show that

$$f(x) = \frac{x}{\sqrt{a^2 + x^2}}$$

is an increasing function of x .

- b. Show that

$$g(x) = \frac{d - x}{\sqrt{b^2 + (d - x)^2}}$$

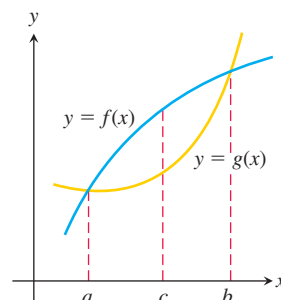
is a decreasing function of x .

- c. Show that

$$\frac{dt}{dx} = \frac{x}{c_1 \sqrt{a^2 + x^2}} - \frac{d - x}{c_2 \sqrt{b^2 + (d - x)^2}}$$

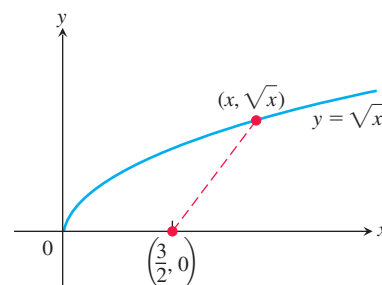
is an increasing function of x .

71. Let $f(x)$ and $g(x)$ be the differentiable functions graphed here. Point c is the point where the vertical distance between the curves is the greatest. Is there anything special about the tangent lines to the two curves at c ? Give reasons for your answer.



72. You have been asked to determine whether the function $f(x) = 3 + 4 \cos x + \cos 2x$ is ever negative.
- Explain why you need to consider values of x only in the interval $[0, 2\pi]$.
 - Is f ever negative? Explain.
73. a. The function $y = \cot x - \sqrt{2} \csc x$ has an absolute maximum value on the interval $0 < x < \pi$. Find it.
- T**b. Graph the function and compare what you see with your answer in part (a).
74. a. The function $y = \tan x + 3 \cot x$ has an absolute minimum value on the interval $0 < x < \pi/2$. Find it.
- T**b. Graph the function and compare what you see with your answer in part (a).
75. a. How close does the curve $y = \sqrt{x}$ come to the point $(3/2, 0)$? (Hint: If you minimize the *square* of the distance, you can avoid square roots.)

- T**b. Graph the distance function $D(x)$ and $y = \sqrt{x}$ together and reconcile what you see with your answer in part (a).



76. a. How close does the semicircle $y = \sqrt{16 - x^2}$ come to the point $(1, \sqrt{3})$?
- T**b. Graph the distance function and $y = \sqrt{16 - x^2}$ together and reconcile what you see with your answer in part (a).

4.7 Newton's Method

For thousands of years, one of the main goals of mathematics has been to find solutions to equations. For linear equations ($ax + b = 0$), and for quadratic equations ($ax^2 + bx + c = 0$), we can explicitly solve for a solution. However, for most equations there is no simple formula that gives the solutions.

In this section we study a numerical method called *Newton's method* or the *Newton–Raphson method*, which is a technique to approximate the solutions to an equation $f(x) = 0$. Newton's method estimates the solutions using tangent lines of the graph of $y = f(x)$ near the points where f is zero. A value of x where f is zero is called a *root* of the function f and a *solution* of the equation $f(x) = 0$. Newton's method is both powerful and efficient, and it has numerous applications in engineering and other fields where solutions to complicated equations are needed.

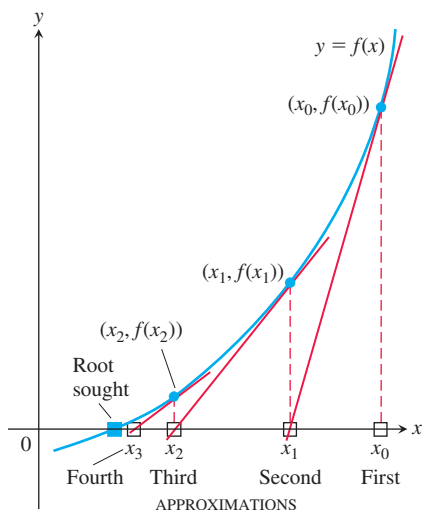


FIGURE 4.48 Newton's method starts with an initial guess x_0 and (under favorable circumstances) improves the guess one step at a time.

Procedure for Newton's Method

The goal of Newton's method for estimating a solution of an equation $f(x) = 0$ is to produce a sequence of approximations that approach the solution. We pick the first number x_0 of the sequence. Then, under favorable circumstances, the method moves step by step toward a point where the graph of f crosses the x -axis (Figure 4.48). At each step the method approximates a zero of f with a zero of one of its linearizations. Here is how it works.

The initial estimate, x_0 , may be found by graphing or just plain guessing. The method then uses the tangent to the curve $y = f(x)$ at $(x_0, f(x_0))$ to approximate the curve, calling the point x_1 where the tangent meets the x -axis (Figure 4.48). The number x_1 is usually a better approximation to the solution than is x_0 . The point x_2 where the tangent to the curve at $(x_1, f(x_1))$ crosses the x -axis is the next approximation in the sequence. We continue, using each approximation to generate the next, until we are close enough to the root to stop.

We can derive a formula for generating the successive approximations in the following way. Given the approximation x_n , the point-slope equation for the tangent line to the curve at $(x_n, f(x_n))$ is

$$y = f(x_n) + f'(x_n)(x - x_n).$$

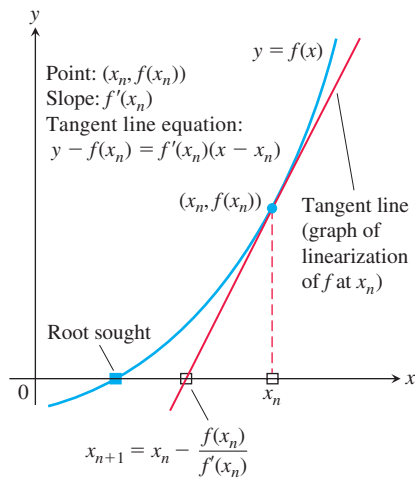


FIGURE 4.49 The geometry of the successive steps of Newton's method. From x_n we go up to the curve and follow the tangent line down to find x_{n+1} .

We can find where it crosses the x -axis by setting $y = 0$ (Figure 4.49):

$$\begin{aligned} 0 &= f(x_n) + f'(x_n)(x - x_n) \\ -\frac{f(x_n)}{f'(x_n)} &= x - x_n \\ x &= x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{If } f'(x_n) \neq 0 \end{aligned}$$

This value of x is the next approximation x_{n+1} . Here is a summary of Newton's method.

Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{if } f'(x_n) \neq 0. \quad (1)$$

Applying Newton's Method

Applications of Newton's method generally involve many numerical computations, making them well suited for computers or calculators. Nevertheless, even when the calculations are done by hand (which may be very tedious), they give a powerful way to find solutions of equations.

In our first example, we find decimal approximations to $\sqrt{2}$ by estimating the positive root of the equation $f(x) = x^2 - 2 = 0$.

EXAMPLE 1 Approximate the positive root of the equation

$$f(x) = x^2 - 2 = 0.$$

Solution With $f(x) = x^2 - 2$ and $f'(x) = 2x$, Equation (1) becomes

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n^2 - 2}{2x_n} \\ &= x_n - \frac{x_n}{2} + \frac{1}{x_n} \\ &= \frac{x_n}{2} + \frac{1}{x_n}. \end{aligned}$$

The equation

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$$

enables us to go from each approximation to the next with just a few keystrokes. With the starting value $x_0 = 1$, we get the results in the first column of the following table. (To five decimal places, or, equivalently, to six digits, $\sqrt{2} = 1.41421$.)

	Error	Number of correct digits
$x_0 = 1$	-0.41421	1
$x_1 = 1.5$	0.08579	1
$x_2 = 1.41667$	0.00246	3
$x_3 = 1.41422$	0.00001	5

Newton's method is used by many software applications to calculate roots because it converges so fast (more about this later). If the arithmetic in the table in Example 1 had been carried to 13 decimal places instead of 5, then going one step further would have given $\sqrt{2}$ correctly to more than 10 decimal places.

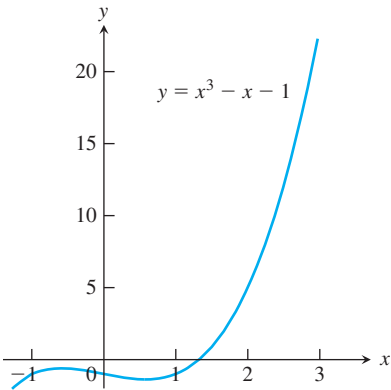


FIGURE 4.50 The graph of $f(x) = x^3 - x - 1$ crosses the x -axis once; this is the root we want to find (Example 2).

EXAMPLE 2 Find the x -coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

Solution The curve crosses the line when $x^3 - x = 1$ or $x^3 - x - 1 = 0$. When does $f(x) = x^3 - x - 1$ equal zero? Since f is continuous on $[1, 2]$ and $f(1) = -1$ while $f(2) = 5$, we know by the Intermediate Value Theorem there is a root in the interval $(1, 2)$ (Figure 4.50).

We apply Newton's method to f with the starting value $x_0 = 1$. The results are displayed in Table 4.1 and Figure 4.51.

At $n = 5$, we come to the result $x_6 = x_5 = 1.3247\,17957$. When $x_{n+1} = x_n$, Equation (1) shows that $f(x_n) = 0$, up to the accuracy of our computation. We have found a solution of $f(x) = 0$ to nine decimal places.

TABLE 4.1 The Result of Applying Newton's Method to $f(x) = x^3 - x - 1$ with $x_0 = 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	1	-1	2	1.5
1	1.5	0.875	5.75	1.3478 26087
2	1.3478 26087	0.1006 82173	4.4499 05482	1.3252 00399
3	1.3252 00399	0.0020 58362	4.2684 68292	1.3247 18174
4	1.3247 18174	0.0000 00924	4.2646 34722	1.3247 17957
5	1.3247 17957	-1.8672E-13	4.2646 32999	1.3247 17957

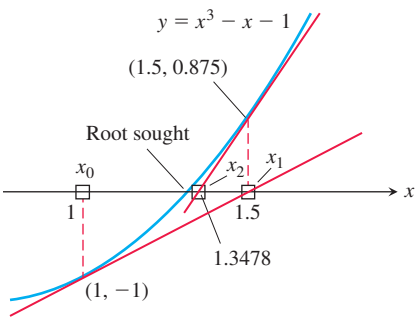


FIGURE 4.51 The first three x -values in Table 4.1 (four decimal places).

In Figure 4.52 we have indicated that the process in Example 2 might have started at the point $B_0(3, 23)$ on the curve, with $x_0 = 3$. Point B_0 is quite far from the x -axis, but the tangent at B_0 crosses the x -axis at about $(2.12, 0)$, so x_1 is still an improvement over x_0 . If we use Equation (1) repeatedly as before, with $f(x) = x^3 - x - 1$ and $f'(x) = 3x^2 - 1$, we obtain the nine-place solution $x_7 = x_6 = 1.3247\,17957$ in seven steps.

Convergence of the Approximations

In Chapter 9 we define precisely the idea of *convergence* for the approximations x_n in Newton's method. Intuitively, we mean that as the number n of approximations increases

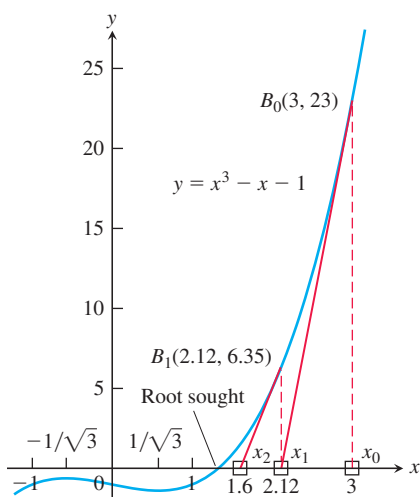


FIGURE 4.52 Any starting value x_0 to the right of $x = 1/\sqrt{3}$ will lead to the root in Example 2.

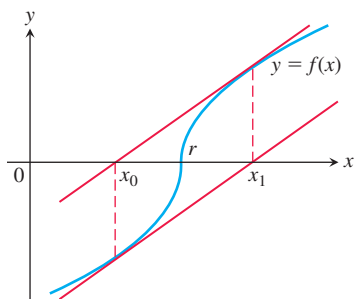


FIGURE 4.53 Newton's method fails to converge. You go from x_0 to x_1 and back to x_0 , never getting any closer to r .

without bound, the values x_n get arbitrarily close to the desired root r . (This notion is similar to the idea of the limit of a function $g(t)$ as t approaches infinity, as defined in Section 2.5.)

In practice, Newton's method usually gives convergence with impressive speed, but this is not guaranteed. One way to test convergence is to begin by graphing the function to estimate a good starting value for x_0 . You can test that you are getting closer to a zero of the function by checking that $|f(x_n)|$ is approaching zero, and you can check that the approximations are converging by evaluating $|x_n - x_{n+1}|$.

Newton's method does not always converge. For instance, if

$$f(x) = \begin{cases} -\sqrt{r-x}, & x < r \\ \sqrt{x-r}, & x \geq r, \end{cases}$$

the graph will be like the one in Figure 4.53. If we begin with $x_0 = r - h$, we get $x_1 = r + h$, and successive approximations go back and forth between these two values. No amount of iteration brings us closer to the root than our first guess.

If Newton's method does converge, it converges to a root. Be careful, however. There are situations in which the method appears to converge but no root is there. Fortunately, such situations are rare.

When Newton's method converges to a root, it may not be the root you have in mind. Figure 4.54 shows two ways this can happen.

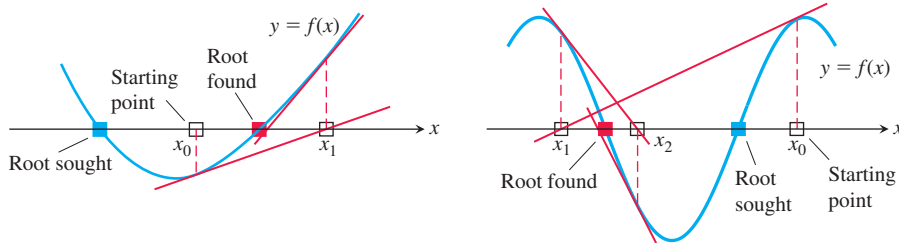


FIGURE 4.54 If you start too far away, Newton's method may miss the root you want.

EXERCISES 4.7

Root Finding

- Use Newton's method to estimate the solutions of the equation $x^2 + x - 1 = 0$. Start with $x_0 = -1$ for the left-hand solution and with $x_0 = 1$ for the solution on the right. Then, in each case, find x_2 .
- Use Newton's method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find x_2 .
- Use Newton's method to estimate the two zeros of the function $f(x) = x^4 + x - 3$. Start with $x_0 = -1$ for the left-hand zero and with $x_0 = 1$ for the zero on the right. Then, in each case, find x_2 .
- Use Newton's method to estimate the two zeros of the function $f(x) = 2x - x^2 + 1$. Start with $x_0 = 0$ for the left-hand zero and with $x_0 = 2$ for the zero on the right. Then, in each case, find x_2 .

- Use Newton's method to find the positive fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = 1$ and find x_2 .
- Use Newton's method to find the negative fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = -1$ and find x_2 .

- Use Newton's method to find an approximate solution of $3 - x = \ln x$. Start with $x_0 = 2$ and find x_2 .
- Use Newton's method to find an approximate solution of $x - 1 = \arctan x$. Start with $x_0 = 1$ and find x_2 .
- Use Newton's method to find an approximate solution of $xe^x = 1$. Start with $x_0 = 0$ and find x_2 .

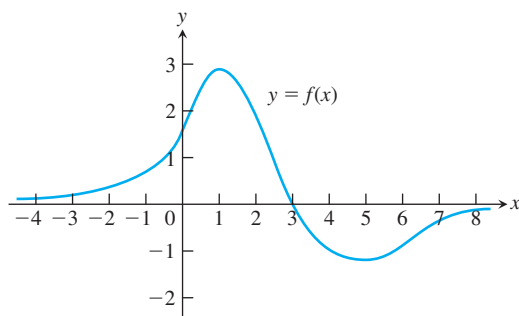
Dependence on Initial Point

- Using the function shown in the figure, and, for each initial estimate x_0 , determine graphically what happens to the sequence of Newton's method approximations

- $x_0 = 0$
- $x_0 = 1$

- c. $x_0 = 2$
e. $x_0 = 5.5$

d. $x_0 = 4$



- 11. Guessing a root** Suppose that your first guess is lucky, in the sense that x_0 is a root of $f(x) = 0$. Assuming that $f'(x_0)$ is defined and is not 0, what happens to x_1 and later approximations?
- 12. Estimating pi** You plan to estimate $\pi/2$ to five decimal places by using Newton's method to solve the equation $\cos x = 0$. Does it matter what your starting value is? Give reasons for your answer.

Theory and Examples

- 13. Oscillation** Show that if $h > 0$, applying Newton's method to

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$

leads to $x_1 = -h$ if $x_0 = h$ and to $x_1 = h$ if $x_0 = -h$. Draw a picture that shows what is going on.

- 14. Approximations that get worse and worse** Apply Newton's method to $f(x) = x^{1/3}$ with $x_0 = 1$ and calculate x_1, x_2, x_3 , and x_4 . Find a formula for $|x_n|$. What happens to $|x_n|$ as $n \rightarrow \infty$? Draw a picture that shows what is going on.
- 15.** Explain why the following four statements ask for the same information:
- Find the roots of $f(x) = x^3 - 3x - 1$.
 - Find the x -coordinates of the intersections of the curve $y = x^3$ with the line $y = 3x + 1$.
 - Find the x -coordinates of the points where the curve $y = x^3 - 3x$ crosses the horizontal line $y = 1$.
 - Find the values of x where the derivative of $g(x) = (1/4)x^4 - (3/2)x^2 - x + 5$ equals zero.

T When solving Exercises 16–34, you may need to use appropriate technology (such as a calculator or a computer).

- 16. Locating a planet** To calculate a planet's space coordinates, we have to solve equations like $x = 1 + 0.5 \sin x$. Graphing the function $f(x) = x - 1 - 0.5 \sin x$ suggests that the function has a root near $x = 1.5$. Use one application of Newton's method to improve this estimate. That is, start with $x_0 = 1.5$ and find x_1 . (The value of the root is 1.49870 to five decimal places.) Remember to use radians.
- 17. Intersecting curves** The curve $y = \tan x$ crosses the line $y = 2x$ between $x = 0$ and $x = \pi/2$. Use Newton's method to find where.
- 18. Real solutions of a quartic** Use Newton's method to find the two real solutions of the equation $x^4 - 2x^3 - x^2 - 2x + 2 = 0$.

- 19. a.** How many solutions does the equation $\sin 3x = 0.99 - x^2$ have?

b. Use Newton's method to find them.

20. Intersection of curves

- a.** Does $\cos 3x$ ever equal x ? Give reasons for your answer.
- b.** Use Newton's method to find where.

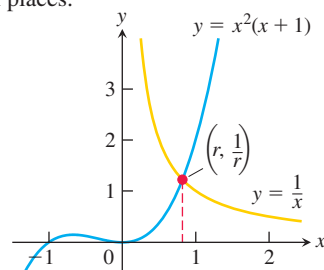
- 21.** Find the four real zeros of the function $f(x) = 2x^4 - 4x^2 + 1$.

- 22. Estimating pi** Estimate π to as many decimal places as your calculator will display by using Newton's method to solve the equation $\tan x = 0$ with $x_0 = 3$.

- 23. Intersection of curves** At what value(s) of x does $\cos x = 2x$?

- 24. Intersection of curves** At what value(s) of x does $\cos x = -x$?

- 25.** The graphs of $y = x^2(x + 1)$ and $y = 1/x$ ($x > 0$) intersect at one point $x = r$. Use Newton's method to estimate the value of r to four decimal places.



- 26.** The graphs of $y = \sqrt{x}$ and $y = 3 - x^2$ intersect at one point $x = r$. Use Newton's method to estimate the value of r to four decimal places.

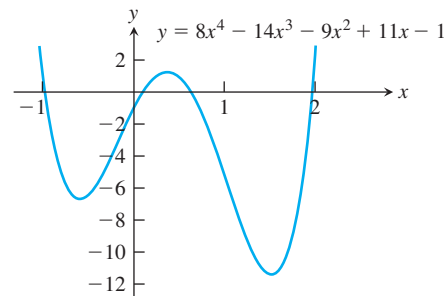
- 27. Intersection of curves** At what value(s) of x does $e^{-x^2} = x^2 - x + 1$?

- 28. Intersection of curves** At what value(s) of x does $\ln(1 - x^2) = x - 1$?

- 29.** Use the Intermediate Value Theorem from Section 2.6 to show that $f(x) = x^3 + 2x - 4$ has a root between $x = 1$ and $x = 2$. Then find the root to five decimal places.

- 30. Factoring a quartic** Find the approximate values of r_1 through r_4 in the factorization

$$8x^4 - 14x^3 - 9x^2 + 11x - 1 = 8(x - r_1)(x - r_2)(x - r_3)(x - r_4).$$

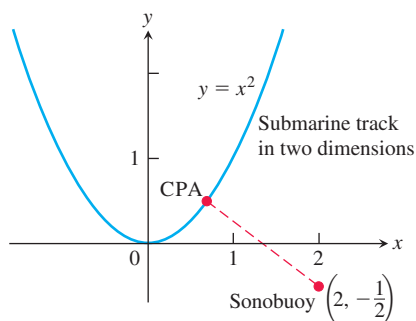


- 31. Converging to different zeros** Use Newton's method to find the zeros of $f(x) = 4x^4 - 4x^2$ using the given starting values.

- $x_0 = -2$ and $x_0 = -0.8$, lying in $(-\infty, -\sqrt{2}/2)$
- $x_0 = -0.5$ and $x_0 = 0.25$, lying in $(-\sqrt{21}/7, \sqrt{21}/7)$
- $x_0 = 0.8$ and $x_0 = 2$, lying in $(\sqrt{2}/2, \infty)$
- $x_0 = -\sqrt{21}/7$ and $x_0 = \sqrt{21}/7$

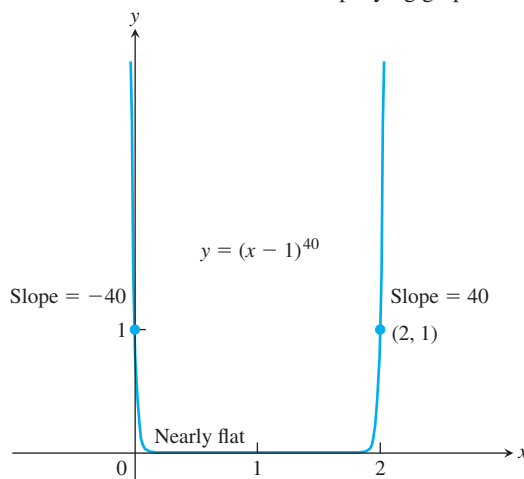
32. The sonobuoy problem In submarine location problems, it is often necessary to find a submarine's closest point of approach (CPA) to a sonobuoy (sound detector) in the water. Suppose that the submarine travels on the parabolic path $y = x^2$ and that the buoy is located at the point $(2, -1/2)$.

- Show that the value of x that minimizes the distance between the submarine and the buoy is a solution of the equation $x = 1/(x^2 + 1)$.
- Solve the equation $x = 1/(x^2 + 1)$ with Newton's method.

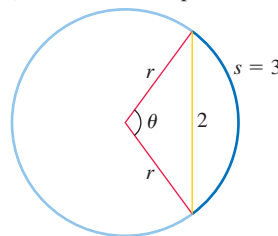


33. Curves that are nearly flat at the root Some curves are so flat that, in practice, Newton's method stops too far from the root to give a useful estimate. Try Newton's method on $f(x) = (x - 1)^{40}$

with a starting value of $x_0 = 2$ to see how close your machine comes to the root $x = 1$. See the accompanying graph.



- 34.** The accompanying figure shows a circle of radius r with a chord of length 2 and an arc s of length 3. Use Newton's method to solve for r and θ (radians) to four decimal places. Assume $0 < \theta < \pi$.



4.8 Antiderivatives

Many problems require that we recover a function from its derivative, or from its rate of change. For instance, the laws of physics tell us the acceleration of an object falling from an initial height, and we can use this to compute its velocity and its height at any time. More generally, starting with a function f , we want to find a function F whose derivative is f . If such a function F exists, it is called an *antiderivative* of f . Antiderivatives are the link connecting the two major elements of calculus: derivatives and definite integrals. Antiderivatives have an important connection to the theory of integrals that is developed in Chapter 5. For this reason the process of taking an antiderivative is also called “integration.”

Finding Antiderivatives

DEFINITION A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I .

The process of recovering a function $F(x)$ from its derivative $f(x)$ is called *antidifferentiation*. We use capital letters such as F to represent an antiderivative of a function f , G to represent an antiderivative of g , and so forth.

EXAMPLE 1 Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$

(b) $g(x) = \cos x$

(c) $h(x) = \frac{1}{x} + 2e^{2x}$

Solution We need to think backward here: What function do we know has a derivative equal to the given function?

(a) $F(x) = x^2$

(b) $G(x) = \sin x$

(c) $H(x) = \ln|x| + e^{2x}$

Each answer can be checked by differentiating. The derivative of $F(x) = x^2$ is $2x$. The derivative of $G(x) = \sin x$ is $\cos x$, and the derivative of $H(x) = \ln|x| + e^{2x}$ is $(1/x) + 2e^{2x}$. ■

The function $F(x) = x^2$ is not the only function whose derivative is $2x$. The function $x^2 + 1$ has the same derivative. So does $x^2 + C$ for any constant C . Are there others?

Corollary 2 of the Mean Value Theorem in Section 4.2 gives the answer: Any two antiderivatives of a function differ by a constant. So the functions $x^2 + C$, where C is an **arbitrary constant**, form *all* the antiderivatives of $f(x) = 2x$. More generally, we have the following result.

THEOREM 8 If F is an antiderivative of f on an interval I , then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

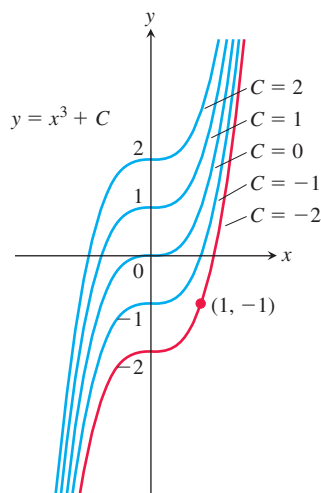


FIGURE 4.55 The curves $y = x^3 + C$ fill the coordinate plane without overlapping. In Example 2, we identify the curve $y = x^3 - 2$ as the one that passes through the given point $(1, -1)$.

Thus the most general antiderivative of f on I is a *family* of functions $F(x) + C$ whose graphs are vertical translations of one another. We can select a particular antiderivative from this family by assigning a specific value to C . Here is an example showing how such an assignment might be made.

EXAMPLE 2 Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

Solution Since the derivative of x^3 is $3x^2$, the general antiderivative

$$F(x) = x^3 + C$$

gives all the antiderivatives of $f(x)$. The condition $F(1) = -1$ determines a specific value for C . Substituting $x = 1$ into $f(x) = x^3 + C$ gives

$$F(1) = (1)^3 + C = 1 + C.$$

Since $F(1) = -1$, solving $1 + C = -1$ for C gives $C = -2$. So

$$F(x) = x^3 - 2$$

is the antiderivative satisfying $F(1) = -1$. Notice that this assignment for C selects the particular curve from the family of curves $y = x^3 + C$ that passes through the point $(1, -1)$ in the plane (Figure 4.55). ■

By working backward from assorted differentiation rules, we can derive formulas and rules for antiderivatives. In each case there is an arbitrary constant C in the general expression representing all antiderivatives of a given function. Table 4.2 gives antiderivative formulas for a number of important functions.

The rules in Table 4.2 are easily verified by differentiating the general antiderivative formula to obtain the function to its left. For example, the derivative of $(\tan kx)/k + C$ is $\sec^2 kx$, whatever the value of the constants C or $k \neq 0$, and this shows that Formula 4 gives the general antiderivative of $\sec^2 kx$.

TABLE 4.2 Antiderivative formulas, k a nonzero constant

Function	General antiderivative	Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$	8. e^{kx}	$\frac{1}{k}e^{kx} + C$
2. $\sin kx$	$-\frac{1}{k}\cos kx + C$	9. $\frac{1}{x}$	$\ln x + C, \quad x \neq 0$
3. $\cos kx$	$\frac{1}{k}\sin kx + C$	10. $\frac{1}{\sqrt{1-k^2x^2}}$	$\frac{1}{k}\arcsin kx + C$
4. $\sec^2 kx$	$\frac{1}{k}\tan kx + C$	11. $\frac{1}{1+k^2x^2}$	$\frac{1}{k}\arctan kx + C$
5. $\csc^2 kx$	$-\frac{1}{k}\cot kx + C$	12. $\frac{1}{x\sqrt{k^2x^2-1}}$	$\operatorname{arcsec} kx + C, \quad kx > 1$
6. $\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$	13. a^{kx}	$\left(\frac{1}{k \ln a}\right)a^{kx} + C, \quad a > 0, a \neq 1$
7. $\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$		

EXAMPLE 3 Find the general antiderivative of each of the following functions.

- (a) $f(x) = x^5$

(d) $i(x) = \cos \frac{x}{2}$
- (b) $g(x) = \frac{1}{\sqrt{x}}$

(e) $j(x) = e^{-3x}$
- (c) $h(x) = \sin 2x$

(f) $k(x) = 2^x$

Solution In each case, we can use one of the formulas listed in Table 4.2.

- (a) $F(x) = \frac{x^6}{6} + C$ Formula 1 with $n = 5$

(b) $g(x) = x^{-1/2}$, so
 $G(x) = \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} + C$ Formula 1 with $n = -1/2$

(c) $H(x) = \frac{-\cos 2x}{2} + C$ Formula 2 with $k = 2$

(d) $I(x) = \frac{\sin(x/2)}{1/2} + C = 2\sin \frac{x}{2} + C$ Formula 3 with $k = 1/2$

(e) $J(x) = -\frac{1}{3}e^{-3x} + C$ Formula 8 with $k = -3$

(f) $K(x) = \left(\frac{1}{\ln 2}\right)2^x + C$ Formula 13 with $a = 2, k = 1$ ■

Other derivative rules also lead to corresponding antiderivative rules. We can add and subtract antiderivatives and multiply them by constants.

TABLE 4.3 Antiderivative linearity rules

	Function	General antiderivative
1. Constant Multiple Rule:	$k f(x)$	$kF(x) + C, k$ a constant
2. Sum or Difference Rule:	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

The formulas in Table 4.3 are easily proved by differentiating the antiderivatives and verifying that the result agrees with the original function.

EXAMPLE 4 Find the general antiderivative of

$$f(x) = \frac{3}{\sqrt{x}} + \sin 2x.$$

Solution We have that $f(x) = 3g(x) + h(x)$ for the functions g and h in Example 3. Since $G(x) = 2\sqrt{x}$ is an antiderivative of $g(x)$ from Example 3b, it follows from the Constant Multiple Rule for antiderivatives that $3G(x) = 3 \cdot 2\sqrt{x} = 6\sqrt{x}$ is an antiderivative of $3g(x) = 3/\sqrt{x}$. Likewise, from Example 3c we know that $H(x) = (-1/2)\cos 2x$ is an antiderivative of $h(x) = \sin 2x$. From the Sum Rule for antiderivatives, we then get that

$$\begin{aligned} F(x) &= 3G(x) + H(x) + C \\ &= 6\sqrt{x} - \frac{1}{2}\cos 2x + C \end{aligned}$$

is the general antiderivative formula for $f(x)$, where C is an arbitrary constant. ■

Initial Value Problems and Differential Equations

Antiderivatives play several important roles in mathematics and its applications. Methods and techniques for finding them are a major part of calculus, and we take up that study in Chapter 8. Finding an antiderivative for a function $f(x)$ is the same problem as finding a function $y(x)$ that satisfies the equation

$$\frac{dy}{dx} = f(x).$$

This is called a **differential equation**, since it is an equation involving an unknown function y that is being differentiated. To solve it, we need a function $y(x)$ that satisfies the equation. This function is found by taking the antiderivative of $f(x)$. We can fix the arbitrary constant arising in the antidifferentiation process by specifying an initial condition

$$y(x_0) = y_0.$$

This condition means the function $y(x)$ has the value y_0 when $x = x_0$. The combination of a differential equation and an initial condition is called an **initial value problem**. Such problems play important roles in all branches of science.

The most general antiderivative $F(x) + C$ of the function $f(x)$ (such as $x^3 + C$ for the function $3x^2$ in Example 2) gives the **general solution** $y = F(x) + C$ of the differential equation $dy/dx = f(x)$. The general solution gives *all* the solutions of the equation (there are infinitely many, one for each value of C). We **solve** the differential equation by finding its general solution. We then solve the initial value problem by finding the **particular solution** that satisfies the initial condition $y(x_0) = y_0$. In Example 2, the function $y = x^3 - 2$ is the particular solution of the differential equation $dy/dx = 3x^2$ satisfying the initial condition $y(1) = -1$.

Antiderivatives and Motion

We have seen that the derivative of the position function of an object gives its velocity, and the derivative of its velocity function gives its acceleration. If we know an object's acceleration, then by finding an antiderivative we can recover the velocity, and from an antiderivative of the velocity we can recover its position function. This procedure was used as an application of Corollary 2 in Section 4.2. Now that we have a terminology and conceptual framework in terms of antiderivatives, we revisit the problem from the point of view of differential equations.

EXAMPLE 5 A hot-air balloon ascending at the rate of 3.6 m/s is at a height 24.5 m above the ground when a package is dropped. How long does it take the package to reach the ground?

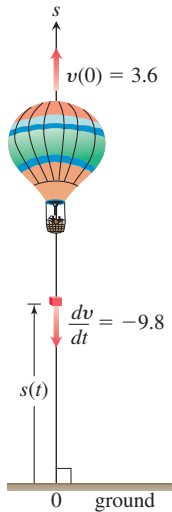


FIGURE 4.56 A package dropped from a rising hot-air balloon (Example 5).

Solution Let $v(t)$ denote the velocity of the package at time t , and let $s(t)$ denote its height above the ground. The acceleration of gravity near the surface of the earth is 9.8 m/s^2 . Assuming no other forces act on the dropped package, we have

$$\frac{dv}{dt} = -9.8. \quad \text{Negative because gravity acts in the direction of decreasing } s$$

This leads to the following initial value problem (Figure 4.56):

$$\text{Differential equation: } \frac{dv}{dt} = -9.8$$

$$\text{Initial condition: } v(0) = 3.6. \quad \text{Balloon initially rising}$$

This is our mathematical model for the package's motion. We solve the initial value problem to obtain the velocity of the package.

1. *Solve the differential equation:* The general formula for an antiderivative of -9.8 is

$$v = -9.8t + C.$$

Having found the general solution of the differential equation, we use the initial condition to find the particular solution that solves our problem.

2. *Evaluate C :*

$$3.6 = -9.8(0) + C \quad \text{Initial condition } v(0) = 3.6$$

$$C = 3.6.$$

The solution of the initial value problem is

$$v = -9.8t + 3.6.$$

Since velocity is the derivative of height, and the height of the package is 24.5 m at time $t = 0$ when it is dropped, we now have a second initial value problem:

$$\text{Differential equation: } \frac{ds}{dt} = -9.8t + 3.6 \quad \text{Set } v = ds/dt \text{ in the previous equation.}$$

$$\text{Initial condition: } s(0) = 24.5.$$

We solve this initial value problem to find the height as a function of t .

1. *Solve the differential equation:* Finding the general antiderivative of $-9.8t + 3.6$ gives

$$s = -4.9t^2 + 3.6t + C.$$

2. *Evaluate C :*

$$24.5 = -4.9(0)^2 + 3.6(0) + C \quad \text{Initial condition } s(0) = 24.5$$

$$C = 24.5.$$

The package's height above ground at time t is

$$s = -4.9t^2 + 3.6t + 24.5.$$

Use the solution: To find how long it takes the package to reach the ground, we set s equal to 0 and solve for t :

$$-4.9t^2 + 3.6t + 24.5 = 0$$

$$t = \frac{-3.6 \pm \sqrt{493.16}}{-9.8} \quad \text{Quadratic formula}$$

$$t \approx -1.90, \quad t \approx 2.63.$$

The package hits the ground about 2.63 s after it is dropped from the balloon. (The negative root has no physical meaning.)

Indefinite Integrals

A special symbol is used to denote the collection of all antiderivatives of a function f .

$$\int_a^b f(x) dx$$

DEFINITION The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x ; it is denoted by

$$\int f(x) dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

After the integral sign in the notation we just defined, the integrand function is always followed by a differential to indicate the variable of integration. We will have more to say about why this is important in Chapter 5. Using this notation, we restate the solutions of Example 1, as follows:

$$\begin{aligned}\int 2x \, dx &= x^2 + C, \\ \int \cos x \, dx &= \sin x + C, \\ \int \left(\frac{1}{x} + 2e^{2x} \right) dx &= \ln|x| + e^{2x} + C.\end{aligned}$$

This notation is related to the main application of antiderivatives, which will be explored in Chapter 5. Antiderivatives play a key role in computing limits of certain infinite sums, an unexpected and wonderfully useful role that is described in a central result of Chapter 5, the Fundamental Theorem of Calculus.

EXAMPLE 6 Evaluate

$$\int (x^2 - 2x + 5) \, dx.$$

Solution If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$\int (x^2 - 2x + 5) \, dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}.$$

If we do not recognize the antiderivative right away, we can generate it term-by-term with the Sum, Difference, and Constant Multiple Rules:

$$\begin{aligned}\int (x^2 - 2x + 5) \, dx &= \int x^2 \, dx - \int 2x \, dx + \int 5 \, dx \\ &= \int x^2 \, dx - 2 \int x \, dx + 5 \int 1 \, dx \\ &= \left(\frac{x^3}{3} + C_1 \right) - 2 \left(\frac{x^2}{2} + C_2 \right) + 5(x + C_3) \\ &= \frac{x^3}{3} + C_1 - x^2 - 2C_2 + 5x + 5C_3.\end{aligned}$$

This formula is more complicated than it needs to be. If we combine C_1 , $-2C_2$, and $5C_3$ into a single arbitrary constant $C = C_1 - 2C_2 + 5C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

and *still* gives all the possible antiderivatives there are. For this reason, we recommend that you go right to the final form even if you elect to integrate term-by-term. Write

$$\begin{aligned}\int (x^2 - 2x + 5) dx &= \int x^2 dx - \int 2x dx + \int 5 dx \\ &= \frac{x^3}{3} - x^2 + 5x + C.\end{aligned}$$

Find the simplest antiderivative you can for each part, and add the arbitrary constant of integration at the end. ■

We conclude this section with a list of basic antidifferentiation formulas in Table 4.4, using the integral sign to indicate an antiderivative.

TABLE 4.4 Integration formulas

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	8. $\int e^x dx = e^x + C$
2. $\int \sin x dx = -\cos x + C$	9. $\int \frac{dx}{x} = \ln x + C \quad (x \neq 0)$
3. $\int \cos x dx = \sin x + C$	10. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$
4. $\int \sec^2 x dx = \tan x + C$	11. $\int \frac{dx}{1+x^2} = \arctan x + C$
5. $\int \csc^2 x dx = -\cot x + C$	12. $\int \frac{dx}{x\sqrt{x^2-1}} = \operatorname{arcsec} x + C \quad (x > 1)$
6. $\int \sec x \tan x dx = \sec x + C$	13. $\int a^x dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$
7. $\int \csc x \cot x dx = -\csc x + C$	

$$\int x^2 dx = \frac{x^3}{3} + C$$

EXERCISES 4.8

Finding Antiderivatives

In Exercises 1–24, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

- a. $2x$
- a. $6x$
- a. $-3x^{-4}$
- a. $2x^{-3}$
- a. $\frac{1}{x^2}$
- a. $-\frac{2}{x^3}$

- b. x^2
- b. x^7
- b. x^{-4}
- b. $\frac{x^{-3}}{2} + x^2$
- b. $\frac{5}{x^2}$
- b. $\frac{1}{2x^3}$

- c. $x^2 - 2x + 1$
- c. $x^7 - 6x + 8$
- c. $x^{-4} + 2x + 3$
- c. $-x^{-3} + x - 1$
- c. $2 - \frac{5}{x^2}$
- c. $x^3 - \frac{1}{x^3}$

- a. $\frac{3}{2}\sqrt{x}$
- a. $\frac{4}{3}\sqrt[3]{x}$
- a. $\frac{2}{3}x^{-1/3}$
- a. $\frac{1}{2}x^{-1/2}$
- a. $\frac{1}{x}$
- a. $\frac{1}{3x}$

- b. $\frac{1}{2\sqrt{x}}$
- b. $\frac{1}{3\sqrt[3]{x}}$
- b. $\frac{1}{3}x^{-2/3}$
- b. $-\frac{1}{2}x^{-3/2}$
- b. $\frac{7}{x}$
- b. $\frac{2}{5x}$

- c. $\sqrt{x} + \frac{1}{\sqrt{x}}$
- c. $\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$
- c. $-\frac{1}{3}x^{-4/3}$
- c. $-\frac{3}{2}x^{-5/2}$
- c. $1 - \frac{5}{x}$
- c. $1 + \frac{4}{3x} - \frac{1}{x^2}$

13. a. $-\pi \sin \pi x$ b. $3 \sin x$ c. $\sin \pi x - 3 \sin 3x$
 14. a. $\pi \cos \pi x$ b. $\frac{\pi}{2} \cos \frac{\pi x}{2}$ c. $\cos \frac{\pi x}{2} + \pi \cos x$
 15. a. $\sec^2 x$ b. $\frac{2}{3} \sec^2 \frac{x}{3}$ c. $-\sec^2 \frac{3x}{2}$
 16. a. $\csc^2 x$ b. $-\frac{3}{2} \csc^2 \frac{3x}{2}$ c. $1 - 8 \csc^2 2x$
 17. a. $\csc x \cot x$ b. $-\csc 5x \cot 5x$ c. $-\pi \csc \frac{\pi x}{2} \cot \frac{\pi x}{2}$
 18. a. $\sec x \tan x$ b. $4 \sec 3x \tan 3x$ c. $\sec \frac{\pi x}{2} \tan \frac{\pi x}{2}$
 19. a. e^{3x} b. e^{-x} c. $e^{x/2}$
 20. a. e^{-2x} b. $e^{4x/3}$ c. $e^{-x/5}$
 21. a. 3^x b. 2^{-x} c. $\left(\frac{5}{3}\right)^x$
 22. a. $x^{\sqrt{3}}$ b. x^π c. $x^{\sqrt{2}-1}$
 23. a. $\frac{2}{\sqrt{1-x^2}}$ b. $\frac{1}{2(x^2+1)}$ c. $\frac{1}{1+4x^2}$
 24. a. $x - \left(\frac{1}{2}\right)^x$ b. $x^2 + 2^x$ c. $\pi^x - x^{-1}$

Finding Indefinite Integrals

In Exercises 25–70, find the most general antiderivative or indefinite integral. You may need to try a solution and then adjust your guess. Check your answers by differentiation.

25. $\int (x+1) dx$ 26. $\int (5-6x) dx$
 27. $\int \left(3t^2 + \frac{t}{2}\right) dt$ 28. $\int \left(\frac{t^2}{2} + 4t^3\right) dt$
 29. $\int (2x^3 - 5x + 7) dx$ 30. $\int (1 - x^2 - 3x^5) dx$
 31. $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx$ 32. $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$
 33. $\int x^{-1/3} dx$ 34. $\int x^{-5/4} dx$
 35. $\int (\sqrt{x} + \sqrt[3]{x}) dx$ 36. $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$
 37. $\int \left(8y - \frac{2}{y^{1/4}}\right) dy$ 38. $\int \left(\frac{1}{7} - \frac{1}{y^{5/4}}\right) dy$
 39. $\int 2x(1 - x^{-3}) dx$ 40. $\int x^{-3}(x+1) dx$
 41. $\int \frac{t\sqrt{t} + \sqrt{t}}{t^2} dt$ 42. $\int \frac{4 + \sqrt{t}}{t^3} dt$
 43. $\int (-2 \cos t) dt$ 44. $\int (-5 \sin t) dt$
 45. $\int 7 \sin \frac{\theta}{3} d\theta$ 46. $\int 3 \cos 5\theta d\theta$
 47. $\int (-3 \csc^2 x) dx$ 48. $\int \left(-\frac{\sec^2 x}{3}\right) dx$
 49. $\int \frac{\csc \theta \cot \theta}{2} d\theta$ 50. $\int \frac{2}{5} \sec \theta \tan \theta d\theta$
 51. $\int (e^{3x} + 5e^{-x}) dx$ 52. $\int (2e^x - 3e^{-2x}) dx$
 53. $\int (e^{-x} + 4^x) dx$ 54. $\int (1.3)^x dx$
 55. $\int (4 \sec x \tan x - 2 \sec^2 x) dx$ 56. $\int \frac{1}{2} (\csc^2 x - \csc x \cot x) dx$
 57. $\int (\sin 2x - \csc^2 x) dx$ 58. $\int (2 \cos 2x - 3 \sin 3x) dx$
 59. $\int \frac{1 + \cos 4t}{2} dt$ 60. $\int \frac{1 - \cos 6t}{2} dt$
 61. $\int \left(\frac{1}{x} - \frac{5}{x^2 + 1}\right) dx$ 62. $\int \left(\frac{2}{\sqrt{1-y^2}} - \frac{1}{y^{1/4}}\right) dy$
 63. $\int 3x^{\sqrt{3}} dx$ 64. $\int x^{\sqrt{2}-1} dx$
 65. $\int (1 + \tan^2 \theta) d\theta$ 66. $\int (2 + \tan^2 \theta) d\theta$
 (Hint: $1 + \tan^2 \theta = \sec^2 \theta$)
 67. $\int \cot^2 x dx$ 68. $\int (1 - \cot^2 x) dx$
 (Hint: $1 + \cot^2 x = \csc^2 x$)
 69. $\int \cos \theta (\tan \theta + \sec \theta) d\theta$ 70. $\int \frac{\csc \theta}{\csc \theta - \sin \theta} d\theta$

Checking Antiderivative Formulas

Verify the formulas in Exercises 71–82 by differentiation.

71. $\int (7x-2)^3 dx = \frac{(7x-2)^4}{28} + C$
 72. $\int (3x+5)^{-2} dx = -\frac{(3x+5)^{-1}}{3} + C$
 73. $\int \sec^2(5x-1) dx = \frac{1}{5} \tan(5x-1) + C$
 74. $\int \csc^2\left(\frac{x-1}{3}\right) dx = -3 \cot\left(\frac{x-1}{3}\right) + C$
 75. $\int \frac{1}{(x+1)^2} dx = -\frac{1}{x+1} + C$
 76. $\int \frac{1}{(x+1)^2} dx = \frac{x}{x+1} + C$
 77. $\int \frac{1}{x+1} dx = \ln|x+1| + C, \quad x \neq -1$
 78. $\int xe^x dx = xe^x - e^x + C$
 79. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$
 80. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
 81. $\int \frac{\arctan x}{x^2} dx = \ln x - \frac{1}{2} \ln(1+x^2) - \frac{\arctan x}{x} + C$
 82. $\int (\arcsin x)^2 dx = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2} \arcsin x + C$
 83. Right, or wrong? Say which for each formula and give a brief reason for each answer.
 a. $\int x \sin x dx = \frac{x^2}{2} \sin x + C$
 b. $\int x \sin x dx = -x \cos x + C$
 c. $\int x \sin x dx = -x \cos x + \sin x + C$
 84. Right, or wrong? Say which for each formula and give a brief reason for each answer.
 a. $\int \tan \theta \sec^2 \theta d\theta = \frac{\sec^3 \theta}{3} + C$
 b. $\int \tan \theta \sec^2 \theta d\theta = \frac{1}{2} \tan^2 \theta + C$
 c. $\int \tan \theta \sec^2 \theta d\theta = \frac{1}{2} \sec^2 \theta + C$

85. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a. $\int (2x + 1)^2 dx = \frac{(2x + 1)^3}{3} + C$

b. $\int 3(2x + 1)^2 dx = (2x + 1)^3 + C$

c. $\int 6(2x + 1)^2 dx = (2x + 1)^3 + C$

86. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a. $\int \sqrt{2x + 1} dx = \sqrt{x^2 + x} + C$

b. $\int \sqrt{2x + 1} dx = \sqrt{x^2 + x} + C$

c. $\int \sqrt{2x + 1} dx = \frac{1}{3}(\sqrt{2x + 1})^3 + C$

87. Right, or wrong? Give a brief reason why.

$$\int \frac{-15(x + 3)^2}{(x - 2)^4} dx = \left(\frac{x + 3}{x - 2}\right)^3 + C$$

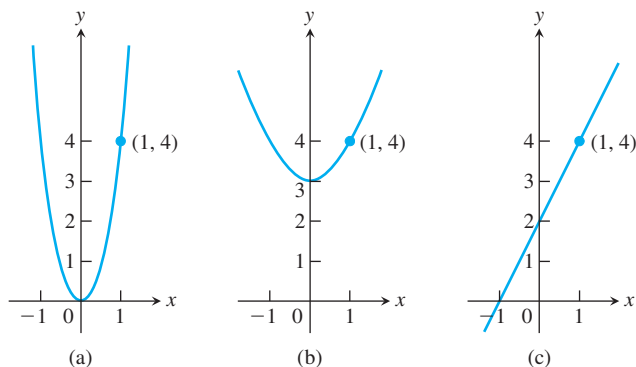
88. Right, or wrong? Give a brief reason why.

$$\int \frac{x \cos(x^2) - \sin(x^2)}{x^2} dx = \frac{\sin(x^2)}{x} + C$$

Initial Value Problems

89. Which of the following graphs shows the solution of the initial value problem

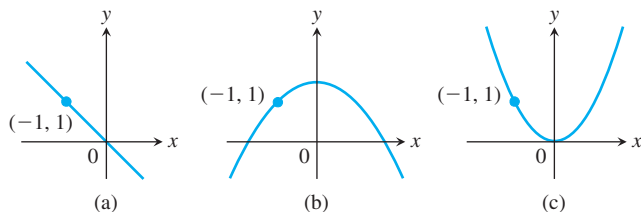
$$\frac{dy}{dx} = 2x, \quad y = 4 \text{ when } x = 1?$$



Give reasons for your answer.

90. Which of the following graphs shows the solution of the initial value problem

$$\frac{dy}{dx} = -x, \quad y = 1 \text{ when } x = -1?$$



Give reasons for your answer.

Solve the initial value problems in Exercises 91–112.

91. $\frac{dy}{dx} = 2x - 7, \quad y(2) = 0$

92. $\frac{dy}{dx} = 10 - x, \quad y(0) = -1$

93. $\frac{dy}{dx} = \frac{1}{x^2} + x, \quad x > 0; \quad y(2) = 1$

94. $\frac{dy}{dx} = 9x^2 - 4x + 5, \quad y(-1) = 0$

95. $\frac{dy}{dx} = 3x^{-2/3}, \quad y(-1) = -5$

96. $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \quad y(4) = 0$

97. $\frac{ds}{dt} = 1 + \cos t, \quad s(0) = 4$

98. $\frac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1$

99. $\frac{dr}{d\theta} = -\pi \sin \pi\theta, \quad r(0) = 0$

100. $\frac{dr}{d\theta} = \cos \pi\theta, \quad r(0) = 1$

101. $\frac{dv}{dt} = \frac{1}{2} \sec t \tan t, \quad v(0) = 1$

102. $\frac{dv}{dt} = 8t + \csc^2 t, \quad v\left(\frac{\pi}{2}\right) = -7$

103. $\frac{dv}{dt} = \frac{3}{t\sqrt{t^2 - 1}}, \quad t > 1, \quad v(2) = 0$

104. $\frac{dv}{dt} = \frac{8}{1 + t^2} + \sec^2 t, \quad v(0) = 1$

105. $\frac{d^2y}{dx^2} = 2 - 6x; \quad y'(0) = 4, \quad y(0) = 1$

106. $\frac{d^2y}{dx^2} = 0; \quad y'(0) = 2, \quad y(0) = 0$

107. $\frac{d^2r}{dt^2} = \frac{2}{t^3}; \quad \left.\frac{dr}{dt}\right|_{t=1} = 1, \quad r(1) = 1$

108. $\frac{d^2s}{dt^2} = \frac{3t}{8}; \quad \left.\frac{ds}{dt}\right|_{t=4} = 3, \quad s(4) = 4$

109. $\frac{d^3y}{dx^3} = 6; \quad y''(0) = -8, \quad y'(0) = 0, \quad y(0) = 5$

110. $\frac{d^3\theta}{dt^3} = 0; \quad \theta''(0) = -2, \quad \theta'(0) = -\frac{1}{2}, \quad \theta(0) = \sqrt{2}$

111. $y^{(4)} = -\sin t + \cos t; \quad y'''(0) = 7, \quad y''(0) = y'(0) = -1, \quad y(0) = 0$

112. $y^{(4)} = -\cos x + 8 \sin 2x; \quad y'''(0) = 0, \quad y''(0) = y'(0) = 1, \quad y(0) = 3$

113. Find the curve $y = f(x)$ in the xy -plane that passes through the point $(9, 4)$ and whose slope at each point is $3\sqrt{x}$.

114. a. Find a curve $y = f(x)$ with the following properties:

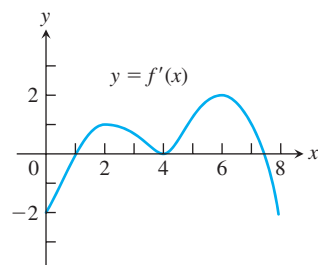
i) $\frac{d^2y}{dx^2} = 6x$

- ii) Its graph passes through the point $(0, 1)$ and has a horizontal tangent line there.

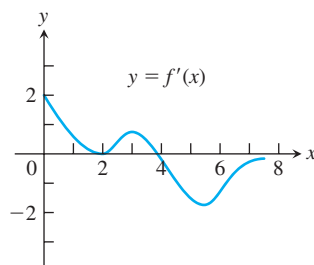
- b. How many curves like this are there? How do you know?

In Exercises 115–118, the graph of f' is given. Assume that $f(0) = 1$ and sketch a possible continuous graph of f .

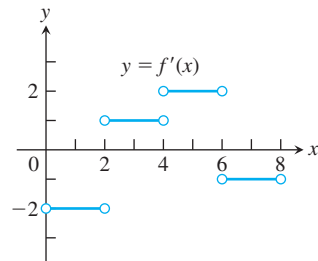
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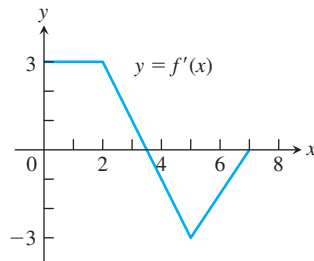
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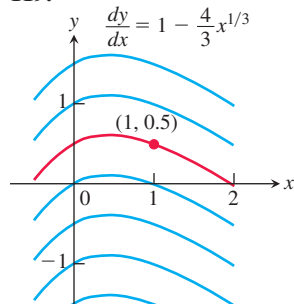
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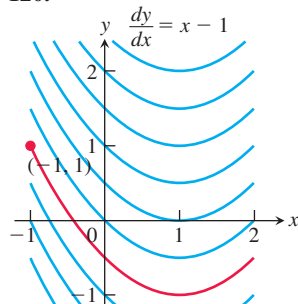
Solution (Integral) Curves

Exercises 119–122 show solution curves of differential equations. In each exercise, find an equation for the curve through the labeled point.

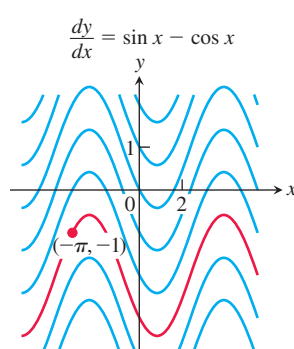
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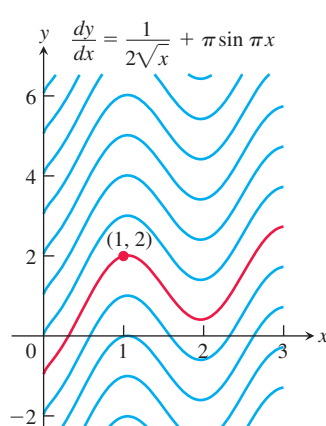
120.



121.



122.



Applications

123. Finding displacement from an antiderivative of velocity

- a. Suppose that the velocity of a body moving along the s -axis is

$$\frac{ds}{dt} = v = 9.8t - 3.$$

- i) Find the body's displacement over the time interval from $t = 1$ to $t = 3$ given that $s = 5$ when $t = 0$.

- ii) Find the body's displacement from $t = 1$ to $t = 3$ given that $s = -2$ when $t = 0$.

- iii) Now find the body's displacement from $t = 1$ to $t = 3$ given that $s = s_0$ when $t = 0$.

- b. Suppose that the position s of a body moving along a coordinate line is a differentiable function of time t . Is it true that once you know an antiderivative of the velocity function ds/dt , you can find the body's displacement from $t = a$ to $t = b$ even if you do not know the body's exact position at either of those times? Give reasons for your answer.

124. **Liftoff from Earth** A rocket lifts off the surface of Earth with a constant acceleration of 20 m/s^2 . How fast will the rocket be going 1 min later?

125. **Stopping a car in time** You are driving along a highway at a steady 108 km/h (30 m/s) when you see an accident ahead and slam on the brakes. What constant deceleration is required to stop your car in 75 m ? To find out, carry out the following steps.

1. Solve the initial value problem

$$\text{Differential equation: } \frac{d^2s}{dt^2} = -k \quad (k \text{ constant})$$

$$\text{Initial conditions: } \frac{ds}{dt} = 30 \text{ and } s = 0 \text{ when } t = 0.$$

Measuring time and distance from when the brakes are applied

2. Find the value of t that makes $ds/dt = 0$. (The answer will involve k .)
3. Find the value of k that makes $s = 75$ for the value of t you found in Step 2.

126. **Stopping a motorcycle** The State of Illinois Cycle Rider Safety Program requires motorcycle riders to be able to brake from 48 km/h (13.3 m/s) to 0 in 13.7 m . What constant deceleration does it take to do that?

127. **Motion along a coordinate line** A particle moves on a coordinate line with acceleration $a = d^2s/dt^2 = 15\sqrt{t} - (3/\sqrt{t})$, subject to the conditions that $ds/dt = 4$ and $s = 0$ when $t = 1$. Find

- a. the velocity $v = ds/dt$ in terms of t .
b. the position s in terms of t .

- T 128. The hammer and the feather** When *Apollo 15* astronaut David Scott dropped a hammer and a feather on the moon to demonstrate that in a vacuum all bodies fall with the same (constant) acceleration, he dropped them from about 1.2 m above the ground. The television footage of the event shows the hammer and the feather falling more slowly than on Earth, where, in a vacuum, they would have taken only half a second to fall the 1.2 m . How long did it take the hammer and feather to fall 1.2 m on the moon? To find out, solve the following initial value problem for s as a function of t . Then find the value of t that makes s equal to 0 .

$$\text{Differential equation: } \frac{d^2s}{dt^2} = -1.6 \text{ m/s}^2$$

$$\text{Initial conditions: } \frac{ds}{dt} = 0 \text{ and } s = 1.2 \text{ when } t = 0$$

- 129. Motion with constant acceleration** The standard equation for the position s of a body moving with a constant acceleration a along a coordinate line is

$$s = \frac{a}{2}t^2 + v_0t + s_0, \quad (1)$$

where v_0 and s_0 are the body's velocity and position at time $t = 0$. Derive this equation by solving the initial value problem

Differential equation: $\frac{d^2s}{dt^2} = a$

Initial conditions: $\frac{ds}{dt} = v_0$ and $s = s_0$ when $t = 0$.

- 130. Free fall near the surface of a planet** For free fall near the surface of a planet where the acceleration due to gravity has a constant magnitude of g length-units/s², Equation (1) in Exercise 129 takes the form

$$s = -\frac{1}{2}gt^2 + v_0t + s_0, \quad (2)$$

where s is the body's height above the surface. The equation has a minus sign because the acceleration acts downward, in the direction of decreasing s . The velocity v_0 is positive if the object is rising at time $t = 0$ and negative if the object is falling.

Instead of using the result of Exercise 129, you can derive Equation (2) directly by solving an appropriate initial value problem. What initial value problem? Solve it to be sure you have the right one, explaining the solution steps as you go along.

- 131.** Suppose that

$$f(x) = \frac{d}{dx}(1 - \sqrt{x}) \quad \text{and} \quad g(x) = \frac{d}{dx}(x + 2).$$

Find:

a. $\int f(x) dx$	b. $\int g(x) dx$
c. $\int [-f(x)] dx$	d. $\int [-g(x)] dx$
e. $\int [f(x) + g(x)] dx$	f. $\int [f(x) - g(x)] dx$

- 132. Uniqueness of solutions** If differentiable functions $y = F(x)$ and $y = g(x)$ both solve the initial value problem

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0,$$

on an interval I , must $F(x) = G(x)$ for every x in I ? Give reasons for your answer.

COMPUTER EXPLORATIONS

Use a CAS to solve the initial value problems in Exercises 133–136. Plot the solution curves.

- 133.** $y' = \cos^2 x + \sin x$, $y(\pi) = 1$
134. $y' = \frac{1}{x} + x$, $y(1) = -1$
135. $y' = \frac{1}{\sqrt{4 - x^2}}$, $y(0) = 2$
136. $y'' = \frac{2}{x} + \sqrt{x}$, $y(1) = 0$, $y'(1) = 0$

CHAPTER 4 Questions to Guide Your Review

- What can be said about the extreme values of a function that is continuous on a closed interval?
- What does it mean for a function to have a local extreme value on its domain? An absolute extreme value? How are local and absolute extreme values related, if at all? Give examples.
- How do you find the absolute extrema of a continuous function on a closed interval? Give examples.
- What are the hypotheses and conclusion of Rolle's Theorem? Are the hypotheses really necessary? Explain.
- What are the hypotheses and conclusion of the Mean Value Theorem? What physical interpretations might the theorem have?
- State the Mean Value Theorem's three corollaries.
- How can you sometimes identify a function $f(x)$ by knowing f' and knowing the value of f at a point $x = x_0$? Give an example.
- What is the First Derivative Test for Local Extreme Values? Give examples of how it is applied.
- How do you test a twice-differentiable function to determine where its graph is concave up or concave down? Give examples.
- What is an inflection point? Give an example. What physical significance do inflection points sometimes have?
- What is the Second Derivative Test for Local Extreme Values? Give examples of how it is applied.
- What do the derivatives of a function tell you about the shape of its graph?
- List the steps you would take to graph a polynomial function. Illustrate with an example.
- What is a cusp? Give examples.
- List the steps you would take to graph a rational function. Illustrate with an example.
- Outline a general strategy for solving max-min problems. Give examples.
- Describe l'Hôpital's Rule. How do you know when to use the rule and when to stop? Give an example.
- How can you sometimes handle limits that lead to indeterminate forms ∞/∞ , $\infty \cdot 0$, and $\infty - \infty$? Give examples.
- How can you sometimes handle limits that lead to indeterminate forms 1^∞ , 0^0 , and ∞^∞ ? Give examples.
- Describe Newton's method for solving equations. Give an example. What is the theory behind the method? What are some of the things to watch out for when you use the method?
- Can a function have more than one antiderivative? If so, how are the antiderivatives related? Explain.
- What is an indefinite integral? How do you evaluate one? What general formulas do you know for finding indefinite integrals?
- How can you sometimes solve a differential equation of the form $dy/dx = f(x)$?
- What is an initial value problem? How do you solve one? Give an example.
- If you know the acceleration of a body moving along a coordinate line as a function of time, what more do you need to know to find the body's position function? Give an example.

CHAPTER 4 Practice Exercises

Finding Extreme Values

In Exercises 1–16, find the extreme values (absolute and local) of the function over its natural domain, and where they occur.

1. $y = 2x^2 - 8x + 9$
2. $y = x^3 - 2x + 4$
3. $y = x^3 + x^2 - 8x + 5$
4. $y = x^3(x - 5)^2$
5. $y = \sqrt{x^2 - 1}$
6. $y = x - 4\sqrt{x}$
7. $y = \frac{1}{\sqrt[3]{1 - x^2}}$
8. $y = \sqrt{3 + 2x - x^2}$
9. $y = \frac{x}{x^2 + 1}$
10. $y = \frac{x + 1}{x^2 + 2x + 2}$
11. $y = e^x + e^{-x}$
12. $y = e^x - e^{-x}$
13. $y = x \ln x$
14. $y = x^2 \ln x$
15. $y = \arccos(x^2)$
16. $y = \sin^{-1}(e^x)$

Extreme Values

17. Does $f(x) = x^3 + 2x + \tan x$ have any local maximum or minimum values? Give reasons for your answer.
18. Does $g(x) = \csc x + 2 \cot x$ have any local maximum values? Give reasons for your answer.
19. Does $f(x) = (7 + x)(11 - 3x)^{1/3}$ have an absolute minimum value? An absolute maximum? If so, find them or give reasons why they fail to exist. List all critical points of f .
20. Find values of a and b such that the function

$$f(x) = \frac{ax + b}{x^2 - 1}$$

has a local extreme value of 1 at $x = 3$. Is this extreme value a local maximum or a local minimum? Give reasons for your answer.

21. Does $g(x) = e^x - x$ have an absolute minimum value? An absolute maximum? If so, find them or give reasons why they fail to exist. List all critical points of g .
22. Does $f(x) = 2e^x/(1 + x^2)$ have an absolute minimum value? An absolute maximum? If so, find them or give reasons why they fail to exist. List all critical points of f .

In Exercises 23 and 24, find the absolute maximum and absolute minimum values of f over the interval.

23. $f(x) = x - 2 \ln x$, $1 \leq x \leq 3$
24. $f(x) = (4/x) + \ln x^2$, $1 \leq x \leq 4$
25. The greatest integer function $f(x) = \lfloor x \rfloor$, defined for all values of x , assumes a local maximum value of 0 at each point of $[0, 1)$. Could any of these local maximum values also be local minimum values of f ? Give reasons for your answer.
26. a. Give an example of a differentiable function f whose first derivative is zero at some point c , even though f has neither a local maximum nor a local minimum at c .
b. How is this consistent with Theorem 2 in Section 4.1? Give reasons for your answer.
27. The function $y = 1/x$ does not take on either a maximum or a minimum on the interval $0 < x < 1$ even though the function is continuous on this interval. Does this contradict the Extreme Value Theorem for continuous functions? Why?
28. What are the maximum and minimum values of the function $y = |x|$ on the interval $-1 \leq x < 1$? Notice that the interval is

not closed. Is this consistent with the Extreme Value Theorem for continuous functions? Why?

- T** 29. A graph that is large enough to show a function's global behavior may fail to reveal important local features. The graph of $f(x) = (x^8/8) - (x^6/2) - x^5 + 5x^3$ is a case in point.
- a. Graph f over the interval $-2.5 \leq x \leq 2.5$. Where does the graph appear to have local extreme values or points of inflection?
 - b. Now factor $f'(x)$ and show that f has a local maximum at $x = \sqrt[3]{5} \approx 1.70998$ and local minima at $x = \pm\sqrt{3} \approx \pm 1.73205$.
 - c. Zoom in on the graph to find a viewing window that shows the presence of the extreme values at $x = \sqrt[3]{5}$ and $x = \sqrt{3}$.

The moral here is that without calculus, the existence of two of the three extreme values would probably have gone unnoticed. On any normal graph of the function, the values would lie close enough together to fall within the dimensions of a single pixel on the screen.

(Source: *Uses of Technology in the Mathematics Curriculum*, by Benny Evans and Jerry Johnson, Oklahoma State University, published in 1990 under a grant from the National Science Foundation, USE-8950044.)

- T** 30. (Continuation of Exercise 29)

- a. Graph $f(x) = (x^8/8) - (2/5)x^5 - 5x - (5/x^2) + 11$ over the interval $-2 \leq x \leq 2$. Where does the graph appear to have local extreme values or points of inflection?
- b. Show that f has a local maximum value at $x = \sqrt[3]{5} \approx 1.2585$ and a local minimum value at $x = \sqrt[3]{2} \approx 1.2599$.
- c. Zoom in to find a viewing window that shows the presence of the extreme values at $x = \sqrt[3]{5}$ and $x = \sqrt[3]{2}$.

The Mean Value Theorem

31. a. Show that $g(t) = \sin^2 t - 3t$ decreases on every interval in its domain.
b. How many solutions does the equation $\sin^2 t - 3t = 5$ have? Give reasons for your answer.
32. a. Show that $y = \tan \theta$ increases on every open interval in its domain.
b. If the conclusion in part (a) is really correct, how do you explain the fact that $\tan \pi = 0$ is less than $\tan(\pi/4) = 1$?
33. a. Show that the equation $x^4 + 2x^2 - 2 = 0$ has exactly one solution on $[0, 1]$.
T b. Find the solution to as many decimal places as you can.
34. a. Show that $f(x) = x/(x + 1)$ increases on every open interval in its domain.
b. Show that $f(x) = x^3 + 2x$ has no local maximum or minimum values.
35. **Water in a reservoir** As a result of a heavy rain, the volume of water in a reservoir increased by million cubic meters in 24 hours. Show that at some instant during that period, the reservoir's volume was increasing at a rate in excess of 500,000 L/min.

36. The formula $F(x) = 3x + C$ gives a different function for each value of C . All of these functions, however, have the same derivative with respect to x , namely $F'(x) = 3$. Are these the only differentiable functions whose derivative is 3? Could there be any others? Give reasons for your answers.

37. Show that

$$\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{d}{dx}\left(-\frac{1}{x+1}\right)$$

even though

$$\frac{x}{x+1} \neq -\frac{1}{x+1}.$$

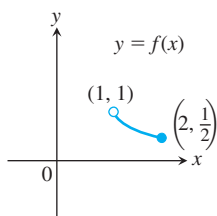
Doesn't this contradict Corollary 2 of the Mean Value Theorem? Give reasons for your answer.

38. Calculate the first derivatives of $f(x) = x^2/(x^2 + 1)$ and $g(x) = -1/(x^2 + 1)$. What can you conclude about the graphs of these functions?

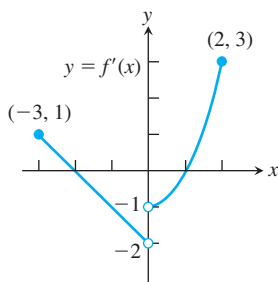
Analyzing Graphs

In Exercises 39 and 40, use the graph to answer the questions.

39. Identify any global extreme values of f and the values of x at which they occur.



40. Estimate the open intervals on which the function $y = f(x)$ is
- increasing.
 - decreasing.
 - Use the given graph of f' to indicate where any local extreme values of the function occur, and whether each extreme is a relative maximum or minimum.

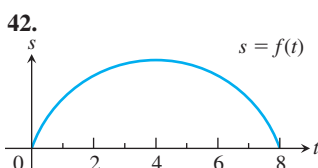
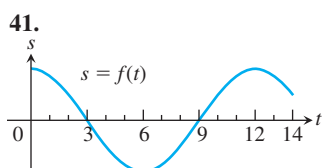


Each of the graphs in Exercises 41 and 42 is the graph of the position function $s = f(t)$ of an object moving on a coordinate line (t represents time). At approximately what times (if any) is each object's

- velocity equal to zero?
- Acceleration equal to zero?

During approximately what time intervals does the object move

- forward?
- Backward?



Graphs and Graphing

Graph the curves in Exercises 43–58.

43. $y = x^2 - (x^3/6)$	44. $y = x^3 - 3x^2 + 3$
45. $y = -x^3 + 6x^2 - 9x + 3$	
46. $y = (1/8)(x^3 + 3x^2 - 9x - 27)$	
47. $y = x^3(8 - x)$	48. $y = x^2(2x^2 - 9)$
49. $y = x - 3x^{2/3}$	50. $y = x^{1/3}(x - 4)$
51. $y = x\sqrt{3 - x}$	52. $y = x\sqrt{4 - x^2}$
53. $y = (x - 3)^2 e^x$	54. $y = xe^{-x^2}$
55. $y = \ln(x^2 - 4x + 3)$	56. $y = \ln(\sin x)$
57. $y = \arcsin\left(\frac{1}{x}\right)$	58. $y = \tan^{-1}\left(\frac{1}{x}\right)$

Each of Exercises 59–64 gives the first derivative of a function $y = f(x)$. (a) At what points, if any, does the graph of f have a local maximum, local minimum, or inflection point? (b) Sketch the general shape of the graph.

59. $y' = 16 - x^2$	60. $y' = x^2 - x - 6$
61. $y' = 6x(x + 1)(x - 2)$	62. $y' = x^2(6 - 4x)$
63. $y' = x^4 - 2x^2$	64. $y' = 4x^2 - x^4$

In Exercises 65–68, graph each function. Then use the function's first derivative to explain what you see.

65. $y = x^{2/3} + (x - 1)^{1/3}$	66. $y = x^{2/3} + (x - 1)^{2/3}$
67. $y = x^{1/3} + (x - 1)^{1/3}$	68. $y = x^{2/3} - (x - 1)^{1/3}$

Sketch the graphs of the rational functions in Exercises 69–76.

69. $y = \frac{x+1}{x-3}$	70. $y = \frac{2x}{x+5}$
71. $y = \frac{x^2+1}{x}$	72. $y = \frac{x^2-x+1}{x}$
73. $y = \frac{x^3+2}{2x}$	74. $y = \frac{x^4-1}{x^2}$
75. $y = \frac{x^2-4}{x^2-3}$	76. $y = \frac{x^2}{x^2-4}$

Using L'Hôpital's Rule

Use l'Hôpital's Rule to find the limits in Exercises 77–88.

77. $\lim_{x \rightarrow 1} \frac{x^2 + 3x - 4}{x - 1}$	78. $\lim_{x \rightarrow 1} \frac{x^a - 1}{x^b - 1}$
79. $\lim_{x \rightarrow \pi} \frac{\tan x}{x}$	80. $\lim_{x \rightarrow 0} \frac{\tan x}{x + \sin x}$
81. $\lim_{x \rightarrow 0} \frac{\sin^2 x}{\tan(x^2)}$	82. $\lim_{x \rightarrow 0} \frac{\sin mx}{\sin nx}$
83. $\lim_{x \rightarrow \pi/2^-} \sec 7x \cos 3x$	84. $\lim_{x \rightarrow 0^+} \sqrt{x} \sec x$
85. $\lim_{x \rightarrow 0} (\csc x - \cot x)$	86. $\lim_{x \rightarrow 0} \left(\frac{1}{x^4} - \frac{1}{x^2} \right)$
87. $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x + 1} - \sqrt{x^2 - x})$	
88. $\lim_{x \rightarrow \infty} \left(\frac{x^3}{x^2 - 1} - \frac{x^3}{x^2 + 1} \right)$	

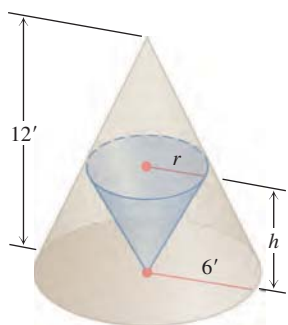
Find the limits in Exercises 89–102.

89. $\lim_{x \rightarrow 0} \frac{10^x - 1}{x}$	90. $\lim_{\theta \rightarrow 0} \frac{3^\theta - 1}{\theta}$
91. $\lim_{x \rightarrow 0} \frac{2^{\sin x} - 1}{e^x - 1}$	92. $\lim_{x \rightarrow 0} \frac{2^{-\sin x} - 1}{e^x - 1}$
93. $\lim_{x \rightarrow 0} \frac{5 - 5 \cos x}{e^x - x - 1}$	94. $\lim_{x \rightarrow 0} \frac{4 - 4e^x}{xe^x}$

95. $\lim_{t \rightarrow 0^+} \frac{t - \ln(1 + 2t)}{t^2}$
96. $\lim_{x \rightarrow 4} \frac{\sin^2(\pi x)}{e^{x-4} + 3 - x}$
97. $\lim_{t \rightarrow 0^+} \left(\frac{e^t}{t} - \frac{1}{t} \right)$
98. $\lim_{y \rightarrow 0^+} e^{-1/y} \ln y$
99. $\lim_{x \rightarrow \infty} \left(1 + \frac{b}{x} \right)^{kx}$
100. $\lim_{x \rightarrow \infty} \left(1 + \frac{2}{x} + \frac{7}{x^2} \right)$
101. $\lim_{x \rightarrow 0} \frac{\cos 2x - 1 - \sqrt{1 - \cos x}}{\sin^2 x}$
102. $\lim_{x \rightarrow 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x^3}$

Optimization

103. The sum of two nonnegative numbers is 36. Find the numbers if
- the difference of their square roots is to be as large as possible.
 - the sum of their square roots is to be as large as possible.
104. The sum of two nonnegative numbers is 20. Find the numbers
- if the product of one number and the square root of the other is to be as large as possible.
 - if one number plus the square root of the other is to be as large as possible.
105. An isosceles triangle has its vertex at the origin and its base parallel to the x -axis with the vertices above the axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have.
106. A customer has asked you to design an open-top rectangular stainless steel vat. It is to have a square base and a volume of 1 m^3 , to be welded from 6-mm-thick plate, and to weigh no more than necessary. What dimensions do you recommend?
107. Find the height and radius of the largest right circular cylinder that can be put in a sphere of radius $\sqrt{3}$.
108. The figure here shows two right circular cones, one upside down inside the other. The two bases are parallel, and the vertex of the smaller cone lies at the center of the larger cone's base. What values of r and h will give the smaller cone the largest possible volume?



109. **Manufacturing tires** Your company can manufacture x hundred grade A tires and y hundred grade B tires a day, where $0 \leq x \leq 4$ and

$$y = \frac{40 - 10x}{5 - x}.$$

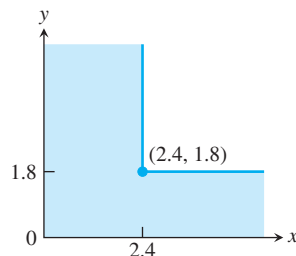
Your profit on a grade A tire is twice your profit on a grade B tire. What is the most profitable number of each kind to make?

110. **Particle motion** The positions of two particles on the s -axis are $s_1 = \cos t$ and $s_2 = \cos(t + \pi/4)$.

- What is the farthest apart the particles ever get?
- When do the particles collide?

- T** 111. **Open-top box** An open-top rectangular box is constructed from a 25-cm-by-40-cm piece of cardboard by cutting squares of equal side length from the corners and folding up the sides. Find analytically the dimensions of the box of largest volume and the maximum volume. Support your answers graphically.

112. **The ladder problem** What is the approximate length (in meters) of the longest ladder you can carry horizontally around the corner of the corridor shown here? Round your answer down to the nearest meter.



Newton's Method

113. Let $f(x) = 3x - x^3$. Show that the equation $f(x) = -4$ has a solution in the interval $[2, 3]$ and use Newton's method to find it.
114. Let $f(x) = x^4 - x^3$. Show that the equation $f(x) = 75$ has a solution in the interval $[3, 4]$ and use Newton's method to find it.

Finding Indefinite Integrals

Find the indefinite integrals (most general antiderivatives) in Exercises 115–138. You may need to try a solution and then adjust your guess. Check your answers by differentiation.

115. $\int (x^3 + 5x - 7) dx$
116. $\int \left(8t^3 - \frac{t^2}{2} + t \right) dt$
117. $\int \left(3\sqrt{t} + \frac{4}{t^2} \right) dt$
118. $\int \left(\frac{1}{2\sqrt{t}} - \frac{3}{t^4} \right) dt$
119. $\int \frac{dr}{(r + 5)^2}$
120. $\int \frac{6dr}{(r - \sqrt{2})^3}$
121. $\int 3\theta\sqrt{\theta^2 + 1} d\theta$
122. $\int \frac{\theta}{\sqrt{7 + \theta^2}} d\theta$
123. $\int x^3(1 + x^4)^{-1/4} dx$
124. $\int (2 - x)^{3/5} dx$
125. $\int \sec^2 \frac{s}{10} ds$
126. $\int \csc^2 \pi s ds$
127. $\int \csc \sqrt{2}\theta \cot \sqrt{2}\theta d\theta$
128. $\int \sec \frac{\theta}{3} \tan \frac{\theta}{3} d\theta$
129. $\int \sin^2 \frac{x}{4} dx$ (Hint: $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$)
130. $\int \cos^2 \frac{x}{2} dx$
131. $\int \left(\frac{3}{x} - x \right) dx$
132. $\int \left(\frac{5}{x^2} + \frac{2}{x^2 + 1} \right) dx$
133. $\int \left(\frac{1}{2}e^t - e^{-t} \right) dt$
134. $\int (5^s + s^5) ds$
135. $\int \theta^{1-\pi} d\theta$
136. $\int 2^{\pi+r} dr$
137. $\int \frac{3}{2x\sqrt{x^2 - 1}} dx$
138. $\int \frac{d\theta}{\sqrt{16 - \theta^2}}$

Initial Value Problems

Solve the initial value problems in Exercises 139–142.

139. $\frac{dy}{dx} = \frac{x^2 + 1}{x^2}, \quad y(1) = -1$
140. $\frac{dy}{dx} = \left(x + \frac{1}{x} \right)^2, \quad y(1) = 1$

141. $\frac{d^2r}{dt^2} = 15\sqrt{t} + \frac{3}{\sqrt{t}}; \quad r'(1) = 8, \quad r(1) = 0$

142. $\frac{d^3r}{dt^3} = -\cos t; \quad r''(0) = r'(0) = 0, \quad r(0) = -1$

Applications and Examples

143. Can the integrations in (a) and (b) both be correct? Explain.

a. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

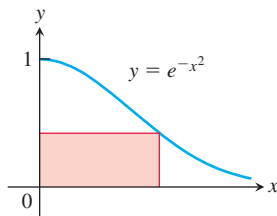
b. $\int \frac{dx}{\sqrt{1-x^2}} = -\int -\frac{dx}{\sqrt{1-x^2}} = -\arccos x + C$

144. Can the integrations in (a) and (b) both be correct? Explain.

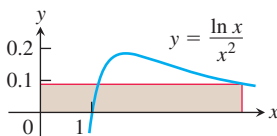
a. $\int \frac{dx}{\sqrt{1-x^2}} = -\int -\frac{dx}{\sqrt{1-x^2}} = -\arccos x + C$

b. $\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{-du}{\sqrt{1-(-u)^2}} \quad \begin{matrix} x = -u \\ dx = -du \end{matrix}$
 $= \int \frac{-du}{\sqrt{1-u^2}}$
 $= \arccos u + C$
 $= \arccos(-x) + C \quad u = -x$

145. The rectangle shown here has one side on the positive y-axis, one side on the positive x-axis, and its upper right-hand vertex on the curve $y = e^{-x^2}$. What dimensions give the rectangle its largest area, and what is that area?



146. The rectangle shown here has one side on the positive y-axis, one side on the positive x-axis, and its upper right-hand vertex on the curve $y = (\ln x)/x^2$. What dimensions give the rectangle its largest area, and what is that area?



In Exercises 147 and 148, find the absolute maximum and minimum values of each function on the given interval.

147. $y = x \ln 2x - x, \quad \left[\frac{1}{2e}, \frac{e}{2}\right]$

148. $y = 10x(2 - \ln x), \quad (0, e^2]$

In Exercises 149 and 150, find the absolute maxima and minima of the functions and give the x-coordinates where they occur.

149. $f(x) = e^{x/\sqrt{x^4+1}}$

150. $g(x) = e^{\sqrt{3-2x-x^2}}$

T 151. Graph the following functions and use what you see to locate and estimate the extreme values, identify the coordinates of the inflection points, and identify the intervals on which the graphs are concave up and concave down. Then confirm your estimates by working with the functions' derivatives.

a. $y = (\ln x)/\sqrt{x}$

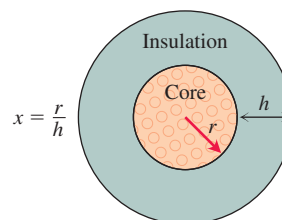
b. $y = e^{-x^2}$

c. $y = (1+x)e^{-x}$

T 152. Graph $f(x) = x \ln x$. Does the function appear to have an absolute minimum value? Confirm your answer with calculus.

T 153. Graph $f(x) = (\sin x)^{\sin x}$ over $[0, 3\pi]$. Explain what you see.

154. A round underwater transmission cable consists of a core of copper wires surrounded by nonconducting insulation. If x denotes the ratio of the radius of the core to the thickness of the insulation, it is known that the speed of the transmission signal is given by the equation $v = x^2 \ln(1/x)$. If the radius of the core is 1 cm, what insulation thickness h will allow the greatest transmission speed?



CHAPTER 4 Additional and Advanced Exercises

Functions and Derivatives

- What can you say about a function whose maximum and minimum values on an interval are equal? Give reasons for your answer.
- Is it true that a discontinuous function cannot have both an absolute maximum value and an absolute minimum value on a closed interval? Give reasons for your answer.
- Can you conclude anything about the extreme values of a continuous function on an open interval? On a half-open interval? Give reasons for your answer.
- Local extrema** Use the sign pattern for the derivative

$$\frac{df}{dx} = 6(x-1)(x-2)^2(x-3)^3(x-3)^4$$

to identify the points where f has local maximum and minimum values.

5. Local extrema

- a. Suppose that the first derivative of $y = f(x)$ is

$$y' = 6(x+1)(x-2)^2.$$

At what points, if any, does the graph of f have a local maximum, local minimum, or point of inflection?

- b. Suppose that the first derivative of $y = f(x)$ is

$$y' = 6(x+1)(x-2).$$

At what points, if any, does the graph of f have a local maximum, local minimum, or point of inflection?

6. If $f'(x) \leq 2$ for all x , what is the most the values of f can increase on $[0, 6]$? Give reasons for your answer.

7. Bounding a function Suppose that f is continuous on $[a, b]$ and that c is an interior point of the interval. Show that if $f'(x) \leq 0$ on $[a, c]$ and $f'(x) \geq 0$ on $(c, b]$, then $f(x)$ is never less than $f(c)$ on $[a, b]$.

8. An inequality

a. Show that $-1/2 \leq x/(1 + x^2) \leq 1/2$ for every value of x .

b. Suppose that f is a function whose derivative is $f'(x) = x/(1 + x^2)$. Use the result in part (a) to show that

$$|f(b) - f(a)| \leq \frac{1}{2}|b - a|$$

for any a and b .

9. The derivative of $f(x) = x^2$ is zero at $x = 0$, but f is not a constant function. Doesn't this contradict the corollary of the Mean Value Theorem that says that functions with zero derivatives are constant? Give reasons for your answer.

10. Extrema and inflection points Let $h = fg$ be the product of two differentiable functions of x .

a. If f and g are positive, with local maxima at $x = a$, and if f' and g' change sign at a , does h have a local maximum at a ?

b. If the graphs of f and g have inflection points at $x = a$, does the graph of h have an inflection point at a ?

In either case, if the answer is yes, give a proof. If the answer is no, give a counterexample.

11. Finding a function Use the following information to find the values of a, b , and c in the formula $f(x) = (x + a)/(bx^2 + cx + 2)$.

a. The values of a, b , and c are either 0 or 1.

b. The graph of f passes through the point $(-1, 0)$.

c. The line $y = 1$ is an asymptote of the graph of f .

12. Horizontal tangent For what value or values of the constant k will the curve $y = x^3 + kx^2 + 3x - 4$ have exactly one horizontal tangent?

Optimization

13. Largest inscribed triangle Points A and B lie at the ends of a diameter of a unit circle and point C lies on the circumference. Is it true that the area of triangle ABC is largest when the triangle is isosceles? How do you know?

14. Proving the second derivative test The Second Derivative Test for Local Maxima and Minima (Section 4.4) says:

a. f has a local maximum value at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$;

b. f has a local minimum value at $x = c$ if $f'(c) = 0$ and $f''(c) > 0$.

To prove statement (a), let $\varepsilon = (1/2)|f''(c)|$. Then use the fact that

$$f''(c) = \lim_{h \rightarrow 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h \rightarrow 0} \frac{f'(c+h)}{h}$$

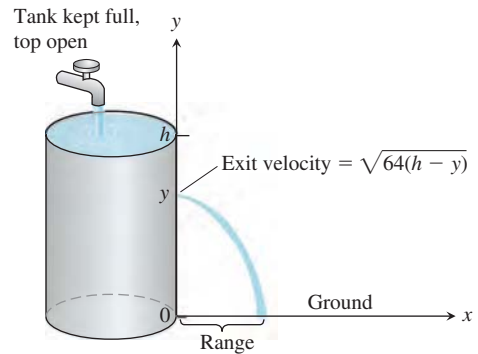
to conclude that for some $\delta > 0$,

$$0 < |h| < \delta \quad \Rightarrow \quad \frac{f'(c+h)}{h} < f''(c) + \varepsilon < 0.$$

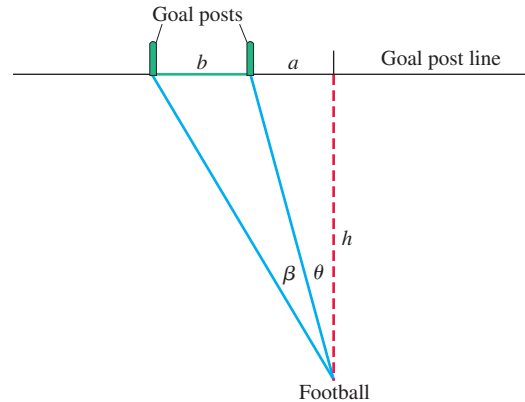
Thus, $f'(c+h)$ is positive for $-\delta < h < 0$ and negative for $0 < h < \delta$. Prove statement (b) in a similar way.

15. Hole in a water tank You want to bore a hole in the side of the tank shown here at a height that will make the stream of water coming out hit the ground as far from the tank as possible. If you drill the hole near the top, where the pressure is low, the water will exit slowly but spend a relatively long time in the air. If you drill the hole near the bottom, the water will exit at a higher velocity but have only a short time to fall. Where is the best place, if any,

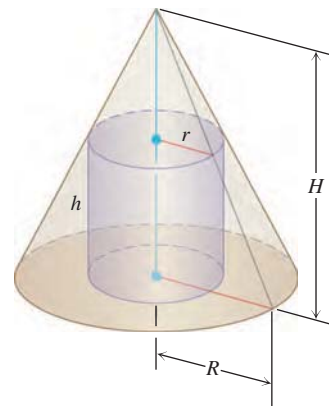
for the hole? (Hint: How long will it take an exiting droplet of water to fall from height y to the ground?)



16. Kicking a field goal An American football player wants to kick a field goal with the ball being on a right hash mark. Assume that the goal posts are b meters apart and that the hash mark line is a distance $a > 0$ meters from the right goal post. (See the accompanying figure.) Find the distance h from the goal post line that gives the kicker his largest angle β . Assume that the football field is flat.

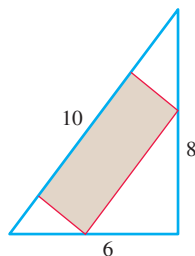


17. A max-min problem with a variable answer Sometimes the solution of a max-min problem depends on the proportions of the shapes involved. As a case in point, suppose that a right circular cylinder of radius r and height h is inscribed in a right circular cone of radius R and height H , as shown here. Find the value of r (in terms of R and H) that maximizes the total surface area of the cylinder (including top and bottom). As you will see, the solution depends on whether $H \leq 2R$ or $H > 2R$.



18. Minimizing a parameter Find the smallest value of the positive constant m that will make $mx - 1 + (1/x)$ greater than or equal to zero for all positive values of x .

19. Determine the dimensions of the rectangle of largest area that can be inscribed in the right triangle in the accompanying figure.



20. A rectangular box with a square base is inscribed in a right circular cone of height 4 and base radius 3. If the base of the box sits on the base of the cone, what is the largest possible volume of the box?

Limits

21. Evaluate the following limits.

$$\begin{array}{ll} \text{a. } \lim_{x \rightarrow 0} \frac{2 \sin 5x}{3x} & \text{b. } \lim_{x \rightarrow 0} \sin 5x \cot 3x \\ \text{c. } \lim_{x \rightarrow 0} x \csc^2 \sqrt{2x} & \text{d. } \lim_{x \rightarrow \pi/2} (\sec x - \tan x) \\ \text{e. } \lim_{x \rightarrow 0} \frac{x - \sin x}{x - \tan x} & \text{f. } \lim_{x \rightarrow 0} \frac{\sin x^2}{x \sin x} \\ \text{g. } \lim_{x \rightarrow 0} \frac{\sec x - 1}{x^2} & \text{h. } \lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 - 4} \end{array}$$

22. L'Hôpital's Rule does not help with the following limits. Find them some other way.

$$\begin{array}{ll} \text{a. } \lim_{x \rightarrow \infty} \frac{\sqrt{x+5}}{\sqrt{x}+5} & \text{b. } \lim_{x \rightarrow \infty} \frac{2x}{x+7\sqrt{x}} \end{array}$$

Theory and Examples

23. Suppose that it costs a company $y = a + bx$ dollars to produce x units per week. It can sell x units per week at a price of $P = c - ex$ dollars per unit. Each of a , b , c , and e represents a positive constant. (a) What production level maximizes the profit? (b) What is the corresponding price? (c) What is the weekly profit at this level of production? (d) At what price should each item be sold to maximize profits if the government imposes a tax of t dollars per item sold? Comment on the difference between this price and the price before the tax.

24. **Estimating reciprocals without division** You can estimate the value of the reciprocal of a number a without ever dividing by a if you apply Newton's method to the function $f(x) = (1/x) - a$. For example, if $a = 3$, the function involved is $f(x) = (1/x) - 3$.

a. Graph $y = (1/x) - 3$. Where does the graph cross the x -axis?

b. Show that the recursion formula in this case is

$$x_{n+1} = x_n(2 - 3x_n),$$

so there is no need for division.

25. To find $x = \sqrt[q]{a}$, we apply Newton's method to $f(x) = x^q - a$. Here we assume that a is a positive real number and q is a positive integer. Show that x_1 is a "weighted average" of x_0 and a/x_0^{q-1} , and find the coefficients m_0, m_1 such that

$$x_1 = m_0 x_0 + m_1 \left(\frac{a}{x_0^{q-1}} \right), \quad m_0 > 0, m_1 > 0, m_0 + m_1 = 1.$$

What conclusion would you reach if x_0 and a/x_0^{q-1} were equal? What would be the value of x_1 in that case?

26. The family of straight lines $y = ax + b$ (a, b arbitrary constants) can be characterized by the relation $y'' = 0$. Find a similar relation satisfied by the family of all circles

$$(x - h)^2 + (y - h)^2 = r^2,$$

where h and r are arbitrary constants. (Hint: Eliminate h and r from the set of three equations including the given one and two obtained by successive differentiation.)

27. **Free fall in the fourteenth century** In the middle of the fourteenth century, Albert of Saxony (1316–1390) proposed a model of free fall, which assumed that the velocity of a falling body was proportional to the distance fallen. It seemed reasonable to think that a body that had fallen 6 m might be moving twice as fast as a body that had fallen 3 m. And besides, none of the instruments in use at the time were accurate enough to prove otherwise. Today we can see just how far off Albert of Saxony's model was by solving the initial value problem implicit in his model. Solve the problem and compare your solution graphically with the equation $s = 4.9t^2$. You will see that it describes a motion that starts too slowly and then becomes too fast to be realistic.

28. **Group testing** During World War II it was necessary to administer blood tests to large numbers of recruits. There are two standard ways to administer a blood test to N people. In method 1, each person is tested separately. In method 2, the blood samples of x people are pooled and tested as one large sample. If the test is negative, this one test is enough for all x people. If the test is positive, then each of the x people is tested separately, requiring a total of $x + 1$ tests. Using the second method and some probability theory it can be shown that, on the average, the total number of tests y will be

$$y = N \left(1 - q^x + \frac{1}{x} \right).$$

With $q = 0.99$ and $N = 1000$, find the integer value of x that minimizes y . Also find the integer value of x that maximizes y . (This second result is not important to the real-life situation.) The group testing method was used in World War II with a savings of 80% over the individual testing method, but not with the given value of q . Group testing has been implemented in diagnosing various diseases, including COVID-19.

29. Assume that the brakes of an automobile produce a constant deceleration of k m/s². (a) Determine what k must be to bring an automobile traveling 108 km/h (30 m/s) to rest in a distance of 30 m from the point where the brakes are applied. (b) With the same k , how far would a car traveling 54 km/hr go before being brought to a stop?
30. Let $f(x)$ and $g(x)$ be two continuously differentiable functions satisfying the relationships $f'(x) = g(x)$ and $f''(x) = -f(x)$. Let $h(x) = f^2(x) + g^2(x)$. If $h(0) = 5$, find $h(10)$.
31. Can there be a curve satisfying the following conditions? d^2y/dx^2 is everywhere equal to zero and, when $x = 0$, $y = 0$ and $dy/dx = 1$. Give a reason for your answer.
32. Find the equation for the curve in the xy -plane that passes through the point $(1, -1)$ if its slope at x is always $3x^2 + 2$.
33. A particle moves along the x -axis. Its acceleration is $a = -t^2$. At $t = 0$, the particle is at the origin. In the course of its motion, it reaches the point $x = b$, where $b > 0$, but no point beyond b . Determine its velocity at $t = 0$.

34. A particle moves with acceleration $a = \sqrt{t} - (1/\sqrt{t})$. Assuming that the velocity $v = 4/3$ and the position $s = -4/15$ when $t = 0$, find
- the velocity v in terms of t .
 - the position s in terms of t .

35. Given $f(x) = ax^2 + 2bx + c$ with $a > 0$. By considering the minimum, prove that $f(x) \geq 0$ for all real x if and only if $b^2 - ac \leq 0$.

36. The Cauchy–Schwarz inequality

- a. In Exercise 35, let

$$f(x) = (a_1x + b_1)^2 + (a_2x + b_2)^2 + \cdots + (a_nx + b_n)^2,$$

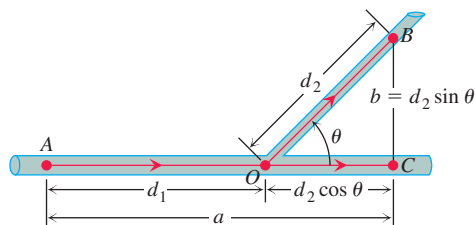
and deduce The Cauchy–Schwarz inequality:

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2).$$

- b. Show that equality holds in The Cauchy–Schwarz inequality only if there exists a real number x that makes a_ix equal $-b_i$ for every value of i from 1 to n .

- 37. The best branching angles for blood vessels and pipes** When a smaller pipe branches off from a larger one in a flow system, we may want it to run off at an angle that is best from some energy-saving point of view. We might require, for instance, that energy loss due to friction be minimized along the section AOB shown in the accompanying figure. In this diagram, B is a given point to be reached by the smaller pipe, A is a point in the larger pipe upstream from B , and O is the point where the branching occurs. A law formulated by Poiseuille states that the loss of energy due to friction in nonturbulent flow is proportional to the length of the path and inversely proportional to the fourth power of the radius. Thus, the loss along AO is $(kd_1)/R^4$ and along OB is $(kd_2)/r^4$, where k is a constant, d_1 is the length of AO , d_2 is the length of OB , R is the radius of the larger pipe, and r is the radius of the smaller pipe. The angle θ is to be chosen to minimize the sum of these two losses:

$$L = k \frac{d_1}{R^4} + k \frac{d_2}{r^4}.$$



In our model, we assume that $AC = a$ and $BC = b$ are fixed. Thus we have the relations

$$d_1 + d_2 \cos \theta = a \quad d_2 \sin \theta = b,$$

so that

$$d_2 = b \csc \theta, \\ d_1 = a - d_2 \cos \theta = a - b \cot \theta.$$

We can express the total loss L as a function of θ :

$$L = k \left(\frac{a - b \cot \theta}{R^4} + \frac{b \csc \theta}{r^4} \right).$$

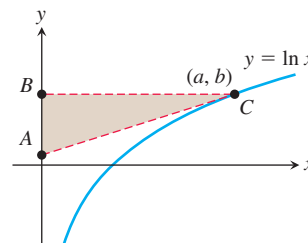
- a. Show that the critical value of θ for which $dL/d\theta$ equals zero is

$$\theta_c = \cos^{-1} \frac{r^4}{R^4}.$$

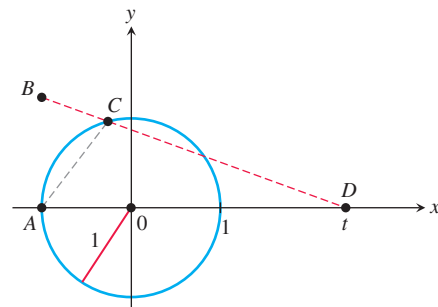
- b. If the ratio of the pipe radii is $r/R = 5/6$ estimate to the nearest degree the optimal branching angle given in part (a).

38. Consider point (a, b) on the graph of $y = \ln x$ and triangle ABC formed by the tangent line at (a, b) , the y -axis, and the line $y = b$. Show that

$$(\text{area triangle } ABC) = \frac{a}{2}.$$



39. Consider the unit circle centered at the origin and with a vertical tangent line passing through point A in the accompanying figure. Assume that the lengths of segments AB and AC are equal, and let point D be the intersection of the x -axis with the line passing through points B and C . Find the limit of t as B approaches A .



CHAPTER 4 Technology Application Projects

Mathematica/Maple Projects

Projects can be found within MyLab Math.

• **Motion Along a Straight Line: Position $\rightarrow$ Velocity $\rightarrow$ Acceleration**

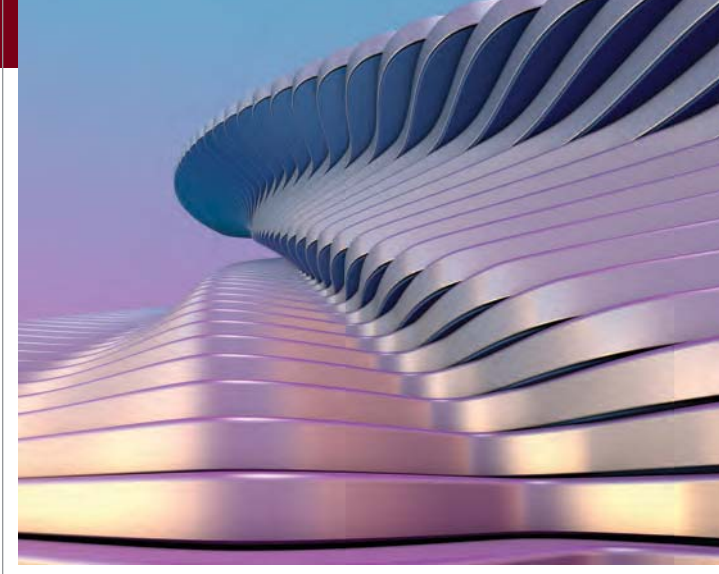
You will observe the shape of a graph through dramatic animated visualizations of the derivative relations among the position, velocity, and acceleration. Figures in the text can be animated.

• **Newton's Method: Estimate π to How Many Places?**

Plot a function, observe a root, pick a starting point near the root, and use Newton's Iteration Procedure to approximate the root to a desired accuracy. The numbers π , e , and $\sqrt{2}$ are approximated.

5

Integrals



OVERVIEW A great achievement of classical geometry was obtaining formulas for the areas and volumes of triangles, spheres, and cones. In this chapter we develop a method, called *integration*, to calculate the areas and volumes of more general shapes. The *definite integral* is the key tool in calculus for defining and calculating areas and volumes. We also use it to compute quantities such as the lengths of curved paths, probabilities, averages, energy consumption, the mass of an object, and the force against a dam's floodgates.

Like the derivative, the definite integral is defined as a limit. The definite integral is a limit of increasingly fine approximations. The idea is to approximate a quantity (such as the area of a curvy region) by dividing it into many small pieces, each of which we can approximate by something simple (such as a rectangle). Summing the contributions of each of the simple pieces gives us an approximation to the original quantity. As we divide the region into more and more pieces, the approximation given by the sum of the pieces will generally improve, converging to the quantity we are measuring. We take a limit as the number of terms increases to infinity, and when the limit exists, the result is a definite integral. We develop this idea in Section 5.3.

We also show that the process of computing these definite integrals is closely connected to finding antiderivatives. This is one of the most important relationships in calculus; it gives us an efficient way to compute definite integrals, providing a simple and powerful method that eliminates the difficulty of directly computing limits of approximations. This connection is captured in the Fundamental Theorem of Calculus.

5.1 Area and Estimating with Finite Sums

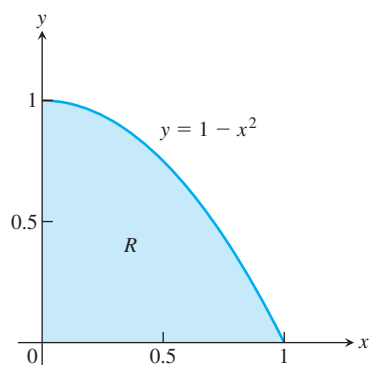


FIGURE 5.1 The area of the shaded region R cannot be found by a simple formula.

The basis for formulating definite integrals is the construction of approximations by finite sums. In this section we consider three examples of this process: finding the area under a graph, the distance traveled by a moving object, and the average value of a function. Although we have yet to define precisely what we mean by the area of a general region in the plane, or the average value of a function over a closed interval, we do have intuitive ideas of what these notions mean. We begin our approach to integration by *approximating* these quantities with simpler finite sums related to these intuitive ideas. We then consider what happens when we take more and more terms in the summation process. In subsequent sections we look at taking the limit of these sums as the number of terms goes to infinity, which leads to a precise definition of the definite integral.

Area

Suppose we want to find the area of the shaded region R that lies above the x -axis, below the graph of $y = 1 - x^2$, and between the vertical lines $x = 0$ and $x = 1$ (see Figure 5.1).

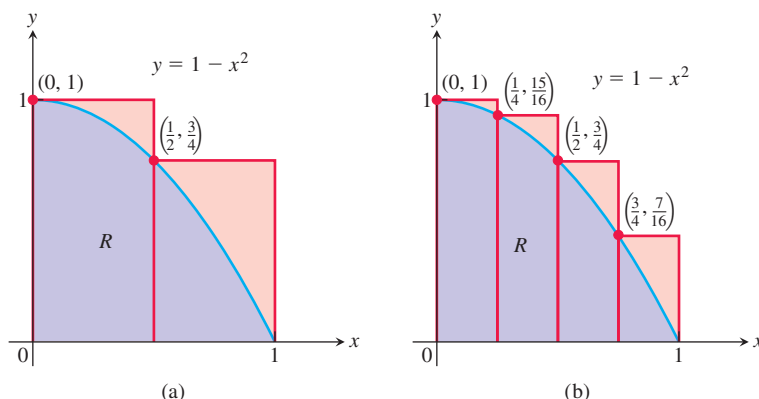


FIGURE 5.2 (a) We get an upper sum approximation of the area of R by using two rectangles containing R . (b) Four rectangles give a better upper sum approximation. Both estimates overshoot the true value for the area by the amount shaded in light red.

Unfortunately, there is no simple geometric formula for calculating the areas of general shapes having curved boundaries like the region R . How, then, can we find the area of R ?

Although we do not yet have a method for determining the exact area of R , we can approximate it in a simple way. Figure 5.2a shows two rectangles that together contain the region R . Each rectangle has width $1/2$ and they have heights 1 and $3/4$ (left to right). The height of each rectangle is the maximum value of the function f in each subinterval. Because the function f is decreasing, the height is its value at the left endpoint of the subinterval of $[0, 1]$ that forms the base of the rectangle. The total area of the two rectangles approximates the area A of the region R :

$$A \approx 1 \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{7}{8} = 0.875.$$

This estimate is larger than the true area A since the two rectangles contain R . We say that 0.875 is an **upper sum** because it is obtained by taking the height of the rectangle corresponding to the maximum (uppermost) value of $f(x)$ over points x lying in the base of each rectangle. In Figure 5.2b, we improve our estimate by using four thinner rectangles, each of width $1/4$, which taken together contain the region R . These four rectangles give the approximation

$$A \approx 1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} = \frac{25}{32} = 0.78125,$$

which is still greater than A since the four rectangles contain R .

Suppose instead we use four rectangles contained *inside* the region R to estimate the area, as in Figure 5.3a. Each rectangle has width $1/4$, as before, but the rectangles are shorter and lie entirely beneath the graph of f . The function $f(x) = 1 - x^2$ is decreasing on $[0, 1]$, so the height of each of these rectangles is given by the value of f at the right endpoint of the subinterval forming its base. The fourth rectangle has zero height and therefore contributes no area. Summing these rectangles, whose heights are the minimum value of $f(x)$ over points x in the rectangle's base, gives a **lower sum** approximation to the area:

$$A \approx \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = \frac{17}{32} = 0.53125.$$

This estimate is smaller than the area A since the rectangles all lie inside of the region R . The true value of A lies somewhere between these lower and upper sums:

$$0.53125 < A < 0.78125.$$

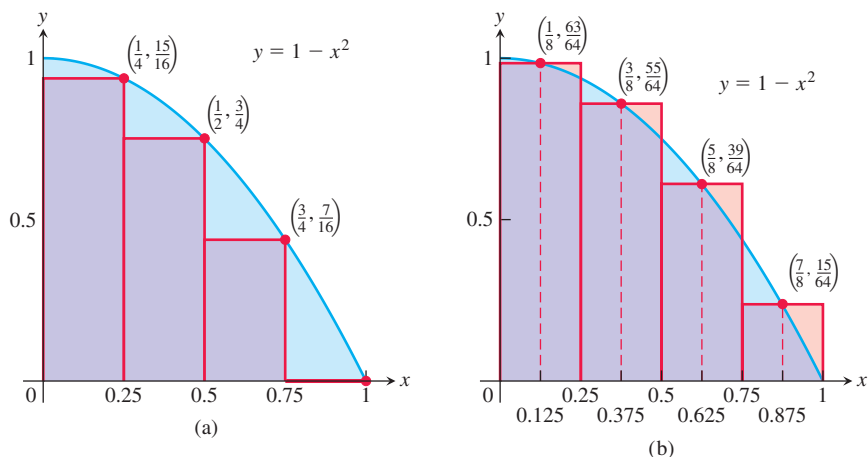


FIGURE 5.3 (a) Rectangles contained in R give a lower sum approximation for the area. This estimate undershoots the true value by the amount shaded in light blue. (b) The midpoint rule uses rectangles whose heights are the values of $y = f(x)$ at the midpoints of their bases. The estimate appears closer to the true value of the area because the light red overshoot areas roughly balance the light blue undershoot areas.

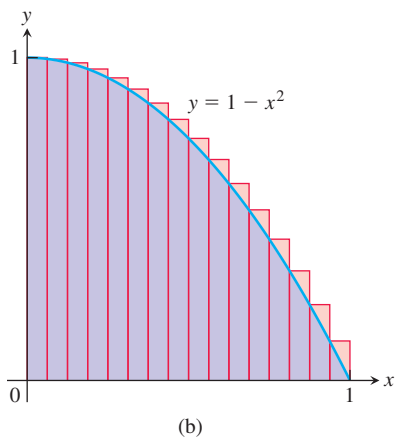
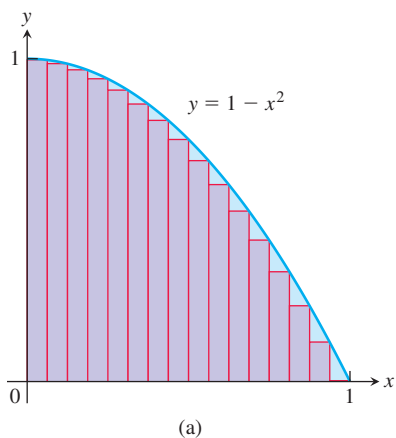


FIGURE 5.4 (a) A lower sum using 16 rectangles of equal width $\Delta x = 1/16$. (b) An upper sum using 16 rectangles.

Considering both lower and upper sum approximations gives us estimates for the area and a bound on the size of the possible error in these estimates since the true value of the area lies somewhere between them. Here the error cannot be greater than the difference $0.78125 - 0.53125 = 0.25$.

Yet another estimate can be obtained by using rectangles whose heights are the values of f at the midpoints of the bases of the rectangles (Figure 5.3b). This method of estimation is called the **midpoint rule** for approximating the area. The midpoint rule gives an estimate that is between a lower sum and an upper sum, but it is not clear whether it overestimates or underestimates the true area. With four rectangles of width $1/4$, as before, the midpoint rule estimates the area of R to be

$$A \approx \frac{63}{64} \cdot \frac{1}{4} + \frac{55}{64} \cdot \frac{1}{4} + \frac{39}{64} \cdot \frac{1}{4} + \frac{15}{64} \cdot \frac{1}{4} = \frac{172}{64} \cdot \frac{1}{4} = 0.671875.$$

In each of the sums that we computed, the interval $[a, b]$ over which the function f is defined was subdivided into n subintervals of equal width (or length) $\Delta x = (b - a)/n$, and f was evaluated at a point in each subinterval: c_1 in the first subinterval, c_2 in the second subinterval, and so on. For the upper sum we chose c_k so that $f(c_k)$ was the maximum value of f in the k th subinterval, for the lower sum we chose it so that $f(c_k)$ was the minimum, and for the midpoint rule we chose c_k to be the midpoint of the k th subinterval. In each case the finite sums have the form

$$f(c_1) \Delta x + f(c_2) \Delta x + f(c_3) \Delta x + \cdots + f(c_n) \Delta x.$$

As we take more and more rectangles, with each rectangle thinner than before, it appears that these finite sums give better and better approximations to the true area of the region R .

Figure 5.4a shows a lower sum approximation for the area of R using 16 rectangles of equal width. The sum of their areas is 0.634765625, which appears close to the true area but is still somewhat smaller since the rectangles lie inside R .

Figure 5.4b shows an upper sum approximation using 16 rectangles of equal width. The sum of their areas is 0.697265625, which is somewhat larger than the true area because the rectangles taken together contain R . The midpoint rule for 16 rectangles gives a total area approximation of 0.6669921875, but it is not immediately clear whether this estimate is larger or smaller than the true area.

Table 5.1 shows the values of upper and lower sum approximations to the area of R , using up to 1000 rectangles. The values of these approximations appear to be approaching $2/3$. In Section 5.2 we will see how to get an exact value of the area of regions such as R by taking a limit as the base width of each rectangle goes to zero and the number of rectangles goes to infinity. With the techniques developed there, we will be able to show that the area of R is exactly $2/3$.

TABLE 5.1 Finite approximations for the area of R

Number of subintervals	Lower sum	Midpoint sum	Upper sum
2	0.375	0.6875	0.875
4	0.5313	0.6719	0.7813
16	0.6348	0.6670	0.6973
50	0.6566	0.6667	0.6766
100	0.66165	0.666675	0.67165
1000	0.6661665	0.66666675	0.6671665

Distance Traveled

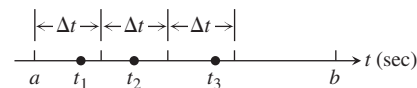
Suppose we know the velocity function $v(t)$ of a car that moves straight down a highway without changing direction, and we want to know how far it traveled between times $t = a$ and $t = b$. The position function $s(t)$ of the car has derivative $v(t)$. If we can find an antiderivative $F(t)$ of $v(t)$, then we can find the car's position function $s(t)$ by setting $s(t) = F(t) + C$. The distance traveled can then be found by calculating the change in position, $s(b) - s(a) = F(b) - F(a)$. However, if the velocity is known only by the readings at various times of a speedometer on the car, then we have no formula for the velocity from which to obtain an antiderivative that gives the position function. So what do we do in this situation?

If we know the velocity at a collection of times $t_1, t_2, \dots, t_n$, we can approximate the distance traveled by using finite sums in a way similar to the area estimates that we discussed before. We first subdivide the interval $[a, b]$ into short time intervals and assume that the velocity on each subinterval is fairly constant. Then we approximate the distance traveled on each time subinterval with the usual distance formula

$$\text{distance} = \text{velocity} \times \text{time}$$

and add the results across $[a, b]$.

Suppose the subdivided interval looks like



with the subintervals all of equal length Δt . Pick a number t_1 in the first interval. If Δt is so small that the velocity barely changes over a short time interval of duration Δt , then the distance traveled in the first time interval is about $v(t_1) \Delta t$. If t_2 is a number in the second interval, the distance traveled in the second time interval is about $v(t_2) \Delta t$. The sum of the distances traveled over all the time intervals is approximated by

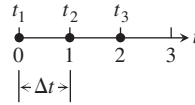
$$v(t_1) \Delta t + v(t_2) \Delta t + \cdots + v(t_n) \Delta t,$$

where n is the total number of subintervals. This sum is only an approximation to the true distance D , but the approximation increases in accuracy as we take more and more subintervals.

EXAMPLE 1 The velocity function of a projectile fired straight into the air is $f(t) = 160 - 9.8t$ m/sec. Use the summation technique just described to estimate how far the projectile rises during the first 3 sec. How close do the sums come to the exact value of 435.9 m? (We will see how to compute the exact value in Section 5.4.)

Solution We explore the results for different numbers of subintervals and different choices of evaluation points. Notice that $f(t)$ is decreasing, so choosing left endpoints gives an upper sum estimate, and choosing right endpoints gives a lower sum estimate.

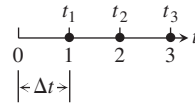
(a) Three subintervals of length 1, with f evaluated at left endpoints giving an upper sum:



With f evaluated at $t = 0, 1$, and 2 , we have

$$\begin{aligned} D &\approx f(t_1) \Delta t + f(t_2) \Delta t + f(t_3) \Delta t \\ &= [160 - 9.8(0)](1) + [160 - 9.8(1)](1) + [160 - 9.8(2)](1) \\ &= 450.6. \end{aligned}$$

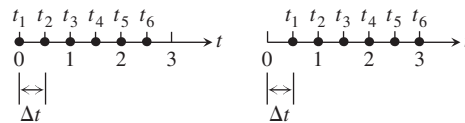
(b) Three subintervals of length 1, with f evaluated at right endpoints giving a lower sum:



With f evaluated at $t = 1, 2$, and 3 , we have

$$\begin{aligned} D &\approx f(t_1) \Delta t + f(t_2) \Delta t + f(t_3) \Delta t \\ &= [160 - 9.8(1)](1) + [160 - 9.8(2)](1) + [160 - 9.8(3)](1) \\ &= 421.2. \end{aligned}$$

(c) With six subintervals of length $1/2$, we get



These estimates give an upper sum using left endpoints: $D \approx 443.25$, and a lower sum using right endpoints: $D \approx 428.55$. These six-interval estimates are somewhat closer than the three-interval estimates. The results improve as the subintervals get shorter.

As we can see in Table 5.2, the left-endpoint upper sums approach the true value 435.9 from above, whereas the right-endpoint lower sums approach it from below. The true value lies between these upper and lower sums. The magnitude of the error in the closest entry is 0.23, a small percentage of the true value.

$$\begin{aligned} \text{Error magnitude} &= |\text{true value} - \text{calculated value}| \\ &= |435.9 - 435.67| = 0.23. \end{aligned}$$

$$\text{Error percentage} = \frac{0.23}{435.9} \approx 0.05\%.$$

It is reasonable to conclude from the table's last entries that the projectile rose about 436 m during its first 3 sec of flight. ■

TABLE 5.2 Travel-distance estimates

Number of subintervals	Length of each subinterval	Upper sum	Lower sum
3	1	450.6	421.2
6	1/2	443.25	428.55
12	1/4	439.58	432.23
24	1/8	437.74	434.06
48	1/16	436.82	434.98
96	1/32	436.36	435.44
192	1/64	436.13	435.67

Displacement Versus Distance Traveled

If an object with position function $s(t)$ moves along a coordinate line without changing direction, we can calculate the total distance it travels from $t = a$ to $t = b$ by summing the distance traveled over small intervals, as in Example 1. If the object reverses direction one or more times during the trip, then we need to use the object's *speed* $|v(t)|$, which is the absolute value of its velocity function, $v(t)$, to find the total distance traveled. Using the velocity itself, as in Example 1, gives instead an estimate of the object's **displacement**, $s(b) - s(a)$, the difference between its initial and final positions. To see the difference, think about what happens when you walk a kilometer from your home and then walk back. The total distance traveled is two kilometers, but your displacement is zero, because you end up back where you started.

To see why using the velocity function in the summation process gives an estimate of the displacement, partition the time interval $[a, b]$ into small enough equal subintervals Δt so that the object's velocity does not change very much from time t_{k-1} to t_k . Then $v(t_k)$ gives a good approximation of the velocity throughout the interval. Accordingly, the change in the object's position coordinate, which is its displacement during the time interval, is about

$$v(t_k) \Delta t.$$

The change is positive if $v(t_k)$ is positive and negative if $v(t_k)$ is negative.

In either case, the distance traveled by the object during the subinterval is about

$$|v(t_k)| \Delta t.$$

The **total distance traveled** over the time interval is approximately the sum

$$|v(t_1)| \Delta t + |v(t_2)| \Delta t + \cdots + |v(t_n)| \Delta t.$$

We will revisit these ideas in Section 5.4.

EXAMPLE 2 In Example 4 in Section 3.4, we analyzed the motion of a heavy rock blown straight up by a dynamite blast. In that example, we found the velocity of the rock at time t was $v(t) = 49 - 9.8t$ m/s. The rock was 78.4 m above the ground 2 s after the explosion, continued upward to reach a maximum height of 122.5 m at 5 s after the explosion, and then fell back down a distance of 44.1 m to reach the height of 78.4 m again at $t = 8$ s after the explosion. (See Figure 5.5.) The total distance traveled in these 8 seconds is $122.5 + 44.1 = 166.6$ m.

Solution If we follow a procedure like the one presented in Example 1, using the velocity function $v(t)$ in the summation process from $t = 0$ to $t = 8$, we obtain an estimate of the rock's height above the ground at time $t = 8$. Starting at time $t = 0$, the rock traveled upward a total of $78.4 + 44.1 = 122.5$ m, but then it peaked and traveled downward

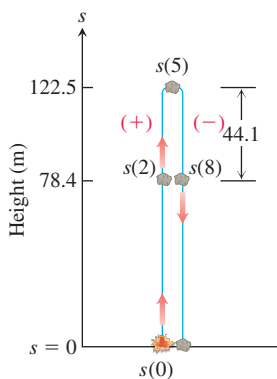


FIGURE 5.5 The rock in Example 2. The height $s = 78.4$ m is reached at $t = 2$ s and $t = 8$ s. The rock falls 44.1 m from its maximum height when $t = 8$.

TABLE 5.3 Velocity function

t	$v(t)$	t	$v(t)$
0	49	4.5	4.9
0.5	44.1	5.0	0
1.0	39.2	5.5	-4.9
1.5	34.3	6.0	-9.8
2.0	29.4	6.5	-14.7
2.5	24.5	7.0	-19.6
3.0	19.6	7.5	-24.5
3.5	14.7	8.0	-29.4
4.0	9.8		

44.1 m ending at a height of 78.4 m at time $t = 8$. The velocity $v(t)$ is positive during the upward travel, but negative while the rock falls back down. When we compute the sum $v(t_1) \Delta t + v(t_2) \Delta t + \cdots + v(t_n) \Delta t$, part of the upward positive distance change is canceled by the negative downward movement, giving in the end an approximation of the displacement from the initial position, equal to a positive change of 78.4 m.

On the other hand, if we use the speed $|v(t)|$, which is the absolute value of the velocity function, then distances traveled while moving up and distances traveled while moving down are both counted positively. Both the total upward motion of 122.5 m and the downward motion of 44.1 m are now counted as positive distances traveled, so the sum $|v(t_1)| \Delta t + |v(t_2)| \Delta t + \cdots + |v(t_n)| \Delta t$ gives us an approximation of 166.6 m, the total distance that the rock traveled from time $t = 0$ to time $t = 8$.

As an illustration of our discussion, we subdivide the interval $[0, 8]$ into 16 subintervals of length $\Delta t = 1/2$ and take the right endpoint of each subinterval as the value of t_k . Table 5.3 shows the values of the velocity function at these endpoints.

Using $v(t)$ in the summation process, we estimate the displacement at $t = 8$:

$$\begin{aligned} &(44.1 + 39.2 + 34.3 + 29.4 + 24.5 + 19.6 + 14.7 + 9.8 + 4.9 \\ &\quad + 0 - 4.9 - 9.8 - 14.7 - 19.6 - 24.5 - 29.4) \cdot \frac{1}{2} = 58.8 \\ &\text{Error magnitude} = 78.4 - 58.8 = 19.6 \end{aligned}$$

Using $|v(t)|$ in the summation process, we estimate the total distance traveled over the time interval $[0, 8]$:

$$\begin{aligned} &(44.1 + 39.2 + 34.3 + 29.4 + 24.5 + 19.6 + 14.7 + 9.8 + 4.9 \\ &\quad + 0 + 4.9 + 9.8 + 14.7 + 19.6 + 24.5 + 29.4) \cdot \frac{1}{2} = 161.7 \\ &\text{Error magnitude} = 166.6 - 161.7 = 4.9 \end{aligned}$$

If we take more and more subintervals of $[0, 8]$ in our calculations, the estimates to the heights 78.4 m and 166.6 m improve, as shown in Table 5.4. ■

TABLE 5.4 Travel estimates for a rock blown straight up during the time interval $[0, 8]$

Number of subintervals	Length of each subinterval	Displacement	Total distance
16	1/2	58.8	161.7
32	1/4	68.6	164.15
64	1/8	73.5	165.375
128	1/16	75.95	165.9875
256	1/32	77.175	166.29375
512	1/64	77.7875	166.446875

Average Value of a Nonnegative Continuous Function

The average value of a collection of n numbers $x_1, x_2, \dots, x_n$ is obtained by adding them together and dividing by n . But what is the average value of a continuous function f on an interval $[a, b]$? Such a function can assume infinitely many values. For example, the temperature at a certain location in a town is a continuous function that goes up and down each day. What does it mean to say that the average temperature in the town over the course of a day is 73 degrees?

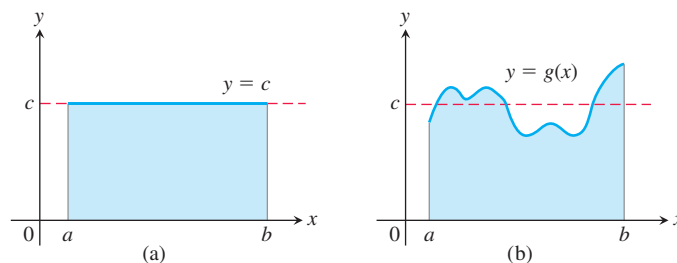


FIGURE 5.6 (a) The average value of $f(x) = c$ on $[a, b]$ is the area of the rectangle divided by $b - a$. (b) The average value of $g(x)$ on $[a, b]$ is the area beneath its graph divided by $b - a$.

When a function is constant, this question is easy to answer. A function with constant value c on an interval $[a, b]$ has average value c . When c is positive, its graph over $[a, b]$ gives a rectangle of height c . The average value of the function can then be interpreted geometrically as the area of this rectangle divided by its width $b - a$ (see Figure 5.6a).

What if we want to find the average value of a nonconstant function, such as the function g in Figure 5.6b? We can think of this graph as a snapshot of the height of some water that is sloshing around in a tank between enclosing walls at $x = a$ and $x = b$. As the water moves, its height over each point changes, but its average height remains the same. To get the average height of the water, we let it settle down until it is level and its height is constant. The resulting height c equals the area under the graph of g divided by $b - a$. We are led to *define* the average value of a nonnegative function on an interval $[a, b]$ to be the area under its graph divided by $b - a$. For this definition to be valid, we need a precise understanding of what is meant by the area under a graph. This will be obtained in Section 5.3, but for now we look at an example.

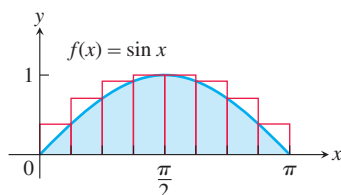


FIGURE 5.7 Approximating the area under $f(x) = \sin x$ between 0 and π to compute the average value of $\sin x$ over $[0, \pi]$, using eight rectangles (Example 3).

EXAMPLE 3 Estimate the average value of the function $f(x) = \sin x$ on the interval $[0, \pi]$.

Solution Looking at the graph of $\sin x$ between 0 and π in Figure 5.7, we can see that its average height is somewhere between 0 and 1. To find the average, we need to calculate the area A under the graph and then divide this area by the length of the interval, $\pi - 0 = \pi$.

We do not have a simple way to determine the area, so we approximate it with finite sums. To get an upper sum approximation, we add the areas of eight rectangles of equal width $\pi/8$ that together contain the region that is beneath the graph of $y = \sin x$ and above the x -axis on $[0, \pi]$. We choose the heights of the rectangles to be the largest value of $\sin x$ on each subinterval. Over a particular subinterval, this largest value may occur at the left endpoint, at the right endpoint, or somewhere between them. We evaluate $\sin x$ at this point to get the height of the rectangle for an upper sum. The sum of the rectangular areas then gives an estimate of the total area (Figure 5.7):

$$\begin{aligned} A &\approx \left(\sin \frac{\pi}{8} + \sin \frac{\pi}{4} + \sin \frac{3\pi}{8} + \sin \frac{\pi}{2} + \sin \frac{\pi}{2} + \sin \frac{5\pi}{8} + \sin \frac{3\pi}{4} + \sin \frac{7\pi}{8} \right) \cdot \frac{\pi}{8} \\ &\approx (0.38 + 0.71 + 0.92 + 1 + 1 + 0.92 + 0.71 + 0.38) \cdot \frac{\pi}{8} = (6.02) \cdot \frac{\pi}{8} \approx 2.364. \end{aligned}$$

To estimate the average value of $\sin x$ on $[0, \pi]$ we divide the estimated area by the length π of the interval and obtain the approximation $2.364/\pi \approx 0.753$.

Since we used an upper sum to approximate the area, this estimate is greater than the actual average value of $\sin x$ over $[0, \pi]$. If we use more and more rectangles, with each rectangle getting thinner and thinner, we get closer and closer to the exact average

TABLE 5.5 Average value of $\sin x$ on $0 \leq x \leq \pi$

Number of subintervals	Upper sum estimate
8	0.75342
16	0.69707
32	0.65212
50	0.64657
100	0.64161
1000	0.63712

value, as shown in Table 5.5. We will show in Section 5.3 that the true average value is $2/\pi \approx 0.63662$.

As before, we could just as well have used rectangles lying under the graph of $y = \sin x$ and calculated a lower sum approximation, or we could have used the midpoint rule. In each case, the approximations are close to the true area if all the rectangles are sufficiently thin. ■

Summary

The area under the graph of a positive function, the distance traveled by a moving object that doesn't change direction, and the average value of a nonnegative function f over an interval can all be approximated by finite sums constructed in a certain way. First we subdivide the interval into subintervals, treating f as if it were constant over each subinterval. Then we multiply the width of each subinterval by the value of f at some point within it and add these products together. If the interval $[a, b]$ is subdivided into n subintervals of equal widths $\Delta x = (b - a)/n$, and if $f(c_k)$ is the value of f at the chosen point c_k in the k th subinterval, this process gives a finite sum of the form

$$f(c_1) \Delta x + f(c_2) \Delta x + f(c_3) \Delta x + \cdots + f(c_n) \Delta x.$$

The choices for the c_k could maximize or minimize the value of f in the k th subinterval, or give some value in between. The true value lies somewhere between the approximations given by upper sums and lower sums. In the examples that we looked at, the finite sum approximations improved as we took more subintervals of smaller width.

EXERCISES 5.1

Area

In Exercises 1–4, apply finite approximations to estimate the area under the graph of the function using

- a. a lower sum with two rectangles of equal width.
 - b. a lower sum with four rectangles of equal width.
 - c. an upper sum with two rectangles of equal width.
 - d. an upper sum with four rectangles of equal width.
- $f(x) = x^2$ between $x = 0$ and $x = 1$.
 - $f(x) = x^3$ between $x = 0$ and $x = 1$.
 - $f(x) = 1/x$ between $x = 1$ and $x = 5$.
 - $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

Using rectangles, each of whose height is given by the value of the function at the midpoint of the rectangle's base (the *midpoint rule*), estimate the area under the graphs of the following functions, using first two and then four rectangles.

- $f(x) = x^2$ between $x = 0$ and $x = 1$.
- $f(x) = x^3$ between $x = 0$ and $x = 1$.
- $f(x) = 1/x$ between $x = 1$ and $x = 5$.
- $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

Distance

9. **Distance traveled** The accompanying table shows the velocity of a model train engine moving along a track for 10 s. Estimate the distance traveled by the engine using 10 subintervals of length 1 with
- a. left-endpoint values.
 - b. right-endpoint values.

Time (s)	Velocity (cm/s)	Time (s)	Velocity (cm/s)
0	0	6	11
1	12	7	6
2	22	8	2
3	10	9	6
4	5	10	0
5	13		

10. **Distance traveled upstream** You are sitting on the bank of a tidal river watching the incoming tide carry a bottle upstream. You record the velocity of the flow every 5 minutes for an hour, with the results shown in the accompanying table. About how far upstream did the bottle travel during that hour? Find an estimate using 12 subintervals of length 5 with

- a. left-endpoint values.
b. right-endpoint values.

Time (min)	Velocity (m/s)	Time (min)	Velocity (m/s)
0	1	35	1.2
5	1.2	40	1.0
10	1.7	45	1.8
15	2.0	50	1.5
20	1.8	55	1.2
25	1.6	60	0
30	1.4		

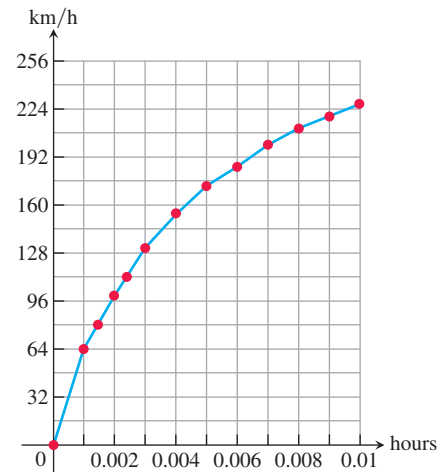
- 11. Length of a road** You and a companion are about to drive a twisty stretch of dirt road in a car whose speedometer works but whose odometer (kilometer counter) is broken. To find out how long this particular stretch of road is, you record the car's velocity at 10-second intervals, with the results shown in the accompanying table. Estimate the length of the road using

- a. left-endpoint values.
b. right-endpoint values.

Time (s)	Velocity (converted to m/s) (36 km/h = 10 m/s)	Time (s)	Velocity (converted to m/s) (30 km/h = 10 m/s)
0	0	70	5
10	15	80	7
20	5	90	12
30	12	100	15
40	10	110	10
50	15	120	12
60	12		

- 12. Distance from velocity data** The accompanying table gives data for the velocity of a vintage sports car accelerating from 0 to 228 km/h in 36 s (10 thousandths of an hour).

Time (h)	Velocity (km/h)	Time (h)	Velocity (km/h)
0.0	0	0.006	187
0.001	64	0.007	201
0.002	100	0.008	212
0.003	132	0.009	220
0.004	154	0.010	228
0.005	174		

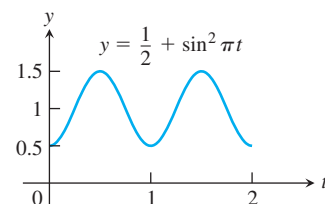


- a. Use rectangles to estimate how far the car traveled during the 36 s it took to reach 228 km/h.
b. Roughly how many seconds did it take the car to reach the halfway point? About how fast was the car going then?
- 13. Free fall with air resistance** An object is dropped straight down from a helicopter. The object falls faster and faster but its acceleration (rate of change of its velocity) decreases over time because of air resistance. The acceleration is measured in m/s^2 and recorded every second after the drop for 5 s, as shown:
- | | | | | | | |
|-----|-----|-------|-------|-------|-------|-------|
| t | 0 | 1 | 2 | 3 | 4 | 5 |
| a | 9.8 | 5.944 | 3.605 | 2.187 | 1.326 | 0.805 |
- a. Find an upper estimate for the speed when $t = 5$.
b. Find a lower estimate for the speed when $t = 5$.
c. Find an upper estimate for the distance fallen when $t = 3$.
- 14. Distance traveled by a projectile** An object is shot straight upward from sea level with an initial velocity of 122.5 m/s.
- a. Assuming that gravity is the only force acting on the object, give an upper estimate for its velocity after 5 s have elapsed. Use $g = 9.8 \text{ m/s}^2$ for the gravitational acceleration.
b. Find a lower estimate for the height attained after 5 s.

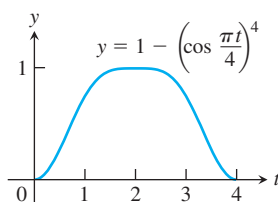
Average Value of a Function

In Exercises 15–18, use a finite sum to estimate the average value of f on the given interval by partitioning the interval into four subintervals of equal length and evaluating f at the subinterval midpoints.

15. $f(x) = x^3$ on $[0, 2]$
16. $f(x) = 1/x$ on $[1, 9]$
17. $f(t) = (1/2) + \sin^2 \pi t$ on $[0, 2]$



T 18. $f(t) = 1 - \left(\cos \frac{\pi t}{4}\right)^4$ on $[0, 4]$



Estimations

19. Water pollution Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the following table.

Time (h)	0	1	2	3	4
Leakage (L/h)	50	70	97	136	190
Time (h)	5	6	7	8	
Leakage (L/h)	265	369	516	720	

- Give an upper and a lower estimate of the total quantity of oil that has escaped after 5 hours.
 - Repeat part (a) for the quantity of oil that has escaped after 8 hours.
 - The tanker continues to leak 720 L/h after the first 8 hours. If the tanker originally contained 25,000 L of oil, approximately how many more hours will elapse in the worst case before all the oil has spilled? In the best case?
- 20. Air pollution** A power plant generates electricity by burning oil. Pollutants produced as a result of the burning process are removed by scrubbers in the smokestacks. Over time, the scrubbers become less efficient and eventually they must be replaced when the amount of pollution released exceeds government standards. Measurements are taken at the end of each month determining the rate at which pollutants are released into the atmosphere, recorded as follows.

Month	Jan	Feb	Mar	Apr	May	Jun
Pollutant release rate (tons/day)	0.20	0.25	0.27	0.34	0.45	0.52

Month	Jul	Aug	Sep	Oct	Nov	Dec
Pollutant release rate (tons/day)	0.63	0.70	0.81	0.85	0.89	0.95

- Assuming a 30-day month and that new scrubbers allow only 0.05 ton/day to be released, give an upper estimate of the total tonnage of pollutants released by the end of June. What is a lower estimate?
 - In the best case, approximately when will a total of 125 tons of pollutants have been released into the atmosphere?
- T** 21. Inscribe a regular n -sided polygon inside a circle of radius 1 and compute the area of the polygon for the following values of n :
- 4 (square)
 - 8 (octagon)
 - 16
 - Compare the areas in parts (a), (b), and (c) with the area inside the circle.
22. (Continuation of Exercise 21.)
- Inscribe a regular n -sided polygon inside a circle of radius 1 and compute the area of one of the n congruent triangles formed by drawing radii to the vertices of the polygon.
 - Compute the limit of the area of the inscribed polygon as $n \rightarrow \infty$.
 - Repeat the computations in parts (a) and (b) for a circle of radius r .

COMPUTER EXPLORATIONS

In Exercises 23–26, use a CAS to perform the following steps.

- Plot the functions over the given interval.
 - Subdivide the interval into $n = 100, 200$, and 1000 subintervals of equal length and evaluate the function at the midpoint of each subinterval.
 - Compute the average value of the function values generated in part (b).
 - Solve the equation $f(x) = (\text{average value})$ for x using the average value calculated in part (c) for the $n = 1000$ partitioning.
23. $f(x) = \sin x$ on $[0, \pi]$
24. $f(x) = \sin^2 x$ on $[0, \pi]$
25. $f(x) = x \sin \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$
26. $f(x) = x \sin^2 \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$

5.2 Sigma Notation and Limits of Finite Sums

While estimating with finite sums in Section 5.1, we encountered sums that had many terms (up to 1000 terms in Table 5.1). In this section we introduce a notation for sums that have a large number of terms. After describing this notation and its properties, we consider what happens as the number of terms in a sum approaches infinity.

Finite Sums and Sigma Notation

Sigma notation enables us to write a sum with many terms in the compact form

$$\sum_{k=1}^n a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n.$$

Σ is the capital Greek letter sigma

The Greek letter Σ (capital sigma, corresponding to our letter S), stands for “sum.” The **index of summation** k tells us where the sum begins (at the number below the Σ symbol) and where it ends (at the number above Σ). Any letter can be used to denote the index, but the letters i , j , k , and n are customary.

The summation symbol (Greek letter sigma)

The index k starts at $k = 1$.

The index k ends at $k = n$.

a_k is a formula for the k th term.

$a_k = a_1 + a_2 + \cdots + a_n$

Thus we can write the sum of the squares of the numbers 1 through 11 as

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 = \sum_{k=1}^{11} k^2,$$

and the sum of $f(i)$ for integers i from 1 to 100 as

$$f(1) + f(2) + f(3) + \cdots + f(100) = \sum_{i=1}^{100} f(i).$$

The starting index does not have to be 1; it can be any integer.

EXAMPLE 1

A sum in sigma notation	The sum written out, one term for each value of k	The value of the sum
$\sum_{k=1}^5 k$	$1 + 2 + 3 + 4 + 5$	15
$\sum_{k=1}^3 (-1)^k k$	$(-1)^1(1) + (-1)^2(2) + (-1)^3(3)$	$-1 + 2 - 3 = -2$
$\sum_{k=1}^2 \frac{k}{k+1}$	$\frac{1}{1+1} + \frac{2}{2+1}$	$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$
$\sum_{k=4}^5 \frac{k^2}{k-1}$	$\frac{4^2}{4-1} + \frac{5^2}{5-1}$	$\frac{16}{3} + \frac{25}{4} = \frac{139}{12}$

EXAMPLE 2 Express the sum $1 + 3 + 5 + 7 + 9$ in sigma notation.

Solution The formula generating the terms depends on what we choose the lower limit of summation to be, but the terms generated remain the same. It is often simplest to choose the starting index to be $k = 0$ or $k = 1$, but we can start with any integer.

$$\text{Starting with } k = 0: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=0}^4 (2k + 1)$$

$$\text{Starting with } k = 1: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=1}^5 (2k - 1)$$

$$\text{Starting with } k = 2: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=2}^6 (2k - 3)$$

$$\text{Starting with } k = -3: \quad 1 + 3 + 5 + 7 + 9 = \sum_{k=-3}^1 (2k + 7)$$

When we have a sum such as

$$\sum_{k=1}^3 (k + k^2),$$

we can rearrange its terms to form two sums:

$$\begin{aligned}\sum_{k=1}^3 (k + k^2) &= (1 + 1^2) + (2 + 2^2) + (3 + 3^2) \\ &= (1 + 2 + 3) + (1^2 + 2^2 + 3^2) \quad \text{Regroup terms.} \\ &= \sum_{k=1}^3 k + \sum_{k=1}^3 k^2.\end{aligned}$$

This illustrates a general rule for finite sums:

$$\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k.$$

This and three other rules are given below. Proofs of these rules can be obtained using mathematical induction (see Appendix A.3).

Algebra Rules for Finite Sums

1. *Sum Rule:*

$$\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$$

2. *Difference Rule:*

$$\sum_{k=1}^n (a_k - b_k) = \sum_{k=1}^n a_k - \sum_{k=1}^n b_k$$

3. *Constant Multiple Rule:*

$$\sum_{k=1}^n c a_k = c \cdot \sum_{k=1}^n a_k \quad (\text{Any number } c)$$

4. *Constant Value Rule:*

$$\sum_{k=1}^n c = n \cdot c \quad (\text{Any number } c)$$

$\sum_{k=1}^{100} 5 = 5 + 5 + 5 + \cdots 5 = 100 \times 5$

EXAMPLE 3 We demonstrate the use of the algebra rules.

(a) $\sum_{k=1}^n (3k - k^2) = 3 \sum_{k=1}^n k - \sum_{k=1}^n k^2$ Difference Rule and Constant Multiple Rule

(b) $\sum_{k=1}^n (-a_k) = \sum_{k=1}^n (-1) \cdot a_k = -1 \cdot \sum_{k=1}^n a_k = -\sum_{k=1}^n a_k$ Constant Multiple Rule

(c) $\sum_{k=1}^3 (k + 4) = \sum_{k=1}^3 k + \sum_{k=1}^3 4$ Sum Rule
 $= (1 + 2 + 3) + (3 \cdot 4)$ Constant Value Rule
 $= 6 + 12 = 18$

(d) $\sum_{k=1}^n \frac{1}{n} = n \cdot \frac{1}{n} = 1$ Constant Value Rule
($1/n$ is constant)

HISTORICAL BIOGRAPHY

Carl Friedrich Gauss
(1777–1855)

Gauss was born in Brunswick, Germany. The list of Gauss's accomplishments in science and mathematics is astonishing, ranging from the invention of the electric telegraph (with Wilhelm Weber in 1833) to the development of a theory of planetary orbits and the development of an accurate theory of non-Euclidean geometry.

To know more, visit the companion Website.

Over the years, people have discovered a variety of formulas for the values of finite sums. The most famous of these are the formula for the sum of the first n positive integers (Gauss is said to have discovered it at age 8) and the formulas for the sums of the squares and cubes of the first n positive integers.

$$1+2+3+4+\dots+100 = \frac{100(101)}{2}$$

EXAMPLE 4 Show that the sum of the first n positive integers is

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}.$$

Solution The formula tells us that the sum of the first 4 positive integers is

$$\frac{(4)(5)}{2} = 10.$$

Addition verifies this prediction:

$$1 + 2 + 3 + 4 = 10.$$

To prove the formula in general, we write out the terms in the sum twice, once forward and once backward.

$$\begin{array}{ccccccc} 1 & + & 2 & + & 3 & + & \cdots + n \\ n & + & (n-1) & + & (n-2) & + & \cdots + 1 \end{array}$$

If we add the two terms in the first column we get $1 + n = n + 1$. Similarly, if we add the two terms in the second column we get $2 + (n-1) = n + 1$. The two terms in any column sum to $n + 1$. When we add the n columns together we get n terms, each equal to $n + 1$, for a total of $n(n + 1)$. Since this is twice the desired quantity, the sum of the first n integers is $n(n + 1)/2$. ■

Formulas for the sums of the squares and cubes of the first n positive integers are proved using mathematical induction (see Appendix A.3). We state them here.

$$\begin{array}{l} \text{Sum of the first } n \text{ squares: } \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6} \\ \text{Sum of the first } n \text{ cubes: } \sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2}\right)^2 \end{array}$$

Limits of Finite Sums

The finite sum approximations that we considered in Section 5.1 became more accurate as the number of terms increased and the subinterval widths (lengths) narrowed. The next example shows how to calculate a limiting value as the widths of the subintervals go to zero and the number of subintervals grows to infinity.

EXAMPLE 5 Find the limiting value of lower sum approximations to the area of the region R below the graph of $y = 1 - x^2$ and above the interval $[0, 1]$ on the x -axis using equal-width rectangles whose widths approach zero and whose number approaches infinity. (See Figure 5.4a.)

Solution We compute a lower sum approximation using n rectangles of equal width Δx and then see what happens as $n \rightarrow \infty$. We start by subdividing $[0, 1]$ into n equal width subintervals

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \dots, \left[\frac{n-1}{n}, \frac{n}{n}\right].$$

Each subinterval has width $\Delta x = 1/n$. The function $1 - x^2$ is decreasing on $[0, 1]$, and its smallest value in a subinterval occurs at the subinterval's right endpoint. So a lower sum is constructed with rectangles whose height over the subinterval $[(k-1)/n, k/n]$ is $f(k/n) = 1 - (k/n)^2$, giving the sum

$$f\left(\frac{1}{n}\right) \cdot \frac{1}{n} + f\left(\frac{2}{n}\right) \cdot \frac{1}{n} + \cdots + f\left(\frac{k}{n}\right) \cdot \frac{1}{n} + \cdots + f\left(\frac{n}{n}\right) \cdot \frac{1}{n}.$$

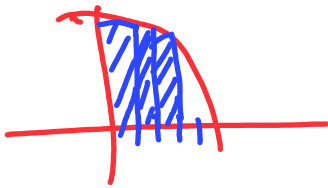
We write this in sigma notation and simplify,

$$\begin{aligned}
 \sum_{k=1}^n f\left(\frac{k}{n}\right) \cdot \frac{1}{n} &= \sum_{k=1}^n \left(1 - \left(\frac{k}{n}\right)^2\right) \frac{1}{n} \\
 &= \sum_{k=1}^n \left(\frac{1}{n} - \frac{k^2}{n^3}\right) \\
 &= \sum_{k=1}^n \frac{1}{n} - \sum_{k=1}^n \frac{k^2}{n^3} && \text{Difference Rule} \\
 &= n \cdot \frac{1}{n} - \frac{1}{n^3} \sum_{k=1}^n k^2 && \text{Constant Value and Constant Multiple Rules} \\
 &= 1 - \left(\frac{1}{n^3}\right) \frac{n(n+1)(2n+1)}{6} && \text{Sum of the first } n \text{ squares} \\
 &= 1 - \frac{2n^3 + 3n^2 + n}{6n^3}. && \text{Numerator expanded}
 \end{aligned}$$

We have obtained an expression for the lower sum that holds for any n . Taking the limit of this expression as $n \rightarrow \infty$, we see that the lower sums converge as the number of subintervals increases and the subinterval widths approach zero:

$$\lim_{n \rightarrow \infty} \left(1 - \frac{2n^3 + 3n^2 + n}{6n^3}\right) = 1 - \frac{2}{6} = \frac{2}{3}.$$

The lower sum approximations converge to $2/3$. A similar calculation shows that the upper sum approximations also converge to $2/3$. Any finite sum approximation $\sum_{k=1}^n f(c_k)(1/n)$ also converges to the same value, $2/3$. This is because it is possible to show that any finite sum approximation is trapped between the lower and upper sum approximations. For this reason we are led to *define* the area of the region R as this limiting value. In Section 5.3 we study the limits of such finite approximations in a general setting. ■



HISTORICAL BIOGRAPHY

Georg Friedrich Bernhard Riemann
(1826–1866)

Riemann was born in Hanover, Germany. His doctorate was obtained under the direction of Gauss in the theory of complex variables. He also worked with physicist Wilhelm Weber. He introduced the foundational ideas of differential geometry and contributed to dynamics, non-Euclidean geometry, and computational physics.

To know more, visit the companion Website.

Riemann Sums

The theory of limits of finite approximations was made precise by the German mathematician Bernhard Riemann. We now introduce the notion of a *Riemann sum*, which underlies the theory of the definite integral that will be presented in the next section.

We begin with an arbitrary bounded function f defined on a closed interval $[a, b]$. Like the function pictured in Figure 5.8, f may have negative as well as positive values. We subdivide the interval $[a, b]$ into subintervals, not necessarily of equal width (or length), and form sums in the same way as for the finite approximations in Section 5.1. To do so, we choose $n - 1$ points $\{x_1, x_2, x_3, \dots, x_{n-1}\}$ between a and b that are in increasing order, so that

$$a < x_1 < x_2 < \cdots < x_{n-1} < b.$$

To make the notation consistent, we set $x_0 = a$ and $x_n = b$, so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The set of all of these points,

$$P = \{x_0, x_1, x_2, \dots, x_{n-1}, x_n\},$$

is called a **partition** of $[a, b]$.

The partition P divides $[a, b]$ into the n closed subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n].$$

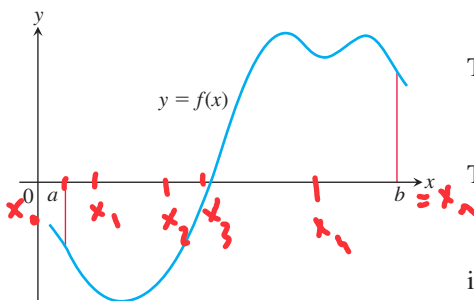


FIGURE 5.8 A typical continuous function $y = f(x)$ over a closed interval $[a, b]$.

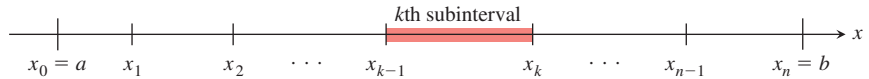
HISTORICAL BIOGRAPHY

Richard Dedekind
(1831–1916)

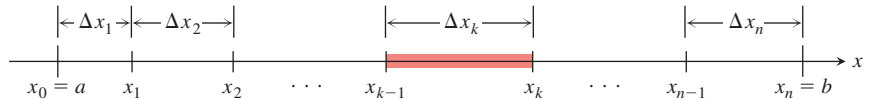
Dedekind grew up in Germany and in 1850 entered the University of Göttingen. There he studied with Bernhard Riemann and Carl Gauss. Like Gauss, Dedekind preferred to study the theoretical aspects of number theory. His work on irrational numbers gave the subject a logical foundation.

To know more, visit the companion Website.

The first of these subintervals is $[x_0, x_1]$, the second is $[x_1, x_2]$, and the k th subinterval is $[x_{k-1}, x_k]$ (where k is an integer between 1 and n).



The width of the first subinterval $[x_0, x_1]$ is denoted Δx_1 , the width of the second $[x_1, x_2]$ is Δx_2 , and the width of the k th subinterval is $\Delta x_k = x_k - x_{k-1}$.



If all n subintervals have equal width, then their common width, which we call Δx , is equal to $(b - a)/n$. Using equal width subintervals is often the simplest choice when doing computations.

In each subinterval we select some point. The point chosen in the k th subinterval $[x_{k-1}, x_k]$ is called c_k . Then on each subinterval, we stand a vertical rectangle that stretches from the x -axis to touch the curve at $(c_k, f(c_k))$. These rectangles can be above or below the x -axis, depending on whether $f(c_k)$ is positive or negative, or on the x -axis if $f(c_k) = 0$ (see Figure 5.9).

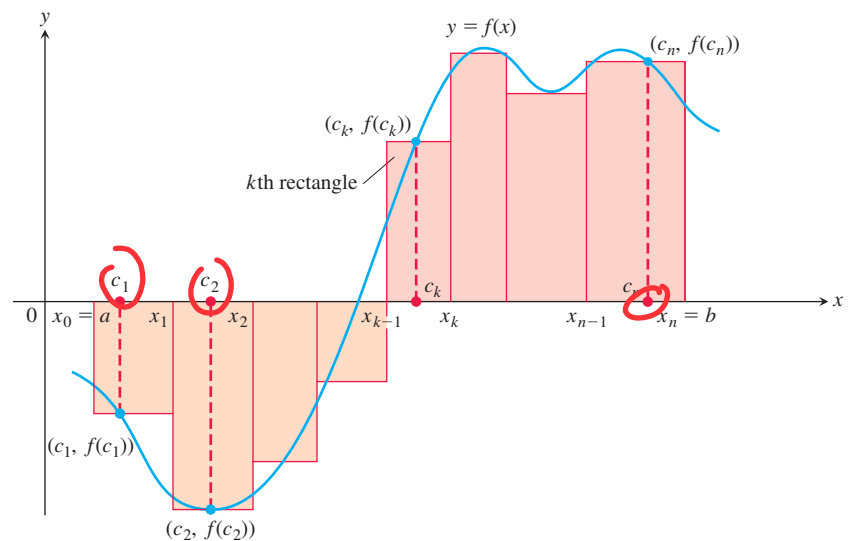


FIGURE 5.9 The rectangles approximate the region between the graph of the function $y = f(x)$ and the x -axis. Figure 5.8 has been repeated and enlarged, the partition of $[a, b]$ and the points c_k have been added, and the corresponding rectangles with heights $f(c_k)$ are shown.

On each subinterval we form the product $f(c_k) \cdot \Delta x_k$. This product is positive, negative, or zero, depending on the sign of $f(c_k)$. When $f(c_k) > 0$, the product $f(c_k) \cdot \Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width Δx_k . When $f(c_k) < 0$, the product $f(c_k) \cdot \Delta x_k$ is a negative number, the negative of the area of a rectangle of width Δx_k that drops from the x -axis to the negative number $f(c_k)$.

Finally, we sum all these products to get

$$f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots \quad \boxed{S_p = \sum_{k=1}^n f(c_k) \Delta x_k}$$

The sum S_p is called a **Riemann sum for f on the interval $[a, b]$** . There are many such sums, depending on the partition P we choose and on the choices of the points c_k in the

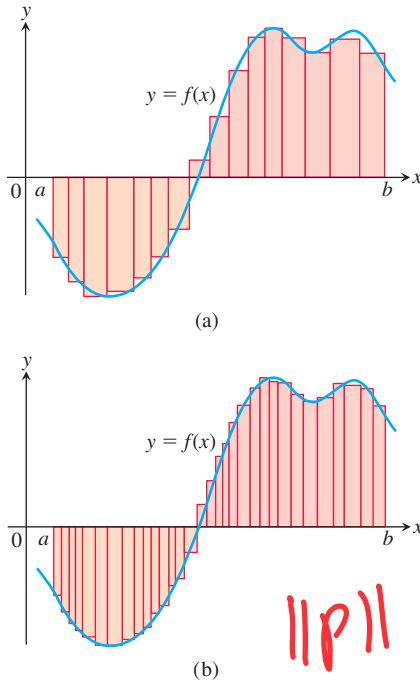


FIGURE 5.10 The curve of Figure 5.9 with rectangles from finer partitions of $[a, b]$. Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of f and the x -axis with increasing accuracy.

subintervals. For instance, we could choose n subintervals all having equal width $\Delta x = (b - a)/n$ to partition $[a, b]$, and then choose the point c_k to be the right-hand endpoint of each subinterval when forming the Riemann sum (as we did in Example 5). This choice leads to the Riemann sum formula

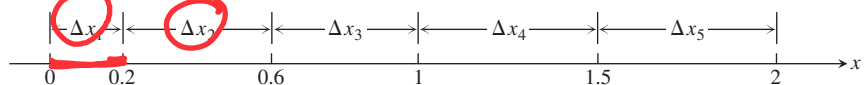
$$S_n = \sum_{k=1}^n f\left(a + k \frac{b-a}{n}\right) \cdot \left(\frac{b-a}{n}\right).$$

$X_k = a + k \Delta x$
 $\Delta x = \frac{b-a}{n}$

Similar formulas can be obtained if instead we choose c_k to be the left-hand endpoint, or the midpoint, of each subinterval.

In the cases in which the subintervals all have equal width $\Delta x = (b - a)/n$, we can make them thinner by simply increasing their number n . When a partition has subintervals of varying widths, we can ensure they are all thin by controlling the width of a widest (longest) subinterval. We define the **norm** of a partition P , written $\|P\|$, to be the largest of all the subinterval widths. If $\|P\|$ is a small number, then all of the subintervals in the partition P have a small width.

EXAMPLE 6 The set $P = \{0, 0.2, 0.6, 1, 1.5, 2\}$ is a partition of $[0, 2]$. There are five subintervals of P : $[0, 0.2]$, $[0.2, 0.6]$, $[0.6, 1]$, $[1, 1.5]$, and $[1.5, 2]$:



The lengths of the subintervals are $\Delta x_1 = 0.2$, $\Delta x_2 = 0.4$, $\Delta x_3 = 0.4$, $\Delta x_4 = 0.5$, and $\Delta x_5 = 0.5$. The longest subinterval length is 0.5, so the norm of the partition is $\|P\| = 0.5$. In this example, there are two subintervals of this length.

Any Riemann sum associated with a partition of a closed interval $[a, b]$ defines rectangles that approximate the region between the graph of a continuous function f and the x -axis. Partitions with norm approaching zero lead to collections of rectangles that approximate this region with increasing accuracy, as suggested by Figure 5.10. We will see in the next section that if the function f is continuous over the closed interval $[a, b]$, then no matter how we choose the partition P and the points c_k in its subintervals, the Riemann sums corresponding to these choices will approach a single limiting value as the subinterval widths (which are controlled by the norm of the partition) approach zero.

EXERCISES 5.2

Sigma Notation

Write the sums in Exercises 1–6 without sigma notation. Then evaluate them.

1. $\sum_{k=1}^2 \frac{6k}{k+1}$

2. $\sum_{k=1}^3 \frac{k-1}{k}$

3. $\sum_{k=1}^4 \cos k\pi$

4. $\sum_{k=1}^5 \sin k\pi$

5. $\sum_{k=1}^3 (-1)^{k+1} \sin \frac{\pi}{k}$

6. $\sum_{k=1}^4 (-1)^k \cos k\pi$

7. Which of the following express $1 + 2 + 4 + 8 + 16 + 32$ in sigma notation?

a. $\sum_{k=1}^6 2^{k-1}$

b. $\sum_{k=0}^5 2^k$

c. $\sum_{k=-1}^4 2^{k+1}$

8. Which of the following express $1 - 2 + 4 - 8 + 16 - 32$ in sigma notation?

a. $\sum_{k=1}^6 (-2)^{k-1}$

b. $\sum_{k=0}^5 (-1)^k 2^k$

c. $\sum_{k=-2}^3 (-1)^{k+1} 2^{k+2}$

9. Which formula is not equivalent to the other two?

a. $\sum_{k=2}^4 \frac{(-1)^{k-1}}{k-1}$

b. $\sum_{k=0}^2 \frac{(-1)^k}{k+1}$

c. $\sum_{k=-1}^1 \frac{(-1)^k}{k+2}$

10. Which formula is not equivalent to the other two?

a. $\sum_{k=1}^4 (k-1)^2$

b. $\sum_{k=-1}^3 (k+1)^2$

c. $\sum_{k=-3}^{-1} k^2$

Express the sums in Exercises 11–16 in sigma notation. The form of your answer will depend on your choice for the starting index.

11. $1 + 2 + 3 + 4 + 5 + 6$

12. $1 + 4 + 9 + 16$

13. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

14. $2 + 4 + 6 + 8 + 10$

$$15. 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \qquad 16. -\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5}$$

Values of Finite Sums

17. Suppose that $\sum_{k=1}^n a_k = -5$ and $\sum_{k=1}^n b_k = 6$. Find the values of

$$\begin{array}{lll} \text{a. } \sum_{k=1}^n 3a_k & \text{b. } \sum_{k=1}^n \frac{b_k}{6} & \text{c. } \sum_{k=1}^n (a_k + b_k) \\ \text{d. } \sum_{k=1}^n (a_k - b_k) & \text{e. } \sum_{k=1}^n (b_k - 2a_k) & \end{array}$$

18. Suppose that $\sum_{k=1}^n a_k = 0$ and $\sum_{k=1}^n b_k = 1$. Find the values of

$$\begin{array}{ll} \text{a. } \sum_{k=1}^n 8a_k & \text{b. } \sum_{k=1}^n 250b_k \\ \text{c. } \sum_{k=1}^n (a_k + 1) & \text{d. } \sum_{k=1}^n (b_k - 1) \end{array}$$

Evaluate the sums in Exercises 19–36.

$$19. \text{ a. } \sum_{k=1}^{10} k \qquad \text{b. } \sum_{k=1}^{10} k^2 \qquad \text{c. } \sum_{k=1}^{10} k^3$$

$$20. \text{ a. } \sum_{k=1}^{13} k \qquad \text{b. } \sum_{k=1}^{13} k^2 \qquad \text{c. } \sum_{k=1}^{13} k^3$$

$$21. \sum_{k=1}^7 (-2k) \qquad 22. \sum_{k=1}^5 \frac{\pi k}{15}$$

$$23. \sum_{k=1}^6 (3 - k^2) \qquad 24. \sum_{k=1}^6 (k^2 - 5)$$

$$25. \sum_{k=1}^5 k(3k + 5) \qquad 26. \sum_{k=1}^7 k(2k + 1)$$

$$27. \sum_{k=1}^5 \frac{k^3}{225} + \left(\sum_{k=1}^5 k \right)^3 \qquad 28. \left(\sum_{k=1}^7 k \right)^2 - \sum_{k=1}^7 \frac{k^3}{4}$$

$$29. \text{ a. } \sum_{k=1}^7 3 \qquad \text{b. } \sum_{k=1}^{500} 7 \qquad \text{c. } \sum_{k=3}^{264} 10$$

$$30. \text{ a. } \sum_{k=9}^{36} k \qquad \text{b. } \sum_{k=3}^{17} k^2 \qquad \text{c. } \sum_{k=18}^{71} k(k - 1)$$

$$31. \text{ a. } \sum_{k=1}^n 4 \qquad \text{b. } \sum_{k=1}^n c \qquad \text{c. } \sum_{k=1}^n (k - 1)$$

$$32. \text{ a. } \sum_{k=1}^n \left(\frac{1}{n} + 2n \right) \qquad \text{b. } \sum_{k=1}^n \frac{c}{n} \qquad \text{c. } \sum_{k=1}^n \frac{k}{n^2}$$

$$33. \sum_{k=1}^{50} [(k + 1)^2 - k^2] \qquad 34. \sum_{k=2}^{20} [\sin(k - 1) - \sin k]$$

$$35. \sum_{k=7}^{30} (\sqrt{k - 4} - \sqrt{k - 3})$$

$$36. \sum_{k=1}^{40} \frac{1}{k(k + 1)} \quad \left(\text{Hint: } \frac{1}{k(k + 1)} = \frac{1}{k} - \frac{1}{k + 1} \right)$$

Riemann Sums

In Exercises 37–40, graph each function $f(x)$ over the given interval. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\sum_{k=1}^4 f(c_k) \Delta x_k$, given that c_k is the (a) left-hand endpoint, (b) right-hand endpoint, (c) midpoint of the k th subinterval. (Make a separate sketch for each set of rectangles.)

$$37. f(x) = x^2 - 1, \quad [0, 2] \qquad 38. f(x) = -x^2, \quad [0, 1]$$

$$39. f(x) = \sin x, \quad [-\pi, \pi]$$

$$40. f(x) = \sin x + 1, \quad [-\pi, \pi]$$

41. Find the norm of the partition $P = \{0, 1.2, 1.5, 2.3, 2.6, 3\}$.

42. Find the norm of the partition $P = \{-2, -1.6, -0.5, 0, 0.8, 1\}$.

Limits of Riemann Sums

For the functions in Exercises 43–50, find a formula for the Riemann sum obtained by dividing the interval $[a, b]$ into n equal subintervals and using the right-hand endpoint for each c_k . Then take a limit of these sums as $n \rightarrow \infty$ to calculate the area under the curve over $[a, b]$.

$$43. f(x) = 1 - x^2 \text{ over the interval } [0, 1].$$

$$44. f(x) = 2x \text{ over the interval } [0, 3].$$

$$45. f(x) = x^2 + 1 \text{ over the interval } [0, 3].$$

$$46. f(x) = 3x^2 \text{ over the interval } [0, 1].$$

$$47. f(x) = x + x^2 \text{ over the interval } [0, 1].$$

$$48. f(x) = 3x + 2x^2 \text{ over the interval } [0, 1].$$

$$49. f(x) = 2x^3 \text{ over the interval } [0, 1].$$

$$50. f(x) = x^2 - x^3 \text{ over the interval } [-1, 0].$$

5.3 The Definite Integral

In this section we consider the limit of general Riemann sums as the norm of the partitions of a closed interval $[a, b]$ approaches zero. This limiting process leads us to the definition of the *definite integral* of a function over a closed interval $[a, b]$.

Definition of the Definite Integral

The definition of the definite integral is based on the fact that for some functions, as the norm of the partitions of $[a, b]$ approaches zero, the values of the corresponding Riemann

sums approach a limiting value J . In particular, this is true for continuous and piecewise-continuous functions. We again use the symbol ε to represent a small positive number, and use it to specify how close to J the Riemann sum must be. The symbol δ is used for a second small positive number that specifies how small the norm of a partition must be in order for the Riemann sum to differ from J by no more than ε . We now define this limit precisely.

DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^n f(c_k) \Delta x_k$ if the following condition is satisfied:

Given any number $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^n f(c_k) \Delta x_k - J \right| < \varepsilon.$$

The definition involves a limiting process in which the norm of the partition goes to zero. When this limit exists, the function f is said to be *integrable* over $[a, b]$.

We have many choices for a partition P with norm going to zero, and many choices of points c_k for each partition. The definite integral exists when we always get the same limit J , no matter what choices are made. When the limit exists we write

$$J = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k,$$

(Handwritten notes: $n \rightarrow \infty$, $\Delta x \rightarrow 0$, and "Riemann Sum" with arrows pointing to the sum and limit respectively.)

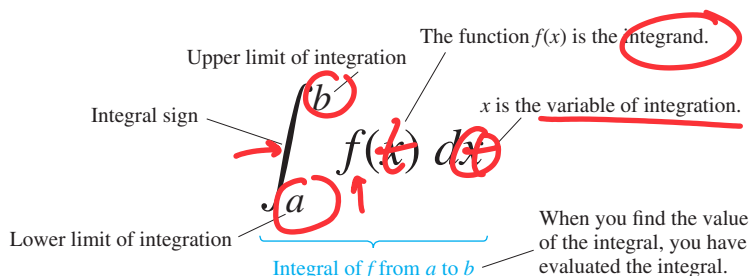
and we say that the *definite integral exists*.

Leibniz introduced a notation for the definite integral that captures its construction as a limit of Riemann sums. He envisioned the finite sums $\sum_{k=1}^n f(c_k) \Delta x_k$ becoming an infinite sum of function values $f(x)$ multiplied by “infinitesimal” subinterval widths dx . The sum symbol $\sum$ is replaced in the limit by the integral symbol $\int$, whose origin is in the letter “S” (for sum). The function values $f(c_k)$ are replaced by a continuous selection of function values $f(x)$. The subinterval widths Δx_k become the differential dx . It is as if we were summing all products of the form $f(x) \cdot dx$ as x goes from a to b . While this notation captures the underlying ideas, it is Riemann’s definition that gives a precise meaning to the definite integral.

If the definite integral exists, then instead of writing J , we write

$$\int_a^b f(x) dx.$$

We read this as “the integral from a to b of f of x dee x ” or sometimes as “the integral from a to b of f of x with respect to x .” The component parts in the integral symbol also have names:



When the definite integral exists, we say that the Riemann sums of f on $[a, b]$ **converge** to the definite integral $J = \int_a^b f(x) dx$ and that f is **integrable** over $[a, b]$.

If we choose all the subintervals in a partition to have equal width $\Delta x = (b - a)/n$, the Riemann sums have the form

$$S_n = \sum_{k=1}^n f(c_k) \Delta x_k = \sum_{k=1}^n f(c_k) \left(\frac{b-a}{n} \right), \quad \Delta x_k = \Delta x = (b-a)/n \text{ for all } k$$

where c_k is any point in the k th subinterval. If the definite integral exists, then these Riemann sums converge to the definite integral of f over $[a, b]$, so

$$J = \int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \left(\frac{b-a}{n} \right). \quad \text{For equal-width subintervals, the condition } \|P\| \rightarrow 0 \text{ is equivalent to } n \rightarrow \infty.$$

If we pick the point c_k to be the right endpoint of the k th subinterval, so that $c_k = a + k \Delta x = a + k(b-a)/n$, then the formula for the definite integral becomes

The Definite Integral as a Limit of Riemann Sums with Equal-Width Subintervals

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f\left(a + k \frac{b-a}{n}\right) \left(\frac{b-a}{n}\right) \quad (1)$$

Equation (1) gives an explicit formula that can be used to compute definite integrals. When the definite integral exists, the Riemann sums coming from other choices of partitions and locations of points c_k will have the same limit as $n \rightarrow \infty$, provided that the norms of the partitions approach zero. Another choice of Riemann sums converging to the definite integral is given by the Midpoint Rule, discussed later in this section.

The value of the definite integral of a function over any particular interval depends on the function, not on the letter we choose to represent its independent variable. If we decide to use t or u instead of x , we simply write the integral as

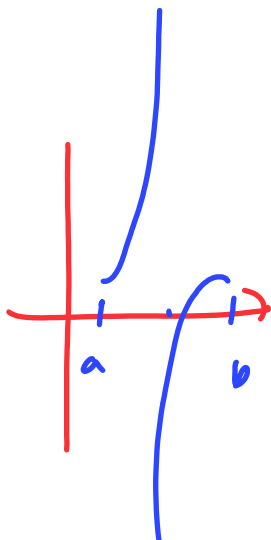
$$\int_a^b f(t) dt \quad \text{or} \quad \int_a^b f(u) du \quad \text{instead of} \quad \int_a^b f(x) dx.$$

No matter how we write the integral, it is still the same number, the limit of the Riemann sums as the norm of the partition approaches zero. Since it does not matter what letter we use, the variable of integration is called a **dummy variable**. In the three integrals given above, the dummy variables are t , u , and x .

Integrable and Nonintegrable Functions

Not every function defined over a closed interval $[a, b]$ is integrable even if the function is bounded. That is, the Riemann sums for some functions might not converge to the same limiting value, or to any value at all. Understanding which functions defined over $[a, b]$ are integrable and which are not requires advanced mathematical analysis, but fortunately most functions that commonly occur in applications are integrable. In particular, every continuous function over $[a, b]$ is integrable over this interval, and so is every function that has no more than a finite number of jump discontinuities on $[a, b]$. (See Figures 1.9 and 1.10. Such functions are called *piecewise continuous functions*, and they are defined in Additional Exercises 11–18 at the end of this chapter.) The following theorem, which is proved in more advanced courses, establishes these results.

THEOREM 1—Continuous Functions Are Integrable If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x) dx$ exists and f is integrable over $[a, b]$.



The idea behind Theorem 1 for continuous functions is given in Exercises 86 and 87. Briefly, when f is continuous, we can choose each c_k so that $f(c_k)$ gives the maximum value of f on the subinterval $[x_{k-1}, x_k]$, resulting in an upper sum. Likewise, we can choose c_k to give the minimum value of f on $[x_{k-1}, x_k]$ to obtain a lower sum. The upper and lower sums can be shown to converge to the same limiting value as the norm of the partition P tends to zero. Moreover, every Riemann sum is trapped between the values of the upper and lower sums, so every Riemann sum converges to the same limit as well. Therefore, the number J in the definition of the definite integral exists, and the continuous function f is integrable over $[a, b]$.

For integrability to fail, a function needs to be sufficiently discontinuous that the region between its graph and the x -axis cannot be approximated well by increasingly thin rectangles. Our first example is a function that is not integrable over a closed interval.

EXAMPLE 1 The function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational,} \end{cases}$$

has no Riemann integral over $[0, 1]$. Underlying this is the fact that between any two numbers there are both a rational number and an irrational number. Thus the function jumps up and down too erratically over $[0, 1]$ to allow the region beneath its graph and above the x -axis to be approximated by rectangles, no matter how thin they are. In fact, we will show that upper sum approximations and lower sum approximations converge to different limiting values.

If we choose a partition P of $[0, 1]$, then the lengths of the intervals in the partition sum to 1; that is, $\sum_{k=1}^n \Delta x_k = 1$. In each subinterval $[x_{k-1}, x_k]$ there is a rational point, say c_k . Because c_k is rational, $f(c_k) = 1$. Since 1 is the maximum value that f can take anywhere, the upper sum approximation for this choice of c_k 's is

$$U = \sum_{k=1}^n f(c_k) \Delta x_k = \sum_{k=1}^n (1) \Delta x_k = 1.$$

As the norm of the partition approaches 0, these upper sum approximations converge to 1 (because each approximation is equal to 1).

On the other hand, we could pick the c_k 's differently and get a different result. Each subinterval $[x_{k-1}, x_k]$ also contains an irrational point c_k , and for this choice $f(c_k) = 0$. Since 0 is the minimum value that f can take anywhere, this choice of c_k gives us the minimum value of f on the subinterval. The corresponding lower sum approximation is

$$L = \sum_{k=1}^n f(c_k) \Delta x_k = \sum_{k=1}^n (0) \Delta x_k = 0.$$

These lower sum approximations converge to 0 as the norm of the partition converges to 0 (because they each equal 0).

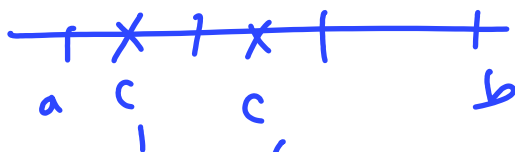
Thus making different choices for the points c_k results in different limits for the corresponding Riemann sums. We conclude that the definite integral of f over the interval $[0, 1]$ does not exist and that f is not integrable over $[0, 1]$. ■

Theorem 1 says nothing about how to *calculate* definite integrals. A method of calculation will be developed in Section 5.4, through a connection of definite integrals to antiderivatives. Meanwhile, finite approximations can be used to calculate an approximation for a definite integral.

The Midpoint Rule

When setting up a Riemann sum

$$\sum_{k=1}^n f(c_k) \Delta x_k$$



$$\Delta x = x_2 - x_1$$

$$c_1 = \frac{x_1 + x_2}{2}$$

we have many options on how to choose a partition and where to choose a point c_k in the k th interval of the partition. The simplest choice for a partition with n subintervals is to take each subinterval to have equal length $\Delta x = (b - a)/n$. In Equation (1) we obtained a formula by choosing c_k at the right endpoint of the k th interval. Setting c_k to be at the middle of the k th interval is often a better choice for approximating the integral using Riemann sums, as we saw in Figure 5.3. The midpoint is a good choice because on a small interval, the graph of a differentiable function can be approximated by a linear function, and the value of a linear function at an interval's midpoint is its average value over that interval.

With these choices, $[x_{k-1}, x_k] = [a + (k - 1)\Delta x, a + k\Delta x]$ is the k th interval in the partition, its midpoint is located at $c_k = (x_{k-1} + x_k)/2$, and its width is $\Delta x = (b - a)/n$. The midpoint rule for forming Riemann sums to approximate a definite integral takes the following form.

Midpoint Rule for Approximating a Definite Integral

$$\int_a^b f(x) dx \approx \sum_{k=1}^n f(c_k) \left(\frac{b-a}{n} \right) = [f(c_1) + f(c_2) + \cdots + f(c_n)] \left(\frac{b-a}{n} \right)$$

with $c_k = \frac{x_{k-1} + x_k}{2}$ and $x_k = a + k \left(\frac{b-a}{n} \right)$.

EXAMPLE 2 Approximate $\int_0^1 \frac{1}{1+x^2} dx$ using the midpoint rule with five intervals of equal length.

Solution The five intervals of the partition are $[0, 0.2]$, $[0.2, 0.4]$, $[0.4, 0.6]$, $[0.6, 0.8]$, and $[0.8, 1.0]$, each of length $1/5$. Their midpoints are at 0.1, 0.3, 0.5, 0.7, and 0.9. Applying the midpoint rule gives

$$\begin{aligned} \int_0^1 \frac{1}{1+x^2} dx &\approx [f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)] \left(\frac{1}{5} \right) \\ &= \left[\frac{1}{1.01} + \frac{1}{1.09} + \frac{1}{1.25} + \frac{1}{1.49} + \frac{1}{1.81} \right] (0.2) \\ &\approx 0.786. \end{aligned}$$

The midpoint rule becomes more powerful when we can combine it with an error bound that tells us how close the approximation it gives with n intervals is to the integral. In Chapter 8 we will examine error bounds in approximations of integrals.

Properties of Definite Integrals

In defining $\int_a^b f(x) dx$ as a limit of sums $\sum_{k=1}^n f(c_k) \Delta x_k$, we moved from left to right across the interval $[a, b]$. What would happen if we instead move right to left, starting with $x_0 = b$ and ending at $x_n = a$? Each Δx_k in the Riemann sum would change its sign, with $x_k - x_{k-1}$ now negative instead of positive. With the same choices of c_k in each subinterval, the sign of any Riemann sum would change, as would the sign of the limit, the integral $\int_b^a f(x) dx$. Since we have not previously given a meaning to integrating backward, we are led to define

$$\int_b^a f(x) dx = -\int_a^b f(x) dx. \quad a \text{ and } b \text{ interchanged}$$

It is convenient to have a definition for the integral over $[a, b]$ when $a = b$, so that we are computing an integral over an interval of zero width. Since $a = b$ gives $\Delta x = 0$, whenever $f(a)$ exists we define

$$\int_a^a f(x) dx = 0. \quad a \text{ is both the lower and the upper limit of integration.}$$

TABLE 5.6 Rules satisfied by definite integrals

1. Order of Integration: $\int_b^a f(x) dx = -\int_a^b f(x) dx$	A definition	
2. Zero Width Interval: $\int_a^a f(x) dx = 0$	A definition when $f(a)$ exists	
3. Constant Multiple: $\int_a^b k f(x) dx = k \int_a^b f(x) dx$	Any constant k	
4. Sum and Difference: $\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$		
5. Additivity: $\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$	$[a, c] \cup [c, b] = [a, b]$	
6. Max-Min Inequality: If f has maximum value $\max f$ and minimum value $\min f$ on $[a, b]$, then $(\min f) \cdot (b - a) \leq \int_a^b f(x) dx \leq (\max f) \cdot (b - a).$		
7. Domination: If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x) dx \geq \int_a^b g(x) dx$. If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x) dx \geq 0$. Special case of Rule 6.		

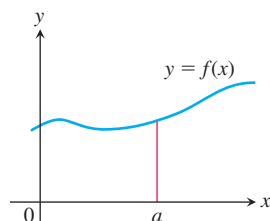
Theorem 2 states some basic properties of integrals, including the two just discussed. These properties, listed in Table 5.6, are very useful for computing integrals. We will refer to them repeatedly to simplify our calculations. Rules 2 through 7 have geometric interpretations, which are shown in Figure 5.11. The graphs in these figures show only positive functions, but the rules apply to general integrable functions, which could take both positive and negative values.

THEOREM 2 When f and g are integrable over the interval $[a, b]$, the definite integral satisfies the rules listed in Table 5.6.

Rules 1 and 2 are definitions, but Rules 3 to 7 of Table 5.6 must be proved. Below we give a proof of Rule 6. Similar proofs can be given to verify the other properties in Table 5.6.

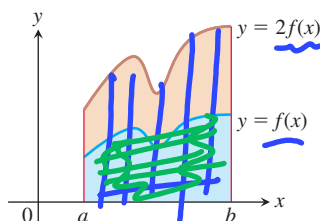
Proof of Rule 6 Rule 6 says that the integral of f over $[a, b]$ is never smaller than the minimum value of f times the length of the interval and never larger than the maximum value of f times the length of the interval. The reason is that for every partition of $[a, b]$ and for every choice of the points c_k ,

$$\begin{aligned}
 (\min f) \cdot (b - a) &= (\min f) \cdot \sum_{k=1}^n \Delta x_k & \sum_{k=1}^n \Delta x_k &= b - a \\
 &= \sum_{k=1}^n (\min f) \cdot \Delta x_k & \text{Constant Multiple Rule} \\
 &\leq \sum_{k=1}^n f(c_k) \Delta x_k & \min f \leq f(c_k) \\
 &\leq \sum_{k=1}^n (\max f) \cdot \Delta x_k & f(c_k) \leq \max f \\
 &= (\max f) \cdot \sum_{k=1}^n \Delta x_k & \text{Constant Multiple Rule} \\
 &= (\max f) \cdot (b - a).
 \end{aligned}$$

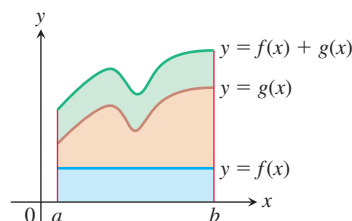


(a) Zero Width Interval:

$$\int_a^a f(x) dx = 0$$

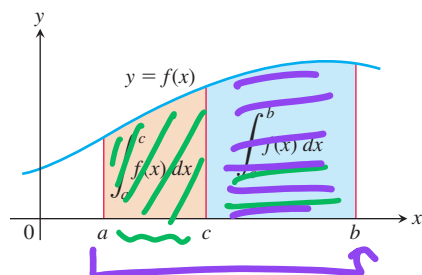
(b) Constant Multiple: ($k = 2$)

$$\int_a^b k f(x) dx = k \int_a^b f(x) dx$$



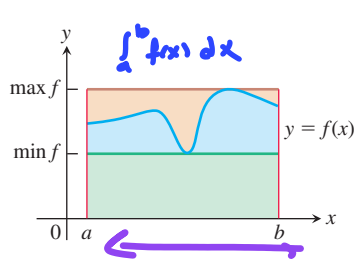
(c) Sum: (areas add)

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$



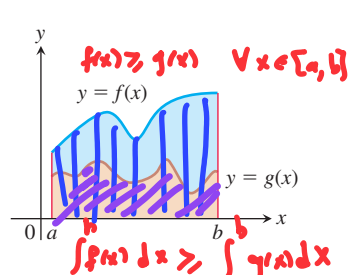
(d) Additivity for Definite Integrals:

$$\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$$



(e) Max-Min Inequality:

$$(\min f) \cdot (b - a) \leq \int_a^b f(x) dx \leq (\max f) \cdot (b - a)$$



(f) Domination:

If $f(x) \geq g(x)$ on $[a, b]$, then

$$\int_a^b f(x) dx \geq \int_a^b g(x) dx.$$

FIGURE 5.11 Geometric interpretations of Rules 2–7 in Table 5.6.

In short, all Riemann sums for f on $[a, b]$ satisfy the inequalities

$$(\min f) \cdot (b - a) \leq \sum_{k=1}^n f(c_k) \Delta x_k \leq (\max f) \cdot (b - a).$$

Hence their limit, which is the integral, satisfies the same inequalities. ■

EXAMPLE 3 To illustrate some of the rules, we suppose that

$$\int_{-1}^1 f(x) dx = 5, \quad \int_1^4 f(x) dx = -2, \quad \text{and} \quad \int_{-1}^1 h(x) dx = 7.$$

Then

$$1. \int_{-1}^4 f(x) dx = -\int_1^4 f(x) dx = -(-2) = 2 \quad \text{Rule 1}$$

$$2. \int_{-1}^1 [2f(x) + 3h(x)] dx = 2 \int_{-1}^1 f(x) dx + 3 \int_{-1}^1 h(x) dx = 2(5) + 3(7) = 31 \quad \text{Rules 3 and 4}$$

$$3. \int_{-1}^4 f(x) dx = \int_{-1}^1 f(x) dx + \int_1^4 f(x) dx = 5 + (-2) = 3 \quad \text{Rule 5} \quad \blacksquare$$

EXAMPLE 4 Show that the value of $\int_0^1 \sqrt{1 + \cos x} dx$ is less than or equal to $\sqrt{2}$.

Solution The Max-Min Inequality for definite integrals (Rule 6) says that $(\min f) \cdot (b - a)$ is a *lower bound* for the value of $\int_a^b f(x) dx$ and that $(\max f) \cdot (b - a)$ is an *upper bound*. The maximum value of $\sqrt{1 + \cos x}$ on $[0, 1]$ is $\sqrt{1 + 1} = \sqrt{2}$, so

$$\int_0^1 \sqrt{1 + \cos x} dx \leq \sqrt{2} \cdot (1 - 0) = \sqrt{2}. \quad \blacksquare$$

$$\textcircled{1} \quad f \geq 0$$

$$\text{Area} = \lim \Sigma$$

$$\textcircled{2} \quad \int_a^b f(x) dx = \lim \Sigma$$

Area Under the Graph of a Nonnegative Function

We now return to the problem that started this chapter, which is defining what we mean by the *area* of a region having a curved boundary. In Section 5.1 we approximated the area under the graph of a nonnegative continuous function using several types of finite sums of areas of rectangles that approximate the region—upper sums, lower sums, and sums using the midpoints of each subinterval—all of which are Riemann sums constructed in special ways. Theorem 1 guarantees that all of these Riemann sums converge to a single definite integral as the norm of the partitions approaches zero and the number of subintervals goes to infinity. As a result, we can now *define* the area under the graph of a nonnegative integrable function to be the value of that definite integral.

DEFINITION If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ is the integral of f from a to b ,

$$A = \int_a^b f(x) dx.$$

For the first time, we have a rigorous definition for the area of a region whose boundary is the graph of a continuous function. We now apply this to a simple example, the area under a straight line, and we verify that our new definition agrees with our previous notion of area.

EXAMPLE 5 Compute $\int_0^b x dx$ and find the area A under $y = x$ over the interval $[0, b]$, $b > 0$.

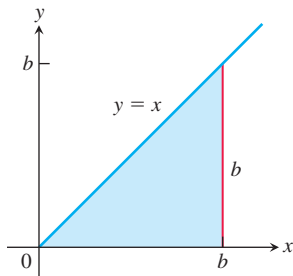


FIGURE 5.12 The region in Example 5 is a triangle.

Solution The region of interest is a triangle (Figure 5.12). We compute the area in two ways.

(a) To compute the definite integral as the limit of Riemann sums, we calculate $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k$ for partitions whose norms go to zero. Theorem 1 tells us that it does not matter how we choose the partitions or the points c_k as long as the norms approach zero. All choices give the exact same limit. So we consider the partition P that subdivides the interval $[0, b]$ into n subintervals of equal width $\Delta x = (b - 0)/n = b/n$, and we choose c_k to be the right endpoint in each subinterval as in formula (1). The partition is $P = \left\{0, \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}, \dots, \frac{nb}{n}\right\}$ and $c_k = \frac{kb}{n}$. So

$$\begin{aligned} \sum_{k=1}^n f(c_k) \Delta x &= \sum_{k=1}^n \frac{kb}{n} \cdot \frac{b}{n} && f(c_k) = c_k \\ &= \sum_{k=1}^n \frac{kb^2}{n^2} \\ &= \frac{b^2}{n^2} \sum_{k=1}^n k && \text{Constant Multiple Rule} \\ &= \frac{b^2}{n^2} \cdot \frac{n(n+1)}{2} && \text{Sum of first } n \text{ integers} \\ &= \frac{b^2}{2} \left(1 + \frac{1}{n}\right). \end{aligned}$$

As $n \rightarrow \infty$ and $\|P\| \rightarrow 0$, this last expression on the right has the limit $b^2/2$. Therefore,

$$\int_0^b x dx = \frac{b^2}{2}.$$

(b) Since the area equals the definite integral for a nonnegative function, we can quickly derive the definite integral by using the formula for the area of a triangle having base

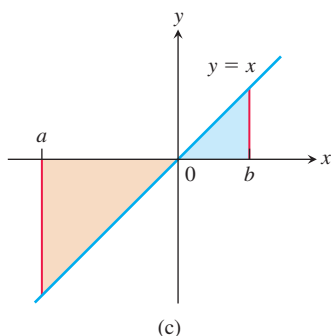
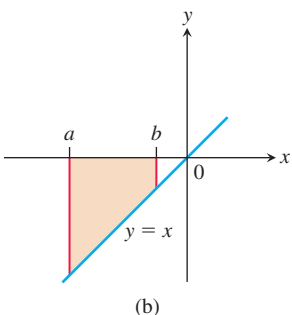
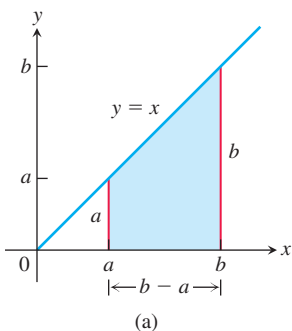


FIGURE 5.13 (a) The area of this trapezoidal region is $A = (b^2 - a^2)/2$. (b) The definite integral in Equation (2) gives the negative of the area of this trapezoidal region. (c) The definite integral in Equation (2) gives the area of the blue triangular region added to the negative of the area of the tan triangular region.

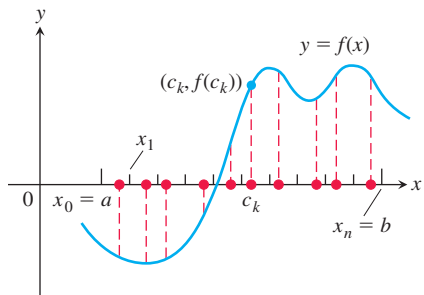


FIGURE 5.14 A sample of values of a function on an interval $[a, b]$.

length b and height $y = b$. The area is $A = (1/2)b \cdot b = b^2/2$. Again we conclude that $\int_0^b x \, dx = b^2/2$. ■

Example 5 can be generalized to integrate $f(x) = x$ over any closed interval $[a, b]$ for which $0 < a < b$.

First write

$$\int_0^b x \, dx = \int_0^a x \, dx + \int_a^b x \, dx \quad \text{Rule 5}$$

Then, by rearranging this equation and applying Example 5, we obtain

$$\int_a^b x \, dx = \int_0^b x \, dx - \int_0^a x \, dx = \frac{b^2}{2} - \frac{a^2}{2}. \quad \text{Example 5}$$

In conclusion, we have the following rule for integrating $f(x) = x$ over the interval $[a, b]$:

$$\int_a^b x \, dx = \frac{b^2}{2} - \frac{a^2}{2}, \quad a < b \quad (2)$$

This computation gives the area of the trapezoid in Figure 5.13a. Equation (2) remains valid when a and b are negative, but the interpretation of the definite integral changes. When $a < b < 0$, the definite integral value $(b^2 - a^2)/2$ is a negative number, the negative of the area of a trapezoid dropping down to the line $y = x$ below the x -axis (Figure 5.13b). When $a < 0$ and $b > 0$, Equation (2) is still valid and the definite integral gives the difference between two areas, the area under the graph and above $[0, b]$ minus the area below $[a, 0]$ and over the graph (Figure 5.13c).

The following results can also be established by using a Riemann sum calculation similar to the one we used in Example 5 (Exercises 63 and 65).

$$\int_a^b c \, dx = c(b - a), \quad c \text{ any constant} \quad (3)$$

$$\int_a^b x^2 \, dx = \frac{b^3}{3} - \frac{a^3}{3}, \quad a < b \quad (4)$$

Average Value of a Continuous Function Revisited

In Section 5.1 we informally introduced the average value of a nonnegative continuous function f over an interval $[a, b]$, leading us to define this average as the area under the graph of $y = f(x)$ divided by $b - a$. In integral notation we write this as

$$\text{Average} = \frac{1}{b - a} \int_a^b f(x) \, dx.$$

This formula gives us a precise definition of the average value of a continuous (or integrable) function, whether it is positive, negative, or both.

Alternatively, we can justify this formula through the following reasoning. We start with the idea from arithmetic that the average of n numbers is their sum divided by n . A continuous function f on $[a, b]$ may have infinitely many values, but we can still sample them in an orderly way. We divide $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$ and evaluate f at a point c_k in each (Figure 5.14). The average of the n sampled values is

$$\begin{aligned} \frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} &= \frac{1}{n} \sum_{k=1}^n f(c_k) \\ &= \frac{\Delta x}{b - a} \sum_{k=1}^n f(c_k) & \Delta x = \frac{b - a}{n}, \text{ so } \frac{1}{n} = \frac{\Delta x}{b - a} \\ &= \frac{1}{b - a} \sum_{k=1}^n f(c_k) \Delta x. & \text{Constant Multiple Rule} \end{aligned}$$

The average of the samples is obtained by dividing a Riemann sum for f on $[a, b]$ by $(b - a)$. As we increase the number of samples and let the norm of the partition approach zero, the average approaches $(1/(b - a)) \int_a^b f(x) dx$. Both points of view lead us to the following definition.

DEFINITION If f is integrable on $[a, b]$, then its **average value on $[a, b]$** , also called its **mean**, is

$$\text{av}(f) = \frac{1}{b - a} \int_a^b f(x) dx.$$

EXAMPLE 6 Find the average value of $f(x) = x$ on $[1, 3]$.

Solution From Equation (2) we see that $\int_1^3 x dx = 9/2 - 1/2 = 4$.

So the average value of f on the interval $[1, 3]$ is $(1/2)(4) = 2$. ■

EXAMPLE 7 Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

Solution We recognize $f(x) = \sqrt{4 - x^2}$ as the function whose graph is the upper semicircle of radius 2 centered at the origin (Figure 5.15).

Since we know the area inside a circle, we do not need to take the limit of Riemann sums. The area between the semicircle and the x -axis from -2 to 2 can be computed using the geometry formula

$$\text{Area} = \frac{1}{2} \cdot \pi r^2 = \frac{1}{2} \cdot \pi (2)^2 = 2\pi.$$

Because f is nonnegative, the area is also the value of the integral of f from -2 to 2 ,

$$\int_{-2}^2 \sqrt{4 - x^2} dx = 2\pi.$$

Therefore, the average value of f is

$$\text{av}(f) = \frac{1}{2 - (-2)} \int_{-2}^2 \sqrt{4 - x^2} dx = \frac{1}{4}(2\pi) = \frac{\pi}{2}.$$

Notice that the average value of f over $[-2, 2]$ is the same as the height of a rectangle over $[-2, 2]$ whose area equals the area of the upper semicircle (see Figure 5.15). ■

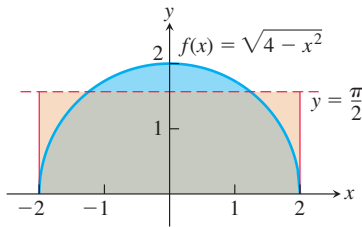


FIGURE 5.15 The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$ (Example 7). The area of the rectangle shown here is $4 \cdot (\pi/2) = 2\pi$, which is also the area of the semicircle.

EXERCISES 5.3

Interpreting Limits of Sums as Integrals

Express the limits in Exercises 1–8 as definite integrals.

- $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n c_k^2 \Delta x_k$, where P is a partition of $[0, 2]$
- $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n 2c_k^3 \Delta x_k$, where P is a partition of $[-1, 0]$
- $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (c_k^2 - 3c_k) \Delta x_k$, where P is a partition of $[-7, 5]$
- $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \left(\frac{1}{c_k} \right) \Delta x_k$, where P is a partition of $[1, 4]$
- $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \frac{1}{1 - c_k} \Delta x_k$, where P is a partition of $[2, 3]$

$$6. \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \sqrt{4 - c_k^2} \Delta x_k, \text{ where } P \text{ is a partition of } [0, 1]$$

$$7. \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (\sec c_k) \Delta x_k, \text{ where } P \text{ is a partition of } [-\pi/4, 0]$$

$$8. \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (\tan c_k) \Delta x_k, \text{ where } P \text{ is a partition of } [0, \pi/4]$$

Using the Definite Integral Rules

9. Suppose that f and g are integrable and that

$$\int_1^2 f(x) dx = -4, \quad \int_1^5 f(x) dx = 6, \quad \int_1^5 g(x) dx = 8.$$

Use the rules in Table 5.6 to find

- a. $\int_2^2 g(x) dx$ b. $\int_5^1 g(x) dx$
 c. $\int_1^2 3f(x) dx$ d. $\int_2^5 f(x) dx$
 e. $\int_1^5 [f(x) - g(x)] dx$ f. $\int_1^5 [4f(x) - g(x)] dx$

10. Suppose that f and h are integrable and that

$$\int_1^9 f(x) dx = -1, \quad \int_7^9 f(x) dx = 5, \quad \int_7^9 h(x) dx = 4.$$

Use the rules in Table 5.6 to find

- a. $\int_1^9 -2f(x) dx$ b. $\int_7^9 [f(x) + h(x)] dx$
 c. $\int_7^9 [2f(x) - 3h(x)] dx$ d. $\int_9^1 f(x) dx$
 e. $\int_1^7 f(x) dx$ f. $\int_9^7 [h(x) - f(x)] dx$

11. Suppose that $\int_1^2 f(x) dx = 5$. Find

- a. $\int_1^2 f(u) du$ b. $\int_1^2 \sqrt{3} f(z) dz$
 c. $\int_2^1 f(t) dt$ d. $\int_1^2 [-f(x)] dx$

12. Suppose that $\int_{-3}^0 g(t) dt = \sqrt{2}$. Find

- a. $\int_0^{-3} g(t) dt$ b. $\int_{-3}^0 g(u) du$
 c. $\int_{-3}^0 [-g(x)] dx$ d. $\int_{-3}^0 \frac{g(r)}{\sqrt{2}} dr$

13. Suppose that f is integrable and that $\int_0^3 f(z) dz = 3$ and $\int_0^4 f(z) dz = 7$. Find

- a. $\int_3^4 f(z) dz$ b. $\int_4^3 f(t) dt$

14. Suppose that h is integrable and that $\int_{-1}^1 h(r) dr = 0$ and $\int_{-1}^3 h(r) dr = 6$. Find

- a. $\int_1^3 h(r) dr$ b. $-\int_3^1 h(u) du$

Using Known Areas to Find Integrals

In Exercises 15–22, graph the integrands and use known area formulas to evaluate the integrals.

15. $\int_{-2}^4 \left(\frac{x}{2} + 3\right) dx$ 16. $\int_{1/2}^{3/2} (-2x + 4) dx$
 17. $\int_{-3}^3 \sqrt{9 - x^2} dx$ 18. $\int_{-4}^0 \sqrt{16 - x^2} dx$
 19. $\int_{-2}^1 |x| dx$ 20. $\int_{-1}^1 (1 - |x|) dx$
 21. $\int_{-1}^1 (2 - |x|) dx$ 22. $\int_{-1}^1 (1 + \sqrt{1 - x^2}) dx$

Use known area formulas to evaluate the integrals in Exercises 23–28.

23. $\int_0^b \frac{x}{2} dx, \quad b > 0$ 24. $\int_0^b 4x dx, \quad b > 0$
 25. $\int_a^b 2s ds, \quad 0 < a < b$ 26. $\int_a^b 3t dt, \quad 0 < a < b$

27. $f(x) = \sqrt{4 - x^2}$ on a. $[-2, 2]$, b. $[0, 2]$

28. $f(x) = 3x + \sqrt{1 - x^2}$ on a. $[-1, 0]$, b. $[-1, 1]$

Evaluating Definite Integrals

Use the results of Equations (2) and (4) to evaluate the integrals in Exercises 29–40.

29. $\int_1^{\sqrt{2}} x dx$ 30. $\int_{0.5}^{2.5} x dx$ 31. $\int_{\pi}^{2\pi} \theta d\theta$
 32. $\int_{\sqrt{2}}^{5\sqrt{2}} r dr$ 33. $\int_0^{\sqrt[3]{7}} x^2 dx$ 34. $\int_0^{0.3} s^2 ds$
 35. $\int_0^{1/2} t^2 dt$ 36. $\int_0^{\pi/2} \theta^2 d\theta$ 37. $\int_a^{2a} x dx$
 38. $\int_a^{\sqrt{3}} x dx$ 39. $\int_0^{\sqrt[3]{b}} x^2 dx$ 40. $\int_0^{3b} x^2 dx$

Use the rules in Table 5.6 and Equations (2)–(4) to evaluate the integrals in Exercises 41–50.

41. $\int_3^1 7 dx$ 42. $\int_0^2 5x dx$
 43. $\int_0^2 (2t - 3) dt$ 44. $\int_0^{\sqrt{2}} (t - \sqrt{2}) dt$
 45. $\int_2^1 \left(1 + \frac{z}{2}\right) dz$ 46. $\int_3^0 (2z - 3) dz$
 47. $\int_1^2 3u^2 du$ 48. $\int_{1/2}^1 24u^2 du$
 49. $\int_0^2 (3x^2 + x - 5) dx$ 50. $\int_1^2 (3x^2 + x - 5) dx$

Finding Area by Definite Integrals

In Exercises 51–54, use a definite integral to find the area of the region between the given curve and the x -axis on the interval $[0, b]$.

51. $y = 3x^2$ 52. $y = \pi x^2$
 53. $y = 2x$ 54. $y = \frac{x}{2} + 1$

Finding Average Value

In Exercises 55–62, graph the function and find its average value over the given interval.

55. $f(x) = x^2 - 1$ on $[0, \sqrt{3}]$ 56. $f(x) = -\frac{x^2}{2}$ on $[0, 3]$
 57. $f(x) = -3x^2 - 1$ on $[0, 1]$ 58. $f(x) = 3x^2 - 3$ on $[0, 1]$
 59. $f(t) = (t - 1)^2$ on $[0, 3]$ 60. $f(t) = t^2 - t$ on $[-2, 1]$
 61. $g(x) = |x| - 1$ on a. $[-1, 1]$, b. $[1, 3]$, and c. $[-1, 3]$
 62. $h(x) = -|x|$ on a. $[-1, 0]$, b. $[0, 1]$, and c. $[-1, 1]$

Definite Integrals as Limits of Sums

Use the method of Example 5a or Equation (1) to evaluate the definite integrals in Exercises 63–70.

63. $\int_a^b c dx$ 64. $\int_0^2 (2x + 1) dx$
 65. $\int_a^b x^2 dx, \quad a < b$ 66. $\int_{-1}^0 (x - x^2) dx$
 67. $\int_{-1}^2 (3x^2 - 2x + 1) dx$ 68. $\int_{-1}^1 x^3 dx$
 69. $\int_a^b x^3 dx, \quad a < b$ 70. $\int_0^1 (3x - x^3) dx$

Theory and Examples

71. What values of a and b , with $a < b$, maximize the value of

$$\int_a^b (x - x^2) dx?$$

(Hint: Where is the integrand positive?)

72. What values of a and b , with $a < b$, minimize the value of

$$\int_a^b (x^4 - 2x^2) dx?$$

73. Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1+x^2} dx.$$

74. (Continuation of Exercise 73.) Use the Max-Min Inequality to find upper and lower bounds for

$$\int_0^{0.5} \frac{1}{1+x^2} dx \quad \text{and} \quad \int_{0.5}^1 \frac{1}{1+x^2} dx.$$

Add these to arrive at an improved estimate of

$$\int_0^1 \frac{1}{1+x^2} dx.$$

75. Show that the value of $\int_0^1 \sin(x^2) dx$ cannot possibly be 2.
 76. Show that the value of $\int_0^1 \sqrt{x+8} dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.
 77. **Integrals of nonnegative functions** Use the Max-Min Inequality to show that if f is integrable, then

$$f(x) \geq 0 \quad \text{on} \quad [a, b] \Rightarrow \int_a^b f(x) dx \geq 0.$$

78. **Integrals of nonpositive functions** Show that if f is integrable, then

$$f(x) \leq 0 \quad \text{on} \quad [a, b] \Rightarrow \int_a^b f(x) dx \leq 0.$$

79. Use the inequality $\sin x \leq x$, which holds for $x \geq 0$, to find an upper bound for the value of $\int_0^1 \sin x dx$.
 80. The inequality $\sec x \geq 1 + (x^2/2)$ holds on $(-\pi/2, \pi/2)$. Use it to find a lower bound for the value of $\int_0^1 \sec x dx$.
 81. If $\text{av}(f)$ really is a typical value of the integrable function $f(x)$ on $[a, b]$, then the constant function $\text{av}(f)$ should have the same integral over $[a, b]$ as f . Does it? That is, does

$$\int_a^b \text{av}(f) dx = \int_a^b f(x) dx?$$

Give reasons for your answer.

82. It would be nice if average values of integrable functions obeyed the following rules on an interval $[a, b]$.

- a. $\text{av}(f + g) = \text{av}(f) + \text{av}(g)$
- b. $\text{av}(kf) = k \text{av}(f)$ (any number k)
- c. $\text{av}(f) \leq \text{av}(g)$ if $f(x) \leq g(x)$ on $[a, b]$.

Do these rules ever hold? Give reasons for your answers.

83. Upper and lower sums for increasing functions

- a. Suppose the graph of a continuous function $f(x)$ rises steadily as x moves from left to right across an interval $[a, b]$. Let P be a partition of $[a, b]$ into n subintervals of equal length $\Delta x = (b - a)/n$. Show by referring to the accompanying

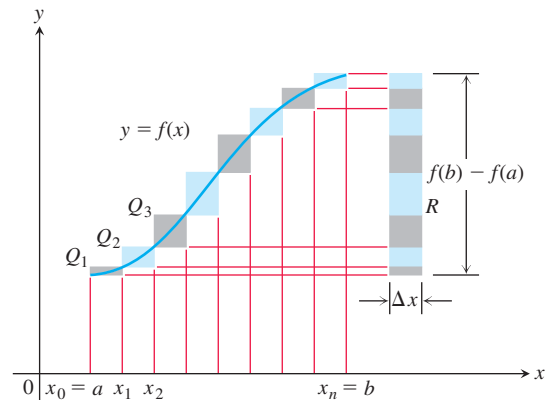
figure that the difference between the upper and lower sums for f on this partition can be represented graphically as the area of a rectangle R whose dimensions are $[f(b) - f(a)]$ by Δx . (Hint: The difference $U - L$ is the sum of areas of rectangles whose diagonals $Q_0Q_1, Q_1Q_2, \dots, Q_{n-1}Q_n$ lie approximately along the curve. There is no overlapping when these rectangles are shifted horizontally onto R .)

- b. Suppose that instead of being equal, the lengths Δx_k of the subintervals of the partition of $[a, b]$ vary in size. Show that

$$U - L \leq |f(b) - f(a)| \Delta x_{\max},$$

where $\Delta x_{\max}$ is the norm of P , and that hence

$$\lim_{\|P\| \rightarrow 0} (U - L) = 0.$$



84. Upper and lower sums for decreasing functions (Continuation of Exercise 83.)

- a. Draw a figure like the one in Exercise 83 for a continuous function $f(x)$ whose values decrease steadily as x moves from left to right across the interval $[a, b]$. Let P be a partition of $[a, b]$ into subintervals of equal length. Find an expression for $U - L$ that is analogous to the one you found for $U - L$ in Exercise 83a.

- b. Suppose that instead of being equal, the lengths Δx_k of the subintervals of P vary in size. Show that the inequality

$$U - L \leq |f(b) - f(a)| \Delta x_{\max}$$

of Exercise 83b still holds and hence $\lim_{\|P\| \rightarrow 0} (U - L) = 0$.

85. Use the formula

$$\begin{aligned} \sin h + \sin 2h + \sin 3h + \cdots + \sin kh \\ = \frac{\cos(h/2) - \cos((k + (1/2))h)}{2 \sin(h/2)} \end{aligned}$$

to find the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi/2$ in two steps:

- a. Partition the interval $[0, \pi/2]$ into n subintervals of equal length and calculate the corresponding upper sum U ; then
- b. Find the limit of U as $n \rightarrow \infty$ and $\Delta x = (b - a)/n \rightarrow 0$.

86. Suppose that f is continuous and nonnegative over $[a, b]$, as in the accompanying figure. By inserting points

$$x_1, x_2, \dots, x_{k-1}, x_k, \dots, x_{n-1}$$

as shown, divide $[a, b]$ into n subintervals of lengths $\Delta x_1 = x_1 - a, \Delta x_2 = x_2 - x_1, \dots, \Delta x_n = b - x_{n-1}$, which need not be equal.

- a. If $m_k = \min \{f(x) \text{ for } x \text{ in the } k\text{th subinterval}\}$, explain the connection between the **lower sum**

$$L = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_n \Delta x_n$$

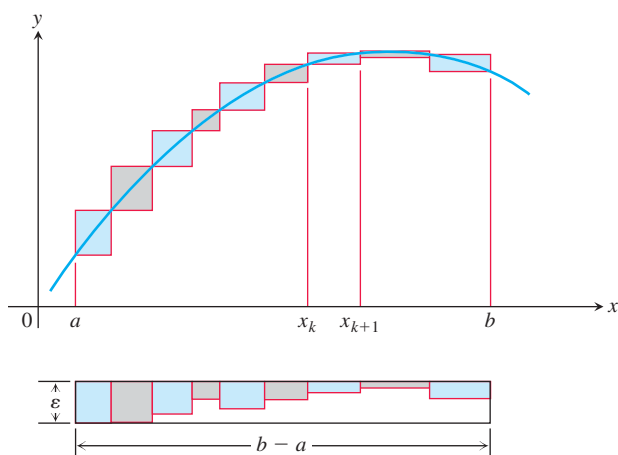
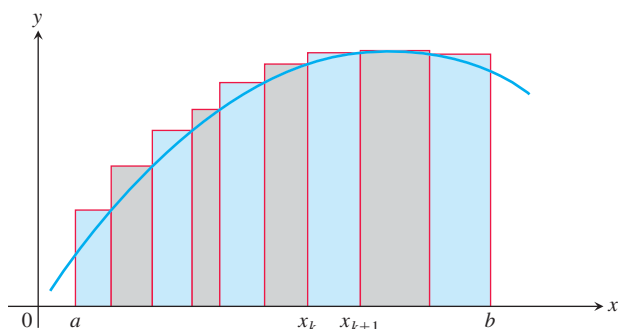
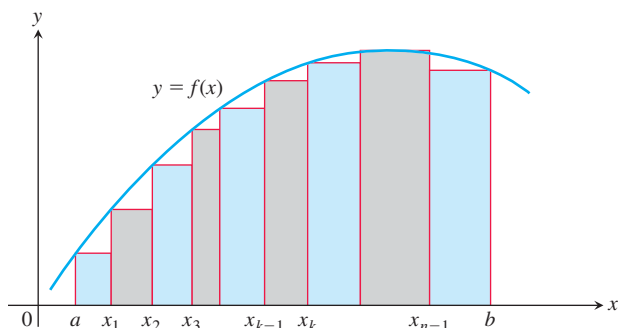
and the shaded regions in the first part of the figure.

- b. If $M_k = \max \{f(x) \text{ for } x \text{ in the } k\text{th subinterval}\}$, explain the connection between the **upper sum**

$$U = M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots + M_n \Delta x_n$$

and the shaded regions in the second part of the figure.

- c. Explain the connection between $U - L$ and the shaded regions along the curve in the third part of the figure.



87. We say f is **uniformly continuous** on $[a, b]$ if, given any $\varepsilon > 0$, there is a $\delta > 0$ such that if x_1, x_2 are in $[a, b]$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \varepsilon$. It can be shown that a continuous function on $[a, b]$ is uniformly continuous. Use this and the figure for Exercise 86 to show that if f is continuous and $\varepsilon > 0$ is given, it is possible to make $U - L \leq \varepsilon \cdot (b - a)$ by making the largest of the Δx_k 's sufficiently small.

88. If you average 48 km/h on a 240-km trip and then return over the same 240 km at the rate of 80 km/h, what is your average speed for the trip? Give reasons for your answer.

89. Integrals of functions that are equal except at one point

Suppose that $f(x)$ is a continuous function over the interval $[a, b]$ and that $g(x)$ is a function on $[a, b]$ such that $g(x) = f(x)$ except at a single point $c \in [a, b]$. Show that $g(x)$ is also integrable over $[a, b]$ and that $\int_a^b g(x) dx = \int_a^b f(x) dx$. (Hint: For a given n , by how much can two different Riemann sums as given in Equation (1) differ?)

90. Some integrable functions that are not continuous

- a. The floor function $f(x) = \lfloor x \rfloor$ gives the greatest integer smaller than or equal to x (Example 5 of Section 1.1). This function is not continuous on the interval $[1, 3]$. Show that f is integrable on $[1, 3]$ and that $\int_1^3 \lfloor x \rfloor dx = 3$.
- b. The function $f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 2 & \text{if } 0 \leq x, \end{cases}$ is not continuous on the interval $[-1, 1]$. Show that f is integrable on $[-1, 1]$ and that $\int_{-1}^1 f(x) dx = 1$.

COMPUTER EXPLORATIONS

If your CAS can draw rectangles associated with Riemann sums, use it to draw rectangles associated with Riemann sums that converge to the integrals in Exercises 91–96. Use $n = 4, 10, 20$, and 50 subintervals of equal length in each case.

91. $\int_0^1 (1 - x) dx = \frac{1}{2}$ 92. $\int_0^1 (x^2 + 1) dx = \frac{4}{3}$
93. $\int_{-\pi}^{\pi} \cos x dx = 0$ 94. $\int_0^{\pi/4} \sec^2 x dx = 1$
95. $\int_{-1}^1 |x| dx = 1$
96. $\int_1^2 \frac{1}{x} dx$ (The integral's value is about 0.693.)

In Exercises 97–104, use a CAS to perform the following steps:

- a. Plot the functions over the given interval.
- b. Partition the interval into $n = 100, 200$, and 1000 subintervals of equal length, and evaluate the function at the midpoint of each subinterval.
- c. Compute the average value of the function values generated in part (b).
- d. Solve the equation $f(x) = (\text{average value})$ for x using the average value calculated in part (c) for the $n = 1000$ partitioning.
97. $f(x) = \sin x$ on $[0, \pi]$
98. $f(x) = \sin^2 x$ on $[0, \pi]$
99. $f(x) = x \sin \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$
100. $f(x) = x \sin^2 \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$
101. $f(x) = xe^{-x}$ on $[0, 1]$
102. $f(x) = e^{-x^2}$ on $[0, 1]$
103. $f(x) = \frac{\ln x}{x}$ on $[2, 5]$
104. $f(x) = \frac{1}{\sqrt{1 - x^2}}$ on $\left[0, \frac{1}{2}\right]$

5.4 The Fundamental Theorem of Calculus

HISTORICAL BIOGRAPHY

Sir Isaac Newton
(1642–1727)

In his youth in England, Newton was interested in mechanical devices and their underlying theories. He even constructed lanterns and windmills that he designed. During the 1670s and 1680s, he built his reputation as a scientific genius. His contributions included the theory of universal gravitation, the laws of motion, methods of calculus, and the composition of white light.

To know more, visit the companion Website.

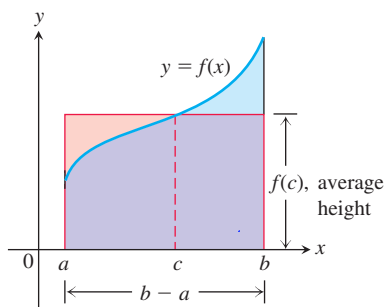


FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b ,

$$f(c)(b - a) = \int_a^b f(x) dx.$$

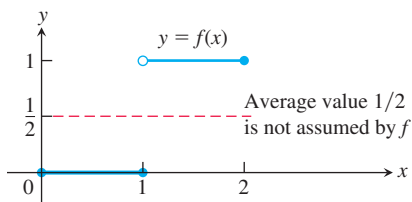


FIGURE 5.17 A discontinuous function need not assume its average value.

In this section we present the Fundamental Theorem of Calculus, which is the central theorem of integral calculus. It connects integration and differentiation, enabling us to compute integrals by using an antiderivative of the integrand function, rather than by taking limits of Riemann sums as we did in Section 5.3. Leibniz and Newton exploited this relationship and started mathematical developments that fueled the scientific revolution for the next 200 years.

Along the way, we will present an integral version of the Mean Value Theorem, which is another important theorem of integral calculus and is used to prove the Fundamental Theorem. We also find that the net change of a function over an interval is the integral of its rate of change, as suggested by Example 2 in Section 5.1.

Mean Value Theorem for Definite Integrals

In the previous section we defined the average value of a continuous function over a closed interval $[a, b]$ to be the definite integral $\int_a^b f(x) dx$ divided by the length or width $b - a$ of the interval. The Mean Value Theorem for Definite Integrals asserts that this average value is *always* taken on at least once by the function f in the interval.

The graph in Figure 5.16 shows a *positive* continuous function $y = f(x)$ defined over the interval $[a, b]$. Geometrically, the Mean Value Theorem says that there is a number c in $[a, b]$ such that the rectangle with height equal to the average value $f(c)$ of the function and base width $b - a$ has exactly the same area as the region beneath the graph of f from a to b .

THEOREM 3—The Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b - a} \int_a^b f(x) dx.$$

Proof If we divide all three expressions in the Max-Min Inequality (Table 5.6, Rule 6) by $(b - a)$, we obtain

$$\min f \leq \frac{1}{b - a} \int_a^b f(x) dx \leq \max f.$$

Since f is continuous, the Intermediate Value Theorem for Continuous Functions (Section 2.6) says that f must assume every value between $\min f$ and $\max f$. It must therefore assume the value $(1/(b - a)) \int_a^b f(x) dx$ at some point c in $[a, b]$. ■

The continuity of f is important here. It is possible for a discontinuous function to never equal its average value (Figure 5.17).

EXAMPLE 1 Show that if f is continuous on $[a, b]$, $a \neq b$, and if

$$\int_a^b f(x) dx = 0,$$

then $f(x) = 0$ at least once in $[a, b]$.

Solution The average value of f on $[a, b]$ is

$$\text{av}(f) = \frac{1}{b - a} \int_a^b f(x) dx = \frac{1}{b - a} \cdot 0 = 0.$$

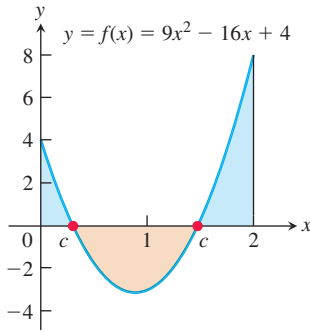


FIGURE 5.18 The function $f(x) = 9x^2 - 16x + 4$ satisfies $\int_0^2 f(x) dx = 0$, and there are two values of c in the interval $[0, 2]$ where $f(c) = 0$.

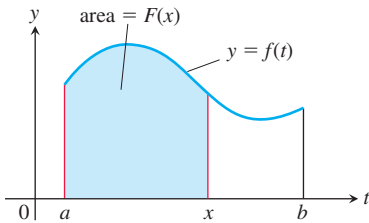


FIGURE 5.19 The function $F(x)$ defined by Equation (1) gives the area under the graph of f from a to x when f is nonnegative and $x > a$.

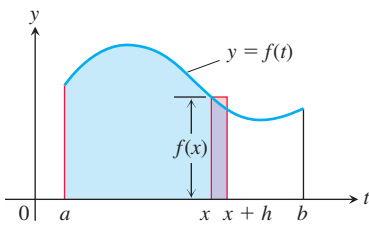


FIGURE 5.20 In Equation (1), $F(x)$ is the area to the left of x . Also, $F(x + h)$ is the area to the left of $x + h$. The difference quotient $[F(x + h) - F(x)]/h$ is then approximately equal to $f(x)$, the height of the rectangle shown here.

By the Mean Value Theorem for Definite Integrals, f assumes this value at some point $c \in [a, b]$. This is illustrated in Figure 5.18 for the function $f(x) = 9x^2 - 16x + 4$ on the interval $[0, 2]$. ■

Fundamental Theorem, Part 1

It can be very difficult to compute definite integrals by taking the limit of Riemann sums. We now develop a powerful new method for evaluating definite integrals, based on using antiderivatives. This method combines the two strands of calculus. One strand involves the idea of taking the limits of finite sums to obtain a definite integral, and the other strand contains derivatives and antiderivatives. They come together in the Fundamental Theorem of Calculus. We begin by considering how to differentiate a certain type of function that is described as an integral.

If $f(t)$ is an integrable function over a finite interval I , then the integral from any fixed number $a \in I$ to another number $x \in I$ defines a new function F whose value at x is

$$F(x) = \int_a^x f(t) dt. \quad (1)$$

For example, if f is nonnegative and x lies to the right of a , then $F(x)$ is the area under the graph from a to x (Figure 5.19). The variable x is the upper limit of integration of an integral, but F is just like any other real-valued function of a real variable. For each value of the input x , there is a single numerical output, in this case the definite integral of f from a to x .

Equation (1) gives a useful way to define new functions (as we will see in Section 7.1), but its key importance is the connection that it makes between integrals and derivatives. If f is a continuous function, then the Fundamental Theorem asserts that F is a differentiable function of x whose derivative is f itself. That is, at each x in the interval $[a, b]$ we have

$$F'(x) = f(x).$$

To gain some insight into why this holds, we look at the geometry behind it.

If $f \geq 0$ on $[a, b]$, then to compute $F'(x)$ from the definition of the derivative, we must take the limit as $h \rightarrow 0$ of the difference quotient

$$\frac{F(x + h) - F(x)}{h}.$$

If $h > 0$, then $F(x + h)$ is the area under the graph of f from a to $x + h$, while $F(x)$ is the area under the graph of f from a to x . Subtracting the two gives us the area under the graph of f between x and $x + h$ (see Figure 5.20). As shown in Figure 5.20, if h is small, the area under the graph of f from x to $x + h$ is approximated by the area of the rectangle whose height is $f(x)$ and whose base is the interval $[x, x + h]$. That is,

$$F(x + h) - F(x) \approx hf(x).$$

Dividing both sides by h , we see that the value of the difference quotient is very close to the value of $f(x)$:

$$\frac{F(x + h) - F(x)}{h} \approx f(x).$$

This approximation improves as h approaches 0. It is reasonable to expect that $F'(x)$, which is the limit of this difference quotient as $h \rightarrow 0$, equals $f(x)$, so that

$$F'(x) = \lim_{h \rightarrow 0} \frac{F(x + h) - F(x)}{h} = f(x).$$

This equation is true even if the function f is not positive, and it forms the first part of the Fundamental Theorem of Calculus.

THEOREM 4—The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t) dt$ is continuous on $[a, b]$ and differentiable on (a, b) , and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x). \quad (2)$$

Before proving Theorem 4, we look at several examples to gain an understanding of what it says. In each of these examples, notice that the independent variable x appears in either the upper or the lower limit of integration (either as part of a formula or by itself). The independent variable on which y depends in these examples is x , while t is merely a dummy variable in the integral.

EXAMPLE 2 Use the Fundamental Theorem to find dy/dx if

(a) $y = \int_a^x (t^3 + 1) dt$

(b) $y = \int_x^5 3t \sin t dt$

(c) $y = \int_1^{x^2} \cos t dt$

(d) $y = \int_{1+3x^2}^4 \frac{1}{2 + e^t} dt$

Solution We calculate the derivatives with respect to the independent variable x .

(a) $\frac{dy}{dx} = \frac{d}{dx} \int_a^x (t^3 + 1) dt = x^3 + 1$ Eq. (2) with $f(t) = t^3 + 1$

(b) $\frac{dy}{dx} = \frac{d}{dx} \int_x^5 3t \sin t dt = \frac{d}{dx} \left(- \int_5^x 3t \sin t dt \right)$ Table 5.6, Rule 1
 $= - \frac{d}{dx} \int_5^x 3t \sin t dt$
 $= -3x \sin x$ Eq. (2) with $f(t) = 3t \sin t$

(c) The upper limit of integration is not x but x^2 . This makes y a composition of the two functions

$$y = \int_1^u \cos t dt \quad \text{and} \quad u = x^2.$$

We must therefore apply the Chain Rule to find dy/dx :

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \left(\frac{d}{du} \int_1^u \cos t dt \right) \cdot \frac{du}{dx} \\ &= \cos u \cdot \frac{du}{dx} && \text{Eq. (2) with } f(t) = \cos t \\ &= (\cos x^2) \cdot 2x \\ &= 2x \cos x^2 \end{aligned}$$

$$\begin{aligned}
\text{(d)} \quad \frac{d}{dx} \int_{1+3x^2}^4 \frac{1}{2+e^t} dt &= \frac{d}{dx} \left(- \int_4^{1+3x^2} \frac{1}{2+e^t} dt \right) && \text{Table 5.6, Rule 1} \\
&= - \frac{d}{dx} \int_4^{1+3x^2} \frac{1}{2+e^t} dt \\
&= - \frac{1}{2+e^{(1+3x^2)}} \frac{d}{dx} (1+3x^2) && \text{Eq. (2) and the Chain Rule} \\
&= - \frac{6x}{2+e^{(1+3x^2)}}
\end{aligned}$$

Proof of Theorem 4 We prove the Fundamental Theorem, Part 1, by applying the definition of the derivative directly to the function $F(x)$, when x and $x+h$ are in (a, b) . This means writing out the difference quotient

$$\frac{F(x+h) - F(x)}{h} \quad (3)$$

and showing that its limit as $h \rightarrow 0$ is the number $f(x)$. Doing so, we find that

$$\begin{aligned}
F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_a^{x+h} f(t) dt - \int_a^x f(t) dt \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt. && \text{Table 5.6, Rule 5}
\end{aligned}$$

According to the Mean Value Theorem for Definite Integrals, there is some point c between x and $x+h$ where $f(c)$ equals the average value of f on the interval $[x, x+h]$. That is, there is some number c in $[x, x+h]$ such that

$$\frac{1}{h} \int_x^{x+h} f(t) dt = f(c). \quad (4)$$

As $h \rightarrow 0$, $x+h$ approaches x , which forces c to approach x also (because c is trapped between x and $x+h$). Since f is continuous at x , $f(c)$ therefore approaches $f(x)$:

$$\lim_{h \rightarrow 0} f(c) = f(x). \quad (5)$$

Hence we have shown that, for any x in (a, b) ,

$$\begin{aligned}
F'(x) &= \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt \\
&= \lim_{h \rightarrow 0} f(c) && \text{Eq. (4)} \\
&= f(x), && \text{Eq. (5)}
\end{aligned}$$

and therefore F is differentiable at x . Since differentiability implies continuity, this also shows that F is continuous on the open interval (a, b) . To complete the proof, we just have to show that F is also continuous at $x = a$ and $x = b$. To do this, we make a very similar argument, except that at $x = a$ we need only consider the one-sided limit as $h \rightarrow 0^+$, and similarly at $x = b$ we need only consider $h \rightarrow 0^-$. This shows that F has a one-sided derivative at $x = a$ and at $x = b$, and therefore Theorem 1 in Section 3.2 implies that F is continuous at those two points. ■

Fundamental Theorem, Part 2 (The Evaluation Theorem)

We now come to the second part of the Fundamental Theorem of Calculus. This part describes how to evaluate definite integrals without having to calculate limits of Riemann sums. Instead we find and evaluate an antiderivative at the upper and lower limits of integration.

THEOREM 4 (Continued)—The Fundamental Theorem of Calculus, Part 2

If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a).$$

Proof Part 1 of the Fundamental Theorem tells us that an antiderivative of f exists, namely

$$G(x) = \int_a^x f(t) dt.$$

Thus, if F is *any* antiderivative of f , then $F(x) = G(x) + C$ for some constant C for $a < x < b$ (by Corollary 2 of the Mean Value Theorem for Derivatives, Section 4.2). Since both F and G are continuous on $[a, b]$, we see that the equality $F(x) = G(x) + C$ also holds when $x = a$ and $x = b$ by taking one-sided limits (as $x \rightarrow a^+$ and $x \rightarrow b^-$).

Evaluating $F(b) - F(a)$, we have

$$\begin{aligned} F(b) - F(a) &= [G(b) + C] - [G(a) + C] \\ &= G(b) - G(a) \\ &= \int_a^b f(t) dt - \int_a^a f(t) dt \\ &= \int_a^b f(t) dt - 0 \\ &= \int_a^b f(t) dt. \end{aligned}$$

The Evaluation Theorem is important because it says that to calculate the definite integral of f over an interval $[a, b]$ we need do only two things:

1. Find an antiderivative F of f , and
2. Calculate the number $F(b) - F(a)$, which is equal to $\int_a^b f(x) dx$.

This process is much easier than computing Riemann sums and finding their limit. The power of the theorem follows from the realization that the definite integral, which is defined by a complicated process involving all of the values of the function f over $[a, b]$, can be found by knowing the values of *any* antiderivative F at only the two endpoints a and b . The usual notation for the difference $F(b) - F(a)$ is

$$F(x) \Big|_a^b \quad \text{or} \quad \left[F(x) \right]_a^b,$$

depending on whether F has one or more terms.

EXAMPLE 3 We calculate several definite integrals using the Evaluation Theorem, rather than by taking limits of Riemann sums.

$$\begin{aligned} \text{(a)} \quad \int_0^\pi \cos x \, dx &= \sin x \Big|_0^\pi \\ &= \sin \pi - \sin 0 = 0 - 0 = 0 \end{aligned}$$

$\frac{d}{dx} \sin x = \cos x$

Riemann
sum
طوبه (ف) نرکانه
X

Area
(ف) نرکانه
X

$$\begin{aligned}
 \text{(b)} \quad \int_{-\pi/4}^0 \sec x \tan x \, dx &= \sec x \Big|_{-\pi/4}^0 & \frac{d}{dx} \sec x &= \sec x \tan x \\
 &= \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx &= \left[x^{3/2} + \frac{4}{x}\right]_1^4 & \frac{d}{dx} \left(x^{3/2} + \frac{4}{x}\right) &= \frac{3}{2}x^{1/2} - \frac{4}{x^2} \\
 &= \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right] \\
 &= [8 + 1] - [5] = 4
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \int_1^2 \frac{1}{x} dx &= \ln|x| \Big|_1^2 & \frac{d}{dx} \ln|x| &= \frac{1}{x} \\
 &= \ln 2 - \ln 1 = \ln 2
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad \int_0^1 \frac{dx}{x^2 + 1} &= \arctan x \Big|_0^1 & \frac{d}{dx} \arctan x &= \frac{1}{x^2 + 1} \\
 &= \arctan 1 - \arctan 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.
 \end{aligned}$$

Exercise 82 offers another proof of the Evaluation Theorem, bringing together the ideas of Riemann sums, the Mean Value Theorem, and the definition of the definite integral.

The Integral of a Rate

We can interpret Part 2 of the Fundamental Theorem in another way. If F is any antiderivative of f , then $F' = f$. The equation in the theorem can then be rewritten as

$$\int_a^b F'(x) \, dx = F(b) - F(a).$$

Now $F'(x)$ represents the rate of change of the function $F(x)$ with respect to x , so the last equation asserts that the integral of F' is just the *net change* in F as x changes from a to b . Formally, we have the following result.

THEOREM 5—The Net Change Theorem

The net change in a differentiable function $F(x)$ over an interval $a \leq x \leq b$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x) \, dx. \quad (6)$$

EXAMPLE 4 Here are several interpretations of the Net Change Theorem.

(a) If $c(x)$ is the cost of producing x units of a certain commodity, then $c'(x)$ is the marginal cost (Section 3.4). From Theorem 5,

$$\int_{x_1}^{x_2} c'(x) \, dx = c(x_2) - c(x_1),$$

which is the cost of increasing production from x_1 units to x_2 units.

- (b) If an object with position function $s(t)$ moves along a coordinate line, its velocity is $v(t) = s'(t)$. Theorem 5 says that

$$\int_{t_1}^{t_2} v(t) dt = s(t_2) - s(t_1),$$

so the integral of velocity is the **displacement** over the time interval $t_1 \leq t \leq t_2$. On the other hand, the integral of the speed $|v(t)|$ is the **total distance traveled** over the time interval. This is consistent with our discussion in Section 5.1. ■

If we rearrange Equation (6) as

$$F(b) = F(a) + \int_a^b F'(x) dx,$$

we see that the Net Change Theorem also says that the final value of a function $F(x)$ over an interval $[a, b]$ equals its initial value $F(a)$ plus its net change over the interval. So if $v(t)$ represents the velocity function of an object moving along a coordinate line, this means that the object's final position $s(t_2)$ over a time interval $t_1 \leq t \leq t_2$ is its initial position $s(t_1)$ plus its net change in position along the line (see Example 4b).

EXAMPLE 5 Consider again our analysis of a heavy rock blown straight up from the ground by a dynamite blast (Example 2, Section 5.1). The velocity of the rock at any time t during its motion was given as $v(t) = 49 - 9.8t$ m/s.

- (a) Find the displacement of the rock during the time period $0 \leq t \leq 8$.
 (b) Find the total distance traveled during this time period.

Solution

- (a) From Example 4b, the displacement is the integral

$$\begin{aligned} \int_0^8 v(t) dt &= \int_0^8 (49 - 9.8t) dt = \left[49t - 4.9t^2 \right]_0^8 \\ &= (49)(8) - (4.9)(64) = 78.4. \end{aligned}$$

This means that the height of the rock is 78.4 m above the ground 8 s after the explosion, which agrees with our conclusion in Example 2, Section 5.1.

- (b) As we noted in Table 5.3, the velocity function $v(t)$ is positive over the time interval $[0, 5]$ and negative over the interval $[5, 8]$. Therefore, from Example 4b, the total distance traveled is the integral

$$\begin{aligned} \int_0^8 |v(t)| dt &= \int_0^5 |v(t)| dt + \int_5^8 |v(t)| dt \\ &= \int_0^5 (49 - 9.8t) dt - \int_5^8 (49 - 9.8t) dt \\ &= \left[49t - 4.9t^2 \right]_0^5 - \left[49t - 4.9t^2 \right]_5^8 \\ &= [(49)(5) - (4.9)(25)] - [(49)(8) - (4.9)(64) - ((49)(5) - (4.9)(25))] \\ &= 122.5 - (-44.1) = 166.6 \end{aligned}$$

Again, this calculation agrees with our conclusion in Example 2, Section 5.1. That is, the total distance of 166.6 m traveled by the rock during the time period $0 \leq t \leq 8$ is (i) the maximum height of 122.5 m it reached over the time interval $[0, 5]$ plus (ii) the additional distance of 44.1 m the rock fell over the time interval $[5, 8]$. ■

$$\frac{d}{dx} \left[\int_a^b x^2 \cos x \, dx \right] =$$

The Relationship Between Integration and Differentiation

The conclusions of the Fundamental Theorem tell us several things. Equation (2) can be rewritten as

$$\frac{d}{dx} \int_a^x f(t) \, dt = f(x),$$

F.T.C Part 1

which says that if you first integrate the function f and then differentiate the result, you get the function f back again. Likewise, replacing b by x and x by t in Equation (6) gives

$$\int_a^x F'(t) \, dt = F(x) - F(a),$$

F.T.C Part 2

so that if you first differentiate the function F and then integrate the result, you get the function F back (adjusted by an integration constant). In a sense, the processes of integration and differentiation are “inverses” of each other. The Fundamental Theorem also says that every continuous function f has an antiderivative F . It shows the importance of finding antiderivatives in order to evaluate definite integrals easily. Furthermore, it says that the differential equation $dy/dx = f(x)$ has a solution (namely, any of the functions $y = F(x) + C$) when f is a continuous function.

Total Area

Area is always a nonnegative quantity. The Riemann sum approximations contain terms such as $f(c_k) \Delta x_k$ that give the area of a rectangle when $f(c_k)$ is positive. When $f(c_k)$ is negative, then the product $f(c_k) \Delta x_k$ is the negative of the rectangle’s area. When we add up such terms for a negative function, we get the negative of the area between the curve and the x -axis. If we then take the absolute value, we obtain the correct positive area.

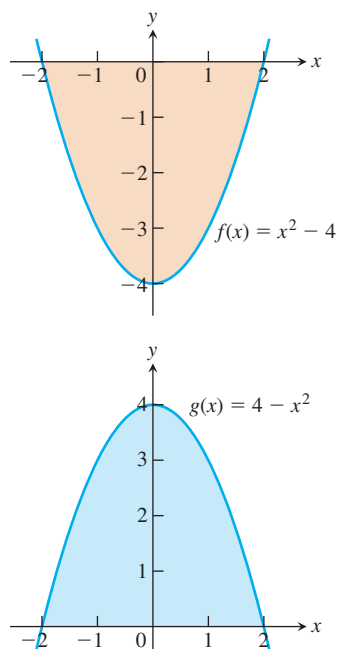


FIGURE 5.21 These graphs enclose the same amount of area with the x -axis, but the definite integrals of the two functions over $[-2, 2]$ differ in sign (Example 6).

EXAMPLE 6 Figure 5.21 shows the graph of $f(x) = x^2 - 4$ and its mirror image $g(x) = 4 - x^2$ reflected across the x -axis. For each function, compute

- the definite integral over the interval $[-2, 2]$, and
- the area between the graph and the x -axis over $[-2, 2]$.

Solution

$$(a) \int_{-2}^2 f(x) \, dx = \left[\frac{x^3}{3} - 4x \right]_{-2}^2 = \left(\frac{8}{3} - 8 \right) - \left(-\frac{8}{3} + 8 \right) = -\frac{32}{3},$$

and

$$\int_{-2}^2 g(x) \, dx = \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{32}{3}.$$

- In both cases, the area between the curve and the x -axis over $[-2, 2]$ is $32/3$ square units. Although the definite integral of $f(x)$ is negative, the area is still positive. ■

To compute the area of the region bounded by the graph of a function $y = f(x)$ and the x -axis when the function takes on both positive and negative values, we must be careful to break up the interval $[a, b]$ into subintervals on which the function doesn’t change sign. Otherwise, we might get cancellation between positive and negative signed areas, leading to an incorrect total. The correct total area is obtained by adding the absolute value of the definite integral over each subinterval where $f(x)$ does not change sign. The term “area” will be taken to mean this *total area*.

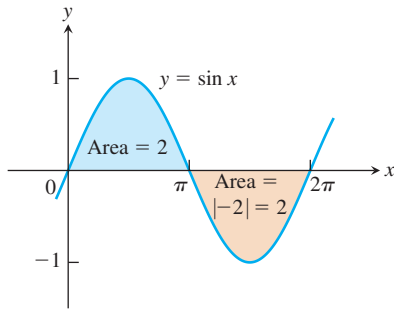


FIGURE 5.22 The total area between $y = \sin x$ and the x -axis for $0 \leq x \leq 2\pi$ is the sum of the absolute values of two integrals (Example 7).

EXAMPLE 7 Figure 5.22 shows the graph of the function $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute

- (a) the definite integral of $f(x)$ over $[0, 2\pi]$,
- (b) the area between the graph of $f(x)$ and the x -axis over $[0, 2\pi]$.

Solution

- (a) The definite integral for $f(x) = \sin x$ is given by

$$\int_0^{2\pi} \sin x \, dx = -\cos x \Big|_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$$

The definite integral is zero because the portions of the graph above and below the x -axis make canceling contributions.

- (b) The area between the graph of $f(x)$ and the x -axis over $[0, 2\pi]$ is calculated by breaking up the domain of $\sin x$ into two pieces: the interval $[0, \pi]$ over which it is non-negative and the interval $[\pi, 2\pi]$ over which it is nonpositive.

$$\int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi} = -[\cos \pi - \cos 0] = -[-1 - 1] = 2$$

$$\int_{\pi}^{2\pi} \sin x \, dx = -\cos x \Big|_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1 - (-1)] = -2$$

The second integral gives a negative value. The area between the graph and the axis is obtained by adding the absolute values,

$$\text{Area} = |2| + |-2| = 4.$$

Summary:

To find the area between the graph of $y = f(x)$ and the x -axis over the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f .
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

EXAMPLE 8 Find the area of the region between the x -axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \leq x \leq 2$.

Solution First find the zeros of f . Since

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x + 1)(x - 2),$$

the zeros are $x = 0$, -1 , and 2 (Figure 5.23). The zeros subdivide $[-1, 2]$ into two subintervals: $[-1, 0]$, on which $f \geq 0$, and $[0, 2]$, on which $f \leq 0$. We integrate f over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^0 (x^3 - x^2 - 2x) \, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^0 = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1 \right] = \frac{5}{12}$$

$$\int_0^2 (x^3 - x^2 - 2x) \, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_0^2 = \left[4 - \frac{8}{3} - 4 \right] - 0 = -\frac{8}{3}$$

The total enclosed area is obtained by adding the absolute values of the calculated integrals.

$$\text{Total enclosed area} = \frac{5}{12} + \left| -\frac{8}{3} \right| = \frac{37}{12}$$

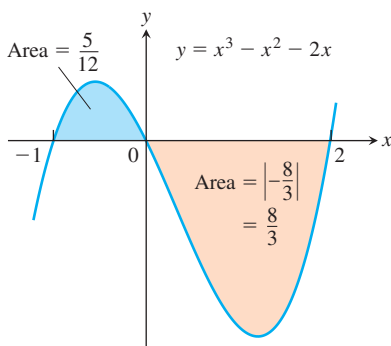


FIGURE 5.23 The region between the curve $y = x^3 - x^2 - 2x$ and the x -axis (Example 8).

EXERCISES 5.4

Evaluating Integrals

Evaluate the integrals in Exercises 1–34.

1. $\int_0^2 x(x-3) dx$
2. $\int_{-1}^1 (x^2 - 2x + 3) dx$
3. $\int_{-2}^2 (x+3)^2 dx$
4. $\int_{-1}^1 x^{299} dx$
5. $\int_1^4 \left(3x^2 - \frac{x^3}{4}\right) dx$
6. $\int_{-2}^3 (x^3 - 2x + 3) dx$
7. $\int_0^1 (x^2 + \sqrt{x}) dx$
8. $\int_1^{32} x^{-6/5} dx$
9. $\int_0^{\pi/3} 2 \sec^2 x dx$
10. $\int_0^{\pi} (1 + \cos x) dx$
11. $\int_{\pi/4}^{3\pi/4} \csc \theta \cot \theta d\theta$
12. $\int_0^{\pi/3} 4 \frac{\sin u}{\cos^2 u} du$
13. $\int_{\pi/2}^0 \frac{1 + \cos 2t}{2} dt$
14. $\int_{-\pi/3}^{\pi/3} \sin^2 t dt$
15. $\int_0^{\pi/4} \tan^2 x dx$
16. $\int_0^{\pi/6} (\sec x + \tan x)^2 dx$
17. $\int_0^{\pi/8} \sin 2x dx$
18. $\int_{-\pi/3}^{-\pi/4} \left(4 \sec^2 t + \frac{\pi}{t^2}\right) dt$
19. $\int_1^{-1} (r+1)^2 dr$
20. $\int_{-\sqrt{3}}^{\sqrt{3}} (t+1)(t^2+4) dt$
21. $\int_{\sqrt{2}}^1 \left(\frac{u^7}{2} - \frac{1}{u^5}\right) du$
22. $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy$
23. $\int_1^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2} ds$
24. $\int_1^8 \frac{(x^{1/3} + 1)(2 - x^{2/3})}{x^{1/3}} dx$
25. $\int_{\pi/6}^{\pi/2} \frac{\sin 2x}{2 \sin x} dx$
26. $\int_0^{\pi/3} (\cos x + \sec x)^2 dx$
27. $\int_{-4}^4 |x| dx$
28. $\int_0^{\pi} \frac{1}{2} (\cos x + |\cos x|) dx$
29. $\int_0^{\ln 2} e^{3x} dx$
30. $\int_1^2 \left(\frac{1}{x} - e^{-x}\right) dx$
31. $\int_0^{1/2} \frac{4}{\sqrt{1-x^2}} dx$
32. $\int_0^{1/\sqrt{3}} \frac{dx}{1+4x^2}$
33. $\int_2^4 x^{\pi-1} dx$
34. $\int_{-1}^0 \pi^{x-1} dx$

In Exercises 35–38, guess an antiderivative for the integrand function. Validate your guess by differentiation, and then evaluate the given definite integral. (*Hint:* Keep the Chain Rule in mind when trying to guess an antiderivative. You will learn how to find such antiderivatives in the next section.)

35. $\int_0^1 x e^{x^2} dx$
36. $\int_1^2 \frac{\ln x}{x} dx$
37. $\int_2^5 \frac{x dx}{\sqrt{1+x^2}}$
38. $\int_0^{\pi/3} \sin^2 x \cos x dx$

Derivatives of Integrals

Find the derivatives in Exercises 39–44.

- a. by evaluating the integral and differentiating the result.
- b. by differentiating the integral directly.

39. $\frac{d}{dx} \int_0^{\sqrt{x}} \cos t dt$
40. $\frac{d}{dx} \int_1^{\sin x} 3t^2 dt$
41. $\frac{d}{dt} \int_0^{t^4} \sqrt{u} du$
42. $\frac{d}{d\theta} \int_0^{\tan \theta} \sec^2 y dy$
43. $\frac{d}{dx} \int_0^{x^3} e^{-t} dt$
44. $\frac{d}{dt} \int_0^{\sqrt{t}} \left(x^4 + \frac{3}{\sqrt{1-x^2}}\right) dx$

Find dy/dx in Exercises 45–56.

45. $y = \int_0^x \sqrt{1+t^2} dt$
46. $y = \int_1^x \frac{1}{t} dt, \quad x > 0$
47. $y = \int_{\sqrt{x}}^0 \sin t^2 dt$
48. $y = x \int_2^{x^2} \sin t^3 dt$
49. $y = \int_{-1}^x \frac{t^2}{t^2+4} dt - \int_3^x \frac{t^2}{t^2+4} dt$
50. $y = \left(\int_0^x (t^3+1)^{10} dt\right)^3$
51. $y = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}, \quad |x| < \frac{\pi}{2}$
52. $y = \int_{\tan x}^0 \frac{dt}{1+t^2}$
53. $y = \int_0^{e^{x^2}} \frac{1}{\sqrt{t}} dt$
54. $y = \int_{2^x}^1 \sqrt[3]{t} dt$
55. $y = \int_0^{\arcsin x} \cos t dt$
56. $y = \int_{-1}^{x^{1/\pi}} \arcsin t dt$

Area

In Exercises 57–60, find the total area between the region and the x -axis.

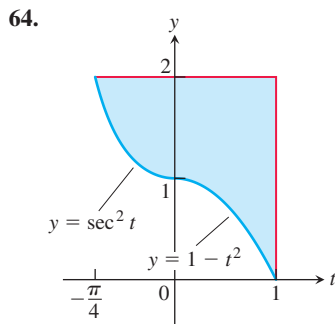
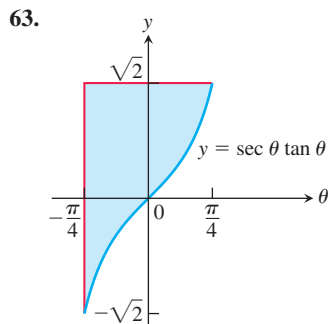
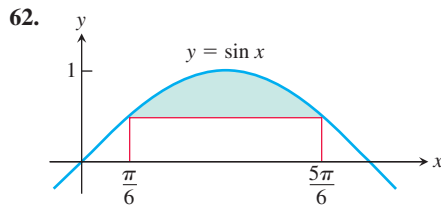
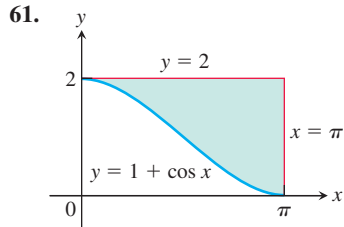
57. $y = -x^2 - 2x, \quad -3 \leq x \leq 2$

58. $y = 3x^2 - 3, \quad -2 \leq x \leq 2$

59. $y = x^3 - 3x^2 + 2x, \quad 0 \leq x \leq 2$

60. $y = x^{1/3} - x, \quad -1 \leq x \leq 8$

Find the areas of the shaded regions in Exercises 61–64.



Initial Value Problems

Each of the following functions solves one of the initial value problems in Exercises 65–68. Which function solves which problem? Give brief reasons for your answers.

a. $y = \int_1^x \frac{1}{t} dt - 3$ b. $y = \int_0^x \sec t dt + 4$

c. $y = \int_{-1}^x \sec t dt + 4$ d. $y = \int_{\pi}^x \frac{1}{t} dt - 3$

65. $\frac{dy}{dx} = \frac{1}{x}, \quad y(\pi) = -3$ 66. $y' = \sec x, \quad y(-1) = 4$

67. $y' = \sec x, \quad y(0) = 4$ 68. $y' = \frac{1}{x}, \quad y(1) = -3$

Express the solutions of the initial value problems in Exercises 69 and 70 in terms of integrals.

69. $\frac{dy}{dx} = \sec x, \quad y(2) = 3$ 70. $\frac{dy}{dx} = \sqrt{1+x^2}, \quad y(1) = -2$

For Exercises 71 and 72, find a function f satisfying each equation.

71. $\int_2^x \sqrt{f(t)} dt = x \ln x$ 72. $f(x) = e^2 + \int_1^x f(t) dt$

Theory and Examples

73. Archimedes' area formula for parabolic arches Archimedes (287–212 B.C.), inventor, military engineer, physicist, and the greatest mathematician of classical times in the Western world, discovered that the area under a parabolic arch is two-thirds the base times the height. Sketch the parabolic arch $y = h - (4h/b^2)x^2$, $-b/2 \leq x \leq b/2$, assuming that h and b are positive. Then use calculus to find the area of the region enclosed between the arch and the x -axis.

74. Show that if k is a positive constant, then the area between the x -axis and one arch of the curve $y = \sin kx$ is $2/k$.

75. Cost from marginal cost The marginal cost of printing a poster when x posters have been printed is

$$\frac{dc}{dx} = \frac{1}{2\sqrt{x}}$$

dollars. Find $c(100) - c(1)$, the cost of printing posters 2–100.

In Exercises 76–78, guess an antiderivative and validate your guess by differentiation. (*Hint:* Keep the Chain Rule in mind when trying to guess an antiderivative. You will learn how to find such antiderivatives in the next section.)

76. Revenue from marginal revenue Suppose that a company's marginal revenue from the manufacture and sale of eggbeaters is

$$\frac{dr}{dx} = 2 - 2/(x+1)^2,$$

where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand eggbeaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

77. The temperature $T(^{\circ}\text{C})$ of a room at time t minutes is given by

$$T = 30 - 2\sqrt{25-t} \quad \text{for } 0 \leq t \leq 25.$$

a. Find the room's temperature when $t = 0$, $t = 16$, and $t = 25$.

b. Find the room's average temperature for $0 \leq t \leq 25$.

78. The height $H(\text{m})$ of a palm tree after growing for t years is given by

$$H = 0.3 - \sqrt{t+1} + 1.5t^{1/3} \quad \text{for } 0 \leq t \leq 8.$$

a. Find the tree's height when $t = 0$, $t = 4$, and $t = 8$.

b. Find the tree's average height for $0 \leq t \leq 8$.

79. Suppose that $\int_1^x f(t) dt = x^2 - 2x + 1$. Find $f(x)$.

80. Find $f(4)$ if $\int_0^x f(t) dt = x \cos \pi x$.

81. Find the linearization of

$$f(x) = 2 - \int_2^{x+1} \frac{9}{1+t} dt$$

at $x = 1$.

82. Find the linearization of

$$g(x) = 3 + \int_1^{x^2} \sec(t-1) dt$$

at $x = -1$.

83. Suppose that f has a positive derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$g(x) = \int_0^x f(t) dt?$$

Give reasons for your answers.

- g is a differentiable function of x .
- g is a continuous function of x .
- The graph of g has a horizontal tangent line at $x = 1$.
- g has a local maximum at $x = 1$.
- g has a local minimum at $x = 1$.
- The graph of g has an inflection point at $x = 1$.
- The graph of dg/dx crosses the x -axis at $x = 1$.

84. Another proof of the Evaluation Theorem

- a. Let $a = x_0 < x_1 < x_2 \cdots < x_n = b$ be any partition of $[a, b]$, and let F be any antiderivative of f . Show that

$$F(b) - F(a) = \sum_{i=1}^n [F(x_i) - F(x_{i-1})].$$

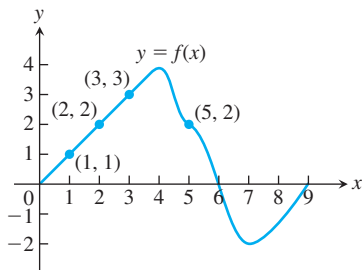
- b. Apply the Mean Value Theorem to each term to show that $F(x_i) - F(x_{i-1}) = f(c_i)(x_i - x_{i-1})$ for some c_i in the interval (x_{i-1}, x_i) . Then show that $F(b) - F(a)$ is a Riemann sum for f on $[a, b]$.
- c. From part (b) and the definition of the definite integral, show that

$$F(b) - F(a) = \int_a^b f(x) dx.$$

85. Suppose that f is the differentiable function shown in the accompanying graph and that the position at time t (in s) of a particle moving along a coordinate axis is

$$s = \int_0^t f(x) dx$$

meters. Use the graph to answer the following questions. Give reasons for your answers.



- What is the particle's velocity at time $t = 5$?
- Is the acceleration of the particle at time $t = 5$ positive or negative?
- What is the particle's position at time $t = 3$?
- At what time during the first 9 seconds does s have its largest value?
- Approximately when is the acceleration zero?
- When is the particle moving toward the origin? Away from the origin?

- g. On which side of the origin does the particle lie at time $t = 9$?

86. Find $\lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} \int_1^x \frac{dt}{\sqrt{t}}$.

COMPUTER EXPLORATIONS

In Exercises 87–90, let $F(x) = \int_a^x f(t) dt$ for the specified function f and interval $[a, b]$. Use a CAS to perform the following steps and answer the questions posed.

- Plot the functions f and F together over $[a, b]$.
- Solve the equation $F'(x) = 0$. What can you see to be true about the graphs of f and F at points where $F'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem coupled with information provided by the first derivative? Explain your answer.
- Over what intervals (approximately) is the function F increasing? Decreasing? What is true about f over those intervals?
- Calculate the derivative f' and plot it together with F . What can you see to be true about the graph of F at points where $f'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem? Explain your answer.

87. $f(x) = x^3 - 4x^2 + 3x$, $[0, 4]$

88. $f(x) = 2x^4 - 17x^3 + 46x^2 - 43x + 12$, $\left[0, \frac{9}{2}\right]$

89. $f(x) = \sin 2x \cos \frac{x}{3}$, $[0, 2\pi]$

90. $f(x) = x \cos \pi x$, $[0, 2\pi]$

In Exercises 91–94, let $F(x) = \int_a^{u(x)} f(t) dt$ for the specified a , u , and f . Use a CAS to perform the following steps and answer the questions posed.

- Find the domain of F .
- Calculate $F'(x)$ and determine its zeros. For what points in its domain is F increasing? Decreasing?
- Calculate $F''(x)$ and determine its zero. Identify the local extrema and the points of inflection of F .
- Using the information from parts (a)–(c), draw a rough hand-sketched $y = F(x)$ over its domain. Then graph $F(x)$ on your CAS to support your sketch.

91. $a = 1$, $u(x) = x^2$, $f(x) = \sqrt{1 - x^2}$

92. $a = 0$, $u(x) = x^2$, $f(x) = \sqrt{1 - x^2}$

93. $a = 0$, $u(x) = 1 - x$, $f(x) = x^2 - 2x - 3$

94. $a = 0$, $u(x) = 1 - x^2$, $f(x) = x^2 - 2x - 3$

In Exercises 95 and 96, assume that f is continuous and $u(x)$ is twice-differentiable.

95. Calculate $\frac{d}{dx} \int_a^{u(x)} f(t) dt$ and check your answer using a CAS.

96. Calculate $\frac{d^2}{dx^2} \int_a^{u(x)} f(t) dt$ and check your answer using a CAS.

5.5 Indefinite Integrals and the Substitution Method

The Fundamental Theorem of Calculus says that a definite integral of a continuous function can be computed directly if we can find an antiderivative of the function. In Section 4.8 we defined the **indefinite integral** of the function f with respect to x as the set of *all* antiderivatives of f , symbolized by $\int f(x) dx$. Since any two antiderivatives of f differ by a constant, the indefinite integral $\int$ notation means that for any antiderivative F of f ,

$$\int f(x) dx = F(x) + C,$$

where C is any arbitrary constant. The connection between antiderivatives and the definite integral stated in the Fundamental Theorem now explains this notation:

$$\begin{aligned}\int_a^b f(x) dx &= F(b) - F(a) = [F(b) + C] - [F(a) + C] \\ &= \left[F(x) + C \right]_a^b = \left[\int f(x) dx \right]_a^b.\end{aligned}$$

When finding the indefinite integral of a function f , remember that it always includes an arbitrary constant C .

We must keep in mind the difference between definite and indefinite integrals. A definite integral $\int_a^b f(x) dx$ is a *number*. An indefinite integral $\int f(x) dx$ is a *function* plus an arbitrary constant C .

So far, we have only been able to find antiderivatives of functions that are clearly recognizable as derivatives. In this section we begin to develop more general techniques for finding antiderivatives of functions.

Substitution: Running the Chain Rule Backwards

If u is a differentiable function of x , and n is any number different from -1 , the Chain Rule tells us that

$$\frac{d}{dx} \left(\frac{u^{n+1}}{n+1} \right) = u^n \frac{du}{dx}.$$

From another point of view, this same equation says that $u^{n+1}/(n+1)$ is one of the antiderivatives of the function $u^n (du/dx)$. Therefore,

$$\int u^n \frac{du}{dx} dx = \frac{u^{n+1}}{n+1} + C. \quad (1)$$

The integral in Equation (1) is equal to the simpler integral

$$\int u^n du = \frac{u^{n+1}}{n+1} + C,$$

which suggests that we can substitute the simpler expression du for $(du/dx) dx$ when computing an integral. Leibniz, one of the founders of calculus, had the insight that indeed this substitution could be done, leading to the *substitution method* for computing integrals. As with differentials, when computing integrals we have

$$du = \frac{du}{dx} dx.$$

EXAMPLE 1 Find the integral $\int (x^3 + x)^5 (3x^2 + 1) dx$.

Solution We set $u = x^3 + x$. Then

$$du = \frac{du}{dx} dx = (3x^2 + 1) dx,$$

so that by substitution we have

$$\begin{aligned}
 \int (x^3 + x)^5 (3x^2 + 1) dx &= \int u^5 du && \text{Let } u = x^3 + x, du = (3x^2 + 1) dx. \\
 &= \frac{u^6}{6} + C && \text{Integrate with respect to } u. \\
 &= \frac{(x^3 + x)^6}{6} + C && \text{Substitute } x^3 + x \text{ for } u.
 \end{aligned}$$

EXAMPLE 2 Find $\int \sqrt{2x + 1} dx$.

Solution The integral does not fit the formula

$$\int u^n du,$$

with $u = 2x + 1$ and $n = 1/2$, because

$$du = \frac{du}{dx} dx = 2 dx,$$

which is not precisely dx . The constant factor 2 is missing from the integral. However, we can introduce this factor after the integral sign if we compensate for it by introducing a factor of $1/2$ in front of the integral sign. So we write

$$\begin{aligned}
 \int \sqrt{2x + 1} dx &= \frac{1}{2} \int \sqrt{\underbrace{2x + 1}_u} \cdot \underbrace{2 dx}_{du} \\
 &= \frac{1}{2} \int u^{1/2} du && \text{Let } u = 2x + 1, du = 2 dx. \\
 &= \frac{1}{2} \frac{u^{3/2}}{3/2} + C && \text{Integrate with respect to } u. \\
 &= \frac{1}{3} (2x + 1)^{3/2} + C. && \text{Substitute } 2x + 1 \text{ for } u.
 \end{aligned}$$

The substitutions in Examples 1 and 2 are instances of the following general rule.

THEOREM 6—The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I , and f is continuous on I , then

$$\int \underbrace{f(g(x))}_{\downarrow} \cdot \underbrace{g'(x)}_{\sim} dx = \int f(u) du.$$

Proof By the Chain Rule, $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$ whenever F is an antiderivative of f , because

$$\begin{aligned}
 \frac{d}{dx} F(g(x)) &= F'(g(x)) \cdot g'(x) && \text{Chain Rule} \\
 &= f(g(x)) \cdot g'(x). && F' = f
 \end{aligned}$$

If we make the substitution $u = g(x)$, then

$$\begin{aligned}
 \int f(g(x)) g'(x) dx &= \int \frac{d}{dx} F(g(x)) dx \\
 &= F(g(x)) + C && \text{Theorem 8 in Chapter 4} \\
 &= F(u) + C && u = g(x) \\
 &= \int F'(u) du && \text{Theorem 8 in Chapter 4} \\
 &= \int f(u) du. && F' = f
 \end{aligned}$$

The use of the variable u in the Substitution Rule is traditional (sometimes it is referred to as u -substitution), but any letter can be used, such as v, t, θ and so forth. The rule provides a method for evaluating an integral of the form $\int f(g(x)) g'(x) dx$, given that the conditions of Theorem 6 are satisfied. The primary challenge is deciding what expression involving x to substitute for in the integrand. The following examples give helpful ideas.

The Substitution Method to evaluate $\int f(g(x)) g'(x) dx$

1. Substitute $u = g(x)$ and $du = (du/dx) dx = g'(x) dx$ to obtain $\int f(u) du$.
2. Integrate with respect to u .
3. Replace u by $g(x)$.

EXAMPLE 3 Find $\int \sec^2(5x + 1) \cdot 5 dx$.

Solution We substitute $u = 5x + 1$ and $du = 5 dx$. Then

$$\begin{aligned}
 \int \sec^2(5x + 1) \cdot 5 dx &= \int \sec^2 u du && \text{Let } u = 5x + 1, du = 5 dx. \\
 &= \tan u + C && \frac{d}{du} \tan u = \sec^2 u \\
 &= \tan(5x + 1) + C. && \text{Substitute } 5x + 1 \text{ for } u.
 \end{aligned}$$

EXAMPLE 4 Find $\int \cos(7\theta + 3) d\theta$.

Solution We let $u = 7\theta + 3$ so that $du = 7 d\theta$. There is a factor of 7 in this formula for du , but there is no corresponding 7 preceding $d\theta$ in the integral. We can compensate for this by multiplying and dividing by 7, using the same procedure as in Example 2. Then

$$\begin{aligned}
 \int \cos(7\theta + 3) d\theta &= \frac{1}{7} \int \cos(7\theta + 3) \cdot 7 d\theta && \text{Place factor } 1/7 \text{ in front of integral.} \\
 &= \frac{1}{7} \int \cos u du && \text{Substitute } u = 7\theta + 3, du = 7 d\theta. \\
 &= \frac{1}{7} \sin u + C && \text{Integrate.} \\
 &= \frac{1}{7} \sin(7\theta + 3) + C. && \text{Replace } u \text{ by } 7\theta + 3.
 \end{aligned}$$

There is another approach to this problem. With $u = 7\theta + 3$ and $du = 7 d\theta$ as before, we solve for $d\theta$ to obtain $d\theta = (1/7) du$. Then the integral becomes

$$\begin{aligned}\int \cos(7\theta + 3) d\theta &= \int \cos u \cdot \frac{1}{7} du && \text{Substitute } u = 7\theta + 3, du = 7 d\theta, \\ &= \frac{1}{7} \sin u + C && \text{and } d\theta = (1/7) du. \\ &= \frac{1}{7} \sin(7\theta + 3) + C. && \text{Integrate.} \\ &&& \text{Replace } u \text{ by } 7\theta + 3.\end{aligned}$$

We can verify this solution by differentiating and checking that we obtain the original function $\cos(7\theta + 3)$. ■

EXAMPLE 5 Sometimes we observe that a power of x appears in the integrand that is one less than the power of x appearing in the argument of a function we want to integrate. This observation immediately suggests we try a substitution for the higher power of x . For example, in the integral below we see that x^3 appears as the exponent of one factor, and this factor is multiplied by x^2 . This suggests trying the substitution $u = x^3$.

$$\begin{aligned}\int x^2 e^{x^3} dx &= \int e^{x^3} \cdot x^2 dx \\ &= \int e^u \cdot \frac{1}{3} du && \text{Substitute } u = x^3, du = 3x^2 dx, \\ &= \frac{1}{3} \int e^u du && (1/3) du = x^2 dx. \\ &= \frac{1}{3} e^u + C && \text{Integrate with respect to } u. \\ &= \frac{1}{3} e^{x^3} + C && \text{Replace } u \text{ by } x^3.\end{aligned}$$

HISTORICAL BIOGRAPHY

George David Birkhoff
(1884–1944)

Birkhoff attended Harvard and the University of Chicago. He received his PhD from Chicago in 1907 for his dissertation on differential equations. He also worked on the four-color problem (colors required to produce a map) and applying mathematics to aesthetics in art, poetry, and music.

To know more, visit the companion Website.

It may happen that an extra factor of x appears in the integrand when we try a substitution $u = g(x)$. In that case, it may be possible to solve the equation $u = g(x)$ for x in terms of u . Replacing the extra factor of x with that expression may then result in an integral that we can evaluate. Here is an example of this situation.

EXAMPLE 6 Evaluate $\int x\sqrt{2x+1} dx$.

Solution Our previous experience with the integral in Example 2 suggests the substitution $u = 2x + 1$ with $du = 2 dx$. Then

$$\sqrt{2x+1} dx = \frac{1}{2} \sqrt{u} du.$$

However, in this example the integrand contains an extra factor of x that multiplies the factor $\sqrt{2x+1}$. To adjust for this, we solve the substitution equation $u = 2x + 1$ for x to obtain $x = (u - 1)/2$ and find that

$$x\sqrt{2x+1} dx = \frac{1}{2}(u - 1) \cdot \frac{1}{2} \sqrt{u} du.$$

The integration now becomes

$$\begin{aligned}
 \int x\sqrt{2x+1} \, dx &= \frac{1}{4} \int (u-1)\sqrt{u} \, du = \frac{1}{4} \int (u-1)u^{1/2} \, du && \text{Substitute.} \\
 &= \frac{1}{4} \int (u^{3/2} - u^{1/2}) \, du && \text{Multiply terms by } u^{1/2}. \\
 &= \frac{1}{4} \left(\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) + C && \text{Integrate.} \\
 &= \frac{1}{10} (2x+1)^{5/2} - \frac{1}{6} (2x+1)^{3/2} + C. && \text{Replace } u \text{ by } 2x+1. \quad \blacksquare
 \end{aligned}$$

EXAMPLE 7 Sometimes we can use trigonometric identities to transform an integral we do not know how to evaluate into one that we can evaluate using the Substitution Rule.

$$\begin{aligned}
 \text{(a)} \quad \int \sin^2 x \, dx &= \int \frac{1 - \cos 2x}{2} \, dx && \sin^2 x = \frac{1 - \cos 2x}{2} \\
 &= \frac{1}{2} \int (1 - \cos 2x) \, dx \\
 &= \frac{1}{2} x - \frac{1}{2} \frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C \\
 \text{(b)} \quad \int \cos^2 x \, dx &= \int \frac{1 + \cos 2x}{2} \, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C && \cos^2 x = \frac{1 + \cos 2x}{2} \\
 \text{(c)} \quad \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx = \int \frac{-du}{u} && u = \cos x, \, du = -\sin x \, dx \\
 &= -\ln |u| + C = -\ln |\cos x| + C \\
 &= \ln \frac{1}{|\cos x|} + C = \ln |\sec x| + C && \text{Reciprocal Rule} \quad \blacksquare
 \end{aligned}$$

EXAMPLE 8 An integrand may require some algebraic manipulation before the substitution method can be applied. This example gives two integrals for which we simplify by multiplying the integrand by an algebraic form equal to 1 before attempting a substitution.

$$\begin{aligned}
 \text{(a)} \quad \int \frac{dx}{e^x + e^{-x}} &= \int \frac{e^x \, dx}{e^{2x} + 1} && \text{Multiply by } (e^x/e^x) = 1. \\
 &= \int \frac{du}{u^2 + 1} && \text{Substitute } u = e^x, \, u^2 = e^{2x}, \\
 &= \arctan u + C && du = e^x \, dx. \\
 &= \arctan(e^x) + C && \text{Integrate with respect to } u. \\
 &&& \text{Replace } u \text{ by } e^x. \\
 \text{(b)} \quad \int \sec x \, dx &= \int (\sec x)(1) \, dx \\
 &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx && \frac{\sec x + \tan x}{\sec x + \tan x} \text{ is equal to } 1. \\
 &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \\
 &= \int \frac{du}{u} && u = \sec x + \tan x, \\
 &= \ln |u| + C = \ln |\sec x + \tan x| + C. && du = (\sec^2 x + \sec x \tan x) \, dx \quad \blacksquare
 \end{aligned}$$

The integrals of $\cot x$ and $\csc x$ are computed in a way similar to the integrals of $\tan x$ and $\sec x$ in Examples 7c and 8b (see Exercises 71 and 72). We summarize the results for these four basic trigonometric integrals here.

Integrals of the tangent, cotangent, secant, and cosecant functions

$$\begin{aligned}\int \tan x \, dx &= \ln |\sec x| + C & \int \sec x \, dx &= \ln |\sec x + \tan x| + C \\ \int \cot x \, dx &= \ln |\sin x| + C & \int \csc x \, dx &= -\ln |\csc x + \cot x| + C\end{aligned}$$

Trying Different Substitutions

The success of the substitution method depends on finding a substitution that changes an integral we cannot directly evaluate into one that we can. Finding the right substitution gets easier with practice and experience. If your first substitution fails, try another substitution, possibly coupled with other algebraic or trigonometric simplifications to the integrand. Several more complicated types of substitutions will be studied in Chapter 8.

EXAMPLE 9 Evaluate $\int \frac{2z \, dz}{\sqrt[3]{z^2 + 1}}$.

Solution We use the substitution method of integration as an exploratory tool. First we substitute for the most troublesome part of the integrand and see how things work out. For the integral here, we might try $u = z^2 + 1$, or we might even press our luck and take u to be the entire cube root. In this example both substitutions turn out to be successful, but that is not always the case. If one substitution does not help, a different substitution may work instead.

Method 1: Substitute $u = z^2 + 1$.

$$\begin{aligned}\int \frac{2z \, dz}{\sqrt[3]{z^2 + 1}} &= \int \frac{du}{u^{1/3}} && \text{Let } u = z^2 + 1, \, du = 2z \, dz. \\ &= \int u^{-1/3} \, du && \text{In the form } \int u^n \, du \\ &= \frac{u^{2/3}}{2/3} + C && \text{Integrate.} \\ &= \frac{3}{2} u^{2/3} + C \\ &= \frac{3}{2} (z^2 + 1)^{2/3} + C && \text{Replace } u \text{ by } z^2 + 1.\end{aligned}$$

Method 2: Substitute $u = \sqrt[3]{z^2 + 1}$ instead.

$$\begin{aligned}\int \frac{2z \, dz}{\sqrt[3]{z^2 + 1}} &= \int \frac{3u^2 \, du}{u} && \text{Let } u = \sqrt[3]{z^2 + 1}, \\ &= 3 \int u \, du && u^2 = z^2 + 1, \, 3u^2 \, du = 2z \, dz. \\ &= 3 \cdot \frac{u^2}{2} + C && \text{Integrate.} \\ &= \frac{3}{2} (z^2 + 1)^{2/3} + C && \text{Replace } u \text{ by } (z^2 + 1)^{1/3}.\end{aligned}$$

EXERCISES 5.5

Evaluating Indefinite Integrals

In Exercises 1–16, make the given substitutions to evaluate the indefinite integrals.

1. $\int 2(2x + 4)^5 dx$, $u = 2x + 4$
2. $\int 7\sqrt{7x - 1} dx$, $u = 7x - 1$
3. $\int 2x(x^2 + 5)^{-4} dx$, $u = x^2 + 5$
4. $\int \frac{4x^3}{(x^4 + 1)^2} dx$, $u = x^4 + 1$
5. $\int (3x + 2)(3x^2 + 4x)^4 dx$, $u = 3x^2 + 4x$
6. $\int \frac{(1 + \sqrt{x})^{1/3}}{\sqrt{x}} dx$, $u = 1 + \sqrt{x}$
7. $\int \sin 3x dx$, $u = 3x$
8. $\int x \sin(2x^2) dx$, $u = 2x^2$
9. $\int \sec 2t \tan 2t dt$, $u = 2t$
10. $\int \left(1 - \cos \frac{t}{2}\right)^2 \sin \frac{t}{2} dt$, $u = 1 - \cos \frac{t}{2}$
11. $\int \frac{9r^2 dr}{\sqrt{1 - r^3}}$, $u = 1 - r^3$
12. $\int 12(y^4 + 4y^2 + 1)^2 (y^3 + 2y) dy$, $u = y^4 + 4y^2 + 1$
13. $\int \sqrt{x} \sin^2(x^{3/2} - 1) dx$, $u = x^{3/2} - 1$
14. $\int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right) dx$, $u = \frac{1}{x}$
15. $\int \csc^2 2\theta \cot 2\theta d\theta$
 - a. Using $u = \cot 2\theta$
 - b. Using $u = \csc 2\theta$
16. $\int \frac{dx}{\sqrt{5x + 8}}$
 - a. Using $u = 5x + 8$
 - b. Using $u = \sqrt{5x + 8}$

Evaluate the integrals in Exercises 17–66.

17. $\int \sqrt{3 - 2s} ds$
18. $\int \frac{1}{\sqrt{5s + 4}} ds$
19. $\int \theta \sqrt{1 - \theta^2} d\theta$
20. $\int 3y \sqrt{7 - 3y^2} dy$
21. $\int \frac{1}{\sqrt{x}(1 + \sqrt{x})^2} dx$
22. $\int \sqrt{\sin x} \cos^3 x dx$
23. $\int \sec^2(3x + 2) dx$
24. $\int \tan^2 x \sec^2 x dx$
25. $\int \sin^5 \frac{x}{3} \cos \frac{x}{3} dx$
26. $\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2} dx$
27. $\int r^2 \left(\frac{r^3}{18} - 1\right)^5 dr$
28. $\int r^4 \left(7 - \frac{r^5}{10}\right)^3 dr$
29. $\int x^{1/2} \sin(x^{3/2} + 1) dx$
30. $\int \csc\left(\frac{v - \pi}{2}\right) \cot\left(\frac{v - \pi}{2}\right) dv$
31. $\int \frac{\sin(2t + 1)}{\cos^2(2t + 1)} dt$
32. $\int \frac{\sec z \tan z}{\sqrt{\sec z}} dz$
33. $\int \frac{1}{t^2} \cos\left(\frac{1}{t} - 1\right) dt$
34. $\int \frac{1}{\sqrt{t}} \cos(\sqrt{t} + 3) dt$
35. $\int \frac{1}{\theta^2} \sin \frac{1}{\theta} \cos \frac{1}{\theta} d\theta$
36. $\int \frac{\cos \sqrt{\theta}}{\sqrt{\theta} \sin^2 \sqrt{\theta}} d\theta$
37. $\int \frac{x}{\sqrt{1 + x}} dx$
38. $\int \sqrt{\frac{x - 1}{x^5}} dx$
39. $\int \frac{1}{x^2} \sqrt{2 - \frac{1}{x}} dx$
40. $\int \frac{1}{x^3} \sqrt{\frac{x^2 - 1}{x^2}} dx$
41. $\int \sqrt{\frac{x^3 - 3}{x^{11}}} dx$
42. $\int \sqrt{\frac{x^4}{x^3 - 1}} dx$
43. $\int x(x - 1)^{10} dx$
44. $\int x \sqrt{4 - x} dx$
45. $\int (x + 1)^2 (1 - x)^5 dx$
46. $\int (x + 5)(x - 5)^{1/3} dx$
47. $\int x^3 \sqrt{x^2 + 1} dx$
48. $\int 3x^5 \sqrt{x^3 + 1} dx$
49. $\int \frac{x}{(x^2 - 4)^3} dx$
50. $\int \frac{x}{(2x - 1)^{2/3}} dx$
51. $\int (\cos x) e^{\sin x} dx$
52. $\int (\sin 2\theta) e^{\sin^2 \theta} d\theta$
53. $\int \frac{1}{\sqrt{x} e^{-\sqrt{x}}} \sec^2(e^{\sqrt{x}} + 1) dx$
54. $\int \frac{1}{x^2} e^{1/x} \sec(1 + e^{1/x}) \tan(1 + e^{1/x}) dx$
55. $\int \frac{dx}{x \ln x}$
56. $\int \frac{\ln \sqrt{t}}{t} dt$
57. $\int \frac{dz}{1 + e^z}$
58. $\int \frac{dx}{x \sqrt{x^4 - 1}}$
59. $\int \frac{5}{9 + 4r^2} dr$
60. $\int \frac{1}{\sqrt{e^{2\theta} - 1}} d\theta$

61. $\int \frac{e^{\arcsin x} dx}{\sqrt{1-x^2}}$

62. $\int \frac{e^{\arccos x} dx}{\sqrt{1-x^2}}$

63. $\int \frac{(\arcsin x)^2 dx}{\sqrt{1-x^2}}$

64. $\int \frac{\sqrt{\arctan x} dx}{1+x^2}$

65. $\int \frac{dy}{(\arctan y)(1+y^2)}$

66. $\int \frac{dy}{(\arcsin y)\sqrt{1-y^2}}$

If you do not know what substitution to make, try reducing the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. You will see what we mean if you try the sequences of substitutions in Exercises 67 and 68.

67. $\int \frac{18 \tan^2 x \sec^2 x}{(2 + \tan^3 x)^2} dx$

- a. $u = \tan x$, followed by $v = u^3$, then by $w = 2 + v$
- b. $u = \tan^3 x$, followed by $v = 2 + u$
- c. $u = 2 + \tan^3 x$

68. $\int \sqrt{1 + \sin^2(x-1)} \sin(x-1) \cos(x-1) dx$

- a. $u = x - 1$, followed by $v = \sin u$, then by $w = 1 + v^2$
- b. $u = \sin(x - 1)$, followed by $v = 1 + u^2$
- c. $u = 1 + \sin^2(x - 1)$

Evaluate the integrals in Exercises 69 and 70.

69. $\int \frac{(2r-1) \cos \sqrt{3(2r-1)^2 + 6}}{\sqrt{3(2r-1)^2 + 6}} dr$

70. $\int \frac{\sin \sqrt{\theta}}{\sqrt{\theta} \cos^3 \sqrt{\theta}} d\theta$

71. Find the integral of $\cot x$ using a substitution like that in Example 7c.

72. Find the integral of $\csc x$ by multiplying by an appropriate form equal to 1, as in Example 8b.

Initial Value Problems

Solve the initial value problems in Exercises 73–78.

73. $\frac{ds}{dt} = 12t(3t^2 - 1)^3, \quad s(1) = 3$

74. $\frac{dy}{dx} = 4x(x^2 + 8)^{-1/3}, \quad y(0) = 0$

75. $\frac{ds}{dt} = 8 \sin^2\left(t + \frac{\pi}{12}\right), \quad s(0) = 8$

76. $\frac{dr}{d\theta} = 3 \cos^2\left(\frac{\pi}{4} - \theta\right), \quad r(0) = \frac{\pi}{8}$

77. $\frac{d^2s}{dt^2} = -4 \sin\left(2t - \frac{\pi}{2}\right), \quad s'(0) = 100, \quad s(0) = 0$

78. $\frac{d^2y}{dx^2} = 4 \sec^2 2x \tan 2x, \quad y'(0) = 4, \quad y(0) = -1$

79. The velocity of a particle moving back and forth on a line is $v = ds/dt = 6 \sin 2t$ m/s for all t . If $s = 0$ when $t = 0$, find the value of s when $t = \pi/2$ s.

80. The acceleration of a particle moving back and forth on a line is $a = d^2s/dt^2 = \pi^2 \cos \pi t$ m/s² for all t . If $s = 0$ and $v = 8$ m/s when $t = 0$, find s when $t = 1$ s.

5.6 Definite Integral Substitutions and the Area Between Curves

There are two methods for evaluating a definite integral by substitution. One method is to find an antiderivative using substitution and then to evaluate the definite integral by applying the Evaluation Theorem. The other method extends the process of substitution directly to *definite* integrals by changing the limits of integration. We will use these methods to compute the area between two curves.

The Substitution Formula

The following formula shows how the limits of integration change when we apply a substitution to an integral.

THEOREM 7—Substitution in Definite Integrals

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

Proof Let F denote any antiderivative of f . Then

$$\begin{aligned}
 \int_a^b f(g(x)) \cdot g'(x) dx &= F(g(x)) \Big|_{x=a}^{x=b} & \frac{d}{dx} F(g(x)) &= F'(g(x)) g'(x) \\
 &= F(g(b)) - F(g(a)) & &= f(g(x)) g'(x) \\
 &= F(u) \Big|_{u=g(a)}^{u=g(b)} \\
 &= \int_{g(a)}^{g(b)} f(u) du. & \text{Fundamental Theorem, Part 2} \quad \blacksquare
 \end{aligned}$$

To use Theorem 7, we make the same u -substitution $u = g(x)$ and $du = g'(x) dx$ that we would use to evaluate the corresponding indefinite integral. We then integrate the transformed integral with respect to u from the value $g(a)$ (the value of u at $x = a$) to the value $g(b)$ (the value of u at $x = b$).

EXAMPLE 1 Evaluate $\int_{-1}^1 3x^2 \sqrt{x^3 + 1} dx$.

Solution We will show how to evaluate the integral using Theorem 7, and how to evaluate it using the original limits of integration.

Method 1: Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem 7.

$$\begin{aligned}
 \int_{-1}^1 3x^2 \sqrt{x^3 + 1} dx &= \int_0^2 \sqrt{u} du & \text{Let } u = x^3 + 1, du = 3x^2 dx. \\
 & & \text{When } x = -1, u = (-1)^3 + 1 = 0. \\
 & & \text{When } x = 1, u = (1)^3 + 1 = 2. \\
 &= \frac{2}{3} u^{3/2} \Big|_0^2 & \text{Evaluate the new definite integral.} \\
 &= \frac{2}{3} [2^{3/2} - 0^{3/2}] = \frac{2}{3} [2\sqrt{2}] = \frac{4\sqrt{2}}{3}
 \end{aligned}$$

Method 2: Transform the integral as an indefinite integral, integrate, change back to x , and use the original x -limits.

$$\begin{aligned}
 \int 3x^2 \sqrt{x^3 + 1} dx &= \int \sqrt{u} du & \text{Let } u = x^3 + 1, du = 3x^2 dx. \\
 &= \frac{2}{3} u^{3/2} + C & \text{Integrate with respect to } u. \\
 &= \frac{2}{3} (x^3 + 1)^{3/2} + C & \text{Replace } u \text{ by } x^3 + 1. \\
 \int_{-1}^1 3x^2 \sqrt{x^3 + 1} dx &= \frac{2}{3} (x^3 + 1)^{3/2} \Big|_{-1}^1 & \text{Use the integral just found, with} \\
 & & \text{limits of integration for } x. \\
 &= \frac{2}{3} [(1^3 + 1)^{3/2} - ((-1)^3 + 1)^{3/2}] \\
 &= \frac{2}{3} [2^{3/2} - 0^{3/2}] = \frac{2}{3} [2\sqrt{2}] = \frac{4\sqrt{2}}{3} \quad \blacksquare
 \end{aligned}$$

Which method is better—evaluating the transformed definite integral with transformed limits using Theorem 7, or transforming the integral, integrating, and transforming back to use the original limits of integration? In Example 1, the first method seems easier, but that is not always the case. Generally, it is best to know both methods and to use whichever one seems better at the time.

EXAMPLE 2 We use the method of transforming the limits of integration.

$$\begin{aligned}
 \text{(a)} \quad \int_{\pi/4}^{\pi/2} \cot \theta \csc^2 \theta \, d\theta &= \int_1^0 u \cdot (-du) \\
 &= -\int_1^0 u \, du \\
 &= -\left[\frac{u^2}{2}\right]_1^0 \\
 &= -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}
 \end{aligned}$$

Let $u = \cot \theta$, $du = -\csc^2 \theta \, d\theta$,
 $-du = \csc^2 \theta \, d\theta$.
 When $\theta = \pi/4$, $u = \cot(\pi/4) = 1$.
 When $\theta = \pi/2$, $u = \cot(\pi/2) = 0$.

$$\begin{aligned}
 \text{(b)} \quad \int_{-\pi/4}^{\pi/4} \tan x \, dx &= \int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos x} \, dx \\
 &= -\int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{du}{u} \\
 &= 0
 \end{aligned}$$

Let $u = \cos x$, $du = -\sin x \, dx$.
 When $x = -\pi/4$, $u = \sqrt{2}/2$.
 When $x = \pi/4$, $u = \sqrt{2}/2$.
 Zero width interval

Definite Integrals of Symmetric Functions

The Substitution Formula in Theorem 7 simplifies the calculation of definite integrals of even and odd functions (Section 1.1) over a symmetric interval $[-a, a]$ (Figure 5.24).

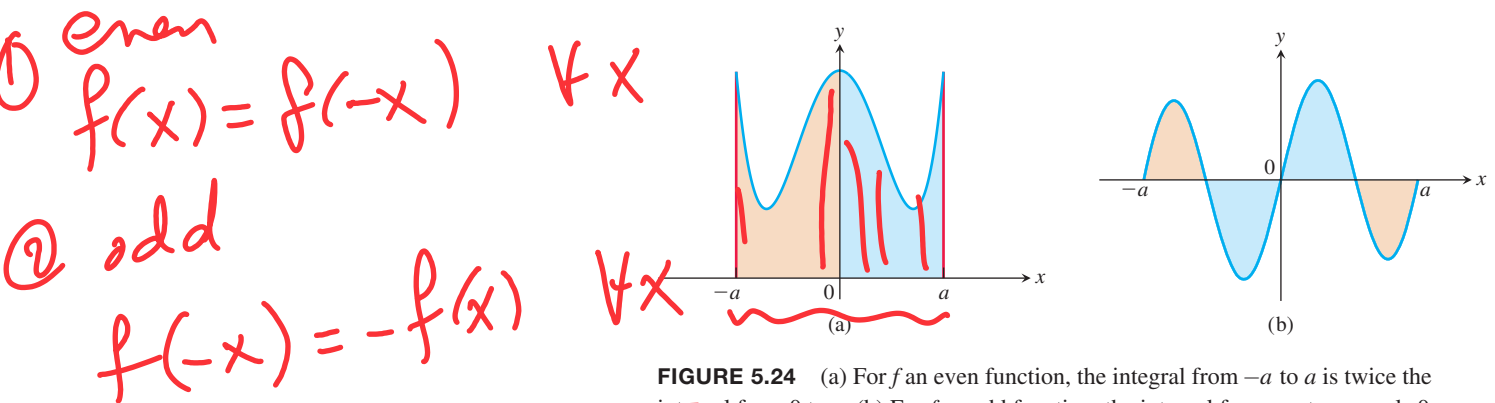


FIGURE 5.24 (a) For f an even function, the integral from $-a$ to a is twice the integral from 0 to a . (b) For f an odd function, the integral from $-a$ to a equals 0.

THEOREM 8 Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx$.

(b) If f is odd, then $\int_{-a}^a f(x) \, dx = 0$.

① even
 $f(x) = f(-x) \quad \forall x$

② odd
 $f(-x) = -f(x) \quad \forall x$

$f(x) = x^4, f(-x) = (-x)^4 = x^4$

$f(x) = x^3, f(-x) = (-x)^3 = -x^3 = -f(x)$

Proof of Part (a)

$$\begin{aligned}
 \int_{-a}^a f(x) dx &= \int_{-a}^0 f(x) dx + \int_0^a f(x) dx && \text{Additivity Rule for Definite Integrals} \\
 &= -\int_0^{-a} f(x) dx + \int_0^a f(x) dx && \text{Order of Integration Rule} \\
 &= -\int_0^a f(-u)(-du) + \int_0^a f(x) dx && \text{Let } u = -x, du = -dx. \\
 & && \text{When } x = 0, u = 0. \\
 & && \text{When } x = -a, u = a. \\
 &= \int_0^a f(-u) du + \int_0^a f(x) dx \\
 &= \int_0^a f(u) du + \int_0^a f(x) dx && f \text{ is even, so } f(-u) = f(u). \\
 &= 2 \int_0^a f(x) dx
 \end{aligned}$$

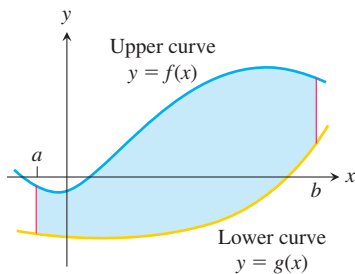


FIGURE 5.25 The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

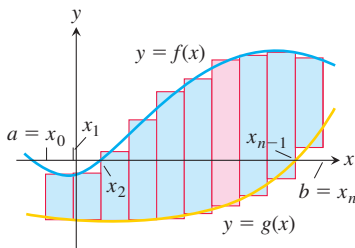


FIGURE 5.26 We approximate the region with rectangles perpendicular to the x -axis.

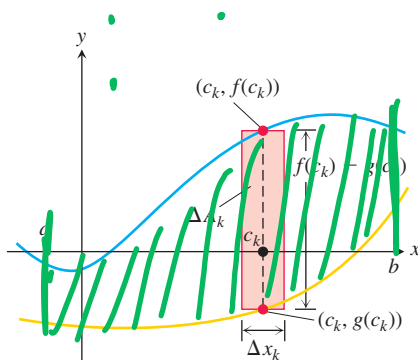


FIGURE 5.27 The area ΔA_k of the k th rectangle is the product of its height, $f(c_k) - g(c_k)$, and its width, Δx_k .

The proof of part (b) is similar, and you are asked to give it in Exercise 120. ■

EXAMPLE 3 Evaluate $\int_{-2}^2 x^4 - 4x^2 + 6 dx$.

Solution Since $f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even on the symmetric interval $[-2, 2]$, so

$$\begin{aligned}
 \int_{-2}^2 x^4 - 4x^2 + 6 dx &= 2 \int_0^2 x^4 - 4x^2 + 6 dx \\
 &= 2 \left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x \right]_0^2 \\
 &= 2 \left(\frac{32}{5} - \frac{32}{3} + 12 \right) = \frac{232}{15}.
 \end{aligned}$$

Areas Between Curves

Suppose we want to find the area of a region that is bounded above by the curve $y = f(x)$, below by the curve $y = g(x)$, and on the left and right by the lines $x = a$ and $x = b$ (Figure 5.25). The region might accidentally have a shape whose area we could find with geometry, but if f and g are arbitrary continuous functions, we usually have to find the area by computing an integral.

To see what the integral should be, we first approximate the region with n vertical rectangles based on a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ (Figure 5.26). The area of the k th rectangle (Figure 5.27) is

$$\Delta A_k = \text{height} \times \text{width} = [f(c_k) - g(c_k)] \Delta x_k.$$

We then approximate the area of the region by adding the areas of the n rectangles:

$$A \approx \sum_{k=1}^n \Delta A_k = \sum_{k=1}^n [f(c_k) - g(c_k)] \Delta x_k. \quad \text{Riemann sum}$$

As $\|P\| \rightarrow 0$, the sums on the right approach the limit $\int_a^b [f(x) - g(x)] dx$ because f and g are continuous. The area of the region is defined to be the value of this integral. That is,

$$A = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n [f(c_k) - g(c_k)] \Delta x_k = \int_a^b [f(x) - g(x)] dx.$$

$$\text{Area} = \int_a^b f(x) - g(x) dx$$

DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves** $y = f(x)$ and $y = g(x)$ from a to b is the integral of $(f - g)$ from a to b :

$$A = \int_a^b [f(x) - g(x)] dx.$$

When applying this definition it is usually helpful to graph the curves. The graph reveals which curve is the upper curve f and which is the lower curve g . It also helps you find the limits of integration if they are not given. You may need to find where the curves intersect to determine the limits of integration, and this may involve solving the equation $f(x) = g(x)$ for values of x . Then you can integrate the function $f - g$ for the area between the intersections.

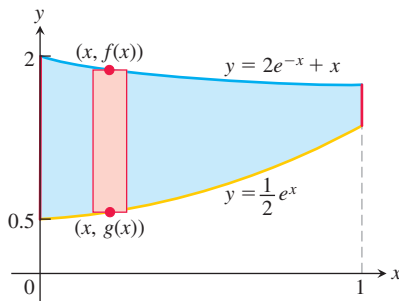


FIGURE 5.28 The region in Example 4 with a typical approximating rectangle.

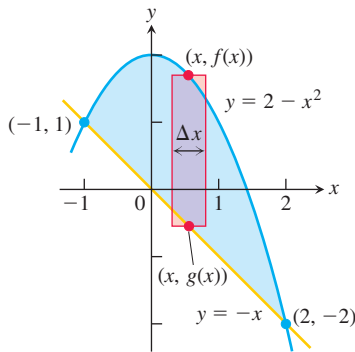


FIGURE 5.29 The region in Example 5 with a typical approximating rectangle from a Riemann sum.

EXAMPLE 4 Find the area of the region bounded above by the curve $y = 2e^{-x} + x$, below by the curve $y = e^x/2$, on the left by $x = 0$, and on the right by $x = 1$.

Solution Figure 5.28 displays the graphs of the curves and the region whose area we want to find. The area between the curves over the interval $0 \leq x \leq 1$ is

$$\begin{aligned} A &= \int_0^1 \left[(2e^{-x} + x) - \frac{1}{2}e^x \right] dx = \left[-2e^{-x} + \frac{1}{2}x^2 - \frac{1}{2}e^x \right]_0^1 \\ &= \left(-2e^{-1} + \frac{1}{2} - \frac{1}{2}e \right) - \left(-2 + 0 - \frac{1}{2} \right) \\ &= 3 - \frac{2}{e} - \frac{e}{2} \approx 0.9051. \end{aligned}$$

EXAMPLE 5 Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution First we sketch the two curves (Figure 5.29). The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x .

$$\begin{aligned} 2 - x^2 &= -x && \text{Equate } f(x) \text{ and } g(x). \\ x^2 - x - 2 &= 0 && \text{Rearrange terms.} \\ (x + 1)(x - 2) &= 0 && \text{Factor.} \\ x = -1, \quad x = 2. &&& \text{Solve.} \end{aligned}$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$. The area between the curves is

$$\begin{aligned} A &= \int_{-1}^2 [f(x) - g(x)] dx = \int_{-1}^2 [(2 - x^2) - (-x)] dx \\ &= \int_{-1}^2 (2 + x - x^2) dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^2 \\ &= \left(4 + \frac{4}{2} - \frac{8}{3} \right) - \left(-2 + \frac{1}{2} - \frac{1}{3} \right) = \frac{9}{2}. \end{aligned}$$

If the formula for a bounding curve changes at one or more points, we subdivide the region into subregions that correspond to the formula changes and apply the formula for the area between curves to each subregion.

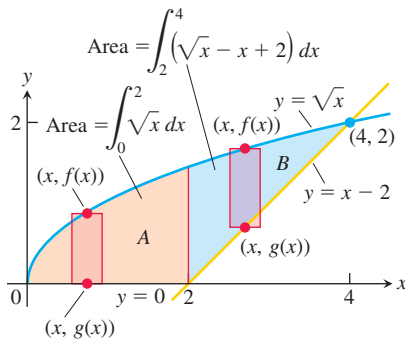


FIGURE 5.30 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 6.

EXAMPLE 6 Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x -axis and the line $y = x - 2$.

Solution Figure 5.30 shows that the region's upper boundary is the graph of $f(x) = \sqrt{x}$. The lower boundary changes from $g(x) = 0$ for $0 \leq x \leq 2$ to $g(x) = x - 2$ for $2 \leq x \leq 4$ (both formulas agree at $x = 2$). We subdivide the region at $x = 2$ into subregions A and B, shown in Figure 5.30.

The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x :

$$\begin{aligned} \sqrt{x} &= x - 2 && \text{Equate } f(x) \text{ and } g(x). \\ x &= (x - 2)^2 = x^2 - 4x + 4 && \text{Square both sides.} \\ x^2 - 5x + 4 &= 0 && \text{Rewrite.} \\ (x - 1)(x - 4) &= 0 && \text{Factor.} \\ x &= 1, \quad x = 4. && \text{Solve.} \end{aligned}$$

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

$$\text{For } 0 \leq x \leq 2: \quad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$$

$$\text{For } 2 \leq x \leq 4: \quad f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2$$

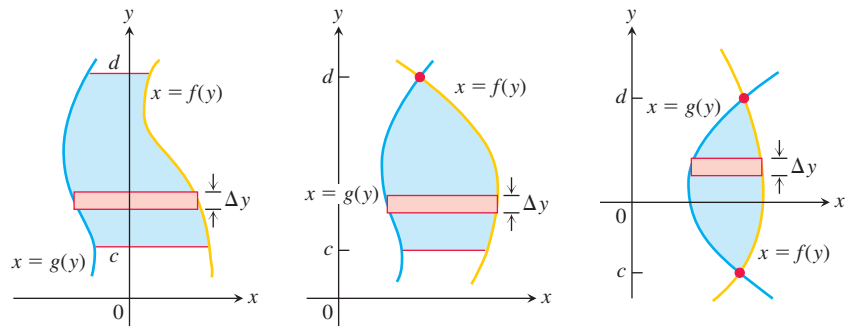
We add the areas of subregions A and B to find the total area:

$$\begin{aligned} \text{Total area} &= \underbrace{\int_0^2 \sqrt{x} \, dx}_{\text{area of A}} + \underbrace{\int_2^4 (\sqrt{x} - x + 2) \, dx}_{\text{area of B}} \\ &= \left[\frac{2}{3} x^{3/2} \right]_0^2 + \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2} + 2x \right]_2^4 \\ &= \frac{2}{3} (2)^{3/2} - 0 + \left(\frac{2}{3} (4)^{3/2} - 8 + 8 \right) - \left(\frac{2}{3} (2)^{3/2} - 2 + 4 \right) \\ &= \frac{2}{3} (8) - 2 = \frac{10}{3}. \end{aligned}$$

Integration with Respect to y

If a region's bounding curves are described by functions of y , the approximating rectangles are horizontal instead of vertical, and the basic formula has y in place of x .

To find the areas of regions like these:



use the formula

$$A = \int_c^d [f(y) - g(y)] \, dy.$$

In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

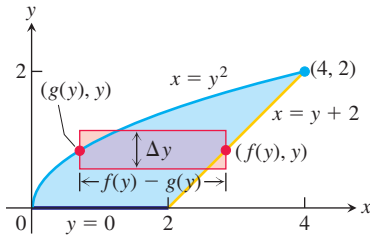


FIGURE 5.31 It takes two integrations to find the area of this region if we integrate with respect to x . It takes only one if we integrate with respect to y (Example 7).

EXAMPLE 7 Find the area of the region in Example 6 by integrating with respect to y .

Solution We first sketch the region and a typical *horizontal* rectangle based on a partition of an interval of y -values (Figure 5.31). The region's right-hand boundary is the line $x = y + 2$, so $f(y) = y + 2$. The left-hand boundary is the curve $x = y^2$, so $g(y) = y^2$. The lower limit of integration is $y = 0$. We find the upper limit by solving $x = y + 2$ and $x = y^2$ simultaneously for y :

$$\begin{aligned} y + 2 &= y^2 && \text{Equate } f(y) = y + 2 \text{ and } g(y) = y^2. \\ y^2 - y - 2 &= 0 && \text{Rewrite.} \\ (y + 1)(y - 2) &= 0 && \text{Factor.} \\ y = -1, \quad y = 2. &&& \text{Solve.} \end{aligned}$$

The upper limit of integration is $b = 2$. (The value $y = -1$ gives a point of intersection *below* the x -axis.)

The area of the region is

$$\begin{aligned} A &= \int_c^d [f(y) - g(y)] dy = \int_0^2 [y + 2 - y^2] dy \\ &= \int_0^2 [2 + y - y^2] dy \\ &= \left[2y + \frac{y^2}{2} - \frac{y^3}{3} \right]_0^2 \\ &= 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}. \end{aligned}$$

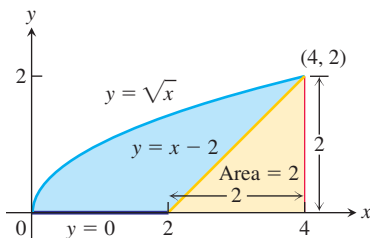


FIGURE 5.32 The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle.

This is the result of Example 6, found with less work. ■

Although it was easier to find the area in Example 6 by integrating with respect to y rather than x (as we did in Example 7), there is an easier way yet. Looking at Figure 5.32, we see that the area we want is the area between the curve $y = \sqrt{x}$ and the x -axis for $0 \leq x \leq 4$, *minus* the area of an isosceles triangle of base and height equal to 2. So by combining calculus with some geometry, we find

$$\begin{aligned} \text{Area} &= \int_0^4 \sqrt{x} dx - \frac{1}{2}(2)(2) \\ &= \frac{2}{3}x^{3/2} \Big|_0^4 - 2 \\ &= \frac{2}{3}(8) - 0 - 2 = \frac{10}{3}. \end{aligned}$$

EXERCISES 5.6

Evaluating Definite Integrals

Use the Substitution Formula in Theorem 7 to evaluate the integrals in Exercises 1–48.

1. a. $\int_0^3 \sqrt{y+1} dy$

b. $\int_{-1}^0 \sqrt{y+1} dy$

2. a. $\int_0^1 r\sqrt{1-r^2} dr$

b. $\int_{-1}^1 r\sqrt{1-r^2} dr$

3. a. $\int_0^{\pi/4} \tan x \sec^2 x dx$

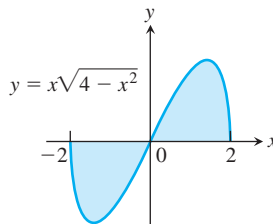
b. $\int_{-\pi/4}^0 \tan x \sec^2 x dx$

4. a. $\int_0^\pi 3 \cos^2 x \sin x \, dx$ b. $\int_{2\pi}^{3\pi} 3 \cos^2 x \sin x \, dx$ 29. $\int_1^2 \frac{2 \ln x}{x} \, dx$ 30. $\int_2^4 \frac{dx}{x \ln x}$
5. a. $\int_0^1 t^3 (1 + t^4)^3 \, dt$ b. $\int_{-1}^1 t^3 (1 + t^4)^3 \, dt$ 31. $\int_2^4 \frac{dx}{x(\ln x)^2}$ 32. $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$
6. a. $\int_0^{\sqrt{7}} t(t^2 + 1)^{1/3} \, dt$ b. $\int_{-\sqrt{7}}^0 t(t^2 + 1)^{1/3} \, dt$ 33. $\int_0^{\pi/2} \tan \frac{x}{2} \, dx$ 34. $\int_{\pi/4}^{\pi/2} \cot t \, dt$
7. a. $\int_{-1}^1 \frac{5r}{(4 + r^2)^2} \, dr$ b. $\int_0^1 \frac{5r}{(4 + r^2)^2} \, dr$ 35. $\int_0^{\pi/3} \tan^2 \theta \cos \theta \, d\theta$ 36. $\int_0^{\pi/12} 6 \tan 3x \, dx$
8. a. $\int_0^1 \frac{10\sqrt{v}}{(1 + v^{3/2})^2} \, dv$ b. $\int_1^4 \frac{10\sqrt{v}}{(1 + v^{3/2})^2} \, dv$ 37. $\int_{-\pi/2}^{\pi/2} \frac{2 \cos \theta \, d\theta}{1 + (\sin \theta)^2}$ 38. $\int_{\pi/6}^{\pi/4} \frac{\csc^2 x \, dx}{1 + (\cot x)^2}$
9. a. $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2 + 1}} \, dx$ b. $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2 + 1}} \, dx$ 39. $\int_0^{\ln \sqrt{3}} \frac{e^x \, dx}{1 + e^{2x}}$ 40. $\int_1^{e^{1/4}} \frac{4 \, dt}{t(1 + \ln^2 t)}$
10. a. $\int_0^1 \frac{x^3}{\sqrt{x^4 + 9}} \, dx$ b. $\int_{-1}^0 \frac{x^3}{\sqrt{x^4 + 9}} \, dx$ 41. $\int_0^1 \frac{4 \, ds}{\sqrt{4 - s^2}}$ 42. $\int_0^{(3/4)\sqrt{2}} \frac{ds}{\sqrt{9 - 4s^2}}$
11. a. $\int_0^1 t\sqrt{4 + 5t} \, dt$ b. $\int_1^9 t\sqrt{4 + 5t} \, dt$ 43. $\int_{\sqrt{2}}^2 \frac{\sec^2(\sec^{-1} x) \, dx}{x\sqrt{x^2 - 1}}$ 44. $\int_{2/\sqrt{3}}^2 \frac{\cos(\sec^{-1} x) \, dx}{x\sqrt{x^2 - 1}}$
12. a. $\int_0^{\pi/6} (1 - \cos 3t) \sin 3t \, dt$ b. $\int_{\pi/6}^{\pi/3} (1 - \cos 3t) \sin 3t \, dt$ 45. $\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2 - 1}}$ 46. $\int_0^3 \frac{y \, dy}{\sqrt{5y + 1}}$
13. a. $\int_0^{2\pi} \frac{\cos z}{\sqrt{4 + 3 \sin z}} \, dz$ b. $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4 + 3 \sin z}} \, dz$ 47. $\int_0^1 \frac{\tan^{-1} x}{1 + x^2} \, dx$ 48. $\int_{-\sqrt{3}}^{1/\sqrt{3}} \frac{\cos(\tan^{-1} x)}{1 + 9x^2} \, dx$
14. a. $\int_{-\pi/2}^0 \left(2 + \tan \frac{t}{2}\right) \sec^2 \frac{t}{2} \, dt$ b. $\int_{-\pi/2}^{\pi/2} \left(2 + \tan \frac{t}{2}\right) \sec^2 \frac{t}{2} \, dt$
15. $\int_0^1 \sqrt{t^5 + 2t} (5t^4 + 2) \, dt$ 16. $\int_1^4 \frac{dy}{2\sqrt{y}(1 + \sqrt{y})^2}$
17. $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta \, d\theta$ 18. $\int_{\pi}^{3\pi/2} \cot^5 \left(\frac{\theta}{6}\right) \sec^2 \left(\frac{\theta}{6}\right) \, d\theta$
19. $\int_0^{\pi} 5(5 - 4 \cos t)^{1/4} \sin t \, dt$ 20. $\int_0^{\pi/4} (1 - \sin 2t)^{3/2} \cos 2t \, dt$
21. $\int_0^1 (4y - y^2 + 4y^3 + 1)^{-2/3} (12y^2 - 2y + 4) \, dy$
22. $\int_0^1 (y^3 + 6y^2 - 12y + 9)^{-1/2} (y^2 + 4y - 4) \, dy$
23. $\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta} \cos^2(\theta^{3/2}) \, d\theta$ 24. $\int_{-1}^{-1/2} t^{-2} \sin^2 \left(1 + \frac{1}{t}\right) \, dt$
25. $\int_0^{\pi/4} (1 + e^{\tan \theta}) \sec^2 \theta \, d\theta$ 26. $\int_{\pi/4}^{\pi/2} (1 + e^{\cot \theta}) \csc^2 \theta \, d\theta$
27. $\int_0^{\pi} \frac{\sin t}{2 - \cos t} \, dt$ 28. $\int_0^{\pi/3} \frac{4 \sin \theta}{1 - 4 \cos \theta} \, d\theta$

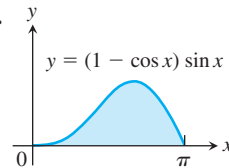
Area

Find the total areas of the shaded regions in Exercises 49–64.

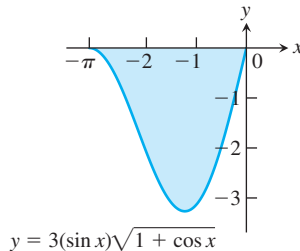
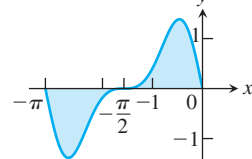
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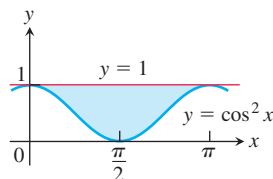
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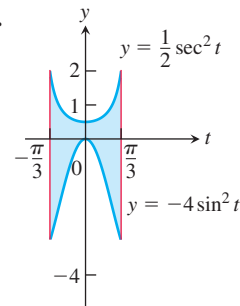
51.

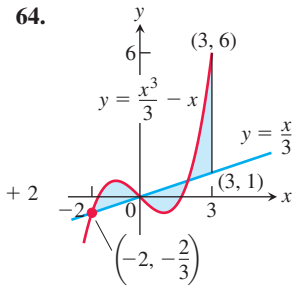
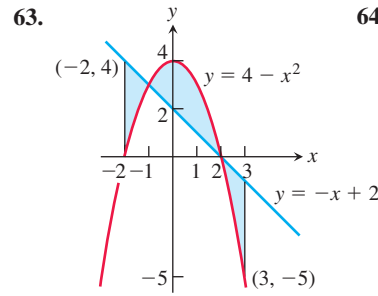
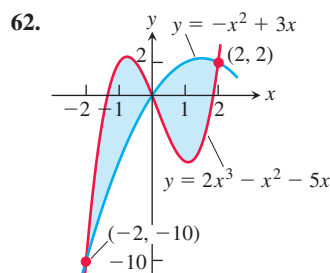
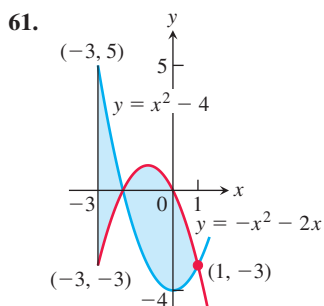
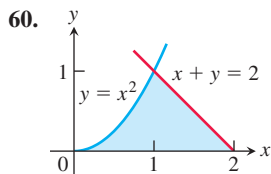
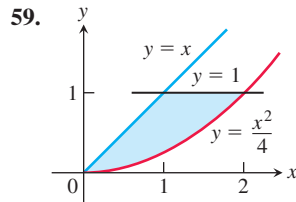
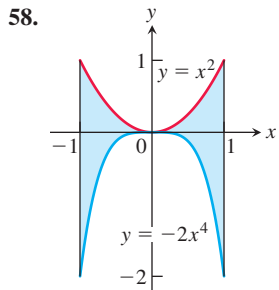
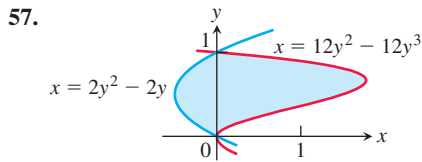
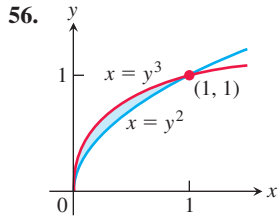
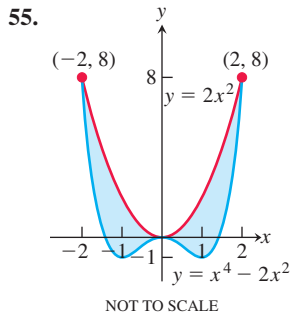
52. $y = \frac{\pi}{2} (\cos x)(\sin(\pi + \pi \sin x))$ 

53.



54.





Find the areas of the regions enclosed by the lines and curves in Exercises 65–74.

65. $y = x^2 - 2$ and $y = 2$

66. $y = 2x - x^2$ and $y = -3$

67. $y = x^4$ and $y = 8x$

68. $y = x^2 - 2x$ and $y = x$

69. $y = x^2$ and $y = -x^2 + 4x$

70. $y = 7 - 2x^2$ and $y = x^2 + 4$

71. $y = x^4 - 4x^2 + 4$ and $y = x^2$

72. $y = x\sqrt{a^2 - x^2}$, $a > 0$, and $y = 0$

73. $y = \sqrt{|x|}$ and $5y = x + 6$ (How many intersection points are there?)

74. $y = |x^2 - 4|$ and $y = (x^2/2) + 4$

Find the areas of the regions enclosed by the lines and curves in Exercises 75–82.

75. $x = 2y^2$, $x = 0$, and $y = 3$

76. $x = y^2$ and $x = y + 2$

77. $y^2 - 4x = 4$ and $4x - y = 16$

78. $x - y^2 = 0$ and $x + 2y^2 = 3$

79. $x + y^2 = 0$ and $x + 3y^2 = 2$

80. $x - y^{2/3} = 0$ and $x + y^4 = 2$

81. $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$

82. $x = y^3 - y^2$ and $x = 2y$

Find the areas of the regions enclosed by the curves in Exercises 83–86.

83. $4x^2 + y = 4$ and $x^4 - y = 1$

84. $x^3 - y = 0$ and $3x^2 - y = 4$

85. $x + 4y^2 = 4$ and $x + y^4 = 1$, for $x \geq 0$

86. $x + y^2 = 3$ and $4x + y^2 = 0$

Find the areas of the regions enclosed by the lines and curves in Exercises 87–94.

87. $y = 2 \sin x$ and $y = \sin 2x$, $0 \leq x \leq \pi$

88. $y = 8 \cos x$ and $y = \sec^2 x$, $-\pi/3 \leq x \leq \pi/3$

89. $y = \cos(\pi x/2)$ and $y = 1 - x^2$

90. $y = \sin(\pi x/2)$ and $y = x$

91. $y = \sec^2 x$, $y = \tan^2 x$, $x = -\pi/4$, and $x = \pi/4$

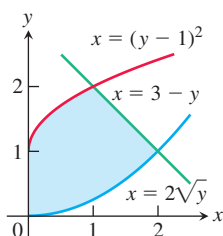
92. $x = \tan^2 y$ and $x = -\tan^2 y$, $-\pi/4 \leq y \leq \pi/4$

93. $x = 3 \sin y \sqrt{\cos y}$ and $x = 0$, $0 \leq y \leq \pi/2$

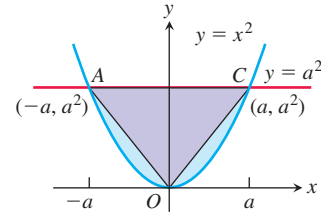
94. $y = \sec^2(\pi x/3)$ and $y = x^{1/3}$, $-1 \leq x \leq 1$

Area Between Curves

95. Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$.
96. Find the area of the propeller-shaped region enclosed by the curves $x - y^{1/3} = 0$ and $x - y^{1/5} = 0$.
97. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x -axis.
98. Find the area of the “triangular” region in the first quadrant bounded on the left by the y -axis and on the right by the curves $y = \sin x$ and $y = \cos x$.
99. Find the area between the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$.
100. Find the area between the curve $y = \tan x$ and the x -axis from $x = -\pi/4$ to $x = \pi/3$.
101. Find the area of the “triangular” region in the first quadrant that is bounded above by the curve $y = e^{2x}$, below by the curve $y = e^x$, and on the right by the line $x = \ln 3$.
102. Find the area of the “triangular” region in the first quadrant that is bounded above by the curve $y = e^{x/2}$, below by the curve $y = e^{-x/2}$, and on the right by the line $x = 2 \ln 2$.
103. Find the area of the region between the curve $y = 2x/(1 + x^2)$ and the interval $-2 \leq x \leq 2$ of the x -axis.
104. Find the area of the region between the curve $y = 2^{1-x}$ and the interval $-1 \leq x \leq 1$ of the x -axis.
105. The region bounded below by the parabola $y = x^2$ and above by the line $y = 4$ is to be partitioned into two subsections of equal area by cutting across it with the horizontal line $y = c$.
 - a. Sketch the region and draw a line $y = c$ across it that looks about right. In terms of c , what are the coordinates of the points where the line and parabola intersect? Add them to your figure.
 - b. Find c by integrating with respect to y . (This puts c in the limits of integration.)
 - c. Find c by integrating with respect to x . (This puts c into the integrand as well.)
106. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to **a.** x , **b.** y .
107. Find the area of the region in the first quadrant bounded on the left by the y -axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.
108. Find the area of the region in the first quadrant bounded on the left by the y -axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y - 1)^2$, and above right by the line $x = 3 - y$.



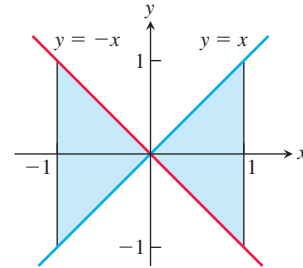
109. The figure here shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.



110. Suppose the area of the region between the graph of a positive continuous function f and the x -axis from $x = a$ to $x = b$ is 4 square units. Find the area between the curves $y = f(x)$ and $y = 2f(x)$ from $x = a$ to $x = b$.
111. Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer.

$$\text{a. } \int_{-1}^1 (x - (-x)) dx = \int_{-1}^1 2x dx$$

$$\text{b. } \int_{-1}^1 (-x - (x)) dx = \int_{-1}^1 -2x dx$$



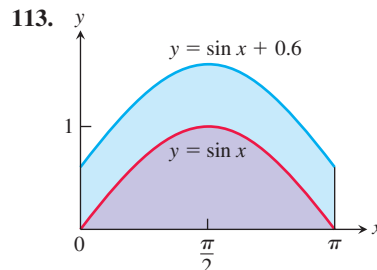
112. True, sometimes true, or never true? The area of the region between the graphs of the continuous functions $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ ($a < b$) is

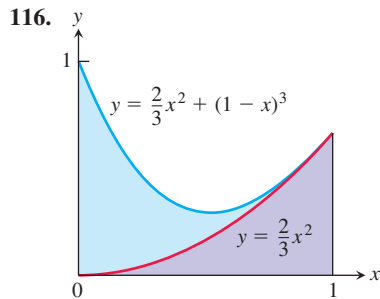
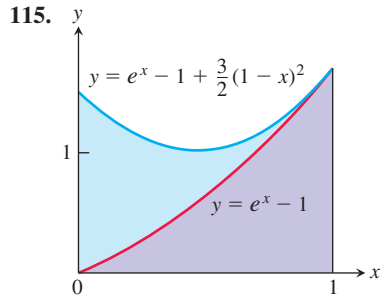
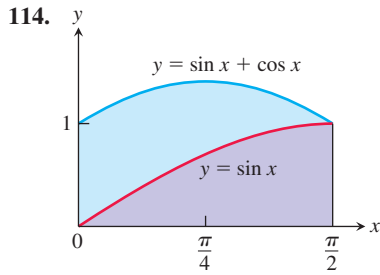
$$\int_a^b [f(x) - g(x)] dx.$$

Give reasons for your answer.

Comparing Areas

Compute the areas of the light blue and dark blue regions in each of Exercises 113 to 116 and determine which is larger, or show that they have equal area.





Theory and Examples

117. Suppose that $F(x)$ is an antiderivative of $f(x) = (\sin x)/x$, $x > 0$. Express

$$\int_1^3 \frac{\sin 2x}{x} dx$$

in terms of F .

118. Show that if f is continuous, then

$$\int_0^1 f(x) dx = \int_0^1 f(1-x) dx.$$

119. Suppose that

$$\int_0^1 f(x) dx = 3.$$

Find

$$\int_{-1}^0 f(x) dx$$

if a. f is odd, b. f is even.

120. a. Show that if f is odd on $[-a, a]$, then

$$\int_{-a}^a f(x) dx = 0.$$

b. Test the result in part (a) with $f(x) = \sin x$ and $a = \pi/2$.

121. If f is a continuous function, find the value of the integral

$$I = \int_0^a \frac{f(x) dx}{f(x) + f(a-x)}$$

by making the substitution $u = a - x$ and adding the resulting integral to I .

122. By using a substitution, prove that for all positive numbers x and y ,

$$\int_x^{xy} \frac{1}{t} dt = \int_1^y \frac{1}{t} dt.$$

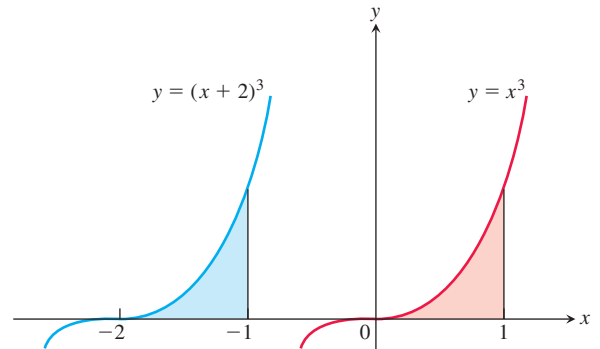
The Shift Property for Definite Integrals A basic property of definite integrals is their invariance under translation, as expressed by the equation

$$\int_a^b f(x) dx = \int_{a-c}^{b-c} f(x+c) dx \quad (1)$$

The equation holds whenever f is integrable and defined for the necessary values of x . For example, in the accompanying figure, show that

$$\int_{-2}^{-1} (x+2)^3 dx = \int_0^1 x^3 dx$$

because the areas of the shaded regions are congruent.



123. Use a substitution to verify Equation (1).

124. For each of the following functions, graph $f(x)$ over $[a, b]$ and $f(x+c)$ over $[a-c, b-c]$ to convince yourself that Equation (1) is reasonable.

- $f(x) = x^2$, $a = 0$, $b = 1$, $c = 1$
- $f(x) = \sin x$, $a = 0$, $b = \pi$, $c = \pi/2$
- $f(x) = \sqrt{x-4}$, $a = 4$, $b = 8$, $c = 5$

COMPUTER EXPLORATIONS

In Exercises 125–128, you will find the area between curves in the plane when you cannot find their points of intersection using simple algebra. Use a CAS to perform the following steps:

- Plot the curves together to see what they look like and how many points of intersection they have.
 - Use the numerical equation solver in your CAS to find all the points of intersection.
 - Integrate $|f(x) - g(x)|$ over consecutive pairs of intersection values.
 - Sum together the integrals found in part (c).
125. $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$, $g(x) = x - 1$
126. $f(x) = \frac{x^4}{2} - 3x^3 + 10$, $g(x) = 8 - 12x$
127. $f(x) = x + \sin(2x)$, $g(x) = x^3$
128. $f(x) = x^2 \cos x$, $g(x) = x^3 - x$

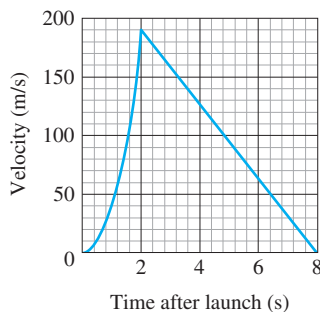
CHAPTER 5 Questions to Guide Your Review

- How can you sometimes estimate quantities like distance traveled, area, and average value with finite sums? Why might you want to do so?
- What is sigma notation? What advantage does it offer? Give examples.
- What is a Riemann sum? Why might you want to consider such a sum?
- What is the norm of a partition of a closed interval?
- What is the definite integral of a function f over a closed interval $[a, b]$? When can you be sure it exists?
- What is the relation between definite integrals and area? Describe some other interpretations of definite integrals.
- What is the average value of an integrable function over a closed interval? Must the function assume its average value? Explain.
- Describe the rules for working with definite integrals (Table 5.6). Give examples.
- What is the Fundamental Theorem of Calculus? Why is it so important? Illustrate each part of the theorem with an example.
- What is the Net Change Theorem? What does it say about the integral of velocity? The integral of marginal cost?
- Discuss how the processes of integration and differentiation can be considered as “inverses” of each other.
- How does the Fundamental Theorem provide a solution to the initial value problem $dy/dx = f(x)$, $y(x_0) = y_0$, when f is continuous?
- How is integration by substitution related to the Chain Rule?
- How can you sometimes evaluate indefinite integrals by substitution? Give examples.
- How does the method of substitution work for definite integrals? Give examples.
- How do you define and calculate the area of the region between the graphs of two continuous functions? Give an example.

CHAPTER 5 Practice Exercises

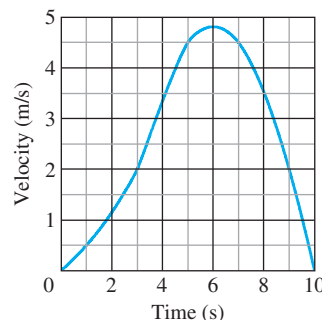
Finite Sums and Estimates

- The accompanying figure shows the graph of the velocity (m/s) of a model rocket for the first 8 s after launch. The rocket accelerated straight up for the first 2 s and then coasted to reach its maximum height at $t = 8$ s.



- Assuming that the rocket was launched from ground level, about how high did it go? (This is the rocket in Section 3.4, Exercise 17, but you do not need to do Exercise 17 to do the exercise here.)
 - Sketch a graph of the rocket's height above ground as a function of time for $0 \leq t \leq 8$.
- The accompanying figure shows the velocity (m/s) of a body moving along the s -axis during the time interval from $t = 0$ to $t = 10$ s. About how far did the body travel during those 10 s?

- Sketch a graph of s as a function of t for $0 \leq t \leq 10$, assuming $s(0) = 0$.



- Suppose that $\sum_{k=1}^{10} a_k = -2$ and $\sum_{k=1}^{10} b_k = 25$. Find the value of

- $\sum_{k=1}^{10} \frac{a_k}{4}$
- $\sum_{k=1}^{10} (b_k - 3a_k)$
- $\sum_{k=1}^{10} (a_k + b_k - 1)$
- $\sum_{k=1}^{10} \left(\frac{5}{2} - b_k \right)$

- Suppose that $\sum_{k=1}^{20} a_k = 0$ and $\sum_{k=1}^{20} b_k = 7$. Find the value of

- $\sum_{k=1}^{20} 3a_k$
- $\sum_{k=1}^{20} (a_k + b_k)$
- $\sum_{k=1}^{20} \left(\frac{1}{2} - \frac{2b_k}{7} \right)$
- $\sum_{k=1}^{20} (a_k - 2)$

Definite Integrals

In Exercises 5–8, express each limit as a definite integral. Then evaluate the integral to find the value of the limit. In each case, P is a partition of the given interval, and the numbers c_k are chosen from the subintervals of P .

5. $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (2c_k - 1)^{-1/2} \Delta x_k$, where P is a partition of $[1, 5]$
6. $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n c_k (c_k^2 - 1)^{1/3} \Delta x_k$, where P is a partition of $[1, 3]$
7. $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \left(\cos\left(\frac{c_k}{2}\right) \right) \Delta x_k$, where P is a partition of $[-\pi, 0]$
8. $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (\sin c_k)(\cos c_k) \Delta x_k$, where P is a partition of $[0, \pi/2]$
9. If $\int_{-2}^2 3f(x) dx = 12$, $\int_{-2}^5 f(x) dx = 6$, and $\int_{-2}^5 g(x) dx = 2$, find the value of each of the following.
 - a. $\int_{-2}^2 f(x) dx$
 - b. $\int_{-2}^5 f(x) dx$
 - c. $\int_5^{-2} g(x) dx$
 - d. $\int_{-2}^5 (-\pi g(x)) dx$
 - e. $\int_{-2}^5 \left(\frac{f(x) + g(x)}{5} \right) dx$
10. If $\int_0^2 f(x) dx = \pi$, $\int_0^2 7g(x) dx = 7$, and $\int_0^1 g(x) dx = 2$, find the value of each of the following.
 - a. $\int_0^2 g(x) dx$
 - b. $\int_1^2 g(x) dx$
 - c. $\int_2^0 f(x) dx$
 - d. $\int_0^2 \sqrt{2} f(x) dx$
 - e. $\int_0^2 (g(x) - 3f(x)) dx$

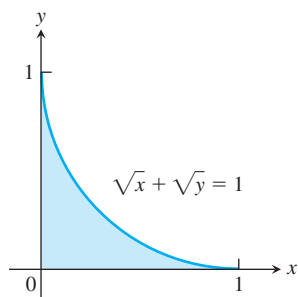
Area

In Exercises 11–14, find the total area of the region between the graph of f and the x -axis.

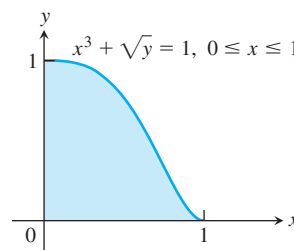
11. $f(x) = x^2 - 4x + 3$, $0 \leq x \leq 3$
12. $f(x) = 1 - (x^2/4)$, $-2 \leq x \leq 3$
13. $f(x) = 5 - 5x^{2/3}$, $-1 \leq x \leq 8$
14. $f(x) = 1 - \sqrt{x}$, $0 \leq x \leq 4$

Find the areas of the regions enclosed by the curves and lines in Exercises 15–26.

15. $y = x$, $y = 1/x^2$, $x = 2$
16. $y = x$, $y = 1/\sqrt{x}$, $x = 2$
17. $\sqrt{x} + \sqrt{y} = 1$, $x = 0$, $y = 0$



18. $x^3 + \sqrt{y} = 1$, $x = 0$, $y = 0$, for $0 \leq x \leq 1$



19. $x = 2y^2$, $x = 0$, $y = 3$
20. $x = 4 - y^2$, $x = 0$
21. $y^2 = 4x$, $y = 4x - 2$
22. $y^2 = 4x + 4$, $y = 4x - 16$
23. $y = \sin x$, $y = x$, $0 \leq x \leq \pi/4$
24. $y = |\sin x|$, $y = 1$, $-\pi/2 \leq x \leq \pi/2$
25. $y = 2 \sin x$, $y = \sin 2x$, $0 \leq x \leq \pi$
26. $y = 8 \cos x$, $y = \sec^2 x$, $-\pi/3 \leq x \leq \pi/3$
27. Find the area of the “triangular” region bounded on the left by $x + y = 2$, on the right by $y = x^2$, and above by $y = 2$.
28. Find the area of the “triangular” region bounded on the left by $y = \sqrt{x}$, on the right by $y = 6 - x$, and below by $y = 1$.
29. Find the extreme values of $f(x) = x^3 - 3x^2$, and find the area of the region enclosed by the graph of f and the x -axis.
30. Find the area of the region cut from the first quadrant by the curve $x^{1/2} + y^{1/2} = a^{1/2}$.
31. Find the total area of the region enclosed by the curve $x = y^{2/3}$ and the lines $x = y$ and $y = -1$.
32. Find the total area of the region between the curves $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq 3\pi/2$.
33. Find the area between the curve $y = 2(\ln x)/x$ and the x -axis from $x = 1$ to $x = e$.
34. a. Show that the area between the curve $y = 1/x$ and the x -axis from $x = 10$ to $x = 20$ is the same as the area between the curve and the x -axis from $x = 1$ to $x = 2$.
b. Show that the area between the curve $y = 1/x$ and the x -axis from ka to kb is the same as the area between the curve and the x -axis from $x = a$ to $x = b$ ($0 < a < b$, $k > 0$).

Initial Value Problems

35. Show that $y = x^2 + \int_1^x \frac{1}{t} dt$ solves the initial value problem

$$\frac{d^2y}{dx^2} = 2 - \frac{1}{x^2}; \quad y'(1) = 3, \quad y(1) = 1.$$

36. Show that $y = \int_0^x (1 + 2\sqrt{\sec t}) dt$ solves the initial value problem

$$\frac{d^2y}{dx^2} = \sqrt{\sec x} \tan x; \quad y'(0) = 3, \quad y(0) = 0.$$

Express the solutions of the initial value problems in Exercises 37 and 38 in terms of integrals.

$$37. \frac{dy}{dx} = \frac{\sin x}{x}, \quad y(5) = -3$$

$$38. \frac{dy}{dx} = \sqrt{2 - \sin^2 x}, \quad y(-1) = 2$$

Solve the initial value problems in Exercises 39–42.

39. $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \quad y(0) = 0$

40. $\frac{dy}{dx} = \frac{1}{x^2 + 1} - 1, \quad y(0) = 1$

41. $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}, \quad x > 1; \quad y(2) = \pi$

42. $\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}}, \quad y(0) = 2$

For Exercises 43 and 44, find a function f that satisfies each equation.

43. $f(x) = 1 + \int_1^x t f(t) dt$ 44. $f(x) = \int_0^x (1 + f(t)^2) dt$

Evaluating Indefinite Integrals

Evaluate the integrals in Exercises 45–76.

45. $\int 2(\cos x)^{-1/2} \sin x dx$ 46. $\int (\tan x)^{-3/2} \sec^2 x dx$

47. $\int (2\theta + 1 + 2 \cos(2\theta + 1)) d\theta$

48. $\int \left(\frac{1}{\sqrt{2\theta - \pi}} + 2 \sec^2(2\theta - \pi) \right) d\theta$

49. $\int \left(t - \frac{2}{t} \right) \left(t + \frac{2}{t} \right) dt$

50. $\int \frac{(t+1)^2 - 1}{t^4} dt$ 51. $\int \sqrt{t} \sin(2t^{3/2}) dt$

52. $\int (\sec \theta \tan \theta) \sqrt{1 + \sec \theta} d\theta$

53. $\int e^x \sec^2(e^x - 7) dx$

54. $\int e^y \csc(e^y + 1) \cot(e^y + 1) dy$

55. $\int (\sec^2 x) e^{\tan x} dx$ 56. $\int (\csc^2 x) e^{\cot x} dx$

57. $\int_{-1}^1 \frac{dx}{3x - 4}$ 58. $\int_1^e \frac{\sqrt{\ln x}}{x} dx$

59. $\int_0^4 \frac{2t}{t^2 - 25} dt$ 60. $\int \frac{\tan(\ln v)}{v} dv$

61. $\int \frac{(\ln x)^{-3}}{x} dx$ 62. $\int \frac{1}{r} \csc^2(1 + \ln r) dr$

63. $\int x 3^{x^2} dx$ 64. $\int 2^{\tan x} \sec^2 x dx$

65. $\int \frac{3 dr}{\sqrt{1 - 4(r-1)^2}}$ 66. $\int \frac{6 dr}{\sqrt{4 - (r+1)^2}}$

67. $\int \frac{dx}{2 + (x-1)^2}$ 68. $\int \frac{dx}{1 + (3x+1)^2}$

69. $\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2 - 4}}$ 70. $\int \frac{dx}{(x+3)\sqrt{(x+3)^2 - 25}}$

71. $\int \frac{e^{\arcsin \sqrt{x}} dx}{2\sqrt{x-x^2}}$ 72. $\int \frac{\sqrt{\arcsin x} dx}{\sqrt{1-x^2}}$

73. $\int \frac{dy}{\sqrt{\arctan y} (1+y^2)}$ 74. $\int \frac{(\arctan x)^2 dx}{1+x^2}$

75. $\int \frac{\sin 2\theta - \cos 2\theta}{(\sin 2\theta + \cos 2\theta)^3} d\theta$ 76. $\int \cos \theta \cdot \sin(\sin \theta) d\theta$

Evaluating Definite Integrals

Evaluate the integrals in Exercises 77–116.

77. $\int_{-1}^1 (3x^2 - 4x + 7) dx$ 78. $\int_0^1 (8s^3 - 12s^2 + 5) ds$

79. $\int_1^2 \frac{4}{v^2} dv$ 80. $\int_1^{27} x^{-4/3} dx$

81. $\int_1^4 \frac{dt}{t\sqrt{t}}$ 82. $\int_1^4 \frac{(1 + \sqrt{u})^{1/2}}{\sqrt{u}} du$

83. $\int_0^1 \frac{36 dx}{(2x+1)^3}$ 84. $\int_0^1 \frac{dr}{\sqrt[3]{(7-5r)^2}}$

85. $\int_{1/8}^1 x^{-1/3} (1 - x^{2/3})^{3/2} dx$ 86. $\int_0^{1/2} x^3 (1 + 9x^4)^{-3/2} dx$

87. $\int_0^\pi \sin^2 5r dr$ 88. $\int_0^{\pi/4} \cos^2 \left(4t - \frac{\pi}{4} \right) dt$

89. $\int_0^{\pi/3} \sec^2 \theta d\theta$ 90. $\int_{\pi/4}^{3\pi/4} \csc^2 x dx$

91. $\int_\pi^{3\pi} \cot^2 \frac{x}{6} dx$ 92. $\int_0^\pi \tan^2 \frac{\theta}{3} d\theta$

93. $\int_{-\pi/3}^0 \sec x \tan x dx$ 94. $\int_{\pi/4}^{3\pi/4} \csc z \cot z dz$

95. $\int_0^{\pi/2} 5(\sin x)^{3/2} \cos x dx$ 96. $\int_{-\pi/2}^{\pi/2} 15 \sin^4 3x \cos 3x dx$

97. $\int_0^{\pi/2} \frac{3 \sin x \cos x}{\sqrt{1 + 3 \sin^2 x}} dx$ 98. $\int_0^{\pi/4} \frac{\sec^2 x}{(1 + 7 \tan x)^{2/3}} dx$

99. $\int_1^4 \left(\frac{x}{8} + \frac{1}{2x} \right) dx$ 100. $\int_1^8 \left(\frac{2}{3x} - \frac{8}{x^2} \right) dx$

101. $\int_{-2}^{-1} e^{-(x+1)} dx$ 102. $\int_{-\ln 2}^0 e^{2w} dw$

103. $\int_0^{\ln 5} e^r (3e^r + 1)^{-3/2} dr$ 104. $\int_0^{\ln 9} e^\theta (e^\theta - 1)^{1/2} d\theta$

105. $\int_1^e \frac{1}{x} (1 + 7 \ln x)^{-1/3} dx$ 106. $\int_1^3 \frac{(\ln(v+1))^2}{v+1} dv$

107. $\int_1^8 \frac{\log_4 \theta}{\theta} d\theta$ 108. $\int_1^e \frac{8 \ln 3 \log_3 \theta}{\theta} d\theta$

109. $\int_{-3/4}^{3/4} \frac{6 dx}{\sqrt{9 - 4x^2}}$ 110. $\int_{-1/5}^{1/5} \frac{6 dx}{\sqrt{4 - 25x^2}}$

111. $\int_{-2}^2 \frac{3 dt}{4 + 3t^2}$ 112. $\int_{\sqrt{3}}^3 \frac{dt}{3 + t^2}$

113. $\int_{1/\sqrt{3}}^1 \frac{dy}{y\sqrt{4y^2 - 1}}$ 114. $\int_{4\sqrt{2}}^8 \frac{24 dy}{y\sqrt{y^2 - 16}}$

115. $\int_{\sqrt{2}/3}^{2/3} \frac{dy}{|y|\sqrt{9y^2 - 1}}$ 116. $\int_{-2/\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \frac{dy}{|y|\sqrt{5y^2 - 3}}$

Average Values

117. Find the average value of $f(x) = mx + b$

a. over $[-1, 1]$

b. over $[-k, k]$

118. Find the average value of

a. $y = \sqrt{3x}$ over $[0, 3]$

b. $y = \sqrt{ax}$ over $[0, a]$

119. Let f be a function that is differentiable on $[a, b]$. In Chapter 2 we defined the average rate of change of f over $[a, b]$ to be

$$\frac{f(b) - f(a)}{b - a}$$

and the instantaneous rate of change of f at x to be $f'(x)$. In this chapter we defined the average value of a function. For the new definition of average to be consistent with the old one, we should have

$$\frac{f(b) - f(a)}{b - a} = \text{average value of } f' \text{ on } [a, b].$$

Is this the case? Give reasons for your answer.

120. Is it true that the average value of an integrable function over an interval of length 2 is half the function's integral over the interval? Give reasons for your answer.
121. a. Verify that $\int \ln x \, dx = x \ln x - x + C$.
b. Find the average value of $\ln x$ over $[1, e]$.
122. Find the average value of $f(x) = 1/x$ on $[1, 2]$.

- T** 123. Compute the average value of the temperature function

$$f(x) = 20 \sin\left(\frac{2\pi}{365}(x - 101)\right) - 4$$

for a 365-day year. (See Exercise 98, Section 3.6.) This is one way to estimate the annual mean air temperature in Fairbanks, Alaska. The National Weather Service's official figure, a numerical average of the daily normal mean air temperatures for the year, is -3.5°C , which is slightly higher than the average value of $f(x)$.

- T** 124. **Specific heat of a gas** Specific heat C_v is the amount of heat required to raise the temperature of one mole (gram molecule) of a gas with constant volume by 1°C . The specific heat of oxygen depends on its temperature T and satisfies the formula

$$C_v = 8.27 + 10^{-5}(26T - 1.87T^2).$$

Find the average value of C_v for $20^\circ\text{C} \leq T \leq 675^\circ\text{C}$ and the temperature at which it is attained.

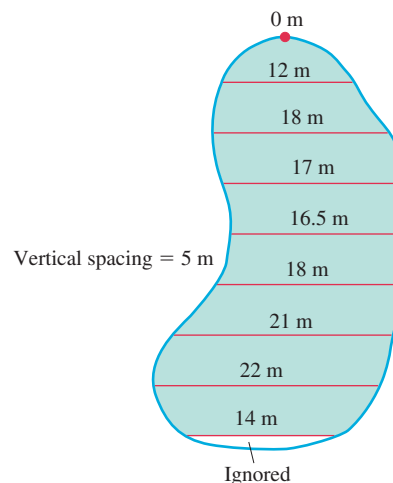
Differentiating Integrals

In Exercises 125–132, find dy/dx .

125. $y = \int_2^x \sqrt{2 + \cos^3 t} \, dt$ 126. $y = \int_2^{7x^2} \sqrt{2 + \cos^3 t} \, dt$
127. $y = \int_x^1 \frac{6}{3 + t^4} \, dt$ 128. $y = \int_{\sec x}^2 \frac{1}{t^2 + 1} \, dt$
129. $y = \int_{\ln x^2}^0 e^{\cos t} \, dt$ 130. $y = \int_1^{e^{\sqrt{x}}} \ln(t^2 + 1) \, dt$
131. $y = \int_0^{\sin^{-1} x} \frac{dt}{\sqrt{1 - 2t^2}}$ 132. $y = \int_{\tan^{-1} x}^{\pi/4} e^{\sqrt{t}} \, dt$

Theory and Examples

133. Is it true that every function $y = f(x)$ that is differentiable on $[a, b]$ is itself the derivative of some function on $[a, b]$? Give reasons for your answer.
134. Suppose that $f(x)$ is an antiderivative of $f(x) = \sqrt{1 + x^4}$. Express $\int_0^1 \sqrt{1 + x^4} \, dx$ in terms of F and give a reason for your answer.
135. Find dy/dx if $y = \int_x^1 \sqrt{1 + t^2} \, dt$. Explain the main steps in your calculation.
136. Find dy/dx if $y = \int_{\cos x}^0 (1/(1 - t^2)) \, dt$. Explain the main steps in your calculation.
137. **A new parking lot** To meet the demand for parking, your town has allocated the area shown here. As the town engineer, you have been asked by the town council to find out if the lot can be built for \$10,000. The cost to clear the land will be \$1.00 a square meter, and the lot will cost \$2.00 a square meter to pave. Can the job be done for \$10,000? Use a lower sum estimate to see. (Answers may vary slightly, depending on the estimate used.)



138. Skydivers A and B are in a helicopter hovering at 2000 m. Skydiver A jumps and descends for 4 s before opening her parachute. The helicopter then climbs to 2200 m and hovers there. Forty-five seconds after A leaves the aircraft, B jumps and descends for 13 s before opening his parachute. Both skydivers descend at 4.9 m/s with parachutes open. Assume that the skydivers fall freely (no effective air resistance) before their parachutes open.
- At what altitude does A's parachute open?
 - At what altitude does B's parachute open?
 - Which skydiver lands first?

CHAPTER 5 Additional and Advanced Exercises

Theory and Examples

1. a. If $\int_0^1 7f(x) \, dx = 7$, does $\int_0^1 f(x) \, dx = 1$?
b. If $\int_0^1 f(x) \, dx = 4$ and $f(x) \geq 0$, does

$$\int_0^1 \sqrt{f(x)} \, dx = \sqrt{4} = 2?$$

Give reasons for your answers.

2. Suppose $\int_{-2}^2 f(x) \, dx = 4$, $\int_2^5 f(x) \, dx = 3$, $\int_{-2}^5 g(x) \, dx = 2$. Which, if any, of the following statements are true?

- a. $\int_5^2 f(x) \, dx = -3$ b. $\int_{-2}^5 (f(x) + g(x)) \, dx = 9$
c. $f(x) \leq g(x)$ on the interval $-2 \leq x \leq 5$

3. Initial value problem Show that

$$y = \frac{1}{a} \int_0^x f(t) \sin a(x-t) dt$$

solves the initial value problem

$$\frac{d^2y}{dx^2} + a^2y = f(x), \quad \frac{dy}{dx} = 0 \text{ and } y = 0 \text{ when } x = 0.$$

(Hint: $\sin(ax - at) = \sin ax \cos at - \cos ax \sin at$.)**4. Proportionality** Suppose that x and y are related by the equation

$$x = \int_0^y \frac{1}{\sqrt{1+4t^2}} dt.$$

Show that d^2y/dx^2 is proportional to y , and find the constant of proportionality.**5.** Find $f(4)$ if

$$\text{a. } \int_0^{x^2} f(t) dt = x \cos \pi x \quad \text{b. } \int_0^{f(x)} t^2 dt = x \cos \pi x.$$

6. Find $f(\pi/2)$ from the following information.i) f is positive and continuous.ii) The area under the curve $y = f(x)$ from $x = 0$ to $x = a$ is

$$\frac{a^2}{2} + \frac{a}{2} \sin a + \frac{\pi}{2} \cos a.$$

7. The area of the region in the xy -plane enclosed by the x -axis, the curve $y = f(x)$, $f(x) \geq 0$, and the lines $x = 1$ and $x = b$ is equal to $\sqrt{b^2 + 1} - \sqrt{2}$ for all $b > 1$. Find $f(x)$.

8. Prove that

$$\int_0^x \left(\int_0^u f(t) dt \right) du = \int_0^x f(u) (x-u) du.$$

(Hint: Express the integral on the right-hand side as the difference of two integrals. Then show that both sides of the equation have the same derivative with respect to x .)

9. Finding a curve Find the equation for the curve in the xy -plane that passes through the point $(1, -1)$ if its slope at x is always $3x^2 + 2$.

10. Shoveling dirt You sling a shovelful of dirt up from the bottom of a hole with an initial velocity of 9.8 m/s. The dirt must rise 5.2 m above the release point to clear the edge of the hole. Is that enough speed to get the dirt out, or had you better duck?

Piecewise Continuous Functions

Although we are mainly interested in continuous functions, many functions in applications are piecewise continuous. A function $f(x)$ is **piecewise continuous on a closed interval I** if f has only finitely many discontinuities in I , the limits

$$\lim_{x \rightarrow c^-} f(x) \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x)$$

exist and are finite at every interior point of I , and the appropriate one-sided limits exist and are finite at the endpoints of I . All piecewise continuous functions are integrable. The points of discontinuity subdivide I into open and half-open subintervals on which f is continuous, and the limit criteria above guarantee that f has a continuous extension to the closure of each subinterval. To integrate a piecewise continuous

function, we integrate the individual extensions and add the results. The integral of

$$f(x) = \begin{cases} 1-x, & -1 \leq x < 0 \\ x^2, & 0 \leq x < 2 \\ -1, & 2 \leq x \leq 3 \end{cases}$$

(Figure 5.33) over $[-1, 3]$ is

$$\begin{aligned} \int_{-1}^3 f(x) dx &= \int_{-1}^0 (1-x) dx + \int_0^2 x^2 dx + \int_2^3 (-1) dx \\ &= \left[x - \frac{x^2}{2} \right]_{-1}^0 + \left[\frac{x^3}{3} \right]_0^2 + \left[-x \right]_2^3 \\ &= \frac{3}{2} + \frac{8}{3} - 1 = \frac{19}{6}. \end{aligned}$$

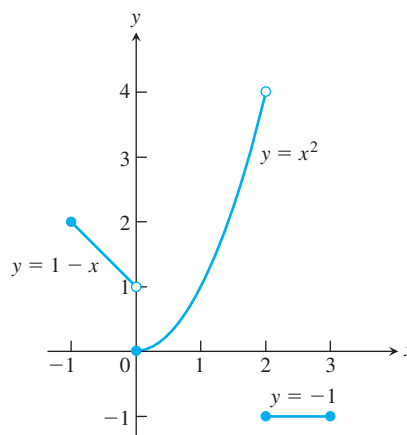


FIGURE 5.33 Piecewise continuous functions like this are integrated piece by piece.

The Fundamental Theorem applies to piecewise continuous functions with the restriction that $(d/dx) \int_a^x f(t) dt$ is expected to equal $f(x)$ only at values of x at which f is continuous. There is a similar restriction on Leibniz's Rule (see Exercises 25–32).

Graph the functions in Exercises 11–16 and integrate them over their domains.

$$\text{11. } f(x) = \begin{cases} x^{2/3}, & -8 \leq x < 0 \\ -4, & 0 \leq x \leq 3 \end{cases}$$

$$\text{12. } f(x) = \begin{cases} \sqrt{-x}, & -4 \leq x < 0 \\ x^2 - 4, & 0 \leq x \leq 3 \end{cases}$$

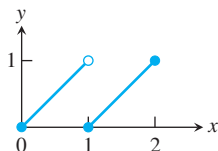
$$\text{13. } g(t) = \begin{cases} t, & 0 \leq t < 1 \\ \sin \pi t, & 1 \leq t \leq 2 \end{cases}$$

$$\text{14. } h(z) = \begin{cases} \sqrt{1-z}, & 0 \leq z < 1 \\ (7z-6)^{-1/3}, & 1 \leq z \leq 2 \end{cases}$$

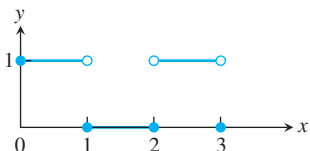
$$\text{15. } f(x) = \begin{cases} 1, & -2 \leq x < -1 \\ 1-x^2, & -1 \leq x < 1 \\ 2, & 1 \leq x \leq 2 \end{cases}$$

$$\text{16. } h(r) = \begin{cases} r, & -1 \leq r < 0 \\ 1-r^2, & 0 \leq r < 1 \\ 1, & 1 \leq r \leq 2 \end{cases}$$

17. Find the average value of the function graphed in the accompanying figure.



18. Find the average value of the function graphed in the accompanying figure.



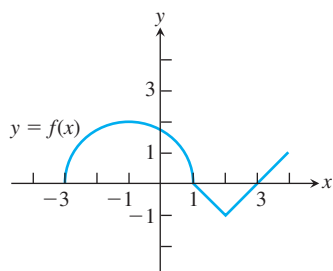
Limits

Find the limits in Exercises 19–22.

19. $\lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}}$ 20. $\lim_{x \rightarrow \infty} \frac{1}{x} \int_0^x \arctan t \, dt$
21. $\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right)$
22. $\lim_{n \rightarrow \infty} \frac{1}{n} (e^{1/n} + e^{2/n} + \cdots + e^{(n-1)/n} + e^{n/n})$

Defining Functions Using the Fundamental Theorem

23. **A function defined by an integral** The graph of a function f consists of a semicircle and two line segments as shown. Let $g(x) = \int_1^x f(t) \, dt$.



- a. Find $g(1)$. b. Find $g(3)$. c. Find $g(-1)$.
- d. Find all values of x on the open interval $(-3, 4)$ at which g has a relative maximum.
- e. Write an equation for the line tangent to the graph of g at $x = -1$.
- f. Find the x -coordinate of each point of inflection of the graph of g on the open interval $(-3, 4)$.
- g. Find the range of g .
24. **A differential equation** Show that both of the following conditions are satisfied by $y = \sin x + \int_x^\pi \cos 2t \, dt + 1$:
- i) $y'' = -\sin x + 2 \sin 2x$
- ii) $y = 1$ and $y' = -2$ when $x = \pi$.

Leibniz's Rule In applications, we sometimes encounter functions defined by integrals that have variable upper limits of integration and variable lower limits of integration at the same time. We can find the derivative of such an integral by a formula called **Leibniz's Rule**.

Leibniz's Rule

If f is continuous on $[a, b]$ and if $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$, then

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(t) \, dt = f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}.$$

To prove the rule, let F be an antiderivative of f on $[a, b]$. Then

$$\int_{u(x)}^{v(x)} f(t) \, dt = F(v(x)) - F(u(x)).$$

Differentiating both sides of this equation with respect to x gives the equation we want:

$$\begin{aligned} \frac{d}{dx} \int_{u(x)}^{v(x)} f(t) \, dt &= \frac{d}{dx} [F(v(x)) - F(u(x))] \\ &= F'(v(x)) \frac{dv}{dx} - F'(u(x)) \frac{du}{dx} \quad \text{Chain Rule} \\ &= f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}. \end{aligned}$$

Use Leibniz's Rule to find the derivatives of the functions in Exercises 25–32.

25. $f(x) = \int_{1/x}^x \frac{1}{t} \, dt$ 26. $f(x) = \int_{\cos x}^{\sin x} \frac{1}{1-t^2} \, dt$
27. $g(y) = \int_{\sqrt{y}}^{2\sqrt{y}} \sin t^2 \, dt$ 28. $g(y) = \int_{\sqrt{y}}^{y^2} \frac{e^t}{t} \, dt$
29. $y = \int_{x^2/2}^{x^2} \ln \sqrt{t} \, dt$ 30. $y = \int_{\sqrt{x}}^{3\sqrt{x}} \ln t \, dt$
31. $y = \int_0^{\ln x} \sin e^t \, dt$ 32. $y = \int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t \, dt$

Theory and Examples

33. Use Leibniz's Rule to find the value of x that maximizes the value of the integral

$$\int_x^{x+3} t(5-t) \, dt.$$

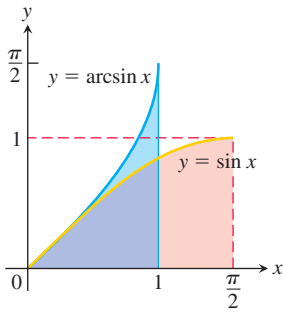
34. For what $x > 0$ does $x^{(x^x)} = (x^x)^x$? Give reasons for your answer.
35. For the two curves $y = 2(\log_2 x)/x$ and $y = 2(\log_4 x)/x$:
- a. Find the area between the first curve and the x -axis from $x = 1$ to $x = e$.
- b. Find the area between the second curve and the x -axis from $x = 1$ to $x = e$.
- c. Consider the two areas obtained in parts (a) and (b). What is the ratio of the larger area to the smaller?
36. a. Find df/dx if

$$f(x) = \int_1^{e^x} \frac{2 \ln t}{t} \, dt.$$

- b. Find $f(0)$.
- c. What can you conclude about the graph of f ? Give reasons for your answer.
37. Find $f'(2)$ if $f(x) = e^{g(x)}$ and $g(x) = \int_2^x \frac{t}{1+t^4} \, dt$.

38. Use the accompanying figure to show that

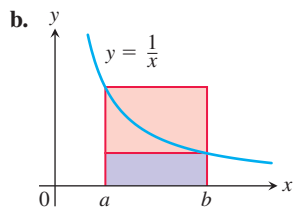
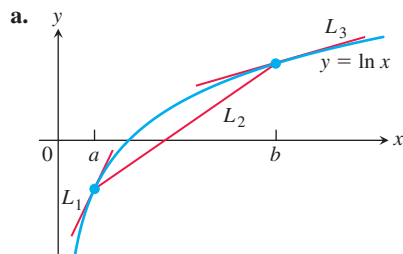
$$\int_0^{\pi/2} \sin x \, dx = \frac{\pi}{2} - \int_0^1 \arcsin x \, dx.$$



39. **Napier's inequality** Here are two pictorial proofs that

$$b > a > 0 \Rightarrow \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}.$$

Explain what is going on in each case.



(Source: Roger B. Nelson, *College Mathematics Journal*, Vol. 24, No. 2, March 1993, p. 165.)

40. **Bound on an integral** Let f be a continuously differentiable function on $[a, b]$ satisfying $\int_a^b f(x) \, dx = 0$.

a. If $c = (a + b)/2$, show that

$$\int_a^b x f(x) \, dx = \int_a^c (x - c) f(x) \, dx + \int_c^b (x - c) f(x) \, dx.$$

b. Let $t = |x - c|$ and $\ell = (b - a)/2$. Show that

$$\int_a^b x f(x) \, dx = \int_0^\ell t(f(c + t) - f(c - t)) \, dt.$$

c. Apply the Mean Value Theorem from Section 4.2 to part (b) to prove that

$$\left| \int_a^b x f(x) \, dx \right| \leq \frac{(b - a)^3}{12} M,$$

where M is the absolute maximum of f' on $[a, b]$.

Approximating Finite Sums with Integrals

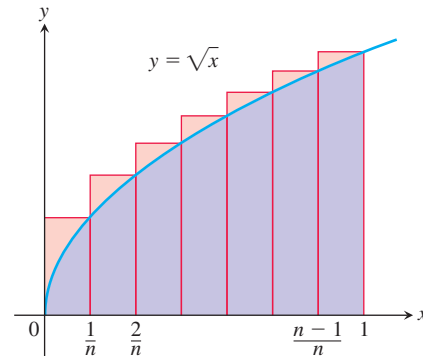
In many applications of calculus, integrals are used to approximate finite sums—the reverse of the usual procedure of using finite sums to approximate integrals.

For example, let's estimate the sum of the square roots of the first n positive integers, $\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}$. The integral

$$\int_0^1 \sqrt{x} \, dx = \frac{2}{3} x^{3/2} \Big|_0^1 = \frac{2}{3}$$

is the limit of the upper sums

$$\begin{aligned} S_n &= \sqrt{\frac{1}{n}} \cdot \frac{1}{n} + \sqrt{\frac{2}{n}} \cdot \frac{1}{n} + \cdots + \sqrt{\frac{n}{n}} \cdot \frac{1}{n} \\ &= \frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^{3/2}}. \end{aligned}$$



Therefore, when n is large, S_n will be close to $2/3$ and we will have

$$\text{Root sum} = \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} = S_n \cdot n^{3/2} \approx \frac{2}{3} n^{3/2}.$$

The following table shows how good the approximation can be.

n	Root sum	$(2/3)n^{3/2}$	Relative error
10	22.468	21.082	$1.386/22.468 \approx 6\%$
50	239.04	235.70	1.4%
100	671.46	666.67	0.7%
1000	21,097	21,082	0.07%

41. Evaluate

$$\lim_{n \rightarrow \infty} \frac{1^5 + 2^5 + 3^5 + \cdots + n^5}{n^6}$$

by showing that the limit is

$$\int_0^1 x^5 \, dx$$

and evaluating the integral.

42. See Exercise 41. Evaluate

$$\lim_{n \rightarrow \infty} \frac{1}{n^4} (1^3 + 2^3 + 3^3 + \cdots + n^3).$$

43. Let $f(x)$ be a continuous function. Express

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right) \right]$$

as a definite integral.

44. Use the result of Exercise 43 to evaluate

a. $\lim_{n \rightarrow \infty} \frac{1}{n^2} (2 + 4 + 6 + \cdots + 2n),$

- b. $\lim_{n \rightarrow \infty} \frac{1}{n^{16}}(1^{15} + 2^{15} + 3^{15} + \cdots + n^{15}),$
 c. $\lim_{n \rightarrow \infty} \frac{1}{n} \left(\sin \frac{\pi}{n} + \sin \frac{2\pi}{n} + \sin \frac{3\pi}{n} + \cdots + \sin \frac{n\pi}{n} \right).$

What can be said about the following limits?

- d. $\lim_{n \rightarrow \infty} \frac{1}{n^{17}}(1^{15} + 2^{15} + 3^{15} + \cdots + n^{15})$
 e. $\lim_{n \rightarrow \infty} \frac{1}{n^{15}}(1^{15} + 2^{15} + 3^{15} + \cdots + n^{15})$
 45. a. Show that the area A_n of an n -sided regular polygon in a circle of radius r is

$$A_n = \frac{nr^2}{2} \sin \frac{2\pi}{n}.$$

- b. Find the limit of A_n as $n \rightarrow \infty$. Is this answer consistent with what you know about the area of a circle?

46. Let

$$S_n = \frac{1^2}{n^3} + \frac{2^2}{n^3} + \cdots + \frac{(n-1)^2}{n^3}.$$

To calculate $\lim_{n \rightarrow \infty} S_n$, show that

$$S_n = \frac{1}{n} \left[\left(\frac{1}{n} \right)^2 + \left(\frac{2}{n} \right)^2 + \cdots + \left(\frac{n-1}{n} \right)^2 \right]$$

and interpret S_n as an approximating sum of the integral

$$\int_0^1 x^2 dx.$$

(Hint: Partition $[0, 1]$ into n intervals of equal length and write out the approximating sum for inscribed rectangles.)

CHAPTER 5 Technology Application Projects

Mathematica/Maple Projects

Projects can be found within MyLab Math.

- **Using Riemann Sums to Estimate Areas, Volumes, and Lengths of Curves**
Visualize and approximate areas and volumes in Part I.
- **Riemann Sums, Definite Integrals, and the Fundamental Theorem of Calculus**
Parts I, II, and III develop Riemann sums and definite integrals. Part IV continues the development of the Riemann sum and definite integral using the Fundamental Theorem to solve problems previously investigated.
- **Rain Catchers, Elevators, and Rockets**
Part I illustrates that the area under a curve is the same as the area of an appropriate rectangle for examples taken from the chapter. You will compute the amount of water accumulating in basins of different shapes as the basin is filled and drained.
- **Motion Along a Straight Line, Part II**
You will observe the shape of a graph through dramatic animated visualizations of the derivative relations among position, velocity, and acceleration. Figures in the text can be animated using this software.
- **Bending of Beams**
Study bent shapes of beams, determine their maximum deflections, concavity, and inflection points, and interpret the results in terms of a beam's compression and tension.

6

Applications of Definite Integrals



OVERVIEW In Chapter 5 we saw that a continuous function over a closed interval has a definite integral, which is the limit of Riemann sum approximations for the function. We found a way to evaluate definite integrals using the Fundamental Theorem of Calculus. We saw that the area under a curve and the area between two curves could be defined and computed as definite integrals. In this chapter we will see some of the many additional applications of definite integrals. We will use the definite integral to define and find volumes, lengths of plane curves, and areas of surfaces of revolution. We will see how integrals are used to solve physical problems involving the work done by a force, and how they give the location of an object's center of mass. The integral arises in these and other applications in which we can approximate a desired quantity by Riemann sums. The limit of those Riemann sums, which is the quantity we seek, is given by a definite integral.

6.1 Volumes Using Cross-Sections

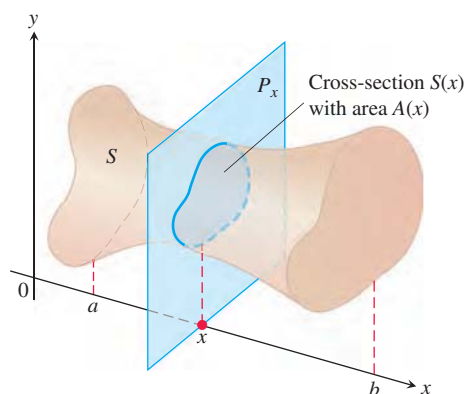


FIGURE 6.1 A cross-section $S(x)$ of the solid S formed by intersecting S with a plane P_x perpendicular to the x -axis through the point x in the interval $[a, b]$.

In this section we define volumes of solids by using the areas of their cross-sections. A **cross-section** of a solid S is the planar region formed by intersecting S with a plane (Figure 6.1). We present three different methods for obtaining the cross-sections appropriate to finding the volume of a particular solid: the method of slicing, the disk method, and the washer method.

Suppose that we want to find the volume of a solid S like the one pictured in Figure 6.1. At each point x in the interval $[a, b]$ we form a cross-section $S(x)$ by intersecting S with a plane perpendicular to the x -axis through the point x , which gives a planar region whose area is $A(x)$. We will show that if A is a continuous function of x , then the volume of the solid S is the definite integral of $A(x)$. This method of computing volumes is known as the method of slicing.

Before showing how this method works, we need to extend the definition of a cylinder from the usual cylinders of classical geometry (which have circular, square, or other regular bases) to cylindrical solids that have more general bases. As shown in Figure 6.2, if the

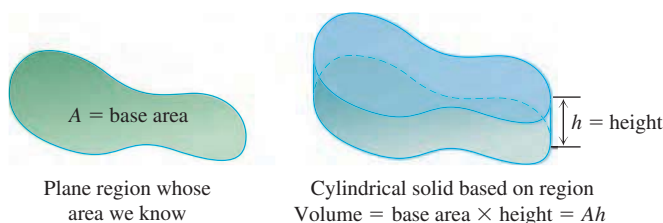


FIGURE 6.2 The volume of a cylindrical solid is equal to its base area times its height.

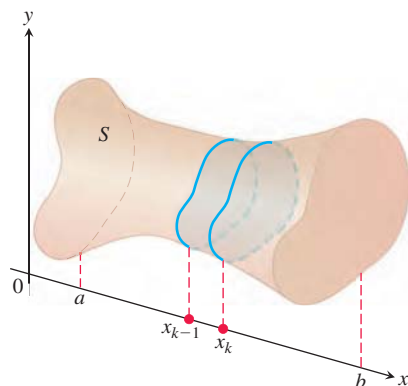


FIGURE 6.3 A typical thin slab in the solid S .

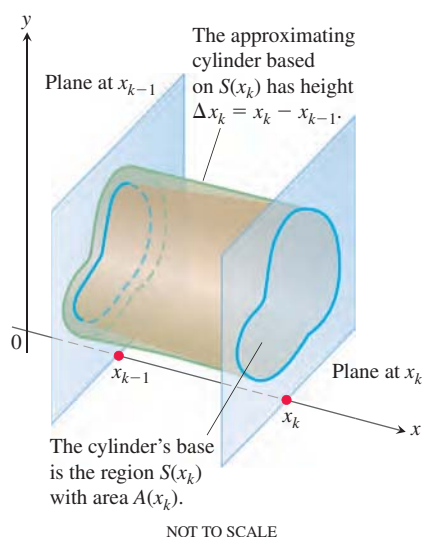


FIGURE 6.4 The solid thin slab in Figure 6.3 is shown enlarged here. It is approximated by the cylindrical solid with base $S(x_k)$ having area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$.

cylindrical solid has a base whose area is A and its height is h , then the volume of the cylindrical solid is

$$\text{Volume} = \text{area} \times \text{height} = A \cdot h.$$

In the method of slicing, the base will be the cross-section of S that has area $A(x)$, and the height will correspond to the width Δx_k of subintervals formed by partitioning the interval $[a, b]$ into finitely many subintervals $[x_{k-1}, x_k]$.

Slicing by Parallel Planes

We partition $[a, b]$ into subintervals of width (length) Δx_k and slice the solid, as we would a loaf of bread, by planes perpendicular to the x -axis at the partition points $a = x_0 < x_1 < \cdots < x_n = b$. These planes slice S into thin “slabs” (like thin slices of a loaf of bread). A typical slab is shown in Figure 6.3. We approximate the slab between the plane at x_{k-1} and the plane at x_k by a cylindrical solid with base area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$ (Figure 6.4). The volume V_k of this cylindrical solid is $A(x_k) \cdot \Delta x_k$, which is approximately the same volume as that of the slab:

$$\text{Volume of the } k\text{th slab} \approx V_k = A(x_k) \Delta x_k.$$

The volume V of the entire solid S is therefore approximated by the sum of these cylindrical volumes,

$$V \approx \sum_{k=1}^n V_k = \sum_{k=1}^n A(x_k) \Delta x_k.$$

This is a Riemann sum for the function $A(x)$ on $[a, b]$. The approximation given by this Riemann sum converges to the definite integral of $A(x)$ as $n \rightarrow \infty$:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n A(x_k) \Delta x_k = \int_a^b A(x) dx.$$

Therefore, we define this definite integral to be the volume of the solid S .

DEFINITION The **volume** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b ,

$$V = \int_a^b A(x) dx.$$

This definition applies whenever $A(x)$ is integrable, and in particular when $A(x)$ is continuous. To apply this definition to calculate the volume of a solid using cross-sections perpendicular to the x -axis, take the following steps:

Calculating the Volume of a Solid

1. Sketch the solid and a typical cross-section.
2. Find a formula for $A(x)$, the area of a typical cross-section.
3. Find the limits of integration.
4. Integrate $A(x)$ to find the volume.

EXAMPLE 1 A pyramid 3 meters high has a square base that is 3 meters on a side. The cross-section of the pyramid perpendicular to the altitude x meters down from the vertex is a square x meters on a side. Find the volume of the pyramid.

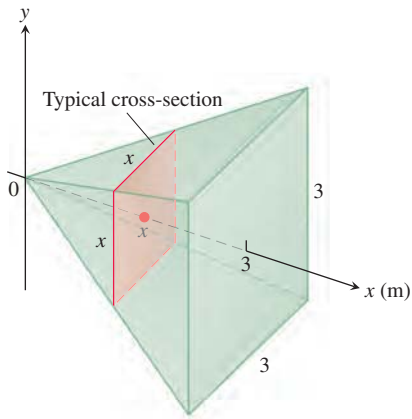


FIGURE 6.5 The cross-sections of the pyramid in Example 1 are squares.

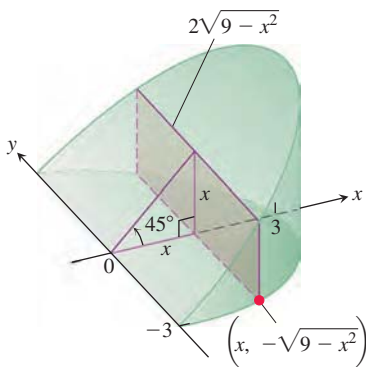


FIGURE 6.6 The wedge of Example 2, sliced perpendicular to the x -axis. The cross-sections are rectangles.

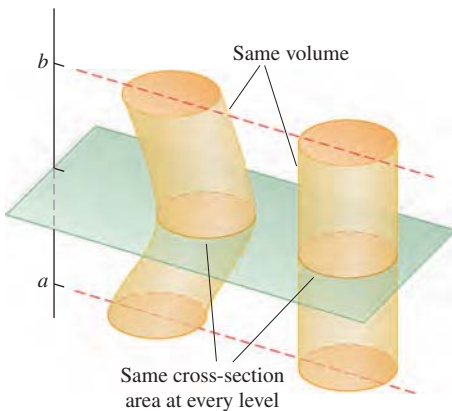


FIGURE 6.7 Cavalieri's principle: These solids have the same volume (imagine each solid as a stack of coins).

Solution

1. *A sketch.* We draw the pyramid with its altitude along the x -axis and its vertex at the origin and include a typical cross-section (Figure 6.5). Note that by positioning the pyramid in this way, we have vertical cross-sections that are squares, whose areas are easy to calculate.

2. *A formula for $A(x)$.* The cross-section at x is a square x meters on a side, so its area is

$$A(x) = x^2.$$

3. *The limits of integration.* The squares lie on the planes from $x = 0$ to $x = 3$.

4. *Integrate to find the volume:*

$$V = \int_0^3 A(x) dx = \int_0^3 x^2 dx = \left. \frac{x^3}{3} \right|_0^3 = 9 \text{ m}^3.$$

EXAMPLE 2 A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

Solution We draw the wedge and sketch a typical cross-section perpendicular to the x -axis (Figure 6.6). The base of the wedge in the figure is the semicircle with $x \geq 0$ that is cut from the circle $x^2 + y^2 = 9$ by the 45° plane when it intersects the y -axis. For any x in the interval $[0, 3]$, the y -values in this semicircular base vary from $y = -\sqrt{9 - x^2}$ to $y = \sqrt{9 - x^2}$. When we slice through the wedge by a plane perpendicular to the x -axis, we obtain a cross-section at x , which is a rectangle of height x whose width extends across the semicircular base. The area of this cross-section is

$$\begin{aligned} A(x) &= (\text{height})(\text{width}) = (x)(2\sqrt{9 - x^2}) \\ &= 2x\sqrt{9 - x^2}. \end{aligned}$$

The rectangles run from $x = 0$ to $x = 3$, so we have

$$\begin{aligned} V &= \int_a^b A(x) dx = \int_0^3 2x\sqrt{9 - x^2} dx \\ &= -\frac{2}{3}(9 - x^2)^{3/2} \Big|_0^3 \\ &= 0 + \frac{2}{3}(9)^{3/2} \\ &= 18. \end{aligned}$$

Let $u = 9 - x^2$ and $du = -2x dx$, integrate, and substitute back.

EXAMPLE 3 Cavalieri's principle says that solids with equal altitudes and identical cross-sectional areas at each height have the same volume (Figure 6.7). This follows immediately from the definition of volume, because the cross-sectional area function $A(x)$ and the interval $[a, b]$ are the same for both solids.

Solids of Revolution: The Disk Method

The solid generated by rotating (or revolving) a planar region about an axis in its plane is called a **solid of revolution**. To find the volume of a solid like the one shown in Figure 6.8, we first observe that the cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, where $R(x)$ is the distance from the axis of revolution to the planar region's boundary. The area is then

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

Therefore, the definition of volume gives us the following formula.

HISTORICAL BIOGRAPHY

Bonaventura Cavalieri
(1598–1647)

Cavalieri was born in Milan, Italy. At a young age, Cavalieri began studying geometry. Later, he studied from and worked with Galileo; the two mathematicians often corresponded through letters. Cavalieri learned the foundations of calculus and developed his ideas on the method of indivisibles, which was his major contribution to mathematics.

To know more, visit the companion Website.

Volume by Disks for Rotation About the x -Axis

$$V = \int_a^b A(x) dx = \int_a^b \pi [R(x)]^2 dx.$$

This method for calculating the volume of a solid of revolution is often called the **disk method** because a cross-section is a circular disk of radius $R(x)$.

EXAMPLE 4 The region between the curve $y = \sqrt{x}$, $0 \leq x \leq 4$, and the x -axis is revolved about the x -axis to generate a solid. Find its volume.

Solution We draw the region and a typical radius (Figure 6.8a) and the generated solid (Figure 6.8b). The volume is

$$\begin{aligned} V &= \int_a^b \pi [R(x)]^2 dx \\ &= \int_0^4 \pi [\sqrt{x}]^2 dx \\ &= \pi \int_0^4 x dx = \pi \left. \frac{x^2}{2} \right|_0^4 = \pi \frac{(4)^2}{2} = 8\pi. \end{aligned}$$

Radius $R(x) = \sqrt{x}$ for rotation around x -axis.

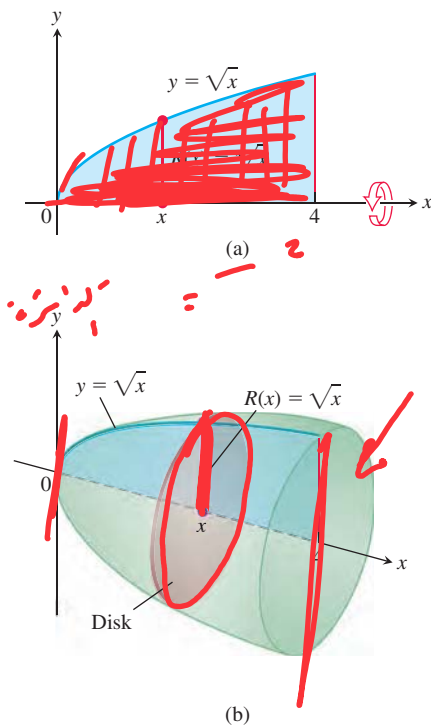


FIGURE 6.8 The region (a) and solid of revolution (b) in Example 4.

EXAMPLE 5 The circle

$$x^2 + y^2 = a^2$$

is rotated about the x -axis to generate a sphere. Find its volume.

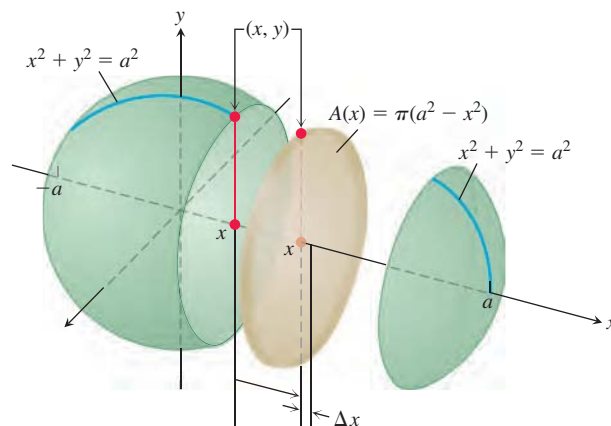


FIGURE 6.9 The sphere generated by rotating the circle $x^2 + y^2 = a^2$ about the x -axis. The radius is $R(x) = y = \sqrt{a^2 - x^2}$ (Example 5).

Solution We imagine the sphere cut into thin slices by planes perpendicular to the x -axis (Figure 6.9). The cross-sectional area at a typical point x between $-a$ and a is

$$A(x) = \pi y^2 = \pi(a^2 - x^2).$$

$R(x) = \sqrt{a^2 - x^2}$ for rotation around x -axis.

Therefore, the volume is

$$V = \int_{-a}^a A(x) dx = \int_{-a}^a \pi(a^2 - x^2) dx = \pi \left[a^2x - \frac{x^3}{3} \right]_{-a}^a = \frac{4}{3} \pi a^3. \quad \blacksquare$$

The axis of revolution in the next example is not the x -axis, but the rule for calculating the volume is the same: Integrate $\pi(\text{radius})^2$ between appropriate limits.

EXAMPLE 6 Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

Solution We draw the region and a typical radius (Figure 6.10a), and the generated solid (Figure 6.10b). The volume is

$$\begin{aligned} V &= \int_1^4 \pi [R(x)]^2 dx \\ &= \int_1^4 \pi [\sqrt{x} - 1]^2 dx && \text{Radius } R(x) = \sqrt{x} - 1 \text{ for rotation around } y = 1. \\ &= \pi \int_1^4 [x - 2\sqrt{x} + 1] dx && \text{Expand integrand.} \\ &= \pi \left[\frac{x^2}{2} - 2 \cdot \frac{2}{3} x^{3/2} + x \right]_1^4 = \frac{7\pi}{6}. && \text{Integrate.} \end{aligned}$$

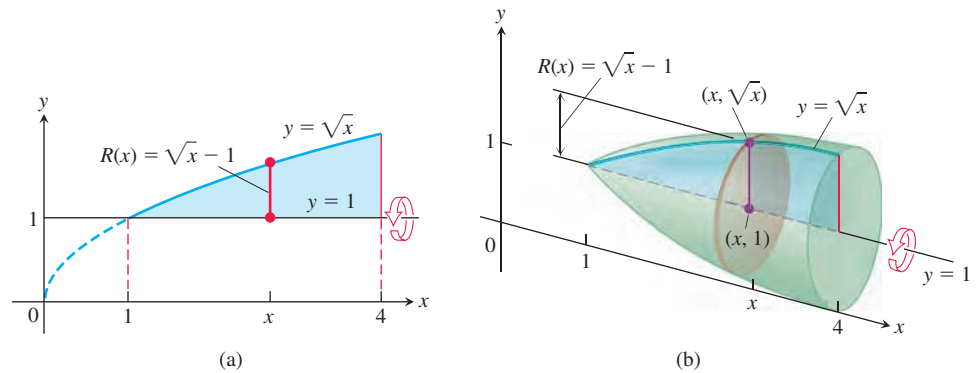


FIGURE 6.10 The region (a) and solid of revolution (b) in Example 6.

To find the volume of a solid generated by revolving a region between the y -axis and a curve $x = R(y)$, $c \leq y \leq d$, about the y -axis, we use the same method with x replaced by y . In this case, the area of the circular cross-section is

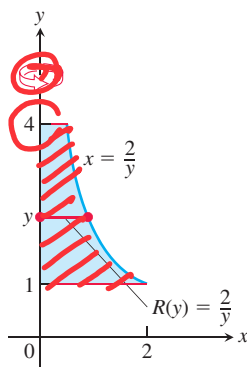
$$A(y) = \pi[\text{radius}]^2 = \pi[R(y)]^2,$$

and the definition of volume gives us the following formula.

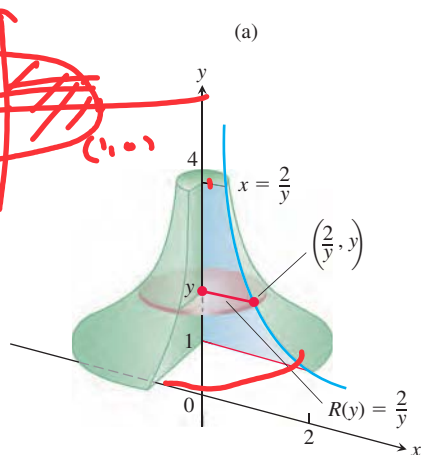
Volume by Disks for Rotation About the y -Axis

$$V = \int_c^d A(y) dy = \int_c^d \pi [R(y)]^2 dy.$$

EXAMPLE 7 Find the volume of the solid generated by revolving the region between the y -axis and the curve $x = 2/y$, $1 \leq y \leq 4$, about the y -axis.



(a)



(b)

FIGURE 6.11 The region (a) and part of the solid of revolution (b) in Example 7.

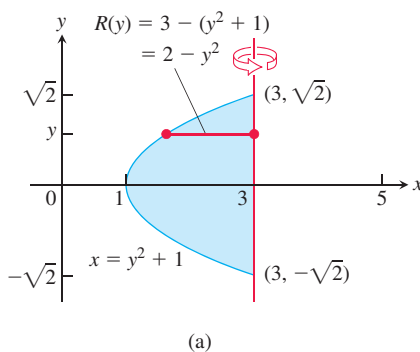
Solution We draw the region and a typical radius (Figure 6.11a) and the generated solid (Figure 6.11b). The volume is

$$\begin{aligned}
 V &= \int_1^4 \pi [R(y)]^2 dy \\
 &= \int_1^4 \pi \left(\frac{2}{y} \right)^2 dy && \text{Radius } R(y) = \frac{2}{y} \text{ for} \\
 & && \text{rotation around } y\text{-axis} \\
 &= \pi \int_1^4 \frac{4}{y^2} dy = 4\pi \left[-\frac{1}{y} \right]_1^4 = 4\pi \left[\frac{3}{4} \right] = 3\pi.
 \end{aligned}$$

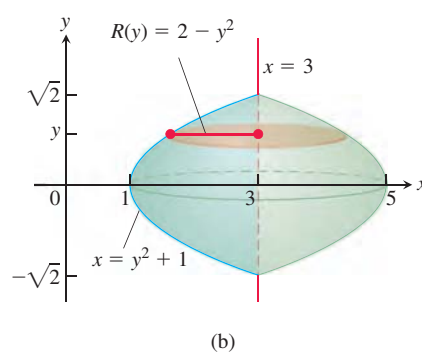
EXAMPLE 8 Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the y -axis.

Solution We draw the region and a typical radius (Figure 6.12a) and the generated solid (Figure 6.12b). Note that the cross-sections are perpendicular to the line $x = 3$ and have y -coordinates from $y = -\sqrt{2}$ to $y = \sqrt{2}$. The volume is

$$\begin{aligned}
 V &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi [R(y)]^2 dy && y = \pm\sqrt{2} \text{ when } x = 3 \\
 &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi [2 - y^2]^2 dy && \text{Radius } R(y) = 3 - (y^2 + 1) \\
 & && \text{for rotation around axis } x = 3. \\
 &= \pi \int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4] dy && \text{Expand integrand.} \\
 &= \pi \left[4y - \frac{4}{3}y^3 + \frac{y^5}{5} \right]_{-\sqrt{2}}^{\sqrt{2}} && \text{Integrate.} \\
 &= \frac{64\pi\sqrt{2}}{15}.
 \end{aligned}$$



(a)



(b)

FIGURE 6.12 The region (a) and solid of revolution (b) in Example 8.

Solids of Revolution: The Washer Method

If the region we revolve to generate a solid does not border on or cross the axis of revolution, then the solid has a hole in it (Figure 6.13). The cross-sections perpendicular to the axis of revolution are *washers* (the purplish circular surface in Figure 6.13) instead of disks. The dimensions of a typical washer are

Outer radius: $R(x)$

Inner radius: $r(x)$

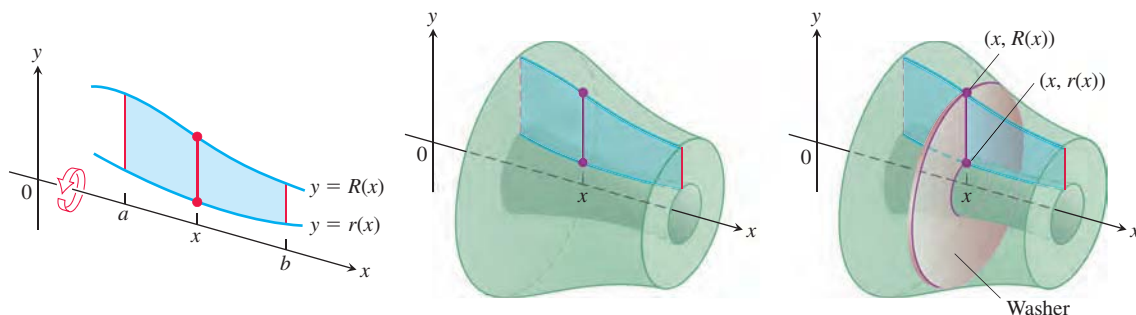


FIGURE 6.13 The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x) dx$ leads to a slightly different formula.

The washer's area is the area of a circle of radius $R(x)$ minus the area of a circle of radius $r(x)$:

$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi([R(x)]^2 - [r(x)]^2).$$

Consequently, the definition of volume in this case gives us the following formula.

Volume by Washers for Rotation About the x -Axis

$$V = \int_a^b A(x) dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2) dx.$$

This method for calculating the volume of a solid of revolution is called the **washer method** because a thin slab of the solid resembles a circular washer with outer radius $R(x)$ and inner radius $r(x)$.

EXAMPLE 9 The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x -axis to generate a solid. Find the volume of the solid.

Solution We draw the region and sketch a line segment across it perpendicular to the axis of revolution (the red segment in Figure 6.14a). We then find the outer and inner radii of the washer that would be swept out by the line segment if it were revolved about the x -axis along with the region. These radii are the distances of the ends of the line segment from the axis of revolution (see Figure 6.14).

$$\text{Outer radius: } R(x) = -x + 3$$

$$\text{Inner radius: } r(x) = x^2 + 1$$

We obtain the limits of integration by finding the x -coordinates of the intersection points of the curve and line in Figure 6.14a.

$$x^2 + 1 = -x + 3$$

$$x^2 + x - 2 = 0$$

$$(x + 2)(x - 1) = 0$$

$$x = -2, \quad x = 1 \quad \text{Limits of integration}$$

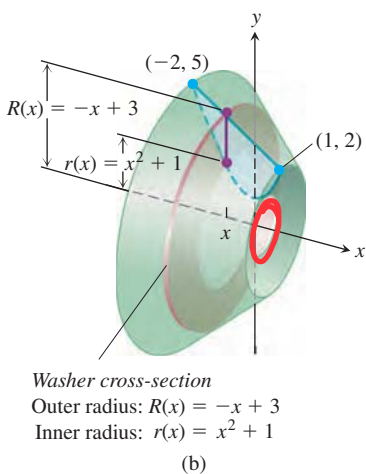
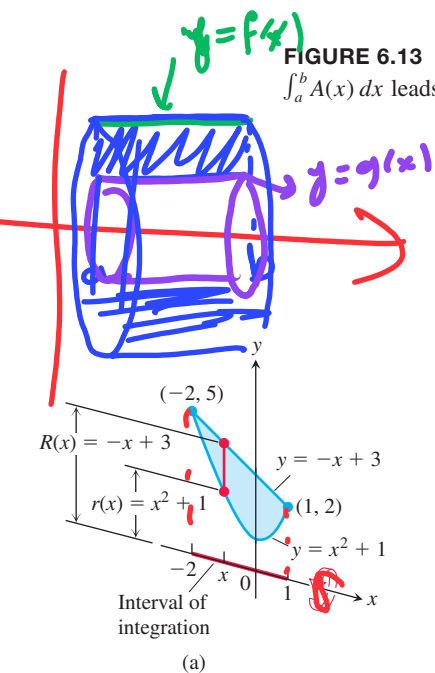
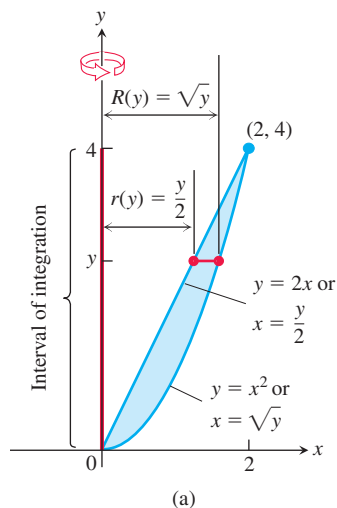


FIGURE 6.14 (a) The region in Example 9 spanned by a line segment perpendicular to the axis of revolution. (b) When the region is revolved about the x -axis, the line segment generates a washer.



The volume is

$$\begin{aligned}
 V &= \int_a^b \pi ([R(x)]^2 - [r(x)]^2) dx && \text{Rotation around } x\text{-axis} \\
 &= \int_{-2}^1 \pi ((-x + 3)^2 - (x^2 + 1)^2) dx && \text{Substitute for radii and limits of integration.} \\
 &= \pi \int_{-2}^1 (8 - 6x - x^2 - x^4) dx && \text{Simplify algebraically.} \\
 &= \pi \left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5} \right]_{-2}^1 = \frac{117\pi}{5}. && \text{Integrate.}
 \end{aligned}$$

To find the volume of a solid formed by revolving a region about the y -axis, we use the same procedure as in Example 9, but integrate with respect to y instead of x . In this situation, the line segment sweeping out a typical washer is perpendicular to the y -axis (the axis of revolution), and the outer and inner radii of the washer are functions of y .

EXAMPLE 10 The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y -axis to generate a solid. Find the volume of the solid.

Solution First we sketch the region and draw a line segment across it perpendicular to the axis of revolution (the y -axis). See Figure 6.15a.

The radii of the washer swept out by the line segment are $R(y) = \sqrt{y}$, $r(y) = y/2$ (Figure 6.15).

The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$. We integrate to find the volume:

$$\begin{aligned}
 V &= \int_c^d \pi ([R(y)]^2 - [r(y)]^2) dy && \text{Rotation around } y\text{-axis} \\
 &= \int_0^4 \pi \left([\sqrt{y}]^2 - \left[\frac{y}{2} \right]^2 \right) dy && \text{Substitute for radii and limits of integration.} \\
 &= \pi \int_0^4 \left(y - \frac{y^2}{4} \right) dy = \pi \left[\frac{y^2}{2} - \frac{y^3}{12} \right]_0^4 = \frac{8}{3}\pi.
 \end{aligned}$$

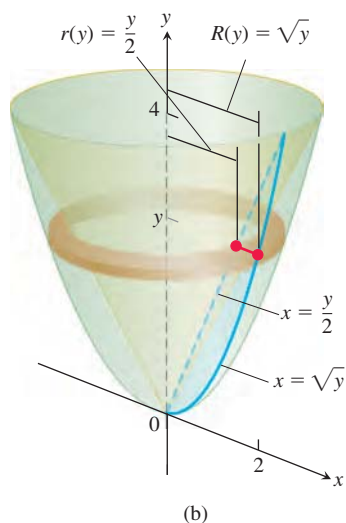


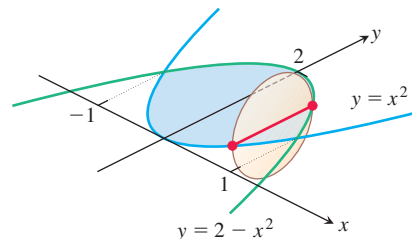
FIGURE 6.15 (a) The region being rotated about the y -axis, the washer radii, and limits of integration in Example 10. (b) The washer swept out by the line segment in part (a).

EXERCISES 6.1

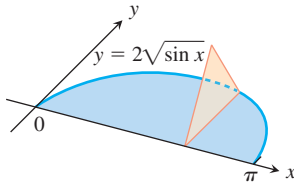
Volumes by Slicing

Find the volumes of the solids in Exercises 1–10.

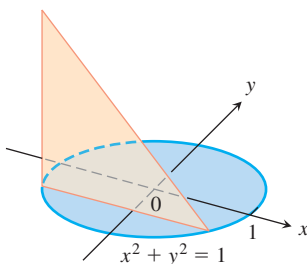
- The solid lies between planes perpendicular to the x -axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the axis on the interval $0 \leq x \leq 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$.
- The solid lies between planes perpendicular to the x -axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x -axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.



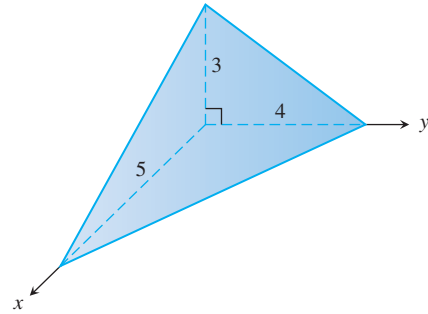
3. The solid lies between planes perpendicular to the x -axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x -axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.
4. The solid lies between planes perpendicular to the x -axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x -axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.
5. The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x -axis. The cross-sections perpendicular to the x -axis are
 - a. equilateral triangles with bases running from the x -axis to the curve as shown in the accompanying figure.



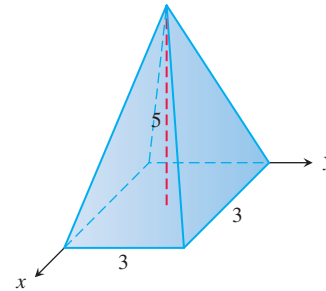
- b. squares with bases running from the x -axis to the curve.
6. The solid lies between planes perpendicular to the x -axis at $x = -\pi/3$ and $x = \pi/3$. The cross-sections perpendicular to the x -axis are
 - a. circular disks with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$.
 - b. squares whose bases run from the curve $y = \tan x$ to the curve $y = \sec x$.
7. The base of a solid is the region bounded by the graphs of $y = 3x$, $y = 6$, and $x = 0$. The cross-sections perpendicular to the x -axis are
 - a. rectangles of height 10.
 - b. rectangles of perimeter 20.
8. The base of a solid is the region bounded by the graphs of $y = \sqrt{x}$ and $y = x/2$. The cross-sections perpendicular to the x -axis are
 - a. isosceles triangles of height 6.
 - b. semicircles with diameters running across the base of the solid.
9. The solid lies between planes perpendicular to the y -axis at $y = 0$ and $y = 2$. The cross-sections perpendicular to the y -axis are circular disks with diameters running from the y -axis to the parabola $x = \sqrt{5}y^2$.
10. The base of the solid is the disk $x^2 + y^2 \leq 1$. The cross-sections by planes perpendicular to the y -axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.



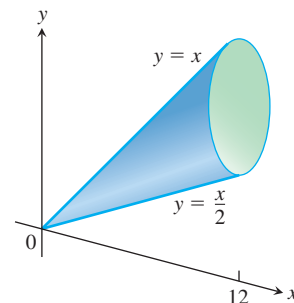
11. Find the volume of the given right tetrahedron. (*Hint:* Consider slices perpendicular to one of the labeled edges.)



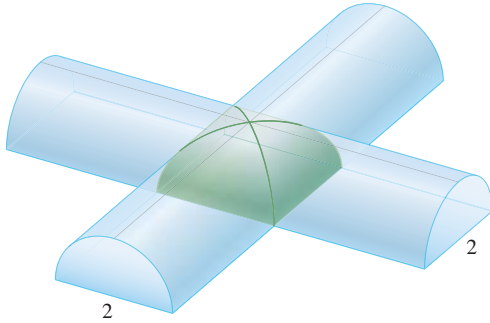
12. Find the volume of the given pyramid, which has a square base of area 9 and height 5.



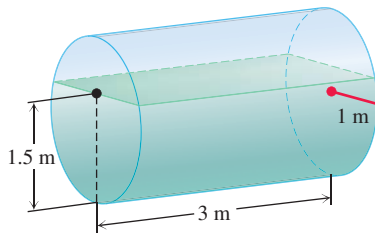
13. **A twisted solid** A square of side length s lies in a plane perpendicular to a line L . One vertex of the square lies on L . As this square moves a distance h along L , the square turns one revolution about L to generate a corkscrew-like column with square cross-sections.
 - a. Find the volume of the column.
 - b. What will the volume be if the square turns twice instead of once? Give reasons for your answer.
14. **Cavalieri's principle** A solid lies between planes perpendicular to the x -axis at $x = 0$ and $x = 12$. The cross-sections by planes perpendicular to the x -axis are circular disks whose diameters run from the line $y = x/2$ to the line $y = x$ as shown in the accompanying figure. Explain why the solid has the same volume as a right circular cone with base radius 3 and height 12.



- 15. Intersection of two half-cylinders** Two half-cylinders of diameter 2 meet at a right angle in the accompanying figure. Find the volume of the solid region common to both half-cylinders. (*Hint:* Consider slices parallel to the base of the solid.)



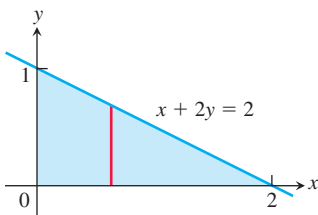
- 16. Gasoline in a tank** A gasoline tank is in the shape of a right circular cylinder (lying on its side) of length 3 m and radius 1 m. Set up an integral that represents the volume of the gas in the tank if it is filled to a depth of 1.5 m. You will learn how to compute this integral in Chapter 8 (or you may use geometry to find its value).



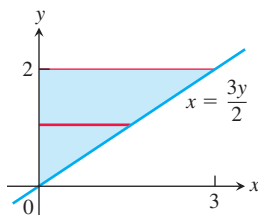
Volumes by the Disk Method

In Exercises 17–20, find the volume of the solid generated by revolving the shaded region about the given axis.

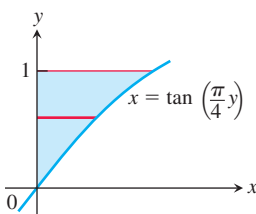
- 17. About the x -axis**



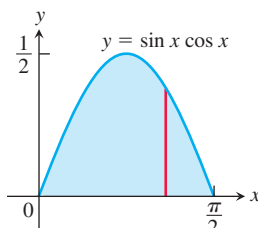
- 18. About the y -axis**



- 19. About the y -axis**



- 20. About the x -axis**



Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 21–30 about the x -axis.

- 21.** $y = x^2$, $y = 0$, $x = 2$ **22.** $y = x^3$, $y = 0$, $x = 2$
23. $y = \sqrt{9 - x^2}$, $y = 0$ **24.** $y = x - x^2$, $y = 0$
25. $y = \sqrt{\cos x}$, $0 \leq x \leq \pi/2$, $y = 0$, $x = 0$
26. $y = \sec x$, $y = 0$, $x = -\pi/4$, $x = \pi/4$
27. $y = e^{-x}$, $y = 0$, $x = 0$, $x = 1$
28. The region between the curve $y = \sqrt{\cot x}$ and the x -axis from $x = \pi/6$ to $x = \pi/2$
29. The region between the curve $y = 1/(2\sqrt{x})$ and the x -axis from $x = 1/4$ to $x = 4$
30. $y = e^{x-1}$, $y = 0$, $x = 1$, $x = 3$

In Exercises 31 and 32, find the volume of the solid generated by revolving the region about the given line.

- 31.** The region in the first quadrant bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \tan x$, and on the left by the y -axis, about the line $y = \sqrt{2}$
32. The region in the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2 \sin x$, $0 \leq x \leq \pi/2$, and on the left by the y -axis, about the line $y = 2$

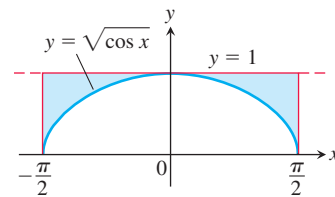
Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 33–38 about the y -axis.

- 33.** The region enclosed by $x = \sqrt{5}y^2$, $x = 0$, $y = -1$, $y = 1$
34. The region enclosed by $x = y^{3/2}$, $x = 0$, $y = 2$
35. The region enclosed by $x = \sqrt{2 \sin 2y}$, $0 \leq y \leq \pi/2$, $x = 0$
36. The region enclosed by $x = \sqrt{\cos(\pi y/4)}$, $-2 \leq y \leq 0$, $x = 0$
37. $x = 2/\sqrt{y+1}$, $x = 0$, $y = 0$, $y = 3$
38. $x = \sqrt{2y}/(y^2 + 1)$, $x = 0$, $y = 1$

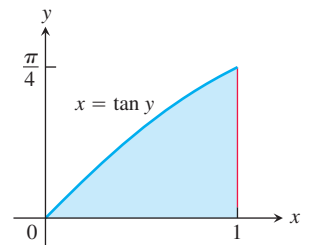
Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 39 and 40 about the indicated axes.

- 39. The x -axis**



- 40. The y -axis**



Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 41–46 about the x -axis.

- 41.** $y = x$, $y = 1$, $x = 0$
42. $y = 2\sqrt{x}$, $y = 2$, $x = 0$
43. $y = x^2 + 1$, $y = x + 3$
44. $y = 4 - x^2$, $y = 2 - x$
45. $y = \sec x$, $y = \sqrt{2}$, $-\pi/4 \leq x \leq \pi/4$
46. $y = \sec x$, $y = \tan x$, $x = 0$, $x = 1$

In Exercises 47–50, find the volume of the solid generated by revolving each region about the y -axis.

47. The region enclosed by the triangle with vertices $(1, 0)$, $(2, 1)$, and $(1, 1)$
48. The region enclosed by the triangle with vertices $(0, 1)$, $(1, 0)$, and $(1, 1)$
49. The region in the first quadrant bounded above by the parabola $y = x^2$, below by the x -axis, and on the right by the line $x = 2$
50. The region in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$, and above by the line $y = \sqrt{3}$

In Exercises 51 and 52, find the volume of the solid generated by revolving each region about the given axis.

51. The region in the first quadrant bounded above by the curve $y = x^2$, below by the x -axis, and on the right by the line $x = 1$, about the line $x = -1$
52. The region in the second quadrant bounded above by the curve $y = -x^3$, below by the x -axis, and on the left by the line $x = -1$, about the line $x = -2$

Volumes of Solids of Revolution

53. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about
 - a. the x -axis.
 - b. the y -axis.
 - c. the line $y = 2$.
 - d. the line $x = 4$.
54. Find the volume of the solid generated by revolving the triangular region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$ about
 - a. the line $x = 1$.
 - b. the line $x = 2$.
55. Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$ about
 - a. the line $y = 1$.
 - b. the line $y = 2$.
 - c. the line $y = -1$.
56. By integration, find the volume of the solid generated by revolving the triangular region with vertices $(0, 0)$, $(b, 0)$, $(0, h)$ about
 - a. the x -axis.
 - b. the y -axis.

Theory and Applications

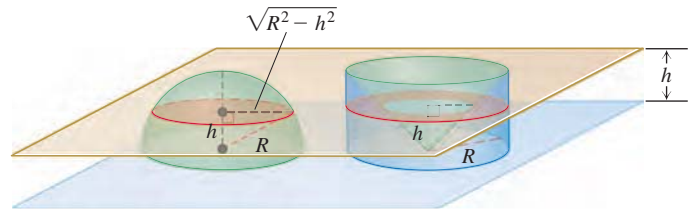
57. **The volume of a torus** The disk $x^2 + y^2 \leq a^2$ is revolved about the line $x = b$ ($b > a$) to generate a solid shaped like a doughnut and called a *torus*. Find its volume.

(Hint: $\int_{-a}^a \sqrt{a^2 - y^2} dy = \pi a^2/2$, since it is the area of a semi-circle of radius a .)

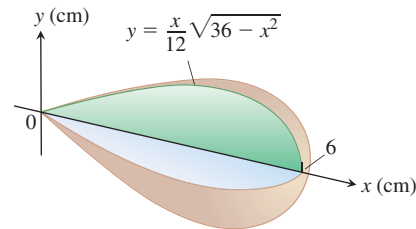
58. **Volume of a bowl** A bowl has a shape that can be generated by revolving the graph of $y = x^2/2$ between $y = 0$ and $y = 5$ about the y -axis.
 - a. Find the volume of the bowl.
 - b. **Related rates** If we fill the bowl with water at a constant rate of 3 cubic units per second, how fast will the water level in the bowl be rising when the water is 4 units deep?
59. **Volume of a bowl**
 - a. A hemispherical bowl of radius a contains water to a depth h . Find the volume of water in the bowl.
 - b. **Related rates** Water runs into a sunken concrete hemispherical bowl of radius 5 m at the rate of $0.2 \text{ m}^3/\text{s}$. How fast is the water level in the bowl rising when the water is 4 m deep?

60. Explain how you could estimate the volume of a solid of revolution by measuring the shadow cast on a table parallel to its axis of revolution by a light shining directly above it.

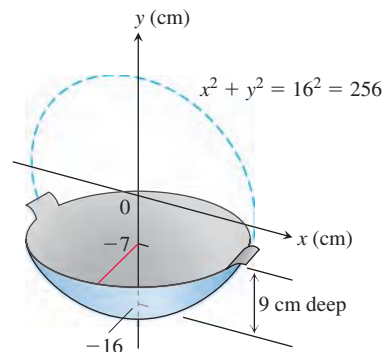
61. **Volume of a hemisphere** Derive the formula $V = (2/3)\pi R^3$ for the volume of a hemisphere of radius R by comparing its cross-sections with the cross-sections of a solid right circular cylinder of radius R and height R from which a solid right circular cone of base radius R and height R has been removed, as suggested by the accompanying figure.



62. **Designing a plumb bob** Having been asked to design a brass plumb bob that will weigh in the neighborhood of 190 g, you decide to shape it like the solid of revolution shown here. Find the plumb bob's volume. If you specify a brass that weighs 8.5 g/cm^3 , how much will the plumb bob weigh (to the nearest gram)?

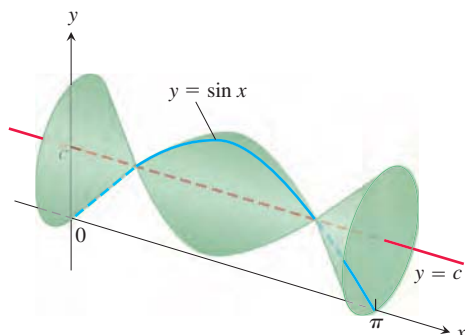


63. **Designing a wok** You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentation at home persuades you that you can get one that holds about 3 L if you make it 9 cm deep and give the sphere a radius of 16 cm. To be sure, you picture the wok as a solid of revolution, as shown here, and calculate its volume with an integral. To the nearest cubic centimeter, what volume do you really get? ($1 \text{ L} = 1000 \text{ cm}^3$)

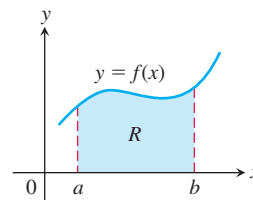


64. **Max-min** The arch $y = \sin x$, $0 \leq x \leq \pi$, is revolved about the line $y = c$, $0 \leq c \leq 1$, to generate the solid in the accompanying figure.
 - a. Find the value of c that minimizes the volume of the solid. What is the minimum volume?
 - b. What value of c in $[0, 1]$ maximizes the volume of the solid?

- T** c. Graph the solid's volume as a function of c , first for $0 \leq c \leq 1$ and then on a larger domain. What happens to the volume of the solid as c moves away from $[0, 1]$? Does this make sense physically? Give reasons for your answers.



65. Consider the region R bounded by the graphs of $y = f(x) > 0$, $x = a > 0$, $x = b > a$, and $y = 0$ (see accompanying figure). If the volume of the solid formed by revolving R about the x -axis is 4π , and the volume of the solid formed by revolving R about the line $y = -1$ is 8π , find the area of R .



66. Consider the region R given in Exercise 65. If the volume of the solid formed by revolving R around the x -axis is 6π , and the volume of the solid formed by revolving R around the line $y = -2$ is 10π , find the area of R .

6.2 Volumes Using Cylindrical Shells

In Section 6.1 we defined the volume of a solid to be the definite integral $V = \int_a^b A(x) dx$, where $A(x)$ is an integrable cross-sectional area of the solid from $x = a$ to $x = b$. The area $A(x)$ was obtained by slicing through the solid with a plane perpendicular to the x -axis. However, this method of slicing is sometimes awkward to apply, as we will illustrate in our first example. To overcome this difficulty, we use the same integral definition for volume, but obtain the area by slicing through the solid in a different way.

Slicing with Cylinders

Suppose we slice through the solid using circular cylinders of increasing radii, like cookie cutters. We slice straight down through the solid so that the axis of each cylinder is parallel to the y -axis. The vertical axis of each cylinder is always the same line, but the radii of the cylinders increase with each slice. In this way the solid is sliced up into thin cylindrical shells of constant thickness that grow outward from their common axis, like circular tree rings. Unrolling a cylindrical shell shows that its volume is approximately that of a rectangular slab with area $A(x)$ and thickness Δx . This slab interpretation allows us to apply the same integral definition for volume as before. The following example provides some insight.

EXAMPLE 1 The region enclosed by the x -axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the ~~vertical line $x = 1$~~ *x -axis* to generate a solid (see Figure 6.16). Find the volume of the solid.

Solution Using the washer method from Section 6.1 would be awkward here because we would need to express the x -values of the left and right sides of the parabola in Figure 6.16a in terms of y . This is because these x -values, which describe the inner and outer radii of a typical washer, are solutions to the equation $y = 3x - x^2$, and this gives a complicated formula for x . Therefore, instead of rotating a horizontal strip of thickness Δy , we rotate a *vertical strip* of thickness Δx . This rotation produces a *cylindrical shell* of height y_k above a point x_k within the base of the vertical strip and of thickness Δx . An example of a cylindrical shell is shown as the orange-shaded region in Figure 6.17. We can think of the cylindrical shell shown in the figure as approximating a slice of the solid obtained by cutting straight down through it, parallel to the axis of revolution, all the way around. We

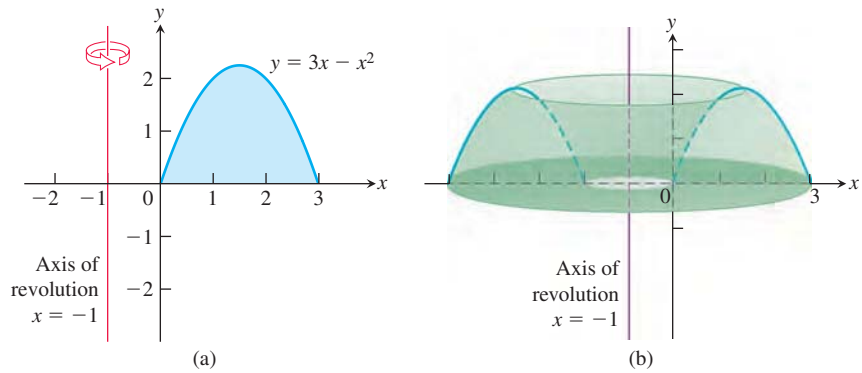


FIGURE 6.16 (a) The graph of the region in Example 1, before revolution. (b) The solid formed when the region in part (a) is revolved about the axis of revolution $x = -1$.

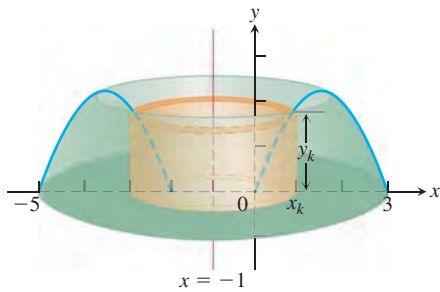


FIGURE 6.17 A cylindrical shell of height y_k obtained by rotating a vertical strip of thickness Δx_k about the line $x = -1$. The outer radius of the cylinder occurs at x_k , where the height of the parabola is $y_k = 3x_k - x_k^2$ (Example 1).

start by cutting close to the inside hole and then cut another cylindrical slice around the enlarged hole, then another, and so on, obtaining n cylinders. The radii of the cylinders gradually increase, and the heights of the cylinders follow the contour of the parabola: shorter to taller, then back to shorter (Figure 6.16a). The sum of the volumes of the shells is a Riemann sum that approximates the volume of the entire solid.

Each shell sits over a subinterval $[x_{k-1}, x_k]$ in the x -axis. The thickness of the shell is $\Delta x_k = x_k - x_{k-1}$. Because the parabola is rotated around the line $x = -1$, the outer radius of the shell is $1 + x_k$. The height of the shell is the height of the parabola at some point in the interval $[x_{k-1}, x_k]$, or approximately $y_k = f(x_k) = 3x_k - x_k^2$. If we unroll this cylinder and flatten it out, it becomes (approximately) a rectangular slab with thickness Δx_k (see Figure 6.18). The height of the rectangular slab is approximately $y_k = 3x_k - x_k^2$, and its length is the circumference of the shell, which is approximately $2\pi \cdot \text{radius} = 2\pi(1 + x_k)$. Hence the volume of the shell is approximately the volume of the rectangular slab, which is

$$\begin{aligned} \Delta V_k &= \text{circumference} \times \text{height} \times \text{thickness} \\ &= 2\pi(1 + x_k) \cdot (3x_k - x_k^2) \cdot \Delta x_k. \end{aligned}$$

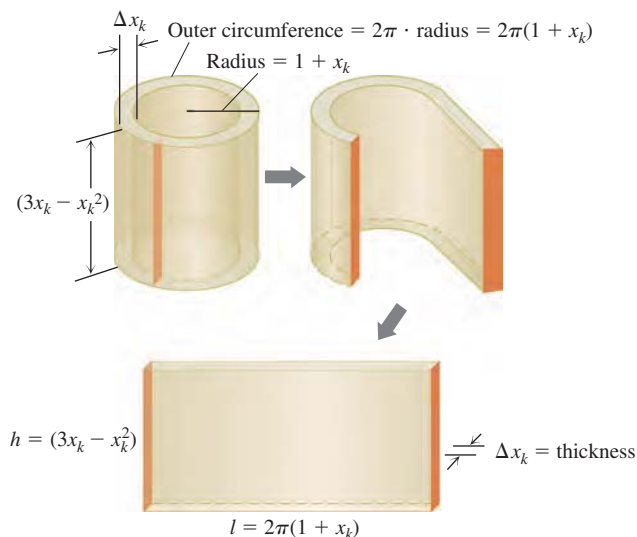


FIGURE 6.18 Cutting and unrolling a cylindrical shell gives a nearly rectangular solid (Example 1).

Summing together the volumes ΔV_k of the individual cylindrical shells over the interval $[0, 3]$ gives the Riemann sum

$$\sum_{k=1}^n \Delta V_k = \sum_{k=1}^n 2\pi(x_k + 1)(3x_k - x_k^2) \Delta x_k.$$

Taking the limit as the thickness $\Delta x_k \rightarrow 0$ and $n \rightarrow \infty$ gives the volume integral

$$\begin{aligned} V &= \lim_{n \rightarrow \infty} \sum_{k=1}^n 2\pi(x_k + 1)(3x_k - x_k^2) \Delta x_k \\ &= \int_0^3 2\pi(x + 1)(3x - x^2) dx \\ &= \int_0^3 2\pi(3x^2 + 3x - x^3 - x^2) dx \\ &= 2\pi \int_0^3 (2x^2 + 3x - x^3) dx \\ &= 2\pi \left[\frac{2}{3}x^3 + \frac{3}{2}x^2 - \frac{1}{4}x^4 \right]_0^3 = \frac{45\pi}{2}. \end{aligned}$$

We now generalize this procedure to a broader class of solids.

The Shell Method

Suppose that the region bounded by the graph of a nonnegative continuous function $y = f(x)$ and the x -axis over the finite closed interval $[a, b]$ lies to the right of the vertical line $x = L$ (see Figure 6.19a). We assume $a \geq L$, so the vertical line may touch the region but cannot pass through it. We generate a solid S by rotating this region about the vertical line L .

Let P be a partition of the interval $[a, b]$ by the points $a = x_0 < x_1 < \cdots < x_n = b$. As usual, we choose a point c_k in each subinterval $[x_{k-1}, x_k]$. In Example 1 we chose c_k to be the endpoint x_k , but now it will be more convenient to let c_k be the midpoint of the subinterval $[x_{k-1}, x_k]$. We approximate the region in Figure 6.19a with rectangles based on this partition of $[a, b]$. A typical approximating rectangle has height $f(c_k)$ and width

$$V = \int_a^b 2\pi x f(x) dx$$

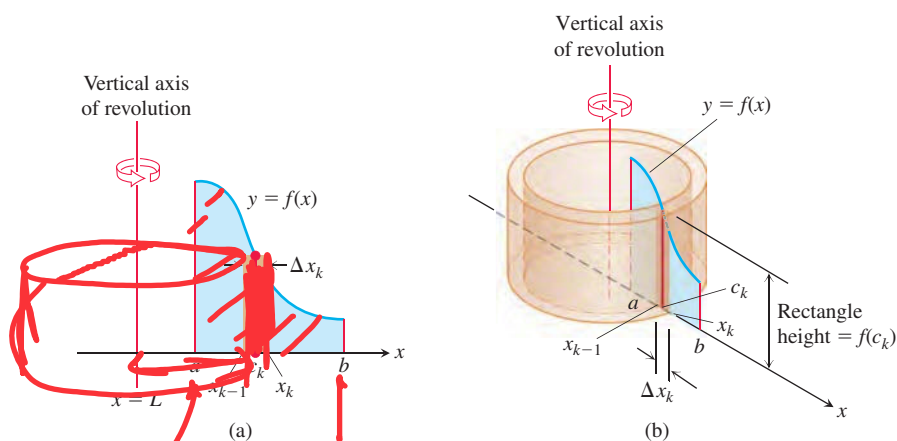


FIGURE 6.19 When the region shown in (a) is revolved about the vertical line $x = L$, a solid is produced which can be sliced into cylindrical shells. A typical shell is shown in (b).

The volume of a cylindrical shell of height h with inner radius r and outer radius R is

$$\pi R^2 h - \pi r^2 h = 2\pi \left(\frac{R+r}{2} \right) (h)(R-r).$$

$\Delta x_k = x_k - x_{k-1}$. If this rectangle is rotated about the vertical line $x = L$, then a shell is swept out, as in Figure 6.19b. A formula from geometry tells us that the volume of the shell swept out by the rectangle is

$$\begin{aligned} \Delta V_k &= 2\pi \times \text{average shell radius} \times \text{shell height} \times \text{thickness} \\ &= 2\pi \cdot (c_k - L) \cdot f(c_k) \cdot \Delta x_k. \quad R = x_k - L \text{ and } r = x_{k-1} - L \end{aligned}$$

We approximate the volume of the solid S by summing the volumes of the shells swept out by the n rectangles:

$$V \approx \sum_{k=1}^n \Delta V_k.$$

The limit of this Riemann sum as each $\Delta x_k \rightarrow 0$ and $n \rightarrow \infty$ gives the volume of the solid as a definite integral:

$$\begin{aligned} V &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta V_k = \int_a^b 2\pi (\text{shell radius})(\text{shell height}) dx \\ &= \int_a^b 2\pi (x - L) f(x) dx. \end{aligned}$$

We refer to the variable of integration, here x , as the **thickness variable**. To emphasize the *process* of the shell method, we state the general formula in terms of the shell radius and shell height. This will allow for rotations about a horizontal line $y = L$ as well.

Shell Formula for Revolution About a Vertical Line

The volume of the solid generated by revolving the region between the x -axis and the graph of a continuous function $y = f(x) \geq 0$, $L \leq a \leq x \leq b$, about a vertical line $x = L$ is

$$V = \int_a^b 2\pi \left(\begin{array}{c} \text{shell} \\ \text{radius} \end{array} \right) \left(\begin{array}{c} \text{shell} \\ \text{height} \end{array} \right) dx.$$

EXAMPLE 2 The region bounded by the curve $y = \sqrt{x}$, the x -axis, and the line $x = 4$ is revolved about the y -axis to generate a solid. Find the volume of the solid.

Solution Sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.20a). Label the segment's height (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.20b, but you need not do that.)

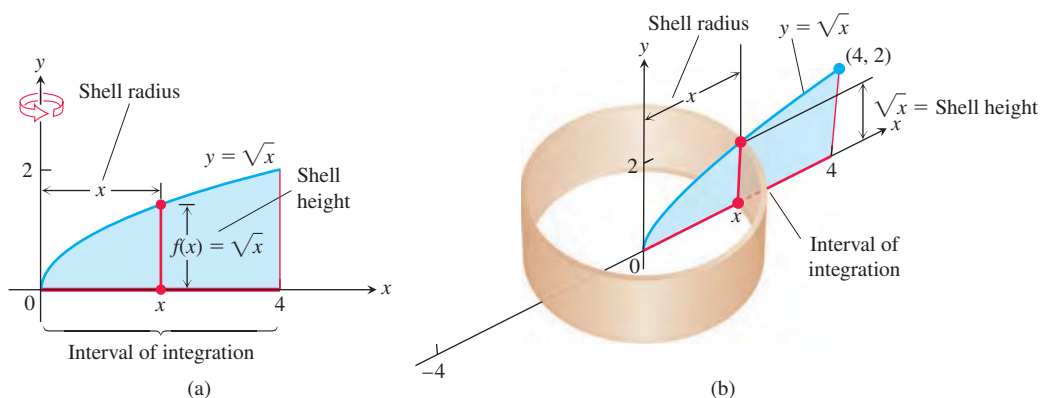


FIGURE 6.20 (a) The region, shell dimensions, and interval of integration in Example 2. (b) The shell swept out by the vertical segment in part (a) with a width Δx .

The shell thickness variable is x , so the limits of integration for the shell formula are $a = 0$ and $b = 4$ (Figure 6.20). The volume is

$$\begin{aligned} V &= \int_a^b 2\pi \left(\begin{array}{c} \text{shell} \\ \text{radius} \end{array} \right) \left(\begin{array}{c} \text{shell} \\ \text{height} \end{array} \right) dx \\ &= \int_0^4 2\pi(x)(\sqrt{x}) dx \\ &= 2\pi \int_0^4 x^{3/2} dx = 2\pi \left[\frac{2}{5} x^{5/2} \right]_0^4 = \frac{128\pi}{5}. \end{aligned}$$

So far, we have used vertical axes of revolution. For horizontal axes, we replace the x 's with y 's.

EXAMPLE 3 The region bounded by the curve $y = \sqrt{x}$, the x -axis, and the line $x = 4$ is revolved about the x -axis to generate a solid. Find the volume of the solid by the shell method.

Solution This is the solid whose volume was found by the disk method in Example 3 of Section 6.1. Now we find its volume by the shell method. First, sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.21a). Label the segment's length (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.21b, but you need not do that.)

In this case, the shell thickness variable is y , so the limits of integration for the shell formula method are $a = 0$ and $b = 2$ (along the y -axis in Figure 6.21). The volume of the solid is

$$\begin{aligned} V &= \int_a^b 2\pi \left(\begin{array}{c} \text{shell} \\ \text{radius} \end{array} \right) \left(\begin{array}{c} \text{shell} \\ \text{height} \end{array} \right) dy \\ &= \int_0^2 2\pi(y)(4 - y^2) dy \\ &= 2\pi \int_0^2 (4y - y^3) dy \\ &= 2\pi \left[2y^2 - \frac{y^4}{4} \right]_0^2 = 8\pi. \end{aligned}$$

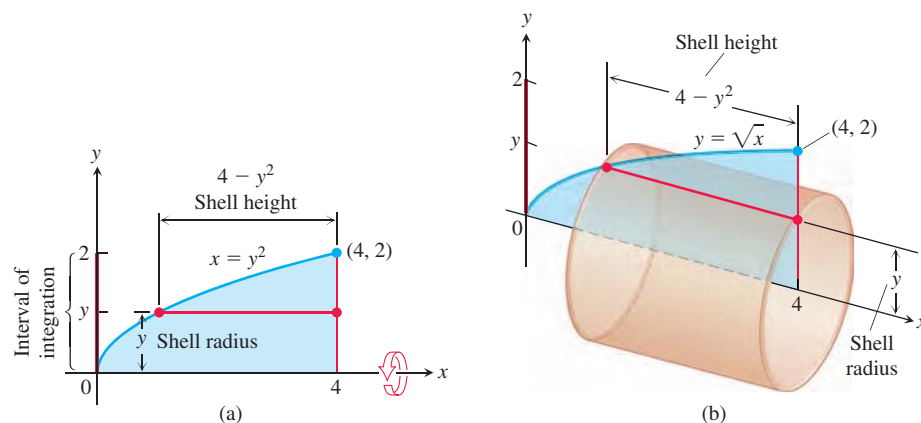


FIGURE 6.21 (a) The region, shell dimensions, and interval of integration in Example 3. (b) The shell swept out by the horizontal segment in part (a) with a width Δy .

Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these.

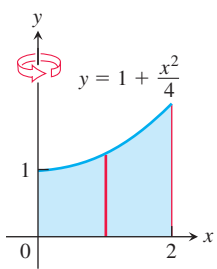
1. Draw the region and sketch a line segment across it *parallel* to the axis of revolution. Label the segment's height or length (shell height) and distance from the axis of revolution (shell radius).
2. Find the limits of integration for the thickness variable.
3. Integrate the product 2π (shell radius) (shell height) with respect to the thickness variable (x or y) to find the volume.

The shell method gives the same answer as the washer method when both are used to calculate the volume of a region. We do not prove that result here, but it is illustrated in Exercises 37 and 38. (Exercise 45 outlines a proof.) Both volume formulas are actually special cases of a general volume formula we will look at when studying double and triple integrals in Chapter 14. That general formula also allows for computing volumes of solids other than those swept out by regions of revolution.

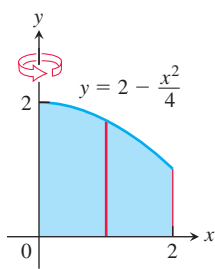
EXERCISES 6.2**Revolution About the Axes**

In Exercises 1–6, use the shell method to find the volumes of the solids generated by revolving the shaded region about the indicated axis.

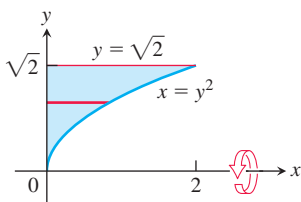
1.



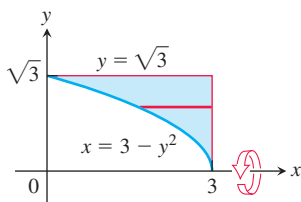
2.



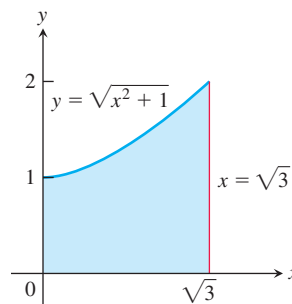
3.



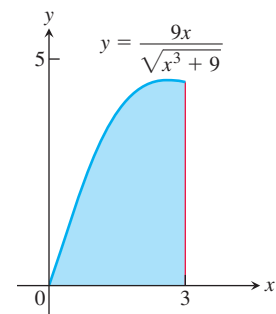
4.



5. The y-axis



6. The y-axis

**Revolution About the y-Axis**

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 7–12 about the y-axis.

7. $y = x$, $y = -x/2$, $x = 2$

8. $y = 2x$, $y = x/2$, $x = 1$

9. $y = x^2$, $y = 2 - x$, $x = 0$, for $x \geq 0$

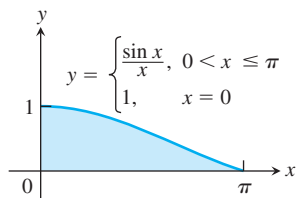
10. $y = 2 - x^2$, $y = x^2$, $x = 0$

11. $y = 2x - 1$, $y = \sqrt{x}$, $x = 0$

12. $y = 3/(2\sqrt{x})$, $y = 0$, $x = 1$, $x = 4$

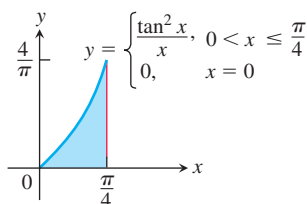
13. Let $f(x) = \begin{cases} (\sin x)/x, & 0 < x \leq \pi \\ 1, & x = 0. \end{cases}$

- a. Show that $x f(x) = \sin x$, $0 \leq x \leq \pi$.
 b. Find the volume of the solid generated by revolving the shaded region about the y -axis in the accompanying figure.



14. Let $g(x) = \begin{cases} (\tan x)^2/x, & 0 < x \leq \pi/4 \\ 0, & x = 0. \end{cases}$

- a. Show that $x g(x) = (\tan x)^2$, $0 \leq x \leq \pi/4$.
 b. Find the volume of the solid generated by revolving the shaded region about the y -axis in the accompanying figure.



Revolution About the x -Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 15–22 about the x -axis.

15. $x = \sqrt{y}$, $x = -y$, $y = 2$
 16. $x = y^2$, $x = -y$, $y = 2$, $y \geq 0$
 17. $x = 2y - y^2$, $x = 0$ 18. $x = 2y - y^2$, $x = y$
 19. $y = |x|$, $y = 1$ 20. $y = x$, $y = 2x$, $y = 2$
 21. $y = \sqrt{x}$, $y = 0$, $y = x - 2$
 22. $y = \sqrt{x}$, $y = 0$, $y = 2 - x$

Revolution About Horizontal and Vertical Lines

In Exercises 23–26, use the shell method to find the volumes of the solids generated by revolving the regions bounded by the given curves about the given lines.

23. $y = 3x$, $y = 0$, $x = 2$
 a. The y -axis b. The line $x = 4$
 c. The line $x = -1$ d. The x -axis
 e. The line $y = 7$ f. The line $y = -2$
 24. $y = x^3$, $y = 8$, $x = 0$
 a. The y -axis b. The line $x = 3$
 c. The line $x = -2$ d. The x -axis
 e. The line $y = 8$ f. The line $y = -1$

25. $y = x + 2$, $y = x^2$

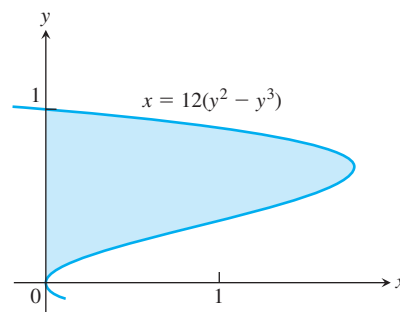
- a. The line $x = 2$ b. The line $x = -1$
 c. The x -axis d. The line $y = 4$

26. $y = x^4$, $y = 4 - 3x^2$

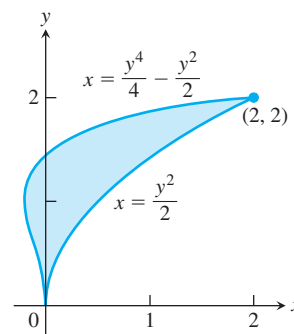
- a. The line $x = 1$ b. The x -axis

In Exercises 27 and 28, use the shell method to find the volumes of the solids generated by revolving the shaded regions about the indicated axes.

27. a. The x -axis b. The line $y = 1$
 c. The line $y = 8/5$ d. The line $y = -2/5$



28. a. The x -axis b. The line $y = 2$
 c. The line $y = 5$ d. The line $y = -5/8$



Choosing the Washer Method or the Shell Method

For some regions, both the washer and shell methods work well for the solid generated by revolving the region about the coordinate axes, but this is not always the case. When a region is revolved about the y -axis, for example, and washers are used, we must integrate with respect to y . It may not be possible, however, to express the integrand in terms of y . In such a case, the shell method allows us to integrate with respect to x instead. Exercises 29 and 30 provide some insight.

29. Compute the volume of the solid generated by revolving the region bounded by $y = x$ and $y = x^2$ about each coordinate axis using
 a. the shell method.
 b. the washer method.

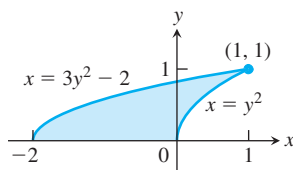
30. Compute the volume of the solid generated by revolving the triangular region bounded by the lines $2y = x + 4$, $y = x$, and $x = 0$ about
- the x -axis using the washer method.
 - the y -axis using the shell method.
 - the line $x = 4$ using the shell method.
 - the line $y = 8$ using the washer method.

In Exercises 31–36, find the volumes of the solids generated by revolving the regions about the given axes. If you think it would be better to use washers in any given instance, feel free to do so.

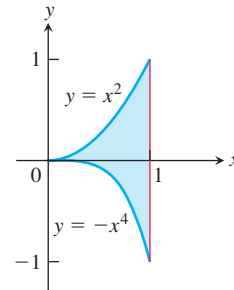
31. The triangle with vertices $(1, 1)$, $(1, 2)$, and $(2, 2)$ about
- the x -axis
 - the y -axis
 - the line $x = 10/3$
 - the line $y = 1$
32. The region bounded by $y = \sqrt{x}$, $y = 2$, $x = 0$ about
- the x -axis
 - the y -axis
 - the line $x = 4$
 - the line $y = 2$
33. The region in the first quadrant bounded by the curve $x = y - y^3$ and the y -axis about
- the x -axis
 - the line $y = 1$
34. The region in the first quadrant bounded by $x = y - y^3$, $x = 1$, and $y = 1$ about
- the x -axis
 - the y -axis
 - the line $x = 1$
 - the line $y = 1$
35. The region bounded by $y = \sqrt{x}$ and $y = x^2/8$ about
- the x -axis
 - the y -axis
36. The region bounded by $y = 2x - x^2$ and $y = x$ about
- the y -axis
 - the line $x = 1$
37. The region in the first quadrant that is bounded above by the curve $y = 1/x^{1/4}$, on the left by the line $x = 1/16$, and below by the line $y = 1$ is revolved about the x -axis to generate a solid. Find the volume of the solid by
- the washer method.
 - the shell method.
38. The region in the first quadrant that is bounded above by the curve $y = 1/\sqrt{x}$, on the left by the line $x = 1/4$, and below by the line $y = 1$ is revolved about the y -axis to generate a solid. Find the volume of the solid by
- the washer method.
 - the shell method.

Theory and Examples

39. The region shown here is to be revolved about the x -axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Explain.



40. The region shown here is to be revolved about the y -axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Give reasons for your answers.



41. A bead is formed from a sphere of radius 5 by drilling through a diameter of the sphere with a drill bit of radius 3.
- Find the volume of the bead.
 - Find the volume of the removed portion of the sphere.
42. A Bundt cake, well known for having a ringed shape, is formed by revolving around the y -axis the region bounded by the graph of $y = \sin(x^2 - 1)$ and the x -axis over the interval $1 \leq x \leq \sqrt{1 + \pi}$. Find the volume of the cake.
43. Derive the formula for the volume of a right circular cone of height h and radius r using an appropriate solid of revolution.
44. Derive the equation for the volume of a sphere of radius r using the shell method.
45. **Equivalence of the washer and shell methods for finding volume** Let f be differentiable and increasing on the interval $a \leq x \leq b$, with $a > 0$, and suppose that f has a differentiable inverse, f^{-1} . Revolve about the y -axis the region bounded by the graph of f and the lines $x = a$ and $y = f(b)$ to generate a solid. Then the values of the integrals given by the washer and shell methods for the volume are identical.

$$\int_{f(a)}^{f(b)} \pi((f^{-1}(y))^2 - a^2) dy = \int_a^b 2\pi x(f(b) - f(x)) dx.$$

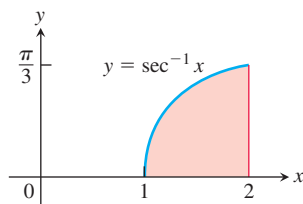
To prove this equality, define

$$W(t) = \int_{f(a)}^{f(t)} \pi((f^{-1}(y))^2 - a^2) dy$$

$$S(t) = \int_a^t 2\pi x(f(t) - f(x)) dx.$$

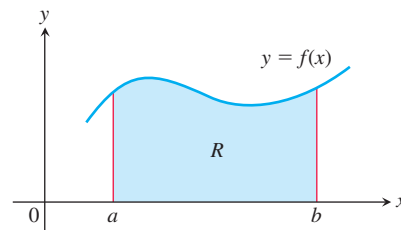
Then show that the functions W and S agree at a point of $[a, b]$ and have identical derivatives on $[a, b]$. As you saw in Section 4.8, Exercise 132, this will guarantee $W(t) = S(t)$ for all t in $[a, b]$. In particular, $W(b) = S(b)$. (Source: "Disks and Shells Revisited" by Walter Carlip, in *American Mathematical Monthly*, Feb. 1991, vol. 98, no. 2, pp. 154–156.)

46. The region between the curve $y = \sec^{-1} x$ and the x -axis from $x = 1$ to $x = 2$ (shown here) is revolved about the y -axis to generate a solid. Find the volume of the solid.



47. Find the volume of the solid generated by revolving the region enclosed by the graphs of $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$ about the y -axis.
48. Find the volume of the solid generated by revolving the region enclosed by the graphs of $y = e^{x/2}$, $y = 1$, and $x = \ln 3$ about the x -axis.

49. Consider the region R bounded by the graphs of $y = f(x) > 0$, $x = a > 0$, and $x = b > a$. If the volume of the solid formed by revolving R about the y -axis is 2π , and the volume formed by revolving R about the line $x = -2$ is 10π , find the area of R .



50. Consider the region R given in Exercise 49. If the area of region R is 1, and the volume of the solid formed by revolving R about the line $x = -3$ is 10π , find the volume of the solid formed by revolving R about the y -axis.

6.3 Arc Length

We know what is meant by the length of a straight-line segment, but without calculus, we have no precise definition of the length of a general winding curve. If the curve is the graph of a continuous function defined over an interval, then we can find the length of the curve using a procedure similar to that we used for defining the area between the curve and the x -axis. We divide the curve into many pieces, and we approximate each piece by a straight-line segment. The sum of the lengths of these segments is an approximation to the total curve length that we seek. The total length of the curve is the limiting value of these approximations as the number of segments goes to infinity.

Length of a Curve $y = f(x)$

Suppose the curve whose length we want to find is the graph of the function $y = f(x)$ from $x = a$ to $x = b$. In order to derive an integral formula for the length of the curve, we assume that f has a continuous derivative at every point of $[a, b]$. Such a function is called **smooth**, and its graph is a **smooth curve** because it does not have any breaks, corners, or cusps.

We partition the interval $[a, b]$ into n subintervals with

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

If $y_k = f(x_k)$, then the corresponding point $P_k(x_k, y_k)$ lies on the curve. Next we connect successive points P_{k-1} and P_k with straight-line segments that, taken together, form a polygonal path whose length approximates the length of the curve (Figure 6.22). If we set $\Delta x_k = x_k - x_{k-1}$ and $\Delta y_k = y_k - y_{k-1}$, then a representative line segment in the path has length

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

(see Figure 6.23), so the length of the curve is approximated by the sum

$$\sum_{k=1}^n L_k = \sum_{k=1}^n \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \quad (1)$$

We expect the approximation to improve as the partition of $[a, b]$ becomes finer. In order to evaluate this limit, we use the Mean Value Theorem, which tells us that there is a point c_k , with $x_{k-1} < c_k < x_k$, such that

$$\Delta y_k = f'(c_k) \Delta x_k.$$

FIGURE 6.22 The length of the polygonal path $P_0P_1P_2\cdots P_n$ approximates the length of the curve $y = f(x)$ from point A to point B .

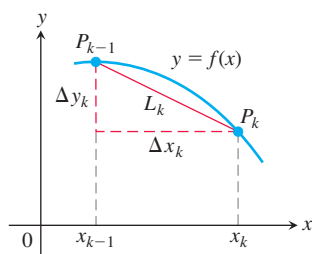
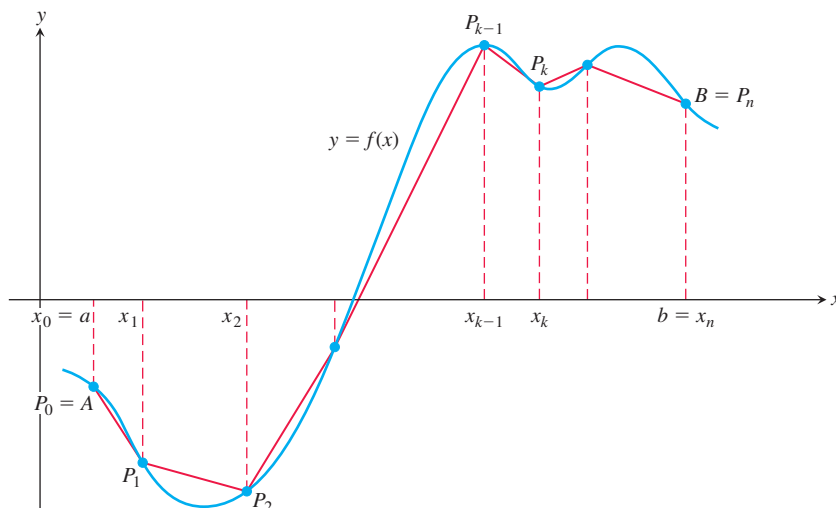


FIGURE 6.23 The arc $P_{k-1}P_k$ of the curve $y = f(x)$ is approximated by the straight-line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

When this is substituted for Δy_k , the sums in Equation (1) take the form

$$\sum_{k=1}^n L_k = \sum_{k=1}^n \sqrt{(\Delta x_k)^2 + (f'(c_k)\Delta x_k)^2} = \sum_{k=1}^n \sqrt{1 + [f'(c_k)]^2} \Delta x_k. \quad (2)$$

This is a Riemann sum whose limit we can evaluate. Because $\sqrt{1 + [f'(x)]^2}$ is continuous on $[a, b]$, the limit of the Riemann sum on the right-hand side of Equation (2) exists and has the value

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n L_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n \sqrt{1 + [f'(c_k)]^2} \Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

We define the length of the curve to be this integral.

DEFINITION If f' is continuous on $[a, b]$, then the **length (arc length)** of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \quad (3)$$

EXAMPLE 1 Find the length of the curve shown in Figure 6.24, which is the graph of the function

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \quad 0 \leq x \leq 1.$$

Solution We use Equation (3) with $a = 0$, $b = 1$, and

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1 \quad \text{If } x = 1, \text{ then } y \approx 0.89.$$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = (2\sqrt{2}x^{1/2})^2 = 8x.$$

The length of the curve over $x = 0$ to $x = 1$ is

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^1 \sqrt{1 + 8x} dx && \text{Eq. (3) with } a = 0, b = 1 \\ &= \frac{2}{3} \cdot \frac{1}{8} (1 + 8x)^{3/2} \Big|_0^1 = \frac{13}{6} \approx 2.17. && \text{Let } u = 1 + 8x, \text{ integrate, and replace } u \text{ by } 1 + 8x. \end{aligned}$$

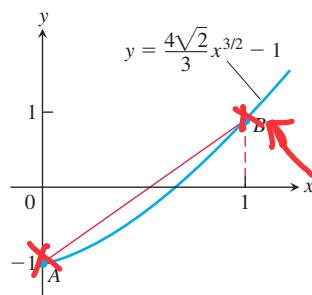


FIGURE 6.24 The length of the curve is slightly larger than the length of the line segment joining points A and B (Example 1).

Notice that the length of the curve is slightly larger than the length of the straight-line segment joining the points $A = (0, -1)$ and $B = (1, 4\sqrt{2}/3 - 1)$ on the curve (see Figure 6.24):

$$2.17 > \sqrt{1^2 + (1.89)^2} \approx 2.14. \quad \text{Decimal approximations}$$

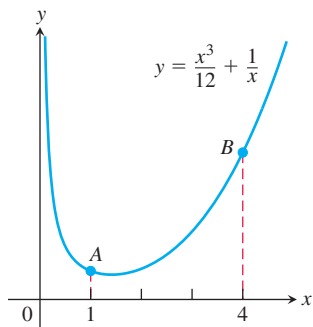


FIGURE 6.25 The curve in Example 2, where $A = (1, 13/12)$ and $B = (4, 67/12)$.

EXAMPLE 2 Find the length of the graph of

$$f(x) = \frac{x^3}{12} + \frac{1}{x}, \quad 1 \leq x \leq 4.$$

Solution A graph of the function is shown in Figure 6.25. To use Equation (3), we find

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

so

$$\begin{aligned} 1 + [f'(x)]^2 &= 1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2 = 1 + \left(\frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}\right) \\ &= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2 \end{aligned}$$

and

$$\sqrt{1 + [f'(x)]^2} = \sqrt{\left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2} = \left|\frac{x^2}{4} + \frac{1}{x^2}\right| = \frac{x^2}{4} + \frac{1}{x^2}. \quad \frac{x^2}{4} + \frac{1}{x^2} > 0$$

The length of the graph over $[1, 4]$ is

$$\begin{aligned} L &= \int_1^4 \sqrt{1 + [f'(x)]^2} dx = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx \\ &= \left[\frac{x^3}{12} - \frac{1}{x}\right]_1^4 = \left(\frac{64}{12} - \frac{1}{4}\right) - \left(\frac{1}{12} - 1\right) = \frac{72}{12} = 6. \end{aligned}$$

EXAMPLE 3 Find the length of the curve

$$y = \frac{1}{2}(e^x + e^{-x}), \quad 0 \leq x \leq 2.$$

Solution We use Equation (3) with $a = 0$, $b = 2$, and

$$y = \frac{1}{2}(e^x + e^{-x})$$

$$\frac{dy}{dx} = \frac{1}{2}(e^x - e^{-x})$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}(e^{2x} - 2 + e^{-2x})$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}(e^{2x} + 2 + e^{-2x}) = \left[\frac{1}{2}(e^x + e^{-x})\right]^2$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\left[\frac{1}{2}(e^x + e^{-x})\right]^2} = \left|\frac{1}{2}(e^x + e^{-x})\right| = \frac{1}{2}(e^x + e^{-x}). \quad \frac{1}{2}(e^x + e^{-x}) > 0$$

The length of the curve from $x = 0$ to $x = 2$ is

$$\begin{aligned} L &= \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^2 \frac{1}{2}(e^x + e^{-x}) dx \\ &= \frac{1}{2} \left[e^x - e^{-x} \right]_0^2 = \frac{1}{2}((e^2 - e^{-2}) - (1 - 1)) \approx 3.63. \end{aligned}$$

Eq. (3) with $a = 0$, $b = 2$

Dealing with Discontinuities in dy/dx

Even if the derivative dy/dx does not exist at some point on a curve, it is possible that dx/dy could exist. This can happen, for example, when a curve has a vertical tangent. In this case, we may be able to find the curve's length by expressing x as a function of y and applying the following analogue of Equation (3).

Formula for the Length of $x = g(y)$, $c \leq y \leq d$

If g' is continuous on $[c, d]$, the length of the curve $x = g(y)$ from $A = (g(c), c)$ to $B = (g(d), d)$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_c^d \sqrt{1 + [g'(y)]^2} dy. \quad (4)$$

EXAMPLE 4 Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.

Solution The derivative

$$\frac{dy}{dx} = \frac{2}{3} \left(\frac{x}{2}\right)^{-1/3} \left(\frac{1}{2}\right) = \frac{1}{3} \left(\frac{2}{x}\right)^{1/3}$$

is not defined at $x = 0$, so we cannot find the curve's length with Equation (3).

We therefore rewrite the equation to express x in terms of y :

$$y = \left(\frac{x}{2}\right)^{2/3}$$

$$y^{3/2} = \frac{x}{2} \quad \text{Raise both sides to the power } 3/2.$$

$$x = 2y^{3/2}. \quad \text{Solve for } x.$$

From this we see that the curve whose length we want is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (see Figure 6.26).

The derivative

$$\frac{dx}{dy} = 2 \left(\frac{3}{2}\right) y^{1/2} = 3y^{1/2}$$

is continuous on $[0, 1]$. We may therefore use Equation (4) to find the curve's length:

$$\begin{aligned} L &= \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_0^1 \sqrt{1 + 9y} dy && \text{Eq. (4) with } c = 0, d = 1 \\ &= \frac{1}{9} \cdot \frac{2}{3} (1 + 9y)^{3/2} \Big|_0^1 && \text{Let } u = 1 + 9y, du/9 = dy, \\ &= \frac{2}{27} (10\sqrt{10} - 1) \approx 2.27. && \text{integrate, and substitute back.} \end{aligned}$$

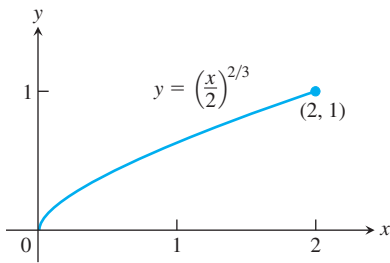


FIGURE 6.26 The graph of $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$ is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Example 4).

~~The Differential Formula for Arc Length~~

If $y = f(x)$ and if f' is continuous on $[a, b]$, then by the Fundamental Theorem of Calculus, we can define a new function

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} dt. \quad (5)$$

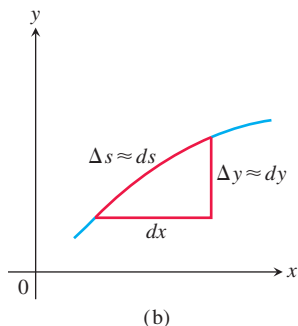
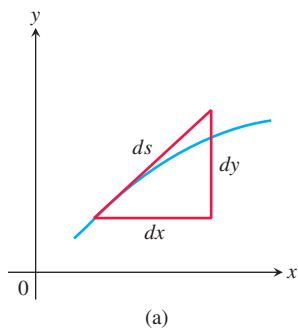


FIGURE 6.27 Diagrams for remembering the equation $ds = \sqrt{dx^2 + dy^2}$.

From Equation (3) and Figure 6.22, we see that this function $s(x)$ is continuous and measures the length along the curve $y = f(x)$ from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$ for each $x \in [a, b]$. The function s is called the **arc length function** for $y = f(x)$. From the Fundamental Theorem, the function s is differentiable on (a, b) and

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Then the differential of arc length is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \quad (6)$$

A useful way to remember Equation (6) is to write

$$ds = \sqrt{dx^2 + dy^2}, \quad (7)$$

which can be integrated between appropriate limits to give the total length of a curve. From this point of view, all the arc length formulas are simply different expressions for the equation $L = \int ds$. Figure 6.27a, which corresponds to Equation (7), can be thought of as a simplified approximation of Figure 6.27b. That is, ds is approximately equal to the exact arc length Δs .

EXAMPLE 5 Find the arc length function for the curve in Example 2, taking $A = (1, 13/12)$ as the starting point (see Figure 6.25).

Solution In the solution to Example 2, we found that

$$\sqrt{1 + [f'(x)]^2} = \frac{x^2}{4} + \frac{1}{x^2}.$$

Therefore the arc length function is given by

$$\begin{aligned} s(x) &= \int_1^x \sqrt{1 + [f'(t)]^2} dt = \int_1^x \left(\frac{t^2}{4} + \frac{1}{t^2} \right) dt \\ &= \left[\frac{t^3}{12} - \frac{1}{t} \right]_1^x = \frac{x^3}{12} - \frac{1}{x} + \frac{11}{12}. \end{aligned}$$

To compute the arc length along the curve from $A = (1, 13/12)$ to $B = (4, 67/12)$, for instance, we simply calculate

$$s(4) = \frac{4^3}{12} - \frac{1}{4} + \frac{11}{12} = 6.$$

This is the same result we obtained in Example 2. ■

EXERCISES 6.3

Finding Lengths of Curves

Find the lengths of the curves in Exercises 1–16. If you have graphing software, you may want to graph these curves to see what they look like.

1. $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$
2. $y = x^{3/2}$ from $x = 0$ to $x = 4$
3. $x = (y^3/3) + 1/(4y)$ from $y = 1$ to $y = 3$

4. $x = (y^{3/2}/3) - y^{1/2}$ from $y = 1$ to $y = 9$
5. $x = (y^4/4) + 1/(8y^2)$ from $y = 1$ to $y = 2$
6. $x = (y^3/6) + 1/(2y)$ from $y = 2$ to $y = 3$
7. $y = (3/4)x^{4/3} - (3/8)x^{2/3} + 5$, $1 \leq x \leq 8$
8. $y = (x^3/3) + x^2 + x + 1/(4x + 4)$, $0 \leq x \leq 2$

9. $y = \ln x - \frac{x^2}{8}$ from $x = 1$ to $x = 2$

10. $y = \frac{x^2}{2} - \frac{\ln x}{4}$ from $x = 1$ to $x = 3$

11. $y = \frac{x^3}{3} + \frac{1}{4x}$, $1 \leq x \leq 3$

12. $y = \frac{x^5}{5} + \frac{1}{12x^3}$, $\frac{1}{2} \leq x \leq 1$

13. $y = \frac{3}{2}x^{2/3} + 1$, $\frac{1}{8} \leq x \leq 1$

14. $y = \frac{1}{2}(e^x + e^{-x})$, $-1 \leq x \leq 1$

15. $x = \int_0^y \sqrt{\sec^4 t - 1} dt$, $-\pi/4 \leq y \leq \pi/4$

16. $y = \int_{-2}^x \sqrt{3t^4 - 1} dt$, $-2 \leq x \leq -1$

T Finding Integrals for Lengths of Curves

In Exercises 17–24, do the following.

- Set up an integral for the length of the curve.
- Graph the curve to see what it looks like.
- Use your grapher's or computer's integral evaluator to find the curve's length numerically.

17. $y = x^2$, $-1 \leq x \leq 2$

18. $y = \tan x$, $-\pi/3 \leq x \leq 0$

19. $x = \sin y$, $0 \leq y \leq \pi$

20. $x = \sqrt{1 - y^2}$, $-1/2 \leq y \leq 1/2$

21. $y^2 + 2y = 2x + 1$ from $(-1, -1)$ to $(7, 3)$

22. $y = \sin x - x \cos x$, $0 \leq x \leq \pi$

23. $y = \int_0^x \tan t dt$, $0 \leq x \leq \pi/6$

24. $x = \int_0^y \sqrt{\sec^2 t - 1} dt$, $-\pi/3 \leq y \leq \pi/4$

Theory and Examples

25. a. Find a curve with a positive derivative through the point $(1, 1)$ whose length integral (Equation 3) is

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}} dx.$$

- b. How many such curves are there? Give reasons for your answer.

26. a. Find a curve with a positive derivative through the point $(0, 1)$ whose length integral (Equation 4) is

$$L = \int_1^2 \sqrt{1 + \frac{1}{y^4}} dy.$$

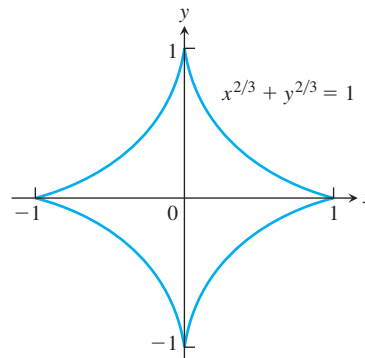
- b. How many such curves are there? Give reasons for your answer.

27. Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t} dt$$

from $x = 0$ to $x = \pi/4$.

28. **The length of an astroid** The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of a family of curves called *astroids* (not “asteroids”) because of their starlike appearance (see the accompanying figure). Find the length of this particular astroid by finding the length of half the first-quadrant portion, $y = (1 - x^{2/3})^{3/2}$, $\sqrt{2}/4 \leq x \leq 1$, and multiplying by 8.



29. **Length of a line segment** Use the arc length formula (Equation 3) to find the length of the line segment $y = 3 - 2x$, $0 \leq x \leq 2$. Check your answer by finding the length of the segment as the hypotenuse of a right triangle.
30. **Circumference of a circle** Set up an integral to find the circumference of a circle of radius r centered at the origin. You will learn how to evaluate the integral in Section 8.3.
31. If $9x^2 = y(y - 3)^2$, show that

$$ds^2 = \frac{(y + 1)^2}{4y} dy^2.$$

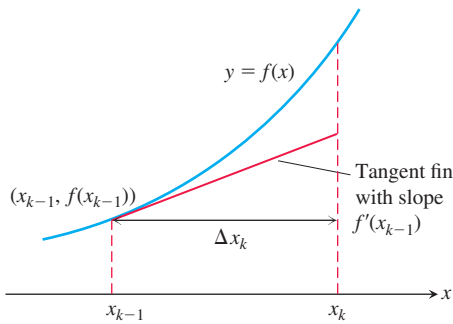
32. If $4x^2 - y^2 = 64$, show that

$$ds^2 = \frac{4}{y^2} (5x^2 - 16) dx^2.$$

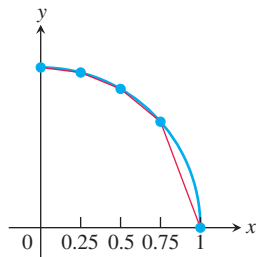
33. Is there a smooth (continuously differentiable) curve $y = f(x)$ whose length over the interval $0 \leq x \leq a$ is always $\sqrt{2}a$? Give reasons for your answer.
34. **Using tangent fins to derive the length formula for curves** Assume that f is smooth on $[a, b]$ and partition the interval $[a, b]$ in the usual way. In each subinterval $[x_{k-1}, x_k]$, construct the *tangent fin* at the point $(x_{k-1}, f(x_{k-1}))$, as shown in the accompanying figure.
- Show that the length of the k th tangent fin over the interval $[x_{k-1}, x_k]$ equals $\sqrt{(\Delta x_k)^2 + (f'(x_{k-1}) \Delta x_k)^2}$.
 - Show that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n (\text{length of } k\text{th tangent fin}) = \int_a^b \sqrt{1 + (f'(x))^2} dx,$$

which is the length L of the curve $y = f(x)$ from a to b .



35. Approximate the arc length of one-quarter of the unit circle (which is $\pi/2$) by computing the length of the polygonal approximation with $n = 4$ segments (see accompanying figure).



36. **Distance between two points** Assume that the two points (x_1, y_1) and (x_2, y_2) lie on the graph of the straight line $y = mx + b$. Use the arc length formula (Equation 3) to find the distance between the two points.

37. Find the arc length function for the graph of $f(x) = 2x^{3/2}$ using $(0, 0)$ as the starting point. What is the length of the curve from $(0, 0)$ to $(1, 2)$?
38. Find the arc length function for the curve in Exercise 8, using $(0, 1/4)$ as the starting point. What is the length of the curve from $(0, 1/4)$ to $(1, 59/24)$?

COMPUTER EXPLORATIONS

In Exercises 39–44, use a CAS to perform the following steps for the given graph of the function over the closed interval.

- Plot the curve together with the polygonal path approximations for $n = 2, 4, 8$ partition points over the interval. (See Figure 6.22.)
 - Find the corresponding approximation to the length of the curve by summing the lengths of the line segments.
 - Evaluate the length of the curve using an integral. Compare your approximations for $n = 2, 4, 8$ with the actual length given by the integral. How does the actual length compare with the approximations as n increases? Explain your answer.
39. $f(x) = \sqrt{1 - x^2}$, $-1 \leq x \leq 1$
40. $f(x) = x^{1/3} + x^{2/3}$, $0 \leq x \leq 2$
41. $f(x) = \sin(\pi x^2)$, $0 \leq x \leq \sqrt{2}$
42. $f(x) = x^2 \cos x$, $0 \leq x \leq \pi$
43. $f(x) = \frac{x-1}{4x^2+1}$, $-\frac{1}{2} \leq x \leq 1$
44. $f(x) = x^3 - x^2$, $-1 \leq x \leq 1$

6.4 Areas of Surfaces of Revolution

When you jump rope, the rope sweeps out a surface in the space around you similar to what is called a *surface of revolution*. The surface surrounds a volume of revolution, and many applications require that we know the area of the surface rather than the volume it encloses. In this section we define areas of surfaces of revolution. More general surfaces are treated in Chapter 15.

Defining Surface Area

If you revolve a region in the plane that is bounded by the graph of a function over an interval, it sweeps out a solid of revolution, as we saw earlier in the chapter. However, if you revolve only the bounding curve itself, it does not sweep out any interior volume but rather a surface that surrounds the solid and forms part of its boundary. Just as we were interested in defining and finding the length of a curve in the last section, we are now interested in defining and finding the area of a surface generated by revolving a curve about an axis.

Before considering general curves, we begin by rotating horizontal and slanted line segments about the x -axis. If we rotate the horizontal line segment AB having length Δx about the x -axis (Figure 6.28a), we generate a cylinder with surface area $2\pi y \Delta x$. This area is the same as that of a rectangle with side lengths Δx and $2\pi y$ (Figure 6.28b). The length $2\pi y$ is the circumference of the circle of radius y generated by rotating the point (x, y) on the line AB about the x -axis.

Suppose the line segment AB has length L and is slanted rather than horizontal. Now when AB is rotated about the x -axis, it generates a frustum of a cone (Figure 6.29a). From

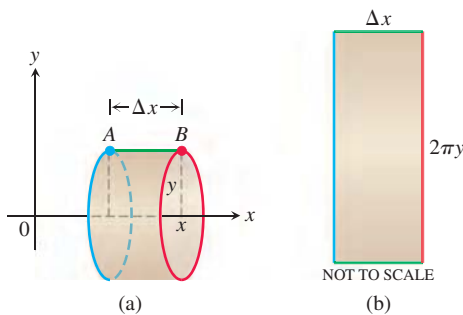


FIGURE 6.28 (a) A cylindrical surface generated by rotating the horizontal line segment AB of length Δx about the x -axis has area $2\pi y \Delta x$. (b) The cut and rolled-out cylindrical surface as a rectangle.

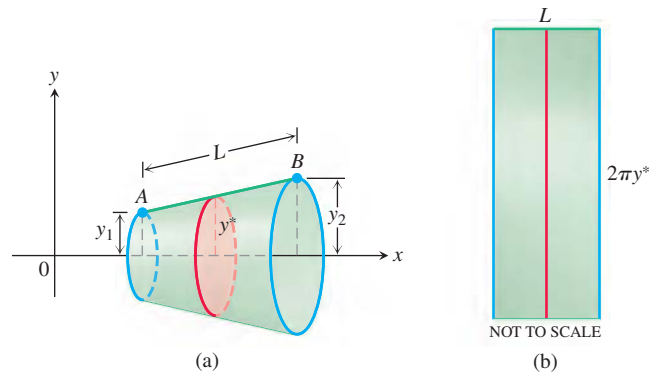


FIGURE 6.29 (a) The frustum of a cone generated by rotating the slanted line segment AB of length L about the x -axis has area $2\pi y^* L$. (b) The area of the rectangle for $y^* = \frac{y_1 + y_2}{2}$, the average height of AB above the x -axis.

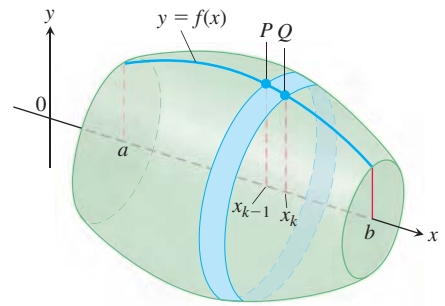


FIGURE 6.30 The surface generated by revolving the graph of a nonnegative function $y = f(x)$, $a \leq x \leq b$, about the x -axis. The surface is a union of bands like the one swept out by the arc PQ .

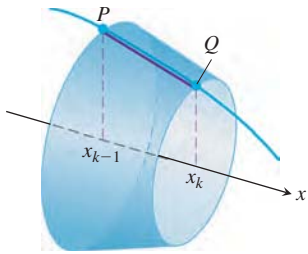


FIGURE 6.31 The line segment joining P and Q sweeps out a frustum of a cone.

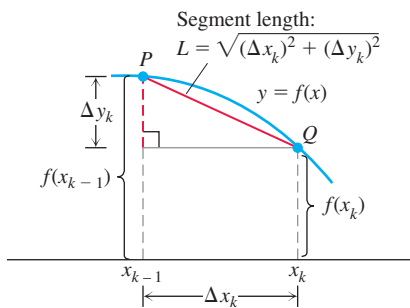


FIGURE 6.32 Dimensions associated with the arc and line segment PQ .

classical geometry, the surface area of this frustum is $2\pi y^* L$, where $y^* = (y_1 + y_2)/2$ is the average height of the slanted segment AB above the x -axis. This surface area is the same as that of a rectangle with side lengths L and $2\pi y^*$ (Figure 6.29b).

Let's build on these geometric principles to define the area of a surface swept out by revolving more general curves about the x -axis. Suppose we want to find the area of the surface swept out by revolving the graph of a nonnegative continuous function $y = f(x)$, $a \leq x \leq b$, about the x -axis. We partition the closed interval $[a, b]$ in the usual way and use the points in the partition to subdivide the graph into short arcs. Figure 6.30 shows a typical arc PQ and the band it sweeps out as part of the graph of f .

As the arc PQ revolves about the x -axis, the line segment joining P and Q sweeps out a frustum of a cone whose axis lies along the x -axis (Figure 6.31). The surface area of this frustum approximates the surface area of the band swept out by the arc PQ . The surface area of the frustum of the cone shown in Figure 6.31 is $2\pi y^* L$, where y^* is the average height of the line segment joining P and Q , and L is its length (just as before). Since $f \geq 0$, from Figure 6.32 we see that the average height of the line segment is $y^* = (f(x_{k-1}) + f(x_k))/2$, and the slant length is $L = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$. Therefore,

$$\begin{aligned} \text{Frustum surface area} &= 2\pi \cdot \frac{f(x_{k-1}) + f(x_k)}{2} \cdot \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \end{aligned}$$

The area of the original surface, being the sum of the areas of the bands swept out by arcs like arc PQ , is approximated by the frustum area sum

$$\sum_{k=1}^n \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \quad (1)$$

We expect the approximation to improve as the partition of $[a, b]$ becomes finer. To find the limit, we first need to find an appropriate substitution for Δy_k . If the function f is differentiable, then by the Mean Value Theorem, there is a point $(c_k, f(c_k))$ on the curve between P and Q where the tangent is parallel to the segment PQ (Figure 6.33). At this point,

$$\begin{aligned} f'(c_k) &= \frac{\Delta y_k}{\Delta x_k}, \\ \Delta y_k &= f'(c_k) \Delta x_k. \end{aligned}$$

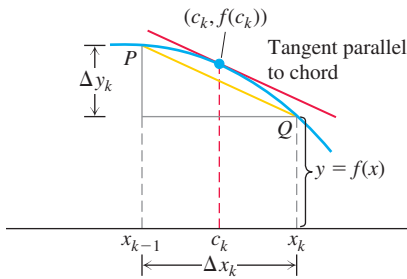


FIGURE 6.33 If f is smooth, the Mean Value Theorem guarantees the existence of a point c_k where the tangent is parallel to segment PQ .

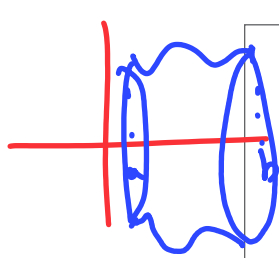
With this substitution for Δy_k , the sums in Equation (1) take the form

$$\begin{aligned} \sum_{k=1}^n \pi(f(x_{k-1}) + f(x_k)) \sqrt{(\Delta x_k)^2 + (f'(c_k) \Delta x_k)^2} \\ = \sum_{k=1}^n \pi(f(x_{k-1}) + f(x_k)) \sqrt{1 + (f'(c_k))^2} \Delta x_k. \end{aligned} \quad (2)$$

These sums are not the Riemann sums of any function because the points x_{k-1} , x_k , and c_k are not the same. However, the points x_{k-1} , x_k , and c_k are very close to each other, and so we expect (and it can be proved) that as the norm of the partition of $[a, b]$ goes to zero, the sums in Equation (2) converge to the integral

$$\int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx.$$

We therefore define this integral to be the area of the surface swept out by the graph of f from a to b .



DEFINITION If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area of the surface** generated by revolving the graph of $y = f(x)$ about the x -axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx. \quad (3)$$

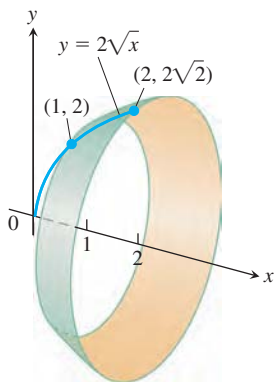


FIGURE 6.34 In Example 1 we calculate the area of this surface.

Note that the square root in Equation (3) is similar to the one that appears in the formula for the arc length of the generating curve in Equation (6) of Section 6.3.

EXAMPLE 1 Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the x -axis (Figure 6.34).

Solution We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{Eq. (3)}$$

with

$$a = 1, \quad b = 2, \quad y = 2\sqrt{x}, \quad \frac{dy}{dx} = \frac{1}{\sqrt{x}}.$$

First, we perform some algebraic manipulation on the radical in the integrand to transform it into an expression that is easier to integrate.

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2} = \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}$$

With these substitutions, we have

$$\begin{aligned} S &= \int_1^2 2\pi \cdot 2\sqrt{x} \cdot \frac{\sqrt{x+1}}{\sqrt{x}} dx = 4\pi \int_1^2 \sqrt{x+1} dx \\ &= 4\pi \cdot \frac{2}{3} (x+1)^{3/2} \Big|_1^2 = \frac{8\pi}{3} (3\sqrt{3} - 2\sqrt{2}). \end{aligned}$$

Revolution About the y-Axis

For revolution about the y-axis, we interchange x and y in Equation (3).

Surface Area for Revolution About the y-Axis

If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the graph of $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2} dy. \quad (4)$$

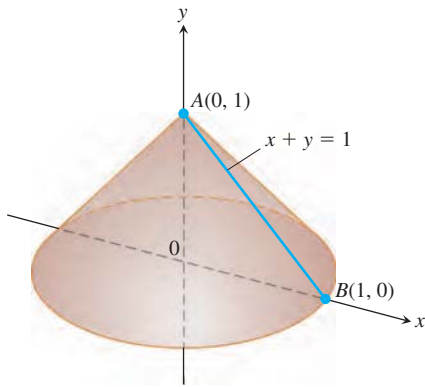


FIGURE 6.35 Revolving line segment AB about the y-axis generates a cone whose lateral surface area we can now calculate in two different ways (Example 2).

EXAMPLE 2 The line segment $x = 1 - y$, $0 \leq y \leq 1$, is revolved about the y-axis to generate the cone in Figure 6.35. Find its lateral surface area (which excludes the base area).

Solution Here we have a calculation we can check with a formula from geometry:

$$\text{Lateral surface area} = \frac{\text{base circumference}}{2} \times \text{slant height} = \pi\sqrt{2}.$$

To see how Equation (4) gives the same result, we take

$$c = 0, \quad d = 1, \quad x = 1 - y, \quad \frac{dx}{dy} = -1,$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2}$$

and calculate

$$\begin{aligned} S &= \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_0^1 2\pi(1 - y)\sqrt{2} dy \\ &= 2\pi\sqrt{2} \left[y - \frac{y^2}{2} \right]_0^1 = 2\pi\sqrt{2} \left(1 - \frac{1}{2} \right) = \pi\sqrt{2}. \end{aligned}$$

EXERCISES 6.4

Finding Integrals for Surface Area

In Exercises 1–8:

- Set up an integral for the area of the surface generated by revolving the given curve about the indicated axis.
 - Graph the curve to see what it looks like. If you can, graph the surface too.
 - Use your utility's integral evaluator to find the surface's area numerically.
- $y = \tan x$, $0 \leq x \leq \pi/4$; x-axis
 - $y = x^2$, $0 \leq x \leq 2$; x-axis
 - $xy = 1$, $1 \leq y \leq 2$; y-axis
 - $x = \sin y$, $0 \leq y \leq \pi$; y-axis
 - $x^{1/2} + y^{1/2} = 3$ from $(4, 1)$ to $(1, 4)$; x-axis
 - $y + 2\sqrt{y} = x$, $1 \leq y \leq 2$; y-axis

$$7. x = \int_0^y \tan t \, dt, \quad 0 \leq y \leq \pi/3; \quad \text{y-axis}$$

$$8. y = \int_1^x \sqrt{t^2 - 1} \, dt, \quad 1 \leq x \leq \sqrt{5}; \quad \text{x-axis}$$

Finding Surface Area

- Find the lateral (side) surface area of the cone generated by revolving the line segment $y = x/2$, $0 \leq x \leq 4$, about the x-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}.$$

- Find the lateral surface area of the cone generated by revolving the line segment $y = x/2$, $0 \leq x \leq 4$, about the y-axis. Check your answer with the geometry formula given in Exercise 9.

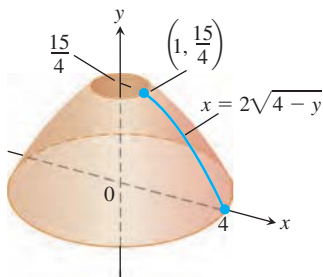
11. Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2)$, $1 \leq x \leq 3$, about the x -axis. Check your result with the geometry formula

Frustum surface area = $\pi(y_1 + y_2) \times \text{slant height}$.

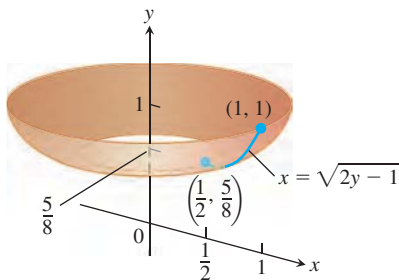
12. Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2)$, $1 \leq x \leq 3$, about the y -axis. Check your result with the geometry formula given in Exercise 11.

Find the areas of the surfaces generated by revolving the curves in Exercises 13–23 about the indicated axes. If you have a grapher, you may want to graph these curves to see what they look like.

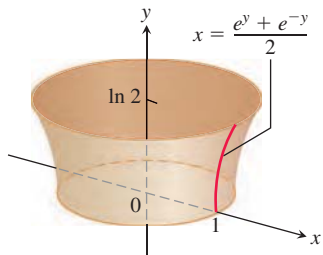
13. $y = x^3/9$, $0 \leq x \leq 2$; x -axis
 14. $y = \sqrt{x}$, $3/4 \leq x \leq 15/4$; x -axis
 15. $y = \sqrt{2x - x^2}$, $0.5 \leq x \leq 1.5$; x -axis
 16. $y = \sqrt{x + 1}$, $1 \leq x \leq 5$; x -axis
 17. $x = y^3/3$, $0 \leq y \leq 1$; y -axis
 18. $x = (1/3)y^{3/2} - y^{1/2}$, $1 \leq y \leq 3$; y -axis
 19. $x = 2\sqrt{4 - y}$, $0 \leq y \leq 15/4$; y -axis



20. $x = \sqrt{2y - 1}$, $5/8 \leq y \leq 1$; y -axis



21. $x = (e^y + e^{-y})/2$, $0 \leq y \leq \ln 2$; y -axis



22. $y = (1/3)(x^2 + 2)^{3/2}$, $0 \leq x \leq \sqrt{2}$; y -axis (Hint: Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dx , and evaluate the integral $S = \int 2\pi x ds$ with appropriate limits.)

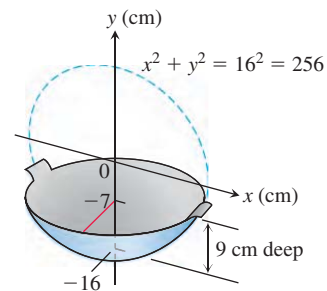
23. $x = (y^4/4) + 1/(8y^2)$, $1 \leq y \leq 2$; x -axis (Hint: Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dy , and evaluate the integral $S = \int 2\pi y ds$ with appropriate limits.)

24. Write an integral for the area of the surface generated by revolving the curve $y = \cos x$, $-\pi/2 \leq x \leq \pi/2$, about the x -axis. In Section 8.3 we will see how to evaluate such integrals.

25. **Testing the new definition** Show that the surface area of a sphere of radius a is still $4\pi a^2$ by using Equation (3) to find the area of the surface generated by revolving the curve $y = \sqrt{a^2 - x^2}$, $-a \leq x \leq a$, about the x -axis.

26. **Testing the new definition** The lateral (side) surface area of a cone of height h and base radius r should be $\pi r \sqrt{r^2 + h^2}$, the semiperimeter of the base times the slant height. Show that this is still the case by finding the area of the surface generated by revolving the line segment $y = (r/h)x$, $0 \leq x \leq h$, about the x -axis.

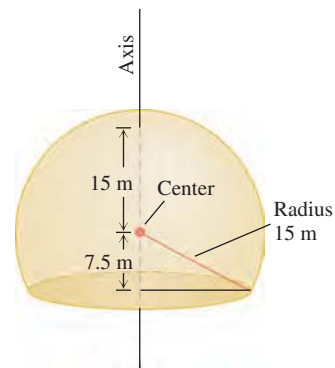
- T** 27. **Enameling woks** Your company decided to put out a deluxe version of a wok you designed. The plan is to coat it inside with white enamel and outside with blue enamel. Each enamel will be sprayed on 0.5 mm thick before baking. (See accompanying figure.) Your manufacturing department wants to know how much enamel to have on hand for a production run of 5000 woks. What do you tell them? (Neglect waste and unused material and give your answer in liters. Remember that $1 \text{ cm}^3 = 1 \text{ mL}$, so $1 \text{ L} = 1000 \text{ cm}^3$.)



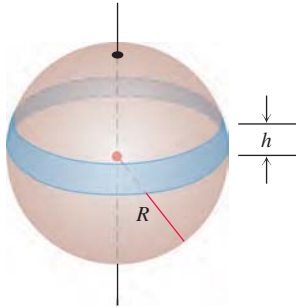
28. Here is a schematic drawing of the 30-m dome used by the U.S. National Weather Service to house radar in Bozeman, Montana.

- a. How much outside surface is there to paint (not counting the bottom)?

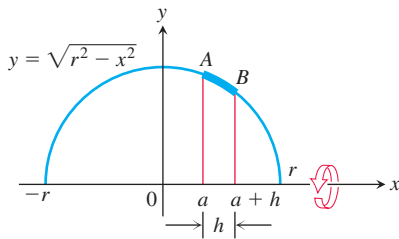
- T** b. Express the answer to the nearest square meter.



29. The shaded band shown here is cut from a sphere of radius R by parallel planes h units apart. Show that the surface area of the band is $2\pi Rh$.



30. **Slicing bread** Did you know that if you cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle $y = \sqrt{r^2 - x^2}$ shown here is revolved about the x -axis to generate a sphere. Let AB be an arc of the semicircle that lies above an interval of length h on the x -axis. Show that the area swept out by AB does not depend on the location of the interval. (It does depend on the length of the interval.)

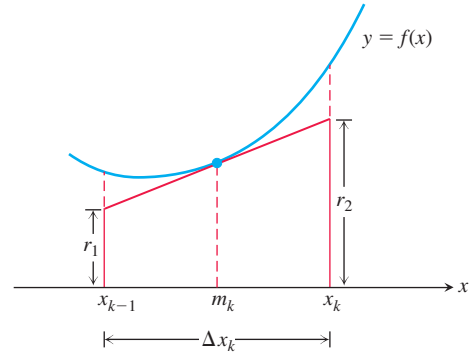


31. **An alternative derivation of the surface area formula** Assume f is smooth on $[a, b]$ and partition $[a, b]$ in the usual way. In the k th subinterval $[x_{k-1}, x_k]$, construct the tangent line to the curve at the midpoint $m_k = (x_{k-1} + x_k)/2$, as in the accompanying figure.

a. Show that

$$r_1 = f(m_k) - f'(m_k) \frac{\Delta x_k}{2} \quad \text{and} \quad r_2 = f(m_k) + f'(m_k) \frac{\Delta x_k}{2}.$$

- b. Show that the length L_k of the tangent line segment in the k th subinterval is $L_k = \sqrt{(\Delta x_k)^2 + (f'(m_k) \Delta x_k)^2}$.

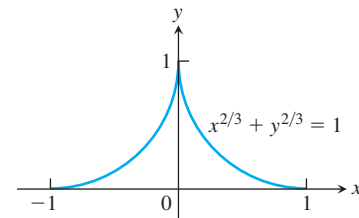


- c. Show that the lateral surface area of the frustum of the cone swept out by the tangent line segment as it revolves about the x -axis is $2\pi f(m_k) \sqrt{1 + (f'(m_k))^2} \Delta x_k$.
- d. Show that the area of the surface generated by revolving $y = f(x)$ about the x -axis over $[a, b]$ is

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\begin{array}{l} \text{lateral surface area} \\ \text{of } k\text{th frustum} \end{array} \right) = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx.$$

32. **The surface of an astroid** Find the area of the surface generated by revolving about the x -axis the portion of the astroid $x^{2/3} + y^{2/3} = 1$ shown in the accompanying figure.

(Hint: Revolve the first-quadrant portion $y = (1 - x^{2/3})^{3/2}$, $0 \leq x \leq 1$, about the x -axis and double your result.)



6.5 Work and Fluid Forces

In everyday life, *work* means an activity that requires muscular or mental effort. In science, the term refers specifically to a force acting on an object and the object's subsequent displacement. This section shows how to calculate work. The applications run from compressing railroad car springs and emptying subterranean tanks to forcing subatomic particles to collide and lifting satellites into orbit.

Work Done by a Constant Force

When an object moves a distance d along a straight line as a result of being acted on by a force of constant magnitude F in the direction of motion, we define the **work** W done by the force on the object with the formula

$$W = Fd \quad (\text{Constant-force formula for work}). \quad (1)$$

From Equation (1) we see that the unit of work in any system is the unit of force multiplied by the unit of distance. In SI units (SI stands for *Système International*, or International System), the unit of force is a newton (N), the unit of distance is a meter (m), and the unit of work is a newton-meter (N · m). This combination appears so often, it has

Joules

The joule, abbreviated J, is named after the English physicist James Prescott Joule (1818–1889). The defining equation is

$$1 \text{ joule} = (1 \text{ newton})(1 \text{ meter}).$$

In symbols, $1 \text{ J} = 1 \text{ N} \cdot \text{m}$.

a special name, the **joule** (J). Taking gravitational acceleration at sea level to be 9.8 m/s^2 , to lift one kilogram one meter requires work of 9.8 joules. This is seen by multiplying the force of 9.8 newtons exerted on one kilogram by the one-meter distance moved.

EXAMPLE 1 Suppose you jack up the side of a 1000-kg car 35 cm to change a tire. The jack applies a constant vertical force of about 5000 N in lifting the side of the car (but because of the mechanical advantage of the jack, the force you apply to the jack itself is only about 150 N). The total work performed by the jack on the car is $5000 \times 0.35 = 1750 \text{ J}$. ■

Work Done by a Variable Force Along a Line

If the force you apply varies along the way, as it will if you are stretching or compressing a spring, the formula $W = Fd$ has to be replaced by an integral formula that takes the variation in F into account.

Suppose that the force performing the work acts on an object moving along a straight line, which we take to be the x -axis. We assume that the magnitude of the force is a continuous function F of the object's position x . We want to find the work done over the interval from $x = a$ to $x = b$. We partition $[a, b]$ in the usual way and choose an arbitrary point c_k in each subinterval $[x_{k-1}, x_k]$. If the subinterval is short enough, the continuous function F will not vary much from x_{k-1} to x_k . The amount of work done across the interval will be about $F(c_k)$ times the distance Δx_k , the same as it would be if F were constant and we could apply Equation (1). The total work done from a to b is therefore approximated by the Riemann sum

$$\text{Work} \approx \sum_{k=1}^n F(c_k) \Delta x_k.$$

We expect the approximation to improve as the norm of the partition goes to zero, so we define the work done by the force from a to b to be the integral of F from a to b :

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n F(c_k) \Delta x_k = \int_a^b F(x) dx.$$

DEFINITION The **work** done by a variable force $F(x)$ in moving an object along the x -axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x) dx. \quad (2)$$

The units of the integral are joules if F is in newtons and x is in meters. So the work done by a force of $F(x) = 1/x^2$ newtons in moving an object along the x -axis from $x = 1 \text{ m}$ to $x = 10 \text{ m}$ is

$$W = \int_1^{10} \frac{1}{x^2} dx = -\frac{1}{x} \Big|_1^{10} = -\frac{1}{10} + 1 = 0.9 \text{ J}.$$

Hooke's Law for Springs: $F = kx$

One calculation for work arises in finding the work required to stretch or compress a spring. **Hooke's Law** says that the force required to hold a stretched or compressed spring x units from its natural (unstressed) length is proportional to x . In symbols,

$$F = kx. \quad (3)$$

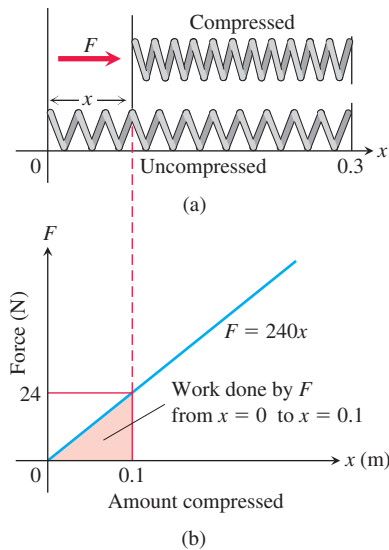


FIGURE 6.36 The force F needed to hold a spring under compression increases linearly as the spring is compressed (Example 2).

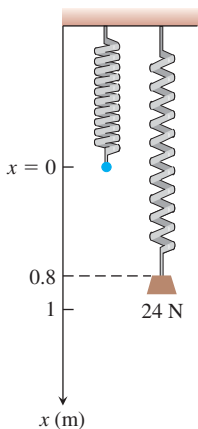


FIGURE 6.37 A 24-N weight stretches this spring 0.8 m beyond its unstressed length (Example 3).

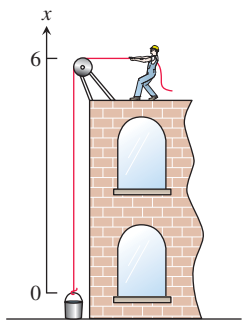


FIGURE 6.38 Lifting the bucket in Example 4.

The constant k , measured in force units per unit length, is a characteristic of the spring, called the **force constant** (or **spring constant**) of the spring. Hooke's Law, Equation (3), gives good results as long as the force doesn't distort the metal in the spring. We assume that the forces in this section are too small to do that.

EXAMPLE 2 Find the work required to compress a spring from its natural length of 30 cm to a length of 20 cm if the force constant is $k = 240$ N/m.

Solution We picture the uncompressed spring laid out along the x -axis with its movable end at the origin and its fixed end at $x = 0.3$ m (Figure 6.36). This enables us to describe the force required to compress the spring from 0 to x with the formula $F = 240x$. To compress the spring from 0 to 0.1 m, the force must increase from

$$F(0) = 240 \cdot 0 = 0 \text{ N} \quad \text{to} \quad F(0.1) = 240 \cdot 0.1 = 24 \text{ N}.$$

The work done by F over this interval is

$$W = \int_0^{0.1} 240x \, dx = 120x^2 \Big|_0^{0.1} = 1.2 \text{ J.} \quad \begin{array}{l} \text{Eq. (2) with} \\ a = 0, b = 0.1, \\ F(x) = 240x \end{array}$$

EXAMPLE 3 A spring has a natural length of 1 m. A force of 24 N holds the spring stretched to a total length of 1.8 m.

- Find the force constant k .
- How much work will it take to stretch the spring from its natural length to a length of 3 m?
- How far will a 45-N force stretch the spring?

Solution

- (a) *The force constant.* We find the force constant from Equation (3). A force of 24 N maintains the spring at a position where it is stretched 0.8 m from its natural length, so

$$\begin{aligned} 24 &= k(0.8) && \text{Eq. (3) with } F = 24, x = 0.8 \\ k &= 24/0.8 = 30 \text{ N/m.} \end{aligned}$$

- (b) *The work to stretch the spring 2 m.* We imagine the unstressed spring hanging along the x -axis with its free end at $x = 0$ (Figure 6.37). The force required to stretch the spring x meters beyond its natural length is the force required to hold the free end of the spring x units from the origin. Hooke's Law with $k = 30$ says that this force is

$$F(x) = 30x.$$

The work done by F on the spring from $x = 0$ m to $x = 2$ m is

$$W = \int_0^2 30x \, dx = 15x^2 \Big|_0^2 = 60 \text{ J.}$$

- (c) *How far will a 45-N force stretch the spring?* We substitute $F = 45$ in the equation $F = 30x$ to find

$$45 = 30x, \quad \text{or} \quad x = 1.5 \text{ m.}$$

A 45-N force will keep the spring stretched 1.5 m beyond its natural length. ■

Lifting Objects and Pumping Liquids from Containers

The work integral is useful for calculating the work done in lifting objects whose weights vary with their elevation.

EXAMPLE 4 A 2-kg bucket is lifted from the ground into the air by pulling in 6 m of rope at a constant speed (Figure 6.38). The rope weighs 0.1 kg/m. How much work was spent lifting the bucket and rope?

Solution The bucket has constant weight, so the work done lifting it alone is

$$\text{weight} \times \text{distance} = 2 \cdot 9.8 \cdot 6 = 117.6 \text{ J.}$$

The weight of the rope varies with the bucket's elevation, because less of it is freely hanging. When the bucket is x m off the ground, the remaining proportion of the rope still being lifted weighs $(0.1 \cdot 9.8) \cdot (6 - x)$ N. So the work in lifting the rope is

$$\begin{aligned} \text{Work on rope} &= \int_0^6 (0.98)(6 - x) dx = \int_0^6 (5.88 - 0.98x) dx \\ &= \left[5.88x - 0.49x^2 \right]_0^6 = 35.28 - 17.64 = 17.64 \text{ J.} \end{aligned}$$

The total work for the bucket and rope combined is

$$117.6 + 17.64 = 135.24 \text{ J.}$$

How much work does it take to pump all or part of the liquid from a container? Engineers often need to know the answer in order to design or choose the right pump, or to compute the cost to transport water or some other liquid from one place to another. To find out how much work is required to pump the liquid, we imagine lifting the liquid out one thin horizontal slab at a time and applying the equation $W = Fd$ to each slab. We then evaluate the integral that this leads to as the slabs become thinner and more numerous.

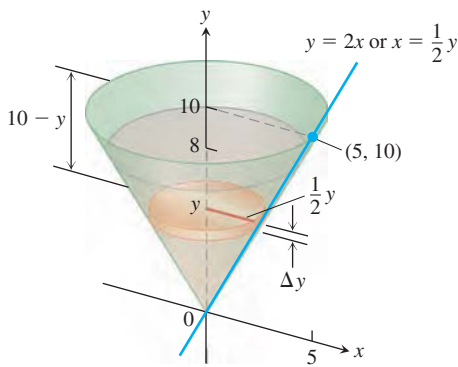


FIGURE 6.39 The olive oil and tank in Example 5.

EXAMPLE 5 The conical tank in Figure 6.39 is filled to within 2 m of the top with olive oil weighing 0.9 g/cm^3 or 8820 N/m^3 . How much work does it take to pump the oil to the rim of the tank?

Solution We imagine the oil divided into thin slabs by planes perpendicular to the y -axis at the points of a partition of the interval $[0, 8]$.

The typical slab between the planes at y and $y + \Delta y$ has a volume of about

$$\Delta V = \pi(\text{radius})^2 (\text{thickness}) = \pi\left(\frac{1}{2}y\right)^2 \Delta y = \frac{\pi}{4}y^2 \Delta y \text{ m}^3.$$

The force $F(y)$ required to lift this slab is equal to its weight,

$$F(y) = 8820 \Delta V = \frac{8820\pi}{4}y^2 \Delta y \text{ N.} \quad \text{Weight} = (\text{weight per unit volume}) \times \text{volume}$$

The distance through which $F(y)$ must act to lift this slab to the level of the rim of the cone is about $(10 - y)$ m, so the work done lifting the slab is about

$$\Delta W = \frac{8820\pi}{4}(10 - y)y^2 \Delta y \text{ J.}$$

Assuming there are n slabs associated with the partition of $[0, 8]$, and that $y = y_k$ denotes the plane associated with the k th slab of thickness Δy_k , we can approximate the work done lifting all of the slabs with the Riemann sum

$$W \approx \sum_{k=1}^n \frac{8820\pi}{4}(10 - y_k)y_k^2 \Delta y_k \text{ J.}$$

The work of pumping the oil to the rim is the limit of these sums as the norm of the partition goes to zero, and the number of slabs tends to infinity:

$$\begin{aligned} W &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{8820\pi}{4}(10 - y_k)y_k^2 \Delta y_k = \int_0^8 \frac{8820\pi}{4}(10 - y)y^2 dy \\ &= \frac{8820\pi}{4} \int_0^8 (10y^2 - y^3) dy \\ &= \frac{8820\pi}{4} \left[\frac{10y^3}{3} - \frac{y^4}{4} \right]_0^8 \approx 4,728,977 \text{ J.} \end{aligned}$$

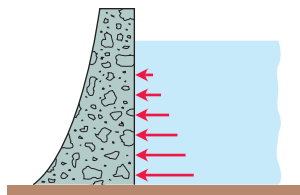


FIGURE 6.40 To withstand the increasing pressure, dams are built thicker as they go down.

Weight-density

A fluid's weight-density w is its weight per unit volume. Typical values (N/m^3) are listed below.

Gasoline	6600
Mercury	133,000
Milk	10,100
Molasses	15,700
Olive oil	8820
Seawater	10,050
Freshwater	9800

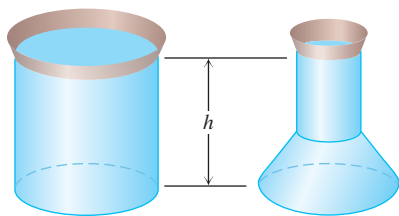


FIGURE 6.41 These containers are filled with water to the same depth and have the same base area. The total force is therefore the same on the bottom of each container. The containers' shapes do not matter here.

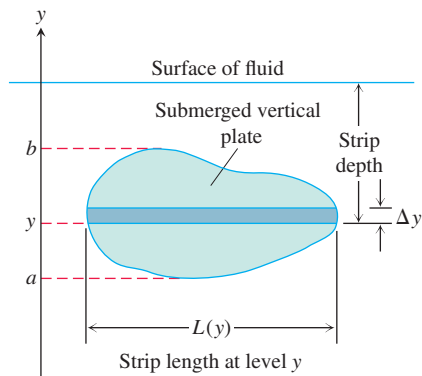


FIGURE 6.42 The force exerted by a fluid against one side of a thin, flat horizontal strip is about $\Delta F = \text{pressure} \times \text{area} = w \times (\text{strip depth}) \times L(y) \Delta y$.

Fluid Pressure and Forces

Dams are built thicker at the bottom than at the top (Figure 6.40) because the pressure against them increases with depth. The pressure at any point on a dam depends only on how far below the surface the point is and not on how much the surface of the dam happens to be tilted at that point. The pressure, in newtons per square meter at a point h meters below the surface, is always $9800h$. The number 9800 is the weight-density of freshwater in newtons per cubic meter. The pressure h meters below the surface of any fluid is the fluid's *weight-density* times h .

The Pressure-Depth Equation

In a fluid that is standing still, the pressure p at depth h is the fluid's weight-density w times h :

$$p = wh. \quad (4)$$

In a container of fluid with a flat horizontal base, the total force exerted by the fluid against the base can be calculated by multiplying the area of the base by the pressure at the base. We can do this because total force equals force per unit area (pressure) times area. (See Figure 6.41.) If F , p , and A are the total force, pressure, and area, then

$$\begin{aligned} F &= \text{total force} = \text{force per unit area} \times \text{area} \\ &= \text{pressure} \times \text{area} = pA && p = wh \text{ from Eq. (4)} \\ &= whA. \end{aligned}$$

Fluid Force on a Constant-Depth Surface

$$F = pA = whA \quad (5)$$

For example, the weight-density of freshwater is 9800 N/m^3 , so the fluid force at the bottom of a $3 \text{ m} \times 6 \text{ m}$ rectangular swimming pool 1 m deep is

$$\begin{aligned} F &= whA = (9800 \text{ N/m}^3)(1 \text{ m})(3 \cdot 6 \text{ m}^2) \\ &= 176,400 \text{ N}. \end{aligned}$$

For a flat plate submerged *horizontally*, like the bottom of the swimming pool just discussed, the downward force acting on its upper face due to liquid pressure is given by Equation (5). If the plate is submerged *vertically*, however, then the pressure against it will be different at different depths and Equation (5) no longer is usable in that form (because h varies).

Suppose we want to know the force exerted by a fluid against one side of a vertical plate submerged in a fluid of weight-density w . To find it, we model the plate as a region extending from $y = a$ to $y = b$ in the xy -plane (Figure 6.42). We partition $[a, b]$ in the usual way and imagine the region to be cut into thin horizontal strips by planes perpendicular to the y -axis at the partition points. The typical strip from y to $y + \Delta y$ is Δy units wide by $L(y)$ units long. We assume $L(y)$ to be a continuous function of y .

The pressure varies across the strip from top to bottom. If the strip is narrow enough, however, the pressure will remain close to its bottom-edge value of $w \times (\text{strip depth})$. The force exerted by the fluid against one side of the strip will be about

$$\begin{aligned} \Delta F &= (\text{pressure along bottom edge}) \times (\text{area}) \\ &= w \cdot (\text{strip depth}) \cdot L(y) \Delta y. \end{aligned}$$

Assume there are n strips associated with the partition of $a \leq y \leq b$ and that y_k is the bottom edge of the k th strip having length $L(y_k)$ and width Δy_k . The force against the entire plate is approximated by summing the forces against each strip, giving the Riemann sum

$$F \approx \sum_{k=1}^n w \cdot (\text{strip depth})_k \cdot L(y_k) \Delta y_k. \quad (6)$$

The sum in Equation (6) is a Riemann sum for a continuous function on $[a, b]$, and we expect the approximations to improve as the norm of the partition goes to zero. The force against the plate is the limit of these sums:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n w \cdot (\text{strip depth})_k \cdot L(y_k) \Delta y_k = \int_a^b w \cdot (\text{strip depth}) \cdot L(y) dy.$$

The Integral for Fluid Force Against a Vertical Flat Plate

Suppose that a plate submerged vertically in fluid of weight-density w runs from $y = a$ to $y = b$ on the y -axis. Let $L(y)$ be the length of the horizontal strip measured from left to right along the surface of the plate at level y . Then the force exerted by the fluid against one side of the plate is

$$F = \int_a^b w \cdot (\text{strip depth}) \cdot L(y) dy. \quad (7)$$

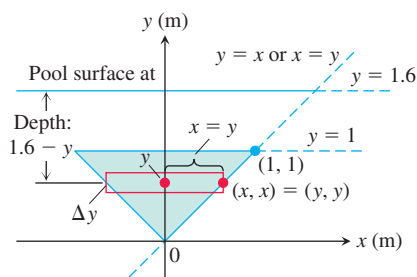


FIGURE 6.43 To find the force on one side of the submerged plate in Example 6, we can use a coordinate system like the one here.

EXAMPLE 6 A flat isosceles right-triangular plate with base 2 m and height 1 m is submerged vertically, base up, 0.6 m below the surface of a swimming pool. Find the force exerted by the water against one side of the plate.

Solution We establish a coordinate system to work in by placing the origin at the plate's bottom vertex and running the y -axis upward along the plate's axis of symmetry (Figure 6.43). The surface of the pool lies along the line $y = 1.6$, and the plate's top edge along the line $y = 1$. The plate's right-hand edge lies along the line $y = x$, with the upper-right vertex at $(1, 1)$. The length of a thin strip at level y is

$$L(y) = 2x = 2y.$$

The depth of the strip beneath the surface is $(1.6 - y)$. The force exerted by the water against one side of the plate is therefore

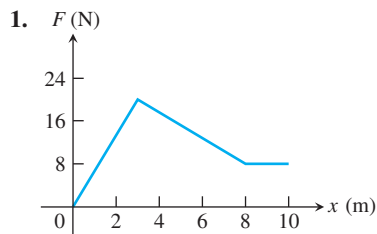
$$\begin{aligned} F &= \int_a^b w \cdot \left(\frac{\text{strip}}{\text{depth}} \right) \cdot L(y) dy && \text{Eq. (7)} \\ &= \int_0^1 9800(1.6 - y)2y dy \\ &= 19,600 \int_0^1 (1.6y - y^2) dy \\ &= 19,600 \left[0.8y^2 - \frac{y^3}{3} \right]_0^1 = 9147 \text{ N}. \end{aligned}$$

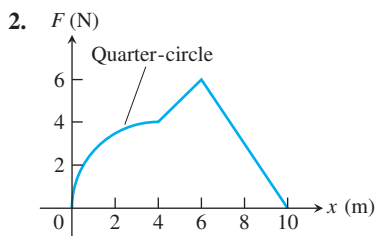
EXERCISES 6.5

For some exercises, a calculator may be helpful when expressing answers in decimal form.

Springs

The graphs of force functions (in newtons) are given in Exercises 1 and 2. How much work is done by each force in moving an object 10 m?





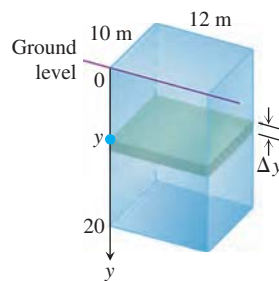
3. **Spring constant** It took 1800 J of work to stretch a spring from its natural length of 2 m to a length of 5 m. Find the spring's force constant.
4. **Stretching a spring** A spring has a natural length of 10 cm. An 800-N force stretches the spring to 14 cm.
- Find the force constant.
 - How much work is done in stretching the spring from 10 cm to 12 cm?
 - How far beyond its natural length will a 1600-N force stretch the spring?
5. **Stretching a rubber band** A force of 2 N will stretch a rubber band 2 cm (0.02 m). Assuming that Hooke's Law applies, how far will a 4-N force stretch the rubber band? How much work does it take to stretch the rubber band this far?
6. **Stretching a spring** If a force of 90 N stretches a spring 1 m beyond its natural length, how much work does it take to stretch the spring 5 m beyond its natural length?
7. **Subway car springs** It takes a force of 96,000 N to compress a coil spring assembly on a New York City Transit Authority subway car from its free height of 20 cm to its fully compressed height of 12 cm.
- What is the assembly's force constant?
 - How much work does it take to compress the assembly the first centimeter? the second centimeter? Answer to the nearest joule.
8. **Bathroom scale** A bathroom scale is compressed 1.5 mm when a 70-kg person stands on it. Assuming that the scale behaves like a spring that obeys Hooke's Law, how much does someone who compresses the scale 3 mm weigh? How much work is done compressing the scale 3 mm?

Work Done by a Variable Force

9. **Lifting a rope** A mountain climber is about to haul up a 50-m length of hanging rope. How much work will it take if the rope weighs 0.624 N/m?
10. **Leaky sandbag** A bag of sand originally weighing 600 N was lifted at a constant rate. As it rose, sand also leaked out at a constant rate. The sand was half gone by the time the bag had been lifted to 6 m. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment.)
11. **Lifting an elevator cable** An electric elevator with a motor at the top has a multistrand cable weighing 60 N/m. When the car is at the first floor, 60 m of cable are paid out, and effectively 0 m are out when the car is at the top floor. How much work does the motor do just lifting the cable when it takes the car from the first floor to the top?
12. **Force of attraction** When a particle of mass m is at $(x, 0)$, it is attracted toward the origin with a force whose magnitude is k/x^2 . If the particle starts from rest at $x = b$ and is acted on by no other forces, find the work done on it by the time it reaches $x = a$, $0 < a < b$.
13. **Leaky bucket** Assume the bucket in Example 4 is leaking. It starts with 8 L of water (78 N) and leaks at a constant rate. It finishes draining just as it reaches the top. How much work was spent lifting the water alone? (*Hint:* Do not include the rope and bucket, and find the proportion of water left at elevation x m.)
14. (*Continuation of Exercise 11*) The workers in Example 4 and Exercise 11 changed to a larger bucket that held 20 L (195 N) of water, but the new bucket had an even larger leak so that it, too, was empty by the time it reached the top. Assuming that the water leaked out at a steady rate, how much work was done lifting the water alone? (Do not include the rope and bucket.)

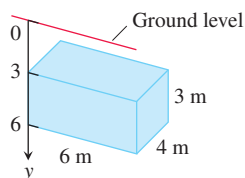
Pumping Liquids from Containers

15. **Pumping water** The rectangular tank shown here, with its top at ground level, is used to catch runoff water. Assume that the water weighs 9800 N/m³.
- How much work does it take to empty the tank by pumping the water back to ground level once the tank is full?
 - If the water is pumped to ground level with a 5-horsepower (hp) motor (work output 3678 W), how long will it take to empty the full tank (to the nearest minute)?
 - Show that the pump in part (b) will lower the water level 10 m (halfway) during the first 266 min of pumping.
 - The weight of water** What are the answers to parts (a) and (b) in a location where water weighs 9780 N/m³? 9820 N/m³?

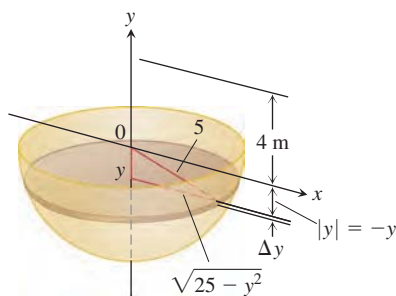


16. **Emptying a cistern** The rectangular cistern (storage tank for rainwater) shown has its top 3 m below ground level. The cistern, currently full, is to be emptied for inspection by pumping its contents to ground level.
- How much work will it take to empty the cistern?
 - How long will it take a 1/2-hp pump, rated at 370 W, to pump the tank dry?
 - How long will it take the pump in part (b) to empty the tank halfway? (It will be less than half the time required to empty the tank completely.)

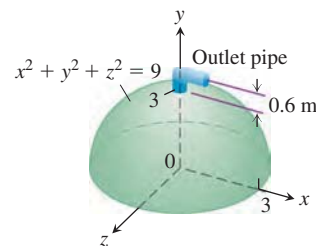
- d. The weight of water** What are the answers to parts (a) through (c) in a location where water weighs 9780 N/m^3 ? 9820 N/m^3 ?



- 17. Pumping oil** How much work would it take to pump oil from the tank in Example 5 to the level of the top of the tank if the tank were completely full?
- 18. Pumping a half-full tank** Suppose that, instead of being full, the tank in Example 5 is only half full. How much work does it take to pump the remaining oil to a level 1 m above the top of the tank?
- 19. Emptying a tank** A vertical right-circular cylindrical tank measures 9 m high and 6 m in diameter. It is full of kerosene weighing 7840 N/m^3 . How much work does it take to pump the kerosene to the level of the top of the tank?
- 20. a. Pumping milk** Suppose that the conical container in Example 5 contains milk (weighing $10,100 \text{ N/m}^3$) instead of olive oil. How much work will it take to pump the contents to the rim?
- b. Pumping oil** How much work will it take to pump the oil in Example 5 to a level 1 m above the cone's rim?
- 21.** The graph of $y = x^2$ on $0 \leq x \leq 2$ is revolved about the y -axis to form a tank that is then filled with salt water from the Dead Sea (weighing approximately $11,500 \text{ N/m}^3$). How much work does it take to pump all of the water to the top of the tank?
- 22.** A right-circular cylindrical tank of height 3 m and radius 1.5 m is lying horizontally and is full of diesel fuel weighing 8300 N/m^3 . How much work is required to pump all of the fuel to a point 4.5 m above the top of the tank?
- 23. Emptying a water reservoir** We model pumping from spherical containers the way we do from other containers, with the axis of integration along the vertical axis of the sphere. Use the figure here to find how much work it takes to empty a full hemispherical water reservoir of radius 5 m by pumping the water to a height of 4 m above the top of the reservoir. Water weighs 9800 N/m^3 .



- 24.** You are in charge of the evacuation and repair of the storage tank shown here. The tank is a hemisphere of radius 3 m and is full of benzene weighing 8800 N/m^3 . A firm you contacted says it can empty the tank for 0.4¢ per joule of work. Find the work required to empty the tank by pumping the benzene to an outlet 0.6 m above the top of the tank. If you have \$5000 budgeted for the job, can you afford to hire the firm?



Work and Kinetic Energy

- 25. Kinetic energy** If a variable force of magnitude $F(x)$ moves an object of mass m along the x -axis from x_1 to x_2 , the object's velocity v can be written as dx/dt (where t represents time). Use Newton's second law of motion $F = m(dv/dt)$ and the Chain Rule

$$\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx}$$

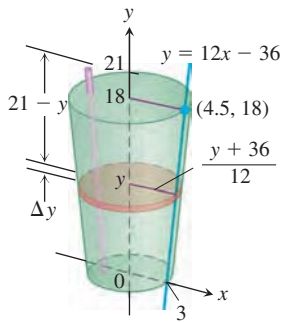
to show that the net work done by the force in moving the object from x_1 to x_2 is

$$W = \int_{x_1}^{x_2} F(x) dx = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2,$$

where v_1 and v_2 are the object's velocities at x_1 and x_2 . In physics, the expression $(1/2)mv^2$ is called the *kinetic energy* of an object of mass m moving with velocity v . Therefore, *the work done by the force equals the change in the object's kinetic energy*, and we can find the work by calculating this change.

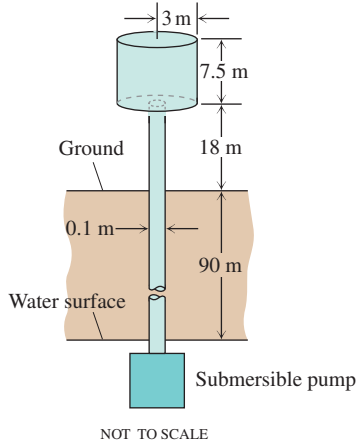
In Exercises 26–30, use the result of Exercise 25.

- 26. Tennis** A 57 g tennis ball was served at 50 m/s (180 km/h). How much work was done on the ball to make it go this fast?
- 27. Baseball** How many joules of work does it take to throw a baseball 144 km/h? A baseball's mass is 150 g.
- 28. Golf** A 50 g golf ball is driven off the tee at a speed of 84 m/s (302.4 km/h). How many joules of work are done on the ball getting it into the air?
- 29. Tennis** At the 2012 Busan Open Challenger Tennis Tournament in Busan, South Korea, the Australian Samuel Groth hit a serve measured at 263 km/ph. How much work was required by Groth to serve a 0.0567-kg tennis ball at that speed?
- 30. Softball** How much work has to be performed on a 200 g softball to pitch it 40 m/s (144 km/h)?
- 31. Drinking a milkshake** The truncated conical container shown here is full of strawberry milkshake, which has a density of 0.8 g/cm^3 . As you can see, the container is 18 cm deep, 6 cm across at the base, and 9 cm across at the top (a standard size at Brigham's in Boston). The straw sticks up 3 cm above the top. About how much work does it take to suck up the milkshake through the straw (neglecting friction)?



Dimensions in centimeters

- 32. Water tower** Your town has decided to drill a well to increase its water supply. As the town engineer, you have determined that a water tower will be necessary to provide the pressure needed for distribution, and you have designed the system shown here. The water is to be pumped from a 90 m well through a vertical 10 cm pipe into the base of a cylindrical tank 6 m in diameter and 7.5 m high. The base of the tank will be 18 m above ground. The pump is a 3-hp pump, rated at 2200 W (J/s). To the nearest hour, how long will it take to fill the tank the first time? (Include the time it takes to fill the pipe.) Assume that water weighs 9800 N/m³.



- 33. Putting a satellite in orbit** The strength of Earth's gravitational field varies with the distance r from Earth's center, and the magnitude of the gravitational force experienced by a satellite of mass m during and after launch is

$$F(r) = \frac{mMG}{r^2}.$$

Here, $M = 5.975 \times 10^{24}$ kg is Earth's mass,

$$G = 6.6720 \times 10^{-11} \text{ N} \cdot \text{m}^2 \text{ kg}^{-2}$$

is the universal gravitational constant, and r is measured in meters. The work it takes to lift a 1000-kg satellite from Earth's surface to a circular orbit 35,780 km above Earth's center is therefore given by the integral

$$\text{Work} = \int_{6,370,000}^{35,780,000} \frac{1000MG}{r^2} dr \text{ joules.}$$

Evaluate the integral. The lower limit of integration is Earth's radius in meters at the launch site. (This calculation does not take into account energy spent lifting the launch vehicle or energy spent bringing the satellite to orbit velocity.)

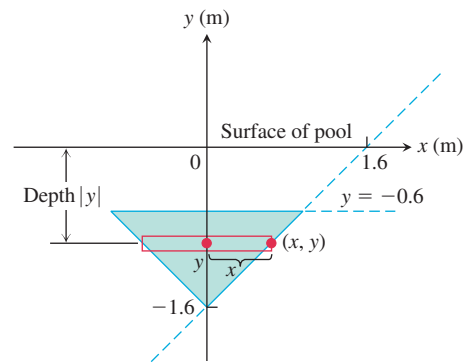
- 34. Forcing electrons together** Two electrons r meters apart repel each other with a force of

$$F = \frac{23 \times 10^{-29}}{r^2} \text{ newtons.}$$

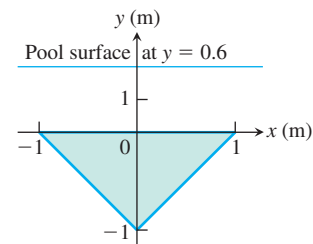
- Suppose one electron is held fixed at the point $(1, 0)$ on the x -axis (units in meters). How much work does it take to move a second electron along the x -axis from the point $(-1, 0)$ to the origin?
- Suppose an electron is held fixed at each of the points $(-1, 0)$ and $(1, 0)$. How much work does it take to move a third electron along the x -axis from $(5, 0)$ to $(3, 0)$?

Finding Fluid Forces

- 35. Triangular plate** Calculate the fluid force on one side of the plate in Example 6 using the coordinate system shown here.



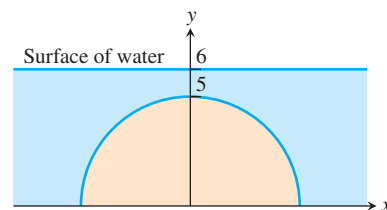
- 36. Triangular plate** Calculate the fluid force on one side of the plate in Example 6 using the coordinate system shown here.



- 37. Rectangular plate** In a pool filled with water to a depth of 3 m, calculate the fluid force on one side of a 0.9 m by 1.2 m rectangular plate if the plate rests vertically at the bottom of the pool

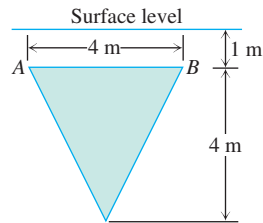
- on its 1.2-m edge.
- on its 0.9-m edge.

- 38. Semicircular plate** Calculate the fluid force on one side of a semicircular plate of radius 5 m that rests vertically on its diameter at the bottom of a pool filled with water to a depth of 6 m.

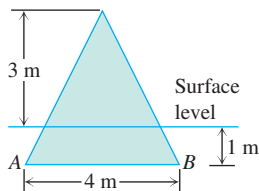


39. Triangular plate The isosceles triangular plate shown here is submerged vertically 1 m below the surface of a freshwater lake.

- Find the fluid force against one face of the plate.
- What would be the fluid force on one side of the plate if the water were seawater instead of freshwater?



40. Rotated triangular plate The plate in Exercise 37 is revolved 180° about line AB so that part of the plate sticks out of the lake, as shown here. What force does the water exert on one face of the plate now?



41. New England Aquarium The viewing portion of the rectangular glass window in a typical fish tank at the New England Aquarium in Boston is 1.6 m wide and runs from 0.01 m below the water's surface to 0.85 m below the surface. Find the fluid force against this portion of the window. The weight-density of seawater is $10,050 \text{ N/m}^3$. (In case you were wondering, the glass is 2 cm thick and the tank walls extend 10 cm above the water to keep the fish from jumping out.)

42. Semicircular plate A semicircular plate 2 m in diameter sticks straight down into freshwater with the diameter along the surface. Find the force exerted by the water on one side of the plate.

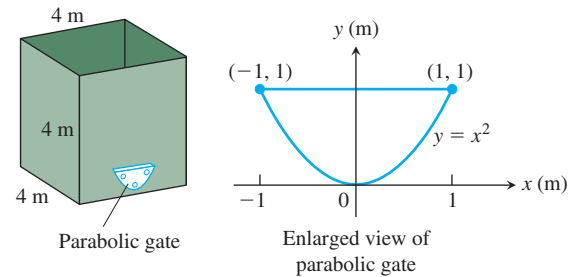
43. Tilted plate Calculate the fluid force on one side of a 1 m by 1 m square plate if the plate is at the bottom of a pool filled with water to a depth of 2 m and

- lying flat on its 1 m by 1 m face.
- resting vertically on a 1 m edge.
- resting on a 1 m edge and tilted at 45° to the bottom of the pool.

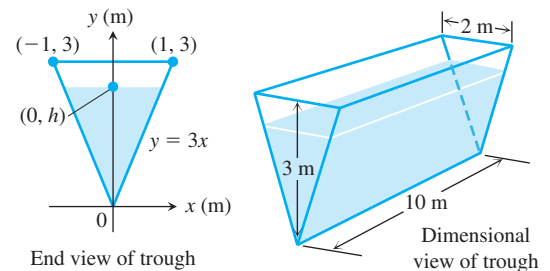
44. Tilted plate Calculate the fluid force on one side of a right-triangular plate with edges 3 m, 4 m, and 5 m if the plate sits at the bottom of a pool filled with water to a depth of 6 m on its 3-m edge and tilted at 60° to the bottom of the pool.

45. The cubical metal tank shown here has a parabolic gate held in place by bolts and designed to withstand a fluid force of 25,000 N without rupturing. The liquid you plan to store has a weight-density of 8000 N/m^3 .

- What is the fluid force on the gate when the liquid is 2 m deep?
- What is the maximum height to which the container can be filled without exceeding the gate's design limitation?



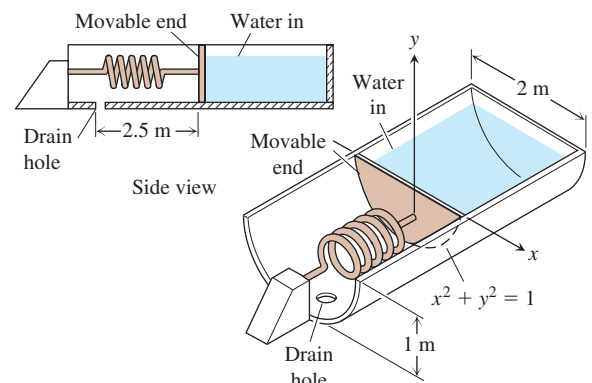
46. The end plates of the trough shown here were designed to withstand a fluid force of 25,000 N. How many cubic meters of water can the tank hold without exceeding this limitation? Round down to the nearest cubic meter. What is the value of h ?



47. A vertical rectangular plate a units long by b units wide is submerged in a fluid of weight-density w with its long edges parallel to the fluid's surface. Find the average value of the pressure along the vertical dimension of the plate. Explain your answer.

48. (Continuation of Exercise 47.) Show that the force exerted by the fluid on one side of the plate is the average value of the pressure (found in Exercise 47) times the area of the plate.

49. Water pours into the tank shown here at the rate of $0.5 \text{ m}^3/\text{min}$. The tank's cross-sections are 2-m-diameter semicircles. One end of the tank is movable, but moving it to increase the volume compresses a spring. The spring constant is $k = 3000 \text{ N/m}$. If the end of the tank moves 2.5 m against the spring, the water will drain out of a safety hole in the bottom at the rate of $0.6 \text{ m}^3/\text{min}$. Will the movable end reach the hole before the tank overflows?



50. Watering trough The vertical ends of a watering trough are squares 1 m on a side.

- Find the fluid force against the ends when the trough is full.
- How many centimeters do you have to lower the water level in the trough to reduce the fluid force by 25%?

6.6 Moments and Centers of Mass

Many structures and mechanical systems behave as if their masses were concentrated at a single point, called the *center of mass* (Figure 6.44). It is important to know how to locate this point, and doing so is basically a mathematical enterprise. Here we consider masses distributed along a line or region in the plane. Masses distributed across a region or curve in three-dimensional space are treated in Chapters 14 and 15.

Masses Along a Line

We develop our mathematical model in stages. The first stage is to imagine masses m_1 , m_2 , and m_3 on a rigid x -axis supported by a fulcrum at the origin.



The resulting system might balance, or it might not, depending on how large the masses are and how they are arranged along the x -axis.

Each mass m_k exerts a downward force $m_k g$ (the weight of m_k) equal to the magnitude of the mass times the acceleration due to gravity. Note that gravitational acceleration is downward, hence negative. Each of these forces has a tendency to turn the x -axis about the origin, the way a child turns a seesaw. This turning effect, called a **torque**, is measured by multiplying the force $m_k g$ by the signed distance x_k from the point of application to the origin. By convention, a positive torque induces a counterclockwise turn. Masses to the left of the origin exert positive (counterclockwise) torque. Masses to the right of the origin exert negative (clockwise) torque.

The sum of the torques measures the tendency of a system to rotate about the origin. This sum is called the **system torque**.

$$\text{System torque} = m_1 g x_1 + m_2 g x_2 + m_3 g x_3 \quad (1)$$

The system will balance if and only if its torque is zero.

If we factor out the g in Equation (1), we see that the system torque is

$$\underbrace{g}_{\text{a feature of the environment}} \cdot \underbrace{(m_1 x_1 + m_2 x_2 + m_3 x_3)}_{\text{a feature of the system}}.$$

Thus, the torque is the product of the gravitational acceleration g , which is a feature of the environment in which the system happens to reside, and the number $(m_1 x_1 + m_2 x_2 + m_3 x_3)$, which is a feature of the system itself.

The number $(m_1 x_1 + m_2 x_2 + m_3 x_3)$ is called the **moment of the system about the origin**. It is the sum of the **moments** $m_1 x_1, m_2 x_2, m_3 x_3$ of the individual masses.

$$M_0 = \text{Moment of system about origin} = \sum m_k x_k$$

(We shift to sigma notation here to allow for sums with more terms.)

We usually want to know where to place the fulcrum to make the system balance; that is, we want to know at what point $\bar{x}$ to place the fulcrum to make the torques add to zero.

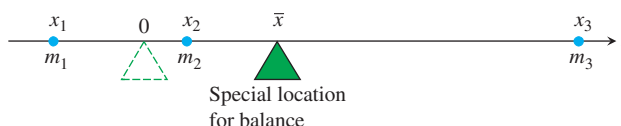


FIGURE 6.44 A wrench gliding on ice turning about its center of mass as the center glides in a vertical line. (Source: Berenice Abbott/ScienceSource)

The torque of each mass about the fulcrum in this special location is

$$\text{Torque of } m_k \text{ about } \bar{x} = \begin{pmatrix} \text{signed distance} \\ \text{of } m_k \text{ from } \bar{x} \end{pmatrix} \begin{pmatrix} \text{downward} \\ \text{force} \end{pmatrix} = (x_k - \bar{x})m_k g.$$

When we write the equation that says that the sum of these torques is zero, we get an equation we can solve for $\bar{x}$:

$$\begin{aligned} \sum (x_k - \bar{x}) m_k g &= 0 && \text{Sum of the torques equals zero.} \\ \bar{x} &= \frac{\sum m_k x_k}{\sum m_k}. && \text{Solved for } \bar{x} \end{aligned}$$

This last equation tells us to find $\bar{x}$ by dividing the system's moment about the origin by the system's total mass:

$$\bar{x} = \frac{\sum m_k x_k}{\sum m_k} = \frac{\text{system moment about origin}}{\text{system mass}}. \quad (2)$$

The point $\bar{x}$ is called the system's **center of mass**.

Thin Wires

Instead of a discrete set of masses arranged in a line, suppose that we have a straight wire or rod located on interval $[a, b]$ on the x -axis. Suppose further that this wire is not homogeneous, but rather the density varies continuously from point to point. If a short segment of a rod containing the point x with length Δx has mass Δm , then the density at x is given by

$$\delta(x) = \lim_{\Delta x \rightarrow 0} \Delta m / \Delta x.$$

We often write this formula in one of the alternative forms $\delta = dm/dx$ and $dm = \delta dx$.

Partition the interval $[a, b]$ into finitely many subintervals $[x_{k-1}, x_k]$. If we take n subintervals and replace the portion of a wire along a subinterval of length Δx_k containing x_k by a point mass located at x_k with mass $\Delta m_k = \delta(x_k) \Delta x_k$, then we obtain a collection of point masses that have approximately the same total mass and same moment as the wire as indicated in Figure 6.45.

The mass M of the wire and the moment M_0 are approximated by the Riemann sums

$$M \approx \sum_{k=1}^n \Delta m_k = \sum_{k=1}^n \delta(x_k) \Delta x_k, \quad M_0 \approx \sum_{k=1}^n x_k \Delta m_k = \sum_{k=1}^n x_k \delta(x_k) \Delta x_k.$$

By taking a limit of these Riemann sums as the length of the intervals in the partition approaches zero, we get integral formulas for the mass and the moment of the wire about the origin. The mass M , moment about the origin M_0 , and center of mass $\bar{x}$ are

$$M = \int_a^b \delta(x) dx, \quad M_0 = \int_a^b x \delta(x) dx, \quad \bar{x} = \frac{M_0}{M} = \frac{\int_a^b x \delta(x) dx}{\int_a^b \delta(x) dx}.$$

EXAMPLE 1 Find the mass M and the center of mass $\bar{x}$ of a rod lying on the x -axis over the interval $[1, 2]$ whose density is given by $\delta(x) = 2 + 3x^2$.

Solution The mass of the rod is obtained by integrating the density,

$$M = \int_1^2 (2 + 3x^2) dx = [2x + x^3]_1^2 = (4 + 8) - (2 + 1) = 9,$$

and the center of mass is

$$\bar{x} = \frac{M_0}{M} = \frac{\int_1^2 x(2 + 3x^2) dx}{9} = \frac{\left[x^2 + \frac{3x^4}{4}\right]_1^2}{9} = \frac{19}{12}.$$

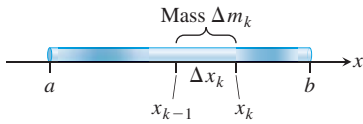


FIGURE 6.45 A rod of varying density can be modeled by a finite number of point masses of mass $\Delta m_k = \delta(x_k) \Delta x_k$ located at points x_k along the rod.

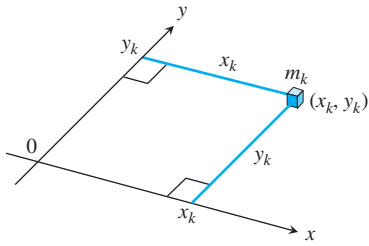


FIGURE 6.46 Each mass m_k has a moment about each axis.

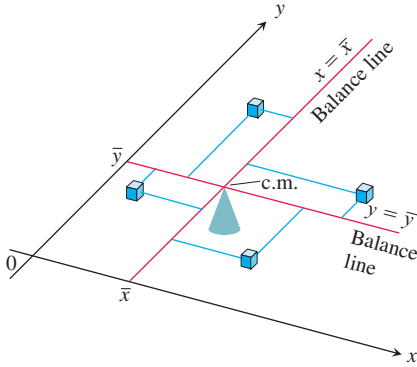


FIGURE 6.47 A two-dimensional array of masses balances on its center of mass.

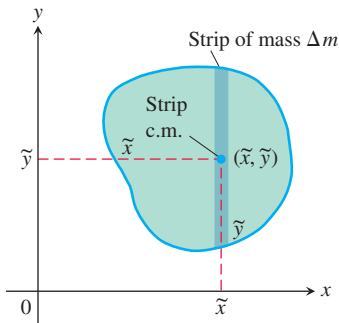


FIGURE 6.48 A plate cut into thin strips parallel to the y -axis. The moment exerted by a typical strip about each axis is the moment its mass Δm would exert if concentrated at the strip's center of mass $(\tilde{x}, \tilde{y})$.

Density of a plate

A material's density is its mass per unit area. For wires, rods, and narrow strips, the density is given in terms of mass per unit length.

Masses Distributed over a Plane Region

Suppose that we have a finite collection of masses located in the plane, with mass m_k at the point (x_k, y_k) (see Figure 6.46). The mass of the system is

$$\text{System mass: } M = \sum m_k.$$

Each mass m_k has a moment about each axis. Its moment about the x -axis is $m_k y_k$, and its moment about the y -axis is $m_k x_k$. The moments of the entire system about the two axes are

$$\text{Moment about } x\text{-axis: } M_x = \sum m_k y_k,$$

$$\text{Moment about } y\text{-axis: } M_y = \sum m_k x_k.$$

The x -coordinate of the system's center of mass is defined to be

$$\bar{x} = \frac{M_y}{M} = \frac{\sum m_k x_k}{\sum m_k}. \quad (3)$$

With this choice of $\bar{x}$, as in the one-dimensional case, the system balances about the line $x = \bar{x}$ (Figure 6.47).

The y -coordinate of the system's center of mass is defined to be

$$\bar{y} = \frac{M_x}{M} = \frac{\sum m_k y_k}{\sum m_k}. \quad (4)$$

With this choice of $\bar{y}$, the system balances about the line $y = \bar{y}$ as well. The torques exerted by the masses about the line $y = \bar{y}$ cancel out. Thus, as far as balance is concerned, the system behaves as if all its mass were at the single point $(\bar{x}, \bar{y})$. We call this point the system's **center of mass (c.m.)**.

Thin, Flat Plates

In many applications, we need to find the center of mass of a thin, flat plate: a disk of aluminum, say, or a triangular sheet of steel. In such cases, we assume the distribution of mass to be continuous, and the formulas we use to calculate $\bar{x}$ and $\bar{y}$ contain integrals instead of finite sums. The integrals arise in the following way.

Imagine that the plate occupying a region in the xy -plane is cut into thin strips parallel to one of the axes (in Figure 6.48, the y -axis). The center of mass of a typical strip is $(\tilde{x}, \tilde{y})$. We treat the strip's mass Δm as if it were concentrated at $(\tilde{x}, \tilde{y})$. The moment of the strip about the y -axis is then $\tilde{x} \Delta m$. The moment of the strip about the x -axis is $\tilde{y} \Delta m$. Equations (3) and (4) then become

$$\bar{x} = \frac{M_y}{M} = \frac{\sum \tilde{x} \Delta m}{\sum \Delta m}, \quad \bar{y} = \frac{M_x}{M} = \frac{\sum \tilde{y} \Delta m}{\sum \Delta m}.$$

These sums are Riemann sums for integrals, and they approach these integrals in the limit as the strips become narrower and narrower. We write these integrals symbolically as

$$\bar{x} = \frac{\int \tilde{x} \, dm}{\int dm} \quad \text{and} \quad \bar{y} = \frac{\int \tilde{y} \, dm}{\int dm}.$$

Moments, Mass, and Center of Mass of a Thin Plate Covering a Region in the xy -Plane

$$\text{Moment about the } x\text{-axis: } M_x = \int \tilde{y} \, dm$$

$$\text{Moment about the } y\text{-axis: } M_y = \int \tilde{x} \, dm$$

$$\text{Mass: } M = \int dm$$

$$\text{Center of mass: } \bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M} \quad (5)$$

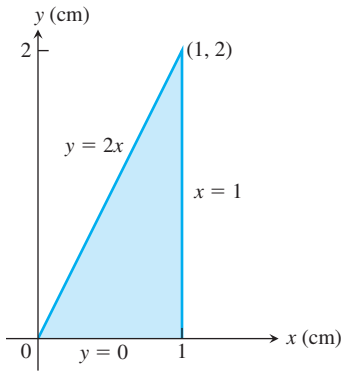


FIGURE 6.49 The plate in Example 2.

Although we do not indicate the limits of integration, the integrals that appear in Equation (5) are definite integrals.

The differential dm in these integrals is the mass of the strip. For this section, we assume the density δ of the plate is a constant or a continuous function of x or of y . Then $dm = \delta dA$, which is the mass per unit area δ times the area dA of the strip.

To evaluate the integrals in Equations (5), we picture the plate in the coordinate plane and sketch a strip of mass parallel to one of the coordinate axes. We then express the strip's mass dm and the coordinates $(\tilde{x}, \tilde{y})$ of the strip's center of mass in terms of x or y . Finally, we integrate $\tilde{y} dm$, $\tilde{x} dm$, and dm between limits of integration determined by the plate's location in the plane.

EXAMPLE 2 The triangular plate shown in Figure 6.49 has a constant density of $\delta = 3 \text{ g/cm}^2$.

- Find the plate's moment M_y about the y -axis.
- Find the plate's mass M .
- Find the x -coordinate of the plate's center of mass (c.m.).

Solution Method 1: Vertical Strips (Figure 6.50)

- (a) The moment M_y : The typical vertical strip has the following relevant data.

center of mass (c.m.):	$(\tilde{x}, \tilde{y}) = (x, x)$
length:	$2x$
width:	dx
area:	$dA = 2x dx$
mass:	$dm = \delta dA = 3 \cdot 2x dx = 6x dx$
distance of c.m. from y -axis:	$\tilde{x} = x$

The moment of the strip about the y -axis is

$$\tilde{x} dm = x \cdot 6x dx = 6x^2 dx.$$

The moment of the plate about the y -axis is therefore

$$M_y = \int \tilde{x} dm = \int_0^1 6x^2 dx = 2x^3 \Big|_0^1 = 2 \text{ g} \cdot \text{cm}.$$

- (b) The plate's mass:

$$M = \int dm = \int_0^1 6x dx = 3x^2 \Big|_0^1 = 3 \text{ g}.$$

- (c) The x -coordinate of the plate's center of mass:

$$\bar{x} = \frac{M_y}{M} = \frac{2 \text{ g} \cdot \text{cm}}{3 \text{ g}} = \frac{2}{3} \text{ cm}.$$

By a similar computation, we could find M_x and $\bar{y} = M_x/M$.

Method 2: Horizontal Strips (Figure 6.51)

- (a) The moment M_y : The y -coordinate of the center of mass of a typical horizontal strip is y (see the figure), so

$$\tilde{y} = y.$$

The x -coordinate is the x -coordinate of the point halfway across the triangle. This makes it the average of $y/2$ (the strip's left-hand x -value) and 1 (the strip's right-hand x -value):

$$\tilde{x} = \frac{(y/2) + 1}{2} = \frac{y}{4} + \frac{1}{2} = \frac{y+2}{4}.$$

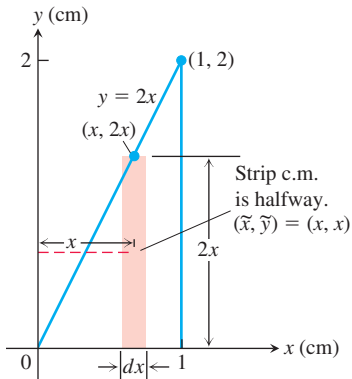


FIGURE 6.50 Modeling the plate in Example 2 with vertical strips.

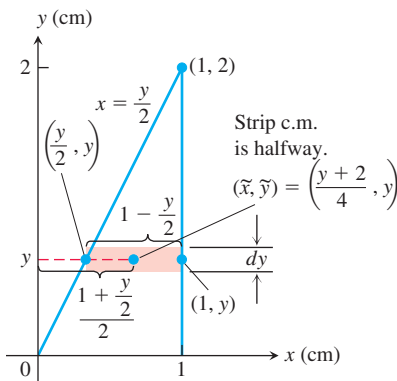


FIGURE 6.51 Modeling the plate in Example 2 with horizontal strips.

We also have

$$\begin{aligned}\text{length:} \quad & 1 - \frac{y}{2} = \frac{2-y}{2} \\ \text{width:} \quad & dy \\ \text{area:} \quad & dA = \frac{2-y}{2} dy \\ \text{mass:} \quad & dm = \delta dA = 3 \cdot \frac{2-y}{2} dy \\ \text{distance of c.m. to } y\text{-axis:} \quad & \tilde{x} = \frac{y+2}{4}.\end{aligned}$$

The moment of the strip about the y -axis is

$$\tilde{x} dm = \frac{y+2}{4} \cdot 3 \cdot \frac{2-y}{2} dy = \frac{3}{8}(4-y^2) dy.$$

The moment of the plate about the y -axis is

$$M_y = \int \tilde{x} dm = \int_0^2 \frac{3}{8}(4-y^2) dy = \frac{3}{8} \left[4y - \frac{y^3}{3} \right]_0^2 = \frac{3}{8} \left(\frac{16}{3} \right) = 2 \text{ g} \cdot \text{cm}.$$

(b) The plate's mass:

$$M = \int dm = \int_0^2 \frac{3}{2}(2-y) dy = \frac{3}{2} \left[2y - \frac{y^2}{2} \right]_0^2 = \frac{3}{2}(4-2) = 3 \text{ g}.$$

(c) The x -coordinate of the plate's center of mass:

$$\bar{x} = \frac{M_y}{M} = \frac{2 \text{ g} \cdot \text{cm}}{3 \text{ g}} = \frac{2}{3} \text{ cm}.$$

By a similar computation, we could find M_x and $\bar{y}$. ■

If the distribution of mass in a thin, flat plate has an axis of symmetry, the center of mass will lie on this axis. If there are two axes of symmetry, the center of mass will lie at their intersection. These facts often help to simplify our work.

EXAMPLE 3 Find the center of mass of a thin plate covering the region bounded above by the parabola $y = 4 - x^2$ and below by the x -axis (Figure 6.52). Assume the density of the plate at the point (x, y) is $\delta = 2x^2$, which is twice the square of the distance from the point to the y -axis.

Solution The mass distribution is symmetric about the y -axis, so $\bar{x} = 0$. We model the distribution of mass with vertical strips, since the density is given as a function of the variable x . The typical vertical strip (see Figure 6.52) has the following relevant data.

$$\begin{aligned}\text{center of mass (c.m.):} \quad & (\tilde{x}, \tilde{y}) = \left(x, \frac{4-x^2}{2} \right) \\ \text{length:} \quad & 4 - x^2 \\ \text{width:} \quad & dx \\ \text{area:} \quad & dA = (4 - x^2) dx \\ \text{mass:} \quad & dm = \delta dA = \delta(4 - x^2) dx \\ \text{distance from c.m. to } x\text{-axis:} \quad & \tilde{y} = \frac{4-x^2}{2}.\end{aligned}$$

The moment of the strip about the x -axis is

$$\tilde{y} dm = \frac{4-x^2}{2} \cdot \delta(4-x^2) dx = \frac{\delta}{2}(4-x^2)^2 dx.$$

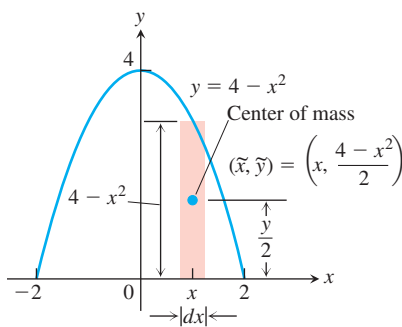


FIGURE 6.52 Modeling the plate in Example 3 with vertical strips.

The moment of the plate about the x -axis is

$$\begin{aligned} M_x &= \int \tilde{y} \, dm = \int_{-2}^2 \frac{\delta}{2} (4 - x^2)^2 \, dx = \int_{-2}^2 x^2 (4 - x^2)^2 \, dx \\ &= \int_{-2}^2 (16x^2 - 8x^4 + x^6) \, dx = \frac{2048}{105}. \end{aligned}$$

The mass of the plate is

$$\begin{aligned} M &= \int dm = \int_{-2}^2 \delta (4 - x^2) \, dx = \int_{-2}^2 2x^2 (4 - x^2) \, dx \\ &= \int_{-2}^2 (8x^2 - 2x^4) \, dx = \frac{256}{15}. \end{aligned}$$

Therefore,

$$\bar{y} = \frac{M_x}{M} = \frac{2048}{105} \cdot \frac{15}{256} = \frac{8}{7}.$$

The plate's center of mass is

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{7}\right).$$

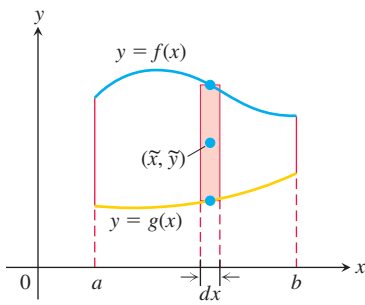


FIGURE 6.53 Modeling the plate bounded by two curves with vertical strips. The strip c.m. is halfway, so $\tilde{y} = \frac{1}{2}[f(x) + g(x)]$.

Plates Bounded by Two Curves

Suppose a plate covers a region that lies between two curves $y = g(x)$ and $y = f(x)$, where $f(x) \geq g(x)$ and $a \leq x \leq b$. The typical vertical strip (see Figure 6.53) has

center of mass (c.m.):	$(\tilde{x}, \tilde{y}) = \left(x, \frac{1}{2}[f(x) + g(x)]\right)$
length:	$f(x) - g(x)$
width:	dx
area:	$dA = [f(x) - g(x)] \, dx$
mass:	$dm = \delta \, dA = \delta [f(x) - g(x)] \, dx.$

The moment of the plate about the y -axis is

$$M_y = \int \tilde{x} \, dm = \int_a^b x \delta [f(x) - g(x)] \, dx,$$

and the moment about the x -axis is

$$\begin{aligned} M_x &= \int \tilde{y} \, dm = \int_a^b \frac{1}{2} [f(x) + g(x)] \cdot \delta [f(x) - g(x)] \, dx \\ &= \int_a^b \frac{\delta}{2} [f^2(x) - g^2(x)] \, dx. \end{aligned}$$

These moments give us the following formulas.

$$\bar{x} = \frac{1}{M} \int_a^b \delta x [f(x) - g(x)] \, dx \quad (6)$$

$$\bar{y} = \frac{1}{M} \int_a^b \frac{\delta}{2} [f^2(x) - g^2(x)] \, dx \quad (7)$$

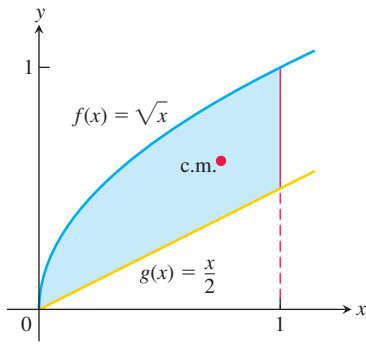


FIGURE 6.54 The region in Example 4.

EXAMPLE 4 Find the center of mass for the thin plate bounded by the curves $g(x) = x/2$ and $f(x) = \sqrt{x}$, $0 \leq x \leq 1$ (Figure 6.54), using Equations (6) and (7) with the density function $\delta(x) = x^2$.

Solution We first compute the mass of the plate, using $dm = \delta[f(x) - g(x)] dx$:

$$M = \int_0^1 x^2 \left(\sqrt{x} - \frac{x}{2} \right) dx = \int_0^1 \left(x^{5/2} - \frac{x^3}{2} \right) dx = \left[\frac{2}{7} x^{7/2} - \frac{1}{8} x^4 \right]_0^1 = \frac{9}{56}.$$

Then, from Equations (6) and (7) we get

$$\begin{aligned} \bar{x} &= \frac{56}{9} \int_0^1 x^2 \cdot x \left(\sqrt{x} - \frac{x}{2} \right) dx \\ &= \frac{56}{9} \int_0^1 \left(x^{7/2} - \frac{x^4}{2} \right) dx \\ &= \frac{56}{9} \left[\frac{2}{9} x^{9/2} - \frac{1}{10} x^5 \right]_0^1 = \frac{308}{405}, \end{aligned}$$

and

$$\begin{aligned} \bar{y} &= \frac{56}{9} \int_0^1 \frac{x^2}{2} \left(x - \frac{x^2}{4} \right) dx \\ &= \frac{28}{9} \int_0^1 \left(x^3 - \frac{x^4}{4} \right) dx \\ &= \frac{28}{9} \left[\frac{1}{4} x^4 - \frac{1}{20} x^5 \right]_0^1 = \frac{28}{45}. \end{aligned}$$

The center of mass is shown in Figure 6.54.

Centroids

The center of mass in Example 4 is not located at the geometric center of the region. This is due to the region's nonuniform density. When the density function is constant, it cancels out of the numerator and denominator of the formulas for $\bar{x}$ and $\bar{y}$. Thus, when the density is constant, the location of the center of mass is a feature of the geometry of the object and not of the material from which it is made. In such cases, engineers may call the center of mass the **centroid** of the shape, as in “Find the centroid of a triangle or a solid cone.” To do so, just set δ equal to 1 and proceed to find $\bar{x}$ and $\bar{y}$ as before, by dividing moments by masses.

EXAMPLE 5 Find the center of mass (centroid) of a thin wire of constant density δ shaped like a semicircle of radius a .

Solution We model the wire with the semicircle $y = \sqrt{a^2 - x^2}$ (Figure 6.55). The distribution of mass is symmetric about the y -axis, so $\bar{x} = 0$. To find $\bar{y}$, we imagine the wire divided into short subarc segments. If $(\tilde{x}, \tilde{y})$ is the center of mass of a subarc and θ is the angle between the x -axis and the radial line joining the origin to $(\tilde{x}, \tilde{y})$, then $\tilde{y} = a \sin \theta$ is a function of the angle θ measured in radians (see Figure 6.55a). The length ds of the subarc containing $(\tilde{x}, \tilde{y})$ subtends an angle of $d\theta$ radians, so $ds = a d\theta$. Thus a typical subarc segment has these relevant data for calculating $\bar{y}$:

length:	$ds = a d\theta$	
mass:	$dm = \delta ds = \delta a d\theta$	Mass per unit length times length
distance of c.m. to x -axis:	$\tilde{y} = a \sin \theta$	

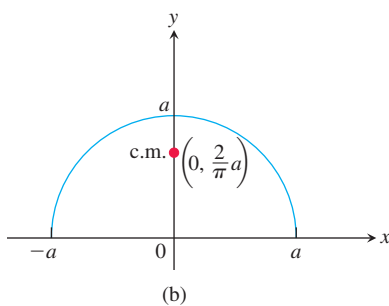
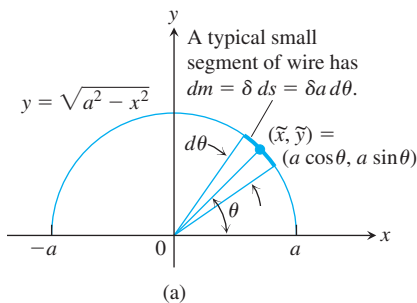


FIGURE 6.55 The semicircular wire in Example 5. (a) The dimensions and variables used in finding the center of mass. (b) The center of mass does not lie on the wire.

Hence,

$$\bar{y} = \frac{\int \tilde{y} \, dm}{\int dm} = \frac{\int_0^\pi a \sin \theta \cdot \delta a \, d\theta}{\int_0^\pi \delta a \, d\theta} = \frac{\delta a^2 [-\cos \theta]_0^\pi}{\delta a \pi} = \frac{2}{\pi} a.$$

The center of mass lies on the axis of symmetry at the point $(0, 2a/\pi)$, about two-thirds of the way up from the origin (Figure 6.55b). Notice how δ cancels in the equation for $\bar{y}$, so we could have set $\delta = 1$ everywhere and obtained the same value for $\bar{y}$. ■

In Example 5 we found the center of mass of a thin wire lying along the graph of a differentiable function in the xy -plane. In Chapter 15 we will learn how to find the center of mass of a wire lying along a more general smooth curve in the plane or in space.

Fluid Forces and Centroids

If we know the location of the centroid of a submerged flat vertical plate (Figure 6.56), we can take a shortcut to find the force against one side of the plate. From Equation (7) in Section 6.5, and the definition of the moment about the x -axis, we have

$$\begin{aligned} F &= \int_a^b w \times (\text{strip depth}) \times L(y) \, dy \\ &= w \int_a^b (\text{strip depth}) \times L(y) \, dy \\ &= w \times (\text{moment about surface level line of region occupied by plate}) \\ &= w \times (\text{depth of plate's centroid}) \times (\text{area of plate}). \end{aligned}$$

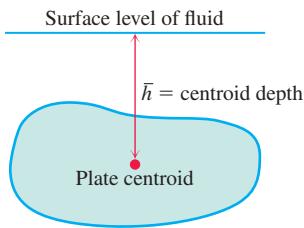


FIGURE 6.56 The force against one side of the plate is $w \cdot \bar{h} \cdot \text{plate area}$.

Fluid Forces and Centroids

The force of a fluid of weight-density w against one side of a submerged flat vertical plate is the product of w , the distance $\bar{h}$ from the plate's centroid to the fluid surface, and the plate's area:

$$F = w\bar{h}A. \quad (8)$$

EXAMPLE 6 A flat isosceles triangular plate with base 2 m and height 1 m is submerged vertically, base up with its vertex at the origin, so that the base is 0.6 m below the surface of a swimming pool. (This is Example 6, Section 6.5.) Use Equation (8) to find the force exerted by the water against one side of the plate.

Solution The centroid of the triangle (Figure 6.43) lies on the y -axis, one-third of the way from the base to the vertex, so $\bar{h} = 2.8/3$ (where $y = 2/3$), since the pool's surface is $y = 1.6$. The triangle's area is

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(2)(1) = 1.$$

Hence,

$$F = w\bar{h}A = (9800)(2.8/3)(1) = 9147 \text{ N.} \quad \blacksquare$$

The Theorems of Pappus

In the fourth century, an Alexandrian Greek named Pappus discovered two formulas that relate centroids to surfaces and solids of revolution. The formulas provide shortcuts to a number of otherwise lengthy calculations.

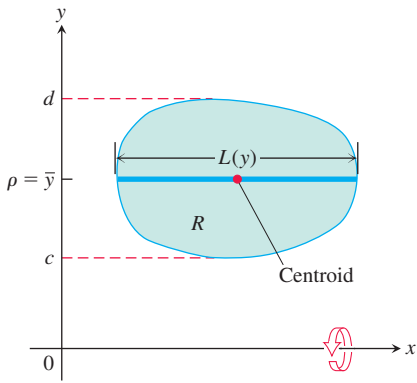


FIGURE 6.57 The region R is to be revolved (once) about the x -axis to generate a solid. A 1700-year-old theorem says that the solid's volume can be calculated by multiplying the region's area by the distance traveled by its centroid during the revolution.

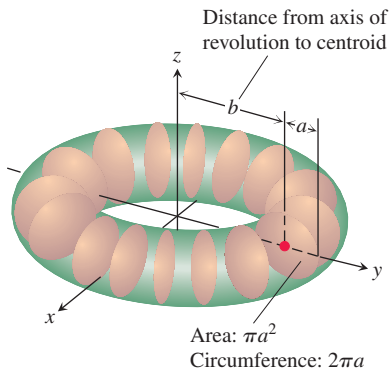


FIGURE 6.58 With Pappus's first theorem, we can find the volume of a torus without having to integrate (Example 7).

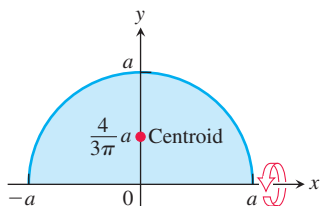


FIGURE 6.59 With Pappus's first theorem, we can locate the centroid of a semicircular region without having to integrate (Example 8).

THEOREM 1—Pappus's Theorem for Volumes

If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$V = 2\pi\rho A. \quad (9)$$

Proof We draw the axis of revolution as the x -axis with the region R in the first quadrant (Figure 6.57). We let $L(y)$ denote the length of the cross-section of R perpendicular to the y -axis at y . We assume $L(y)$ to be continuous.

By the method of cylindrical shells, the volume of the solid generated by revolving the region about the x -axis is

$$V = \int_c^d 2\pi(\text{shell radius})(\text{shell height}) dy = 2\pi \int_c^d y L(y) dy. \quad (10)$$

The y -coordinate of R 's centroid is

$$\bar{y} = \frac{\int_c^d \tilde{y} dA}{A} = \frac{\int_c^d y L(y) dy}{A}, \quad \tilde{y} = y, dA = L(y) dy$$

so that

$$\int_c^d y L(y) dy = A\bar{y}.$$

Substituting $A\bar{y}$ for the last integral in Equation (10) gives $V = 2\pi\bar{y}A$. With ρ equal to $\bar{y}$, we have $V = 2\pi\rho A$. ■

EXAMPLE 7 Find the volume of the torus (doughnut) generated by revolving a circular disk of radius a about an axis in its plane at a distance $b \geq a$ from its center (Figure 6.58).

Solution We apply Pappus's Theorem for volumes. The centroid of a disk is located at its center, the area is $A = \pi a^2$, and $\rho = b$ is the distance from the centroid to the axis of revolution (see Figure 6.58). Substituting these values into Equation (9), we find the volume of the torus to be

$$V = 2\pi(b)(\pi a^2) = 2\pi^2 b a^2. \quad \blacksquare$$

The next example shows how we can use Equation (9) in Pappus's Theorem to find one of the coordinates of the centroid of a plane region of known area A when we also know the volume V of the solid generated by revolving the region about the other coordinate axis. That is, if $\bar{y}$ is the coordinate we want to find, we revolve the region around the x -axis so that $\bar{y} = \rho$ is the distance from the centroid to the axis of revolution. The idea is that the rotation generates a solid of revolution whose volume V is an already known quantity. Then we can solve Equation (9) for ρ , which is the value of the centroid's coordinate $\bar{y}$.

EXAMPLE 8 Locate the centroid of a semicircular region of radius a .

Solution We consider the region between the semicircle $y = \sqrt{a^2 - x^2}$ (Figure 6.59) and the x -axis and imagine revolving the region about the x -axis to generate a solid sphere. By symmetry, the x -coordinate of the centroid is $\bar{x} = 0$. With $\bar{y} = \rho$ in Equation (9), we have

$$\bar{y} = \frac{V}{2\pi A} = \frac{(4/3)\pi a^3}{2\pi(1/2)\pi a^2} = \frac{4}{3\pi}a. \quad \blacksquare$$

THEOREM 2—Pappus's Theorem for Surface Areas

If an arc of a smooth plane curve is revolved once about a line in the plane that does not cut through the arc's interior, then the area of the surface generated by the arc equals the length L of the arc times the distance traveled by the arc's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$S = 2\pi\rho L. \quad (11)$$

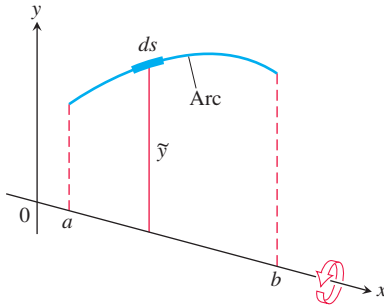


FIGURE 6.60 Figure for proving Pappus's Theorem for surface area. The arc length differential ds is given by Equation (6) in Section 6.3.

The proof we give assumes that we can model the axis of revolution as the x -axis and the arc as the graph of a continuously differentiable function of x .

Proof We draw the axis of revolution as the x -axis with the arc extending from $x = a$ to $x = b$ in the first quadrant (Figure 6.60). The area of the surface generated by the arc is

$$S = \int_{x=a}^{x=b} 2\pi y \, ds = 2\pi \int_{x=a}^{x=b} y \, ds. \quad (12)$$

The y -coordinate of the arc's centroid is

$$\bar{y} = \frac{\int_{x=a}^{x=b} \tilde{y} \, ds}{\int_{x=a}^{x=b} ds} = \frac{\int_{x=a}^{x=b} y \, ds}{L}. \quad \text{L = } \int ds \text{ is the arc's length and } \tilde{y} = y.$$

Hence

$$\int_{x=a}^{x=b} y \, ds = \bar{y}L.$$

Substituting $\bar{y}L$ for the last integral in Equation (12) gives $S = 2\pi\bar{y}L$. With ρ equal to $\bar{y}$, we have $S = 2\pi\rho L$. ■

EXAMPLE 9 Use Pappus's area theorem to find the surface area of the torus in Example 7.

Solution From Figure 6.58, the surface of the torus is generated by revolving a circle of radius a about the z -axis, and $b \geq a$ is the distance from the centroid to the axis of revolution. The arc length of the smooth curve generating this surface of revolution is the circumference of the circle, so $L = 2\pi a$. Substituting these values into Equation (11), we find the surface area of the torus to be

$$S = 2\pi(b)(2\pi a) = 4\pi^2 ba. \quad \blacksquare$$

EXERCISES 6.6

Mass of a wire

In Exercises 1–6, find the mass M and center of mass $\bar{x}$ of the linear wire covering the given interval and having the given density $\delta(x)$.

1. $1 \leq x \leq 4$, $\delta(x) = \sqrt{x}$

2. $-3 \leq x \leq 3$, $\delta(x) = 1 + 3x^2$

3. $0 \leq x \leq 3$, $\delta(x) = \frac{1}{x+1}$

4. $1 \leq x \leq 2$, $\delta(x) = \frac{8}{x^3}$

5. $\delta(x) = \begin{cases} 4, & 0 \leq x \leq 2 \\ 5, & 2 < x \leq 3 \end{cases}$

6. $\delta(x) = \begin{cases} 2-x, & 0 \leq x < 1 \\ x, & 1 \leq x \leq 2 \end{cases}$

Thin Plates with Constant Density

In Exercises 7–20, find the center of mass of a thin plate of constant density δ covering the given region.

7. The region bounded by the parabola $y = x^2$ and the line $y = 4$

8. The region bounded by the parabola $y = 25 - x^2$ and the x -axis
9. The region bounded by the parabola $y = x - x^2$ and the line $y = -x$
10. The region enclosed by the parabolas $y = x^2 - 3$ and $y = -2x^2$
11. The region bounded by the y -axis and the curve $x = y - y^3$, $0 \leq y \leq 1$
12. The region bounded by the parabola $x = y^2 - y$ and the line $y = x$
13. The region bounded by the x -axis and the curve $y = \cos x$, $-\pi/2 \leq x \leq \pi/2$
14. The region between the curve $y = \sec^2 x$, $-\pi/4 \leq x \leq \pi/4$ and the x -axis
- T** 15. The region between the curve $y = 1/x$ and the x -axis from $x = 1$ to $x = 2$. Give the coordinates to two decimal places.
16. a. The region cut from the first quadrant by the circle $x^2 + y^2 = 9$
b. The region bounded by the x -axis and the semicircle $y = \sqrt{9 - x^2}$
Compare your answer in part (b) with the answer in part (a).
17. The region in the first and fourth quadrants enclosed by the curves $y = 1/(1 + x^2)$ and $y = -1/(1 + x^2)$ and by the lines $x = 0$ and $x = 1$
18. The region bounded by the parabolas $y = 2x^2 - 4x$ and $y = 2x - x^2$
19. The region between the curve $y = 1/x$ and the x -axis from $x = 1$ to $x = 16$
20. The region bounded above by the curve $y = 1/x^3$, below by the curve $y = -1/x^3$, and on the left and right by the lines $x = 1$ and $x = a > 1$. Also, find $\lim_{a \rightarrow \infty} x$.
21. Consider a region bounded by the graphs of $y = x^4$ and $y = x^5$. Show that the center of mass lies outside the region.
22. Consider a thin plate of constant density δ lies in the region bounded by the graphs of $y = \sqrt{x}$ and $x = 2y$. Find the plate's
a. moment about the x -axis.
b. moment about the y -axis.
c. moment about the line $x = 5$.
d. moment about the line $x = -1$.
e. moment about the line $y = 2$.
f. moment about the line $y = -3$.
g. mass.
h. center of mass.

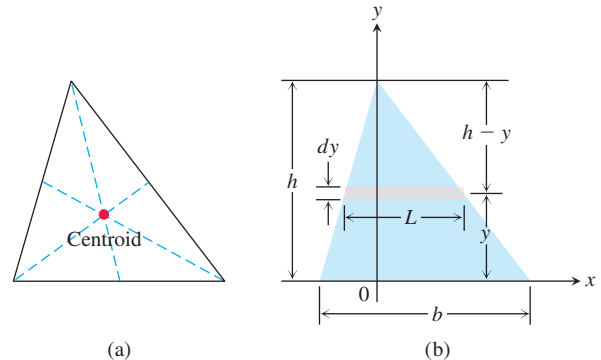
Thin Plates with Varying Density

23. Find the center of mass of a thin plate covering the region between the x -axis and the curve $y = 2/x^2$, $1 \leq x \leq 2$, if the plate's density at the point (x, y) is $\delta(x) = x^2$.
24. Find the center of mass of a thin plate covering the region bounded below by the parabola $y = x^2$ and above by the line $y = x$ if the plate's density at the point (x, y) is $\delta(x) = 12x$.
25. The region bounded by the curves $y = \pm 4/\sqrt{x}$ and the lines $x = 1$ and $x = 4$ is revolved about the y -axis to generate a solid.
a. Find the volume of the solid.

- b. Find the center of mass of a thin plate covering the region if the plate's density at the point (x, y) is $\delta(x) = 1/x$.
- c. Sketch the plate and show the center of mass in your sketch.
26. The region between the curve $y = 2/x$ and the x -axis from $x = 1$ to $x = 4$ is revolved about the x -axis to generate a solid.
a. Find the volume of the solid.
b. Find the center of mass of a thin plate covering the region if the plate's density at the point (x, y) is $\delta(x) = \sqrt{x}$.
c. Sketch the plate and show the center of mass in your sketch.

Centroids of Triangles

27. **The centroid of a triangle lies at the intersection of the triangle's medians** You may recall that the point inside a triangle that lies one-third of the way from each side toward the opposite vertex is the point where the triangle's three medians intersect. Show that the centroid lies at the intersection of the medians by showing that it too lies one-third of the way from each side toward the opposite vertex. To do so, take the following steps.
i) Stand one side of the triangle on the x -axis as in part (b) of the accompanying figure. Express dm in terms of L and dy .
ii) Use similar triangles to show that $L = (b/h)(h - y)$. Substitute this expression for L in your formula for dm .
iii) Show that $\bar{y} = h/3$.
iv) Extend the argument to the other sides.



Use the result in Exercise 27 to find the centroids of the triangles whose vertices appear in Exercises 28–32. Assume $a, b > 0$.

28. $(-1, 0), (1, 0), (0, 3)$
29. $(0, 0), (1, 0), (0, 1)$
30. $(0, 0), (a, 0), (0, a)$
31. $(0, 0), (a, 0), (0, b)$
32. $(0, 0), (a, 0), (a/2, b)$

Thin Wires

33. **Constant density** Find the moment about the x -axis of a wire of constant density that lies along the curve $y = \sqrt{x}$ from $x = 0$ to $x = 2$.
34. **Constant density** Find the moment about the x -axis of a wire of constant density that lies along the curve $y = x^3$ from $x = 0$ to $x = 1$.
35. **Variable density** Suppose that the density of the wire in Example 5 is $\delta = k \sin \theta$ (k constant). Find the center of mass.
36. **Variable density** Suppose that the density of the wire in Example 5 is $\delta = 1 + k |\cos \theta|$ (k constant). Find the center of mass.

Plates Bounded by Two Curves

In Exercises 37–40, find the centroid of the thin plate bounded by the graphs of the given functions. Use Equations (6) and (7) with $\delta = 1$ and $M = \text{area of the region covered by the plate}$.

37. $g(x) = x^2$ and $f(x) = x + 6$

38. $g(x) = x^2(x + 1)$, $f(x) = 2$, and $x = 0$

39. $g(x) = x^2(x - 1)$ and $f(x) = x^2$

40. $g(x) = 0$, $f(x) = 2 + \sin x$, $x = 0$, and $x = 2\pi$

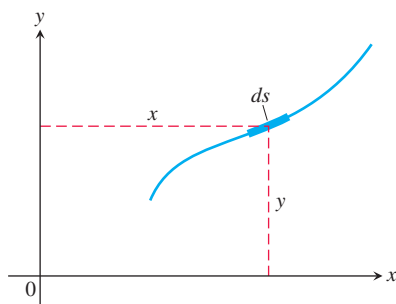
(Hint: $\int x \sin x \, dx = \sin x - x \cos x + C$.)

Theory and Examples

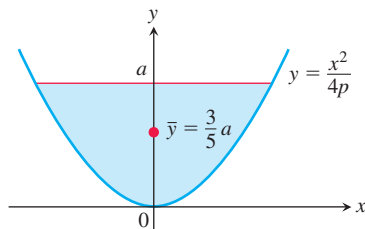
Verify the statements and formulas in Exercises 41 and 42.

41. The coordinates of the centroid of a differentiable plane curve are

$$\bar{x} = \frac{\int x \, ds}{\text{length}}, \quad \bar{y} = \frac{\int y \, ds}{\text{length}}.$$



42. Whatever the value of $p > 0$ in the equation $y = x^2/(4p)$, the y -coordinate of the centroid of the parabolic segment shown here is $\bar{y} = (3/5)a$.



The Theorems of Pappus

43. The square region with vertices $(0, 2)$, $(2, 0)$, $(4, 2)$, and $(2, 4)$ is revolved about the x -axis to generate a solid. Find the volume and surface area of the solid.

44. Use a theorem of Pappus to find the volume generated by revolving about the line $x = 5$ the triangular region bounded by the coordinate axes and the line $2x + y = 6$ (see Exercise 27).

45. Find the volume of the torus generated by revolving the circle $(x - 2)^2 + y^2 = 1$ about the y -axis.

46. Use the theorems of Pappus to find the lateral surface area and the volume of a right-circular cone.

47. Use Pappus's Theorem for surface area and the fact that the surface area of a sphere of radius a is $4\pi a^2$ to find the centroid of the semicircle $y = \sqrt{a^2 - x^2}$.

48. As found in Exercise 47, the centroid of the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 2a/\pi)$. Find the area of the surface swept out by revolving the semicircle about the line $y = a$.

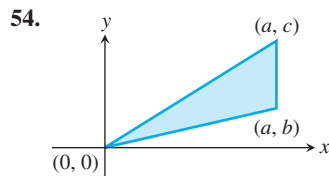
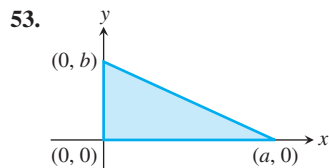
49. The area of the region R enclosed by the semiellipse $y = (b/a)\sqrt{a^2 - x^2}$ and the x -axis is $(1/2)\pi ab$, and the volume of the ellipsoid generated by revolving R about the x -axis is $(4/3)\pi ab^2$. Find the centroid of R . Notice that the location is independent of a .

50. As found in Example 8, the centroid of the region enclosed by the x -axis and the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 4a/3\pi)$. Find the volume of the solid generated by revolving this region about the line $y = -a$.

51. The region of Exercise 50 is revolved about the line $y = x - a$ to generate a solid. Find the volume of the solid.

52. As found in Exercise 47, the centroid of the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 2a/\pi)$. Find the area of the surface generated by revolving the semicircle about the line $y = x - a$.

In Exercises 53 and 54, use a theorem of Pappus to find the centroid of the given triangle. Use the fact that the volume of a cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$.



CHAPTER 6 Questions to Guide Your Review

- How do you define and calculate the volumes of solids by the method of slicing? Give an example.
- How are the disk and washer methods for calculating volumes derived from the method of slicing? Give examples of volume calculations by these methods.
- Describe the method of cylindrical shells. Give an example.
- How do you find the length of the graph of a smooth function over a closed interval? Give an example. What about functions that do not have continuous first derivatives?
- How do you define and calculate the area of the surface swept out by revolving the graph of a smooth function $y = f(x)$, $a \leq x \leq b$, about the x -axis? Give an example.

6. How do you define and calculate the work done by a variable force directed along a portion of the x -axis? How do you calculate the work it takes to pump a liquid from a tank? Give examples.
7. What is a center of mass? What is a centroid?

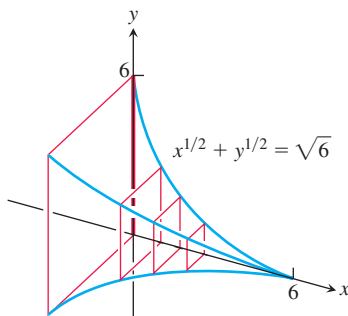
8. How do you locate the center of mass of a thin flat plate of material? Give an example.
9. How do you locate the center of mass of a thin plate bounded by two curves $y = f(x)$ and $y = g(x)$ over $a \leq x \leq b$?

CHAPTER 6 Practice Exercises

Volumes

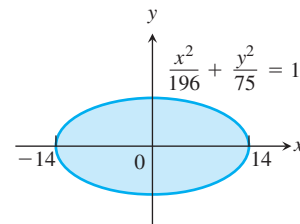
Find the volumes of the solids in Exercises 1–18.

1. The solid lies between planes perpendicular to the x -axis at $x = 0$ and $x = 1$. The cross-sections perpendicular to the x -axis between these planes are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = \sqrt{x}$.
2. The base of the solid is the region in the first quadrant between the line $y = x$ and the parabola $y = 2\sqrt{x}$. The cross-sections of the solid perpendicular to the x -axis are equilateral triangles whose bases stretch from the line to the curve.
3. The solid lies between planes perpendicular to the x -axis at $x = \pi/4$ and $x = 5\pi/4$. The cross-sections between these planes are circular disks whose diameters run from the curve $y = 2\cos x$ to the curve $y = 2\sin x$.
4. The solid lies between planes perpendicular to the x -axis at $x = 0$ and $x = 6$. The cross-sections between these planes are squares whose bases run from the x -axis up to the curve $x^{1/2} + y^{1/2} = \sqrt{6}$.

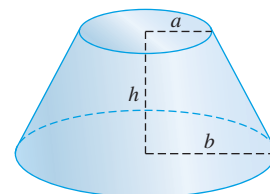


5. The solid lies between planes perpendicular to the x -axis at $x = 0$ and $x = 4$. The cross-sections of the solid perpendicular to the x -axis between these planes are circular disks whose diameters run from the curve $x^2 = 4y$ to the curve $y^2 = 4x$.
6. The base of the solid is the region bounded by the parabola $y^2 = 4x$ and the line $x = 1$ in the xy -plane. Each cross-section perpendicular to the x -axis is an equilateral triangle with one edge in the plane. (The triangles all lie on the same side of the plane.)
7. Find the volume of the solid generated by revolving the region bounded by the x -axis, the curve $y = 3x^4$, and the lines $x = 1$ and $x = -1$ about (a) the x -axis; (b) the y -axis; (c) the line $x = 1$; (d) the line $y = 3$.
8. Find the volume of the solid generated by revolving the “triangular” region bounded by the curve $y = 4/x^3$ and the lines $x = 1$ and $y = 1/2$ about (a) the x -axis; (b) the y -axis; (c) the line $x = 2$; (d) the line $y = 4$.

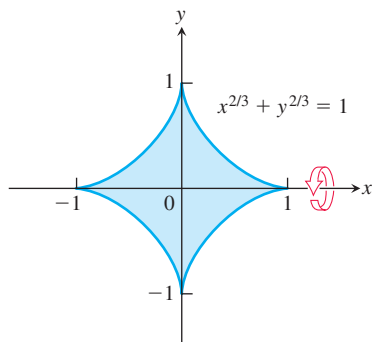
9. Find the volume of the solid generated by revolving the region bounded on the left by the parabola $x = y^2 + 1$ and on the right by the line $x = 5$ about (a) the x -axis; (b) the y -axis; (c) the line $x = 5$.
10. Find the volume of the solid generated by revolving the region bounded by the parabola $y^2 = 4x$ and the line $y = x$ about (a) the x -axis; (b) the y -axis; (c) the line $x = 4$; (d) the line $y = 4$.
11. Find the volume of the solid generated by revolving the “triangular” region bounded by the x -axis, the line $x = \pi/3$, and the curve $y = \tan x$ in the first quadrant about the x -axis.
12. Find the volume of the solid generated by revolving the region bounded by the curve $y = \sin x$ and the lines $x = 0$, $x = \pi$, and $y = 2$ about the line $y = 2$.
13. Find the volume of the solid generated by revolving the region bounded by the curve $x = e^{y^2}$ and the lines $y = 0$, $x = 0$, and $y = 1$ about the x -axis.
14. Find the volume of the solid generated by revolving about the x -axis the region bounded by $y = 2\tan x$, $y = 0$, $x = -\pi/4$, and $x = \pi/4$. (The region lies in the first and third quadrants and resembles a skewed bowtie.)
15. **Volume of a solid sphere hole** A round hole of radius $\sqrt{3}$ m is bored through the center of a solid sphere of radius 2 m. Find the volume of material removed from the sphere.
16. **Volume of a football** A football resembles the surface of revolution obtained by revolving the ellipse shown here around the x -axis. Find the football’s volume to the nearest cubic centimeter.



17. Set up and evaluate an integral to find the volume of the given circular frustum of height h and radii a and b .



18. The graph of $x^{2/3} + y^{2/3} = 1$ is called an astroid and is given below. Find the volume of the solid formed by revolving the region enclosed by the astroid about the x -axis.



Lengths of Curves

Find the lengths of the curves in Exercises 19–24.

19. $y = x^{1/2} - (1/3)x^{3/2}$, $1 \leq x \leq 4$
20. $x = y^{2/3}$, $1 \leq y \leq 8$
21. $y = x^2 - (\ln x)/8$, $1 \leq x \leq 2$
22. $x = (y^3/12) + (1/y)$, $1 \leq y \leq 2$
23. $y = \arcsin x - \sqrt{1 - x^2}$, $0 \leq x \leq \frac{3}{4}$
24. $y = \frac{2}{3}x^{3/2} - 1$, $0 \leq x \leq 1$

Areas of Surfaces of Revolution

In Exercises 25–28, find the areas of the surfaces generated by revolving the curves about the given axes.

25. $y = \sqrt{2x + 1}$, $0 \leq x \leq 3$; x -axis
26. $y = x^3/3$, $0 \leq x \leq 1$; x -axis
27. $x = \sqrt{4y - y^2}$, $1 \leq y \leq 2$; y -axis
28. $x = \sqrt{y}$, $2 \leq y \leq 6$; y -axis

Work

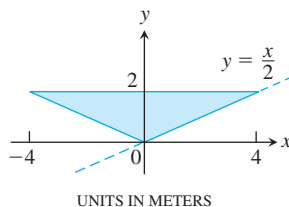
29. **Lifting equipment** A rock climber is about to haul up 100 N of equipment that has been hanging beneath her on 40 m of rope that weighs 0.8 N/m. How much work will it take? (*Hint*: Solve for the rope and equipment separately, then add.)
30. **Leaky tank truck** You drove an 4000-L tank truck of water from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 1500-m elevation change in 50 min. Assuming that the water leaked out at a steady rate, how much work was spent in carrying water to the top? Do not count the work done in getting yourself and the truck there. Water weighs 9.8 N/L.
31. **Earth's attraction** The force of attraction on an object below Earth's surface is directly proportional to its distance from Earth's center. Find the work done in moving a weight of w N located a km below Earth's surface up to the surface itself. Assume Earth's radius is a constant r km.
32. **Garage door spring** A force of 200 N will stretch a garage door spring 0.8 m beyond its unstressed length. How far will a 300-N force stretch the spring? How much work does it take to stretch the spring this far from its unstressed length?
33. **Pumping a reservoir** A reservoir shaped like a right-circular cone, point down, 20 m across the top and 8 m deep, is full of water. How much work does it take to pump the water to a level 6 m above the top?
34. **Pumping a reservoir** (*Continuation of Exercise 33.*) The reservoir is filled to a depth of 5 m, and the water is to be pumped to the same level as the top. How much work does it take?
35. **Pumping a conical tank** A right-circular conical tank, point down, with top radius 5 m and height 10 m, is filled with a liquid whose weight-density is 9000 N/m^3 . How much work does it take to pump the liquid to a point 2 m above the tank? If the pump is driven by a motor rated at 41,250 J/s, how long will it take to empty the tank?
36. **Pumping a cylindrical tank** A storage tank is a right-circular cylinder 6 m long and 2.5 m in diameter with its axis horizontal. If the tank is half full of olive oil weighing 8950 N/m^3 , find the work done in emptying it through a pipe that runs from the bottom of the tank to an outlet that is 2 m above the top of the tank.
37. Assume that a spring does not follow Hooke's Law. Instead, the force required to stretch the spring x m from its natural length is $F(x) = 5x^{3/2}$ N. How much work does it take to
 - a. stretch the spring 2 m from its natural length?
 - b. stretch the spring from an initial 1 m past its natural length to 3 m past its natural length?
38. Assume that a spring does not follow Hooke's Law. Instead, the force required to stretch the spring x m from its natural length is $F(x) = k\sqrt{5 + x^2}$ N.
 - a. If a 3-N force stretches the spring 2 m, find the value of k .
 - b. How much work is required to stretch the spring 1 m from its natural length?

Centers of Mass and Centroids

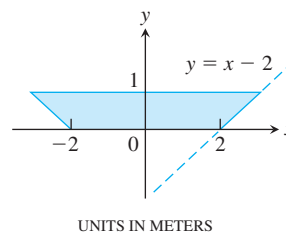
39. Find the centroid of a thin, flat plate covering the region enclosed by the parabolas $y = 2x^2$ and $y = 3 - x^2$.
40. Find the centroid of a thin, flat plate covering the region enclosed by the x -axis, the lines $x = 2$ and $x = -2$, and the parabola $y = x^2$.
41. Find the centroid of a thin, flat plate covering the "triangular" region in the first quadrant bounded by the y -axis, the parabola $y = x^2/4$, and the line $y = 4$.
42. Find the centroid of a thin, flat plate covering the region enclosed by the parabola $y^2 = x$ and the line $x = 2y$.
43. Find the center of mass of a thin, flat plate covering the region enclosed by the parabola $y^2 = x$ and the line $x = 2y$ if the density function is $\delta(y) = 1 + y$. (Use horizontal strips.)
44. a. Find the center of mass of a thin plate of constant density covering the region between the curve $y = 3/x^{3/2}$ and the x -axis from $x = 1$ to $x = 9$.
b. Find the plate's center of mass if, instead of being constant, the density is $\delta(x) = x$. (Use vertical strips.)

Fluid Force

- 45. Trough of water** The vertical triangular plate shown here is the end plate of a trough full of water ($w = 9800$). What is the fluid force against the plate?



- 46. Trough of maple syrup** The vertical trapezoidal plate shown here is the end plate of a trough full of maple syrup weighing $11,000 \text{ N/m}^3$. What is the force exerted by the syrup against the end plate of the trough when the syrup is 0.5 m deep?



- 47. Force on a parabolic gate** A flat vertical gate in the face of a dam is shaped like the parabolic region between the curve $y = 4x^2$ and the line $y = 4$, with measurements in meters. The top of the gate lies 5 m below the surface of the water. Find the force exerted by the water against the gate ($w = 9800$).

- T 48.** You plan to store mercury ($w = 133,350 \text{ N/m}^3$) in a vertical rectangular tank with a 0.3 m square base side whose interior side wall can withstand a total fluid force of $150,000 \text{ N}$. About how many cubic meters of mercury can you store in the tank at any one time?

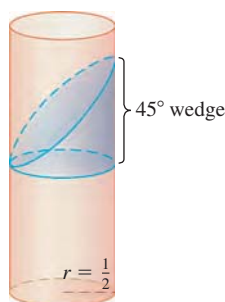
CHAPTER 6 Additional and Advanced Exercises

Volume and Length

- A solid is generated by revolving about the x -axis the region bounded by the graph of the positive continuous function $y = f(x)$, the x -axis, the fixed line $x = a$, and the variable line $x = b$, $b > a$. Its volume, for all b , is $b^2 - ab$. Find $f(x)$.
- A solid is generated by revolving about the x -axis the region bounded by the graph of the positive continuous function $y = f(x)$, the x -axis, and the lines $x = 0$ and $x = a$. Its volume, for all $a > 0$, is $a^2 + a$. Find $f(x)$.
- Suppose that the increasing function $f(x)$ is smooth for $x \geq 0$ and that $f(0) = a$. Let $s(x)$ denote the length of the graph of f from $(0, a)$ to $(x, f(x))$, $x > 0$. Find $f(x)$ if $s(x) = Cx$ for some constant C . What are the allowable values for C ?
- a. Show that for $0 < \alpha \leq \pi/2$,

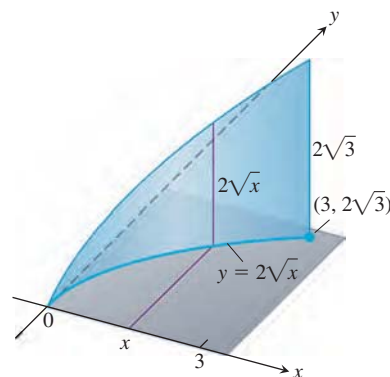
$$\int_0^\alpha \sqrt{1 + \cos^2 \theta} d\theta > \sqrt{\alpha^2 + \sin^2 \alpha}.$$

- Generalize the result in part (a).
- Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x$ and $y = x^2$ about the line $y = x$.
 - Consider a right-circular cylinder of diameter 1. Form a wedge by making one slice parallel to the base of the cylinder completely through the cylinder, and another slice at an angle of 45° to the first slice and intersecting the first slice at the opposite edge of the cylinder (see accompanying diagram). Find the volume of the wedge.

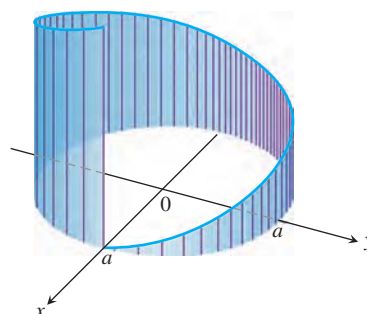


Surface Area

- At points on the curve $y = 2\sqrt{x}$, line segments of length $h = y$ are drawn perpendicular to the xy -plane. (See accompanying figure.) Find the area of the surface formed by these perpendiculars from $(0, 0)$ to $(3, 2\sqrt{3})$.



- At points on a circle of radius a , line segments are drawn perpendicular to the plane of the circle, the perpendicular at each point P being of length ks , where s is the length of the arc of the circle measured counterclockwise from $(a, 0)$ to P , and k is a positive constant, as shown here. Find the area of the surface formed by the perpendiculars along the arc beginning at $(a, 0)$ and extending once around the circle.



Work

9. A particle of mass m starts from rest at time $t = 0$ and is moved along the x -axis with constant acceleration a from $x = 0$ to $x = h$ against a variable force of magnitude $F(t) = t^2$. Find the work done.
10. **Work and kinetic energy** Suppose a 50-g golf ball is placed on a vertical spring with force constant $k = 2 \text{ N/cm}$. The spring is compressed 15 cm and released. About how high does the ball go (measured from the spring's rest position)?

Centers of Mass

11. Find the centroid of the region bounded below by the x -axis and above by the curve $y = 1 - x^n$, n an even positive integer. What is the limiting position of the centroid as $n \rightarrow \infty$?
12. If you haul a telephone pole on a two-wheeled carriage behind a truck, you want the wheels to be 1 m or so behind the pole's center of mass to provide an adequate "tongue" weight. The 12-m wooden telephone poles used by Verizon have a 66-cm circumference at the top and a 104-cm circumference at the base. About how far from the top is the center of mass?
13. Suppose that a thin metal plate of area A and constant density δ occupies a region R in the xy -plane, and let M_y be the plate's moment about the y -axis. Show that the plate's moment about the line $x = b$ is
- $M_y - b\delta A$ if the plate lies to the right of the line, and
 - $b\delta A - M_y$ if the plate lies to the left of the line.
14. Find the center of mass of a thin plate covering the region bounded by the curve $y^2 = 4ax$ and the line $x = a$, $a = \text{positive constant}$, if the density at (x, y) is directly proportional to (a) x , (b) $|y|$.
15. a. Find the centroid of the region in the first quadrant bounded by two concentric circles and the coordinate axes, if the circles have radii a and b , $0 < a < b$, and their centers are at the origin.
b. Find the limits of the coordinates of the centroid as $a \rightarrow b$ and discuss the meaning of the result.
16. A triangular corner is cut from a square 40 cm on a side. The area of the triangle removed is 400 cm^2 . If the centroid of the remaining region is 22 cm from one side of the original square, how far is it from the remaining sides?

Fluid Force

17. A triangular plate ABC is submerged in water with its plane vertical. The side AB , 4 m long, is 6 m below the surface of the water, while the vertex C is 2 m below the surface. Find the force exerted by the water on one side of the plate.
18. A vertical rectangular plate is submerged in a fluid with its top edge parallel to the fluid's surface. Show that the force exerted by the fluid on one side of the plate equals the average value of the pressure up and down the plate times the area of the plate.

CHAPTER 6 Technology Application Projects**Mathematica/Maple Projects**

Projects can be found within MyLab Math.

- Using Riemann Sums to Estimate Areas, Volumes, and Lengths of Curves**

Visualize and approximate areas and volumes in **Part I** and **Part II**: Volumes of Revolution; and **Part III**: Lengths of Curves.

- Modeling a Bungee Cord Jump**

Collect data (or use data previously collected) to build and refine a model for the force exerted by a jumper's bungee cord. Use the work-energy theorem to compute the distance fallen for a given jumper and a given length of bungee cord.

7

Integrals and Transcendental Functions



OVERVIEW Our treatment of the logarithmic and exponential functions has been rather informal. In this chapter, we give a rigorous analytic approach to the definitions and properties of these functions. We also introduce the hyperbolic functions and their inverses. Like the trigonometric functions, these functions belong to the class of *transcendental functions*.

7.1 The Logarithm Defined as an Integral

In Chapter 1, we introduced the natural logarithm function $\ln x$ as the inverse of the exponential function e^x . The function e^x was chosen as that function in the family of general exponential functions a^x , $a > 0$, whose graph has slope 1 as it crosses the y -axis. The function a^x was presented intuitively, however, based on its graph at rational values of x .

In this section we recreate the theory of logarithmic and exponential functions from an entirely different point of view. Here we define these functions analytically and derive their behaviors. To begin, we use the Fundamental Theorem of Calculus to define the natural logarithm function $\ln x$ as an integral. We quickly develop its properties, including the algebraic, geometric, and analytic properties with which we are already familiar. Next we introduce the function e^x as the inverse function of $\ln x$, and establish its properties. Defining $\ln x$ as an integral and e^x as its inverse is an indirect approach that gives an elegant and powerful way to obtain and validate the key properties of logarithmic and exponential functions.

Definition of the Natural Logarithm Function

The natural logarithm of a positive number x , written as $\ln x$, is the value of an integral. The appropriate integral is suggested by our earlier results in Chapter 5.

DEFINITION The **natural logarithm** is the function given by

$$\ln x = \int_1^x \frac{1}{t} dt, \quad x > 0.$$

From the Fundamental Theorem of Calculus, we know that $\ln x$ is a continuous function. Geometrically, if $x > 1$, then $\ln x$ is the area under the curve $y = 1/t$ from $t = 1$ to $t = x$ (Figure 7.1). For $0 < x < 1$, $\ln x$ gives the negative of the area under the curve from x to 1,

and the function is not defined for $x \leq 0$. From the Zero Width Interval Rule for definite integrals, we also have

$$\ln 1 = \int_1^1 \frac{1}{t} dt = 0.$$

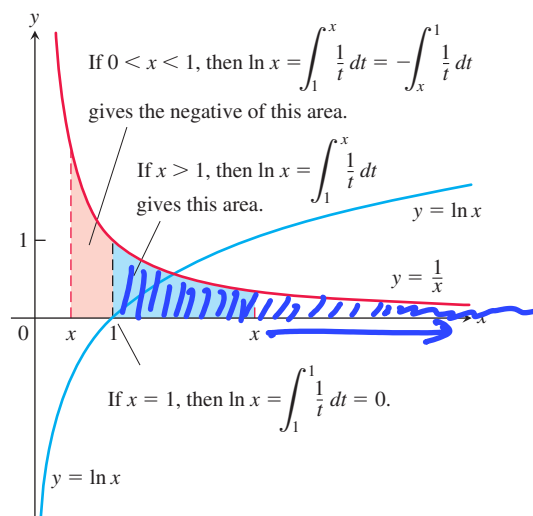


FIGURE 7.1 The graph of $y = \ln x$ and its relation to the function $y = 1/x$, $x > 0$. The graph of the logarithm rises above the x -axis as x moves from 1 to the right, and it falls below the axis as x moves from 1 to the left.

Notice that we show the graph of $y = 1/x$ in Figure 7.1 but use $y = 1/t$ in the integral. Using x for everything would have us writing

$$\ln x = \int_1^x \frac{1}{x} dx,$$

with x meaning two different things. So we change the variable of integration to t .

By using rectangles to obtain finite approximations of the area under the graph of $y = 1/t$ and over the interval between $t = 1$ and $t = x$, as in Section 5.1, we can approximate the values of the function $\ln x$. Several values are given in Table 7.1. There is an important number between $x = 2$ and $x = 3$ whose natural logarithm equals 1. This number, which we now define, exists because $\ln x$ is a continuous function and therefore satisfies the Intermediate Value Theorem on the interval $[2, 3]$.

DEFINITION The **number e** is the number in the domain of the natural logarithm that satisfies

$$\ln(e) = \int_1^e \frac{1}{t} dt = 1.$$

Interpreted geometrically, the number e corresponds to the point on the x -axis for which the area under the graph of $y = 1/t$ and above the interval $[1, e]$ equals the area of the unit square. That is, the area of the region shaded blue in Figure 7.1 is 1 square unit when $x = e$. We will see further on that this is the same number $e \approx 2.718281828$ we have encountered before.

TABLE 7.1 Typical 2-place values of $\ln x$

x	$\ln x$
0	undefined
0.05	-3.00
0.5	-0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

The Derivative of $y = \ln x$

By the first part of the Fundamental Theorem of Calculus (Section 5.4),

$$\frac{d}{dx} \ln x = \frac{d}{dx} \int_1^x \frac{1}{t} dt = \frac{1}{x},$$

so we have

$$\frac{d}{dx} \ln x = \frac{1}{x}, \quad x > 0. \quad (1)$$

Therefore, the function $y = \ln x$ is a solution to the initial value problem $dy/dx = 1/x$, $x > 0$, with $y(1) = 0$. Notice that the derivative is always positive.

If u is a differentiable function of x whose values are positive so that $\ln u$ is defined, then by applying the Chain Rule, we obtain

$$\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}, \quad u > 0. \quad (2)$$

The derivative of $\ln |x|$ can be found just as in Example 3(c) of Section 3.8, giving

$$\frac{d}{dx} \ln |x| = \frac{1}{x}, \quad x \neq 0. \quad (3)$$

Moreover, if b is any constant with $bx > 0$, Equation (2) gives

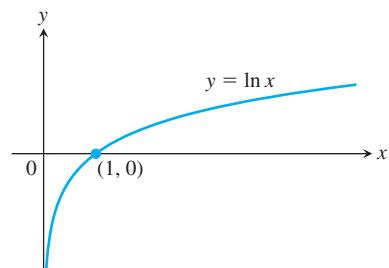
$$\frac{d}{dx} \ln bx = \frac{1}{bx} \cdot \frac{d}{dx}(bx) = \frac{1}{bx}(b) = \frac{1}{x}.$$

The Graph and Range of $\ln x$

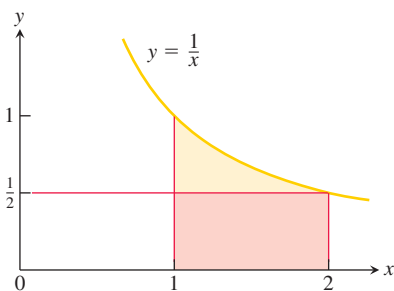
The derivative $d(\ln x)/dx = 1/x$ is positive for $x > 0$, so $\ln x$ is an increasing function of x . The second derivative, $-1/x^2$, is negative, so the graph of $\ln x$ is concave down. (See Figure 7.2a.)

The function $\ln x$ has the following familiar algebraic properties, which we stated in Section 1.5. In Section 4.2 we showed these properties are a consequence of Corollary 2 of the Mean Value Theorem, and those derivations still apply.

1. $\ln bx = \ln b + \ln x$	2. $\ln \frac{b}{x} = \ln b - \ln x$
3. $\ln \frac{1}{x} = -\ln x$	4. $\ln x^r = r \ln x$, r rational



(a)



(b)

FIGURE 7.2 (a) The graph of the natural logarithm. (b) The rectangle of height $y = 1/2$ fits beneath the graph of $y = 1/x$ for the interval $1 \leq x \leq 2$.

We can estimate the value of $\ln 2$ by considering the area under the graph of $y = 1/x$ and above the interval $[1, 2]$. In Figure 7.2(b) a rectangle of height $1/2$ over the interval

$[1, 2]$ fits under the graph. Therefore, the area under the graph, which is $\ln 2$, is greater than the area of the rectangle, which is $1/2$. So $\ln 2 > 1/2$. Knowing this we have

$$\ln 2^n = n \ln 2 > n\left(\frac{1}{2}\right) = \frac{n}{2}.$$

This result shows that $\ln(2^n) \rightarrow \infty$ as $n \rightarrow \infty$. Since $\ln x$ is an increasing function, it follows that

$$\lim_{x \rightarrow \infty} \ln x = \infty. \quad \text{ln } x \text{ is increasing and not bounded above.}$$

We also have

$$\lim_{x \rightarrow 0^+} \ln x = \lim_{t \rightarrow \infty} \ln \frac{1}{t} = \lim_{t \rightarrow \infty} (-\ln t) = -\infty. \quad x = 1/t$$

We defined $\ln x$ for $x > 0$, so the domain of $\ln x$ is the set of positive real numbers. The above discussion and the Intermediate Value Theorem show that its range is the entire real line, giving the familiar graph of $y = \ln x$ shown in Figure 7.2(a).

The Integral $\int 1/x \, dx$

Equation (3) leads to the following integral formula:

$$\int \frac{1}{x} dx = \ln|x| + C. \quad (4)$$

If u is a differentiable function that is never zero, then

$$\int \frac{1}{u} du = \ln|u| + C. \quad (5)$$

Equation (5) applies anywhere on the domain of $1/u$, which is the set of points where $u \neq 0$. It says that integrals that have the form $\int \frac{du}{u}$ lead to logarithms. Whenever $u = f(x)$ is a differentiable function that is never zero, we have that $du = f'(x) dx$ and

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C.$$

EXAMPLE 1 We rewrite an integral so that it has the form $\int \frac{du}{u}$.

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} \frac{4 \cos \theta}{3 + 2 \sin \theta} d\theta &= \int_1^5 \frac{2}{u} du & u = 3 + 2 \sin \theta, \quad du = 2 \cos \theta d\theta, \\ &= 2 \ln|u| \Big|_1^5 & u(-\pi/2) = 1, \quad u(\pi/2) = 5 \\ &= 2 \ln|5| - 2 \ln|1| = 2 \ln 5 \end{aligned}$$

Note that $u = 3 + 2 \sin \theta$ is always positive on $[-\pi/2, \pi/2]$, so Equation (5) applies. ■

The Inverse of $\ln x$ and the Number e

The function $\ln x$, being an increasing function of x with domain $(0, \infty)$ and range $(-\infty, \infty)$, has an inverse $\ln^{-1} x$ with domain $(-\infty, \infty)$ and range $(0, \infty)$. The graph of $\ln^{-1} x$ is the graph of $\ln x$ reflected across the line $y = x$. As you can see in Figure 7.3,

$$\lim_{x \rightarrow \infty} \ln^{-1} x = \infty \quad \text{and} \quad \lim_{x \rightarrow -\infty} \ln^{-1} x = 0.$$

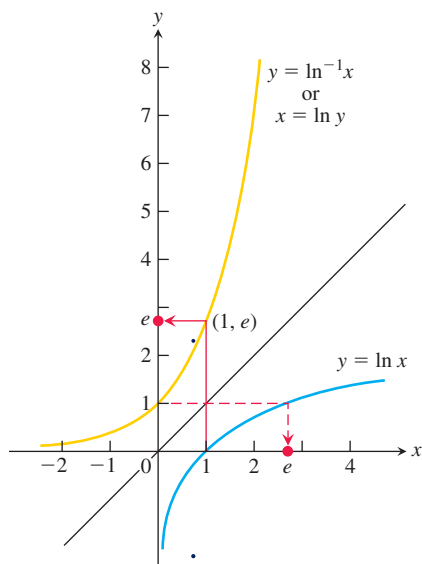


FIGURE 7.3 The graphs of $y = \ln x$ and $y = \ln^{-1} x = \exp x$. The number e is $\ln^{-1} 1 = \exp(1)$.

The notations $\ln^{-1} x$, $\exp x$, and e^x all refer to the natural exponential function.

The inverse function $\ln^{-1} x$ is also denoted by $\exp x$. We have not yet established that $\exp x$ is an exponential function, only that $\exp x$ is the inverse of the function $\ln x$. We will now show that $\ln^{-1} x = \exp x$ is, in fact, the exponential function with base e .

The number e was defined to satisfy the equation $\ln(e) = 1$, so $e = \exp(1)$. We can raise the number e to a rational power r using algebra:

$$e^2 = e \cdot e, \quad e^{-2} = \frac{1}{e^2}, \quad e^{1/2} = \sqrt{e}, \quad e^{2/3} = \sqrt[3]{e^2},$$

and so on. Since e is positive, e^r is positive too. Therefore, e^r has a logarithm. When we take the logarithm, we find that if r is rational, then

$$\ln e^r = r \ln e = r \cdot 1 = r.$$

Applying the function $\ln^{-1}$ to both sides of the equation $\ln e^r = r$, we find that

$$e^r = \exp r \quad \text{for } r \text{ rational.} \quad \exp \text{ is } \ln^{-1}. \quad (6)$$

Thus $\exp r$ coincides with the exponential function e^r for all rational values of r . We have not yet found a way to give an exact meaning to e^x for x irrational, but we can use Equation (6) to do so. The function $\exp x = \ln^{-1} x$ has domain $(-\infty, \infty)$, so it is defined for every x . We have $\exp r = e^r$ for r rational by Equation (6), and we now define e^x to equal $\exp x$ for all x .

DEFINITION For every real number x , we define the **natural exponential function** to be $e^x = \exp x$.

For the first time we have a precise meaning for a number raised to an irrational power. Usually the exponential function is denoted by e^x rather than $\exp x$. Since $\ln x$ and e^x are inverses of one another, we have the following relations.

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \quad (\text{all } x > 0)$$

$$\ln(e^x) = x \quad (\text{all } x)$$

The Derivative and Integral of e^x

The exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero. We calculate its derivative by using Theorem 3 of Section 3.8 and our knowledge of the derivative of $\ln x$. Let

$$f(x) = \ln x \quad \text{and} \quad y = e^x = \ln^{-1} x = f^{-1}(x).$$

Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} e^x = \frac{d}{dx} \ln^{-1} x \\ &= \frac{d}{dx} f^{-1}(x) \\ &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3, Section 3.8} \\ &= \frac{1}{f'(e^x)} && f^{-1}(x) = e^x \\ &= \frac{1}{\left(\frac{1}{e^x}\right)} && f'(z) = \frac{1}{z} \text{ with } z = e^x \\ &= e^x. \end{aligned}$$

Typical values of e^x

x	e^x (rounded)
-1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

That is, for $y = e^x$, we find that $dy/dx = e^x$, so the natural exponential function e^x is its own derivative, just as we claimed in Section 3.3.

$$\frac{d}{dx}e^x = e^x. \quad (7)$$

We will see in the next section that the only functions that behave this way are constant multiples of e^x . The Chain Rule extends the derivative result in the usual way to a more general form: If u is any differentiable function of x , then

$$\frac{d}{dx}e^u = e^u \frac{du}{dx}. \quad (8)$$

Since $e^x > 0$, its derivative is also everywhere positive, so it is an increasing and continuous function for all x , having limits

$$\lim_{x \rightarrow -\infty} e^x = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} e^x = \infty.$$

It follows that the x -axis ($y = 0$) is a horizontal asymptote of the graph $y = e^x$ (see Figure 7.3).

Equation (7) also tells us the indefinite integral of e^x .

$$\int e^x dx = e^x + C$$

If $f(x) = e^x$, then we see from Equation (7) that $f'(0) = e^0 = 1$. That is, the exponential function e^x has slope 1 as it crosses the y -axis at $x = 0$. This agrees with our assertion for the natural exponential in Section 3.3.

Logarithms and Laws of Exponents

The familiar algebraic properties of logarithms and exponential functions were stated in Section 1.5. We now show that these follow from the definition of the logarithm as an integral that we have used. The properties of logarithms are stated in Theorem 1.

THEOREM 1—Algebraic Properties of the Natural Logarithm

For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. Product Rule: $\ln bx = \ln b + \ln x$

2. Quotient Rule: $\ln \frac{b}{x} = \ln b - \ln x$

3. Reciprocal Rule: $\ln \frac{1}{x} = -\ln x$ Rule 2 with $b = 1$

4. Power Rule: $\ln x^r = r \ln x$

The proof uses the Mean Value Theorem, which was established in Section 4.2.

Proof that $\ln bx = \ln b + \ln x$ The argument starts by observing that $\ln bx$ and $\ln x$ have the same derivative:

$$\frac{d}{dx} \ln(bx) = \frac{b}{bx} = \frac{1}{x} = \frac{d}{dx} \ln x.$$

According to Corollary 2 of the Mean Value Theorem, the functions must differ by a constant, which means that

$$\ln bx = \ln x + C$$

for some C .

Since this last equation holds for all positive values of x , it must hold for $x = 1$. Hence,

$$\begin{aligned} \ln(b \cdot 1) &= \ln 1 + C \\ \ln b &= 0 + C && \ln 1 = 0 \\ C &= \ln b. \end{aligned}$$

By substituting, we conclude

$$\ln bx = \ln b + \ln x. \quad \blacksquare$$

Proof that $\ln x^r = r \ln x$ We use the same-derivative argument again. For all positive values of x ,

$$\begin{aligned} \frac{d}{dx} \ln x^r &= \frac{1}{x^r} \frac{d}{dx} (x^r) && \text{Chain Rule} \\ &= \frac{1}{x^r} r x^{r-1} && \text{Derivative Power Rule} \\ &= r \cdot \frac{1}{x} = \frac{d}{dx} (r \ln x). \end{aligned}$$

Since $\ln x^r$ and $r \ln x$ have the same derivative,

$$\ln x^r = r \ln x + C$$

for some constant C . Taking x to be 1 identifies C as zero, and we're done. $\blacksquare$

You are asked to prove the Quotient Rule for logarithms,

$$\ln\left(\frac{b}{x}\right) = \ln b - \ln x,$$

in Exercises 63. The Reciprocal Rule, $\ln(1/x) = -\ln x$, is a special case of the Quotient Rule, obtained by taking $b = 1$ and noting that $\ln 1 = 0$.

We now state the algebraic properties of exponential functions.

THEOREM 2—Laws of Exponents for e^x

For all numbers x and y , the natural exponential function e^x obeys the following laws.

$$1. e^x e^y = e^{x+y}$$

$$2. e^{-x} = \frac{1}{e^x}$$

$$3. \frac{e^x}{e^y} = e^{x-y}$$

$$4. (e^x)^y = e^{xy} = (e^y)^x$$

Proof that $e^x e^y = e^{x+y}$

$$\begin{aligned}
 e^x e^y &= e^{\ln(e^x e^y)} & u &= e^{\ln u}. \\
 &= e^{\ln(e^x) + \ln(e^y)} & \text{Product formula for } \ln \\
 &= e^{x+y}. & \ln e^u &= u.
 \end{aligned}$$

Theorem 1 and the inverse relationship between the logarithmic and exponential functions also imply the other algebraic properties of the exponential function (see Exercise 67).

The General Exponential Function a^x

Since $a = e^{\ln a}$ for any positive number a , we can express a^x as $(e^{\ln a})^x = e^{x \ln a}$. We therefore make the following definition, consistent with what we stated in Section 1.5.

DEFINITION For any numbers $a > 0$ and x , the exponential function with base a is given by

$$a^x = e^{x \ln a}.$$

When $a = e$, the definition gives $a^x = e^{x \ln a} = e^{x \ln e} = e^{x \cdot 1} = e^x$. Similarly, the power function $f(x) = x^r$ is defined for positive x by the formula $x^r = e^{r \ln x}$, which holds for any real number r , rational or irrational.

Theorem 2 is also valid for a^x , the exponential function with base a . For example,

$$\begin{aligned}
 a^x \cdot a^y &= e^{x \ln a} \cdot e^{y \ln a} & \text{Definition of } a^x \\
 &= e^{x \ln a + y \ln a} & \text{Law 1} \\
 &= e^{(x+y) \ln a} & \text{Common factor } \ln a \\
 &= a^{x+y}. & \text{Definition of } a^x
 \end{aligned}$$

Starting with the definition $a^x = e^{x \ln a}$, $a > 0$, we get the derivative

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = (\ln a) e^{x \ln a} = (\ln a) a^x,$$

so

$$\frac{d}{dx} a^x = a^x \ln a.$$

Alternatively, we get the same derivative rule by applying logarithmic differentiation:

$$\begin{aligned}
 y &= a^x \\
 \ln y &= x \ln a & \text{Take logarithms.} \\
 \frac{1}{y} \frac{dy}{dx} &= \ln a & \text{Differentiate with respect to } x. \\
 \frac{dy}{dx} &= y \ln a = a^x \ln a.
 \end{aligned}$$

With the Chain Rule, we get a more general form, as in Section 3.8: If $a > 0$ and u is a differentiable function of x , then a^u is a differentiable function of x and

$$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}.$$

The General Power Function

x^r is the function $e^{r \ln x}$.

e^x

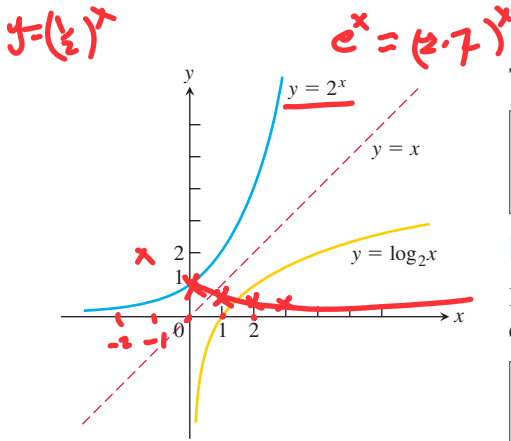


FIGURE 7.4 The graph of 2^x and its inverse, $\log_2 x$.

The integral equivalent of this derivative rule is

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

Logarithms with Base a

If a is any positive number other than 1, the function a^x is one-to-one and has a nonzero derivative at every point. It therefore has a differentiable inverse.

DEFINITION For any positive number $a \neq 1$, the **logarithm of x with base a** , denoted by $\log_a x$, is the inverse function of a^x .

The graph of $y = \log_a x$ can be obtained by reflecting the graph of $y = a^x$ across the 45° line $y = x$ (Figure 7.4). When $a = e$, we have $\log_e x = \ln x$. Since $\log_a x$ and a^x are inverses of one another, composing them in either order gives the identity function.

Inverse Equations for a^x and $\log_a x$

$$\begin{aligned} a^{\log_a x} &= x & (x > 0) \\ \log_a(a^x) &= x & (\text{all } x) \end{aligned}$$

As stated in Section 1.5, the function $\log_a x$ is just a numerical multiple of $\ln x$. We see this from the following derivation.

TABLE 7.2 Rules for base a logarithms

For any numbers $x > 0$ and $y > 0$,

1. Product Rule:

$$\log_a xy = \log_a x + \log_a y$$

2. Quotient Rule:

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

3. Reciprocal Rule:

$$\log_a \frac{1}{y} = -\log_a y$$

4. Power Rule:

$$\log_a x^y = y \log_a x$$

$$\begin{aligned} y &= \log_a x && \text{Defining equation for } y \\ a^y &= x && \text{Equivalent equation} \\ \ln a^y &= \ln x && \text{Natural log of both sides} \\ y \ln a &= \ln x && \text{Algebra Rule 4 for natural log} \\ y &= \frac{\ln x}{\ln a} && \text{Solve for } y. \end{aligned}$$

$$\log_a x = \frac{\ln x}{\ln a}$$

It then follows easily that the arithmetic rules satisfied by $\log_a x$ are the same as the ones for $\ln x$. These rules, given in Table 7.2, can be proved by dividing the corresponding rules for the natural logarithm function by $\ln a$. For example,

$$\ln xy = \ln x + \ln y \quad \text{Rule 1 for natural logarithms ...}$$

$$\frac{\ln xy}{\ln a} = \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a} \quad \text{... divided by } \ln a \text{ ...}$$

$$\log_a xy = \log_a x + \log_a y. \quad \text{... gives Rule 1 for base } a \text{ logarithms.}$$

Derivatives and Integrals Involving $\log_a x$

To find derivatives or integrals involving base a logarithms, we can convert them to natural logarithms. In particular, differentiating $\log_a x$ gives

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx} \left(\frac{\ln x}{\ln a} \right) = \frac{1}{\ln a} \frac{d}{dx}(\ln x) = \frac{1}{\ln a} \cdot \frac{1}{x},$$

so

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}.$$

$\log_a x$

$$\log x = \ln x$$

$$\log x = \frac{\ln x}{\ln a}$$

$$\log_e x = \frac{\ln x}{\ln e} = \frac{\ln x}{1} = \ln x$$

If u is a positive differentiable function of x , then

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u} \frac{du}{dx}.$$

EXAMPLE 2 We illustrate the derivative and integral results.

$$(a) \frac{d}{dx} \log_{10}(3x + 1) = \frac{1}{\ln 10} \cdot \frac{1}{3x + 1} \frac{d}{dx}(3x + 1) = \frac{3}{(\ln 10)(3x + 1)}$$

$$(b) \int \frac{\log_2 x}{x} dx = \frac{1}{\ln 2} \int \frac{\ln x}{x} dx \quad \log_2 x = \frac{\ln x}{\ln 2}$$

$$= \frac{1}{\ln 2} \int u du \quad u = \ln x, \quad du = \frac{1}{x} dx$$

$$= \frac{1}{\ln 2} \frac{u^2}{2} + C = \frac{1}{\ln 2} \frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2 \ln 2} + C$$

Transcendental Numbers and Transcendental Functions

Numbers that are solutions of polynomial equations with rational coefficients are called **algebraic**: -2 is algebraic because it satisfies the equation $x + 2 = 0$, and $\sqrt{3}$ is algebraic because it satisfies the equation $x^2 - 3 = 0$. Numbers such as e and π that are not algebraic are called **transcendental**.

We call a function $y = f(x)$ algebraic if it satisfies an equation of the form

$$P_n y^n + \cdots + P_1 y + P_0 = 0$$

in which the P 's are polynomials in x with rational coefficients. The function $y = 1/\sqrt{x+1}$ is algebraic because it satisfies the equation $(x+1)y^2 - 1 = 0$. Here the polynomials are $P_2 = x+1$, $P_1 = 0$, and $P_0 = -1$. Functions that are not algebraic are called transcendental.

Summary

In this section we used calculus to give precise definitions of the logarithmic and exponential functions. This approach is somewhat different from our earlier treatments of the polynomial, rational, and trigonometric functions. There we first defined the function and then we studied its derivatives and integrals. Here we started with an integral from which the functions of interest were obtained. The motivation behind this approach was to address mathematical difficulties that arise when we attempt to define functions such as a^x for any real number x , rational or irrational. Defining $\ln x$ as the integral of the function $1/t$ from $t = 1$ to $t = x$ enabled us to define all of the exponential and logarithmic functions and then to derive their key algebraic and analytic properties.

EXERCISES 7.1

Integration

Evaluate the integrals in Exercises 1–46.

- $\int_{-3}^{-2} \frac{dx}{x}$
- $\int_{-1}^0 \frac{3 dx}{3x - 2}$
- $\int \frac{2y dy}{y^2 - 25}$
- $\int \frac{8r dr}{4r^2 - 5}$
- $\int \frac{3 \sec^2 t}{6 + 3 \tan t} dt$
- $\int \frac{\sec y \tan y}{2 + \sec y} dy$
- $\int \frac{dx}{2\sqrt{x} + 2x}$
- $\int \frac{\sec x dx}{\sqrt{\ln(\sec x + \tan x)}}$
- $\int_{\ln 2}^{\ln 3} e^x dx$
- $\int 8e^{(x+1)} dx$
- $\int_1^4 \frac{(\ln x)^3}{2x} dx$
- $\int \frac{\ln(\ln x)}{x \ln x} dx$
- $\int_{\ln 4}^{\ln 9} e^{x/2} dx$
- $\int \tan x \ln(\cos x) dx$
- $\int \frac{e^{-\sqrt{r}}}{\sqrt{r}} dr$
- $\int \frac{e^{-1/x^2}}{x^3} dx$
- $\int 2te^{-t^2} dt$
- $\int \frac{\ln x dx}{x\sqrt{\ln^2 x + 1}}$
- $\int \frac{e^{1/x}}{x^2} dx$
- $\int \frac{e^{-1/x^2}}{x^3} dx$
- $\int e^{\sec \pi t} \sec \pi t \tan \pi t dt$
- $\int_0^{\pi/2} e^{\csc(\pi+t)} \csc(\pi+t) \cot(\pi+t) dt$
- $\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^v \cos e^v dv$
- $\int_0^{\sqrt{\ln \pi}} 2xe^{x^2} \cos(e^{x^2}) dx$
- $\int \frac{e^r}{1+e^r} dr$
- $\int \frac{dx}{1+e^x}$
- $\int_0^1 2^{-\theta} d\theta$
- $\int_{-2}^0 5^{-\theta} d\theta$
- $\int_1^{\sqrt{2}} x 2^{(x^2)} dx$
- $\int_1^4 \frac{2\sqrt{x}}{\sqrt{x}} dx$
- $\int_0^{\pi/2} 7^{\cos t} \sin t dt$
- $\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} \sec^2 t dt$
- $\int_2^4 x^{2x} (1 + \ln x) dx$
- $\int_1^2 \frac{2 \ln x}{x} dx$
- $\int_0^3 (\sqrt{2} + 1)x^{\sqrt{2}} dx$
- $\int_1^e x^{(\ln 2)-1} dx$
- $\int \frac{\log_{10} x}{x} dx$
- $\int_1^4 \frac{2 \log_2 x}{x} dx$
- $\int_1^e \frac{2 \ln 10 \log_{10} x}{x} dx$
- $\int_0^{10} \frac{\log_{10}(10x)}{x} dx$
- $\int_0^9 \frac{2 \log_{10}(x+1)}{x+1} dx$
- $\int_2^3 \frac{2 \log_2(x-1)}{x-1} dx$

$$45. \int \frac{dx}{x \log_{10} x} \quad 46. \int \frac{dx}{x(\log_8 x)^2}$$

Initial Value Problems

Solve the initial value problems in Exercises 47–52.

47. $\frac{dy}{dt} = e^t \sin(e^t - 2)$, $y(\ln 2) = 0$
48. $\frac{dy}{dt} = e^{-t} \sec^2(\pi e^{-t})$, $y(\ln 4) = 2/\pi$
49. $\frac{d^2y}{dx^2} = 2e^{-x}$, $y(0) = 1$ and $y'(0) = 0$
50. $\frac{d^2y}{dt^2} = 1 - e^{2t}$, $y(1) = -1$ and $y'(1) = 0$
51. $\frac{dy}{dx} = 1 + \frac{1}{x}$, $y(1) = 3$
52. $\frac{d^2y}{dx^2} = \sec^2 x$, $y(0) = 0$ and $y'(0) = 1$

Theory and Applications

53. The region between the curve $y = 1/x^2$ and the x -axis from $x = 1/2$ to $x = 2$ is revolved about the y -axis to generate a solid. Find the volume of the solid.
54. In Section 6.2, Exercise 6, we revolved about the y -axis the region between the curve $y = 9x/\sqrt{x^3 + 9}$ and the x -axis from $x = 0$ to $x = 3$ to generate a solid of volume 36π . What volume do you get if you revolve the region about the x -axis instead? (See Section 6.2, Exercise 6, for a graph.)

Find the lengths of the curves in Exercises 55 and 56.

55. $y = (x^2/8) - \ln x$, $4 \leq x \leq 8$
56. $x = (y/4)^2 - 2 \ln(y/4)$, $4 \leq y \leq 12$

T 57. **The linearization of $\ln(1+x)$ at $x = 0$** Instead of approximating $\ln x$ near $x = 1$, we approximate $\ln(1+x)$ near $x = 0$. We get a simpler formula this way.

- a. Derive the linearization $\ln(1+x) \approx x$ at $x = 0$.
- b. Estimate to five decimal places the error involved in replacing $\ln(1+x)$ by x on the interval $[0, 0.1]$.
- c. Graph $\ln(1+x)$ and x together for $0 \leq x \leq 0.5$. Use different colors, if available. At what points does the approximation of $\ln(1+x)$ seem best? Least good? By reading coordinates from the graphs, find as good an upper bound for the error as your grapher will allow.

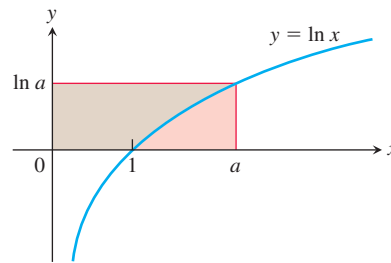
58. The linearization of e^x at $x = 0$

- a. Derive the linear approximation $e^x \approx 1+x$ at $x = 0$.
- T** b. Estimate to five decimal places the magnitude of the error involved in replacing e^x by $1+x$ on the interval $[0, 0.2]$.
- T** c. Graph e^x and $1+x$ together for $-2 \leq x \leq 2$. Use different colors, if available. On what intervals does the approximation appear to overestimate e^x ? Underestimate e^x ?

59. Show that for any number $a > 1$

$$\int_1^a \ln x \, dx + \int_0^{\ln a} e^y \, dy = a \ln a,$$

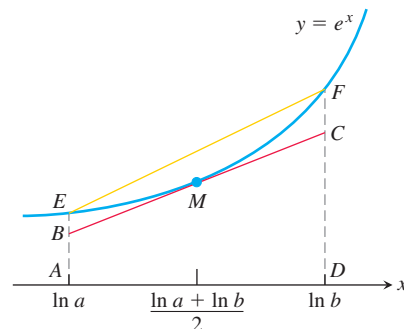
as suggested by the accompanying figure.



60. The geometric, logarithmic, and arithmetic mean inequality

- a. Show that the graph of e^x is concave up over every interval of x -values.
- b. Show, by reference to the accompanying figure, that if $0 < a < b$, then

$$e^{(\ln a + \ln b)/2} \cdot (\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x \, dx < \frac{e^{\ln a} + e^{\ln b}}{2} \cdot (\ln b - \ln a).$$



NOT TO SCALE

- c. Use the inequality in part (b) to conclude that

$$\sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}.$$

This inequality says that the geometric mean of two positive numbers is less than their logarithmic mean, which in turn is less than their arithmetic mean.

61. Use Figure 7.1 and appropriate areas to show that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}.$$

62. Partition the interval $[1, 2]$ into n equal parts. Then use Figure 7.1 and appropriate partition points and areas to show that

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} < \ln 2 < \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1}.$$

63. Use the same-derivative argument, as was done to prove the Product and Power Rules for logarithms, to prove the Quotient Rule property.

64. Use the same-derivative argument to prove the identities

$$\text{a. } \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \quad \text{b. } \sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$$

65. Starting with the equation $e^x e^y = e^{x+y}$, derived in the text, show that $e^{-x} = 1/e^x$ for any real number x . Then show that $e^x/e^y = e^{x-y}$ for any numbers x and y .

66. Show that $(e^x)^y = e^{xy} = (e^y)^x$ for any numbers x and y .
67. Show that properties (2), (3), and (4) of Theorem 2 follow from Theorem 1 and the inverse relationship between the logarithmic and exponential functions.

68. Alternative proof that $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$:

- a. Let $x > 0$ be given, and use Figure 7.1 to show that

$$\frac{1}{x+1} < \int_x^{x+1} \frac{1}{t} dt < \frac{1}{x}.$$

- b. Conclude from part (a) that

$$\frac{1}{x+1} < \ln\left(1 + \frac{1}{x}\right) < \frac{1}{x}.$$

- c. Conclude from part (b) that

$$e^{\frac{1}{x+1}} < \left(1 + \frac{1}{x}\right)^x < e.$$

- d. Conclude from part (c) that

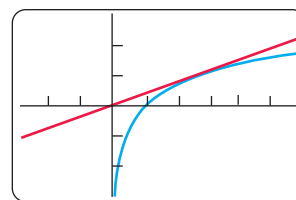
$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e.$$

Grapher Explorations

T When solving Exercises 69–76, you may need to use appropriate technology (such as a graphing calculator or a computer).

69. Graph $\ln x$, $\ln 2x$, $\ln 4x$, $\ln 8x$, and $\ln 16x$ (as many as you can) together for $0 < x \leq 10$. What is going on? Explain.
70. Graph $y = \ln|\sin x|$ in the window $0 \leq x \leq 22$, $-2 \leq y \leq 0$. Explain what you see. How could you change the formula to turn the arches upside down?
71. a. Graph $y = \sin x$ and the curves $y = \ln(a + \sin x)$ for $a = 2, 4, 8, 20$, and 50 together for $0 \leq x \leq 23$.
b. Why do the curves flatten as a increases? (Hint: Find an a -dependent upper bound for $|y'|$.)
72. Does the graph of $y = \sqrt{x} - \ln x$, $x > 0$, have an inflection point? Try to answer the question (a) by graphing, (b) by using calculus.
73. The equation $x^2 = 2^x$ has three solutions: $x = 2$, $x = 4$, and one other. Estimate the third solution as accurately as you can by graphing.
74. Could $x^{\ln 2}$ possibly be the same as $2^{\ln x}$ for some $x > 0$? Graph the two functions and explain what you see.
75. Which is bigger, π^e or e^π ? Calculators have taken some of the mystery out of this once-challenging question. (Go ahead and check; you will see that it is a surprisingly close call.) You can answer the question without a calculator, though.

- a. Find an equation for the line through the origin tangent to the graph of $y = \ln x$.



$[-3, 6]$ by $[-3, 3]$

- b. Give an argument based on the graphs of $y = \ln x$ and the tangent line to explain why $\ln x < x/e$ for all positive $x \neq e$.
- c. Show that $\ln(x^e) < x$ for all positive $x \neq e$.
- d. Conclude that $x^e < e^x$ for all positive $x \neq e$.
- e. So which is bigger, π^e or e^π ?
76. **A decimal representation of e** Find e to as many decimal places as you can by solving the equation $\ln x = 1$ using Newton's method in Section 4.7.

Calculations with Other Bases

T 77. Most scientific calculators have keys for $\log_{10} x$ and $\ln x$. To find logarithms to other bases, we use the equation $\log_a x = (\ln x)/(\ln a)$.

Find the following logarithms to five decimal places.

- a. $\log_3 8$
b. $\log_7 0.5$
c. $\log_{20} 17$
d. $\log_{0.5} 7$
e. $\ln x$, given that $\log_{10} x = 2.3$
f. $\ln x$, given that $\log_2 x = 1.4$
g. $\ln x$, given that $\log_2 x = -1.5$
h. $\ln x$, given that $\log_{10} x = -0.7$
78. **Conversion factors**
- a. Show that the equation for converting base 10 logarithms to base 2 logarithms is
- $$\log_2 x = \frac{\ln 10}{\ln 2} \log_{10} x.$$
- b. Show that the equation for converting base a logarithms to base b logarithms is
- $$\log_b x = \frac{\ln a}{\ln b} \log_a x.$$

7.2 Exponential Change and Separable Differential Equations

Exponential functions increase or decrease very rapidly with changes in the independent variable. They describe growth or decay in many natural and industrial situations. The variety of models based on these functions partly accounts for their importance.

Exponential Change

In modeling many real-world situations, a quantity y increases or decreases at a rate proportional to its size at a given time t . Examples of such quantities include the size of a

population, the amount of a decaying radioactive material, and the temperature difference between a hot object and its surrounding medium. Such quantities are said to undergo **exponential change**.

If the amount present at time $t = 0$ is called y_0 , then we can find y as a function of t by solving the following initial value problem:

$$\text{Differential equation: } \frac{dy}{dt} = ky \quad (1a)$$

$$\text{Initial condition: } y = y_0 \text{ when } t = 0. \quad (1b)$$

If y is positive and increasing, then k is positive, and we use Equation (1a) to say that the rate of growth is proportional to what has already been accumulated. If y is positive and decreasing, then k is negative, and we use Equation (1a) to say that the rate of decay is proportional to the amount still left.

The constant function $y = 0$ is a solution of Equation (1a) if $y_0 = 0$. To find the solutions when y_0 is not zero, we divide Equation (1a) by y :

$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dt} &= k && y \neq 0 \\ \int \frac{1}{y} \frac{dy}{dt} dt &= \int k dt && \text{Integrate with respect to } t. \\ \ln |y| &= kt + C && \int (1/u) du = \ln |u| + C \\ |y| &= e^{kt+C} && \text{Exponentiate.} \\ |y| &= e^C \cdot e^{kt} && e^{a+b} = e^a \cdot e^b \\ y &= \pm e^C e^{kt} && \text{If } |y| = r, \text{ then } y = \pm r. \\ y &= Ae^{kt}. && A \text{ is a shorter name for } \pm e^C. \end{aligned}$$

By allowing A to take on the value 0 in addition to all possible values $\pm e^C$, we can include the solution $y = 0$ (which occurs when $y_0 = 0$) in the formula.

We find the value of A for the initial value problem by solving for A when $y = y_0$ and $t = 0$:

$$y_0 = Ae^{k \cdot 0} = A.$$

The solution of the initial value problem

$$\frac{dy}{dt} = ky, \quad y(0) = y_0$$

is

$$y = y_0 e^{kt}. \quad (2)$$

Quantities changing in this way are said to undergo **exponential growth** if $k > 0$ and **exponential decay** if $k < 0$. The number k is called the **rate constant** of the change. (See Figure 7.5.)

The derivation of Equation (2) shows also that the only functions that are their own derivatives ($k = 1$) are constant multiples of the exponential function.

Before presenting several examples of exponential change, let us consider the process we used to derive it.

Separable Differential Equations

Exponential change is modeled by a differential equation of the form $dy/dx = ky$, where k is a nonzero constant. More generally, suppose we have a differential equation of the form

$$\frac{dy}{dx} = f(x, y), \quad (3)$$

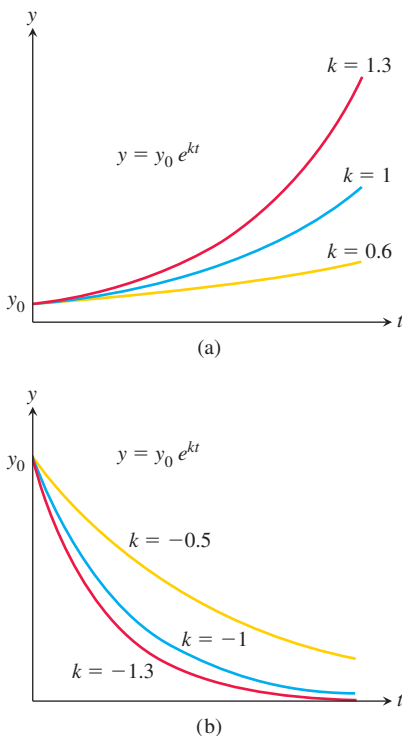


FIGURE 7.5 Graphs of (a) exponential growth and (b) exponential decay. As $|k|$ increases, the growth ($k > 0$) or decay ($k < 0$) intensifies.

where f is a function of *both* the independent and dependent variables. A **solution** of the equation is a differentiable function $y = y(x)$ defined on an interval of x -values (perhaps infinite) such that

$$\frac{d}{dx}y(x) = f(x, y(x))$$

on that interval. That is, when $y(x)$ and its derivative $y'(x)$ are substituted into the differential equation, the resulting equation is true for all x in the solution interval. The **general solution** is a solution $y(x)$ that contains all possible solutions, and it always contains an arbitrary constant.

Equation (3) is **separable** if f can be expressed as a product of a function of x and a function of y . The differential equation then has the form

$$\frac{dy}{dx} = g(x) H(y). \quad \begin{array}{l} g \text{ is a function of } x; \\ H \text{ is a function of } y. \end{array}$$

Then collect all of the y factors so that they are together with dy , and likewise collect all of the x factors together with dx :

$$\frac{1}{H(y)} dy = g(x) dx.$$

Now we integrate both sides of this equation:

$$\int \frac{1}{H(y)} dy = \int g(x) dx. \quad (4)$$

After completing the integrations, we obtain the solution y , defined implicitly as a function of x .

The justification that we can integrate both sides in Equation (4) in this way is based on the Substitution Rule (Section 5.5):

$$\begin{aligned} \int \frac{1}{H(y)} dy &= \int \frac{1}{H(y(x))} \frac{dy}{dx} dx \\ &= \int \frac{1}{H(y(x))} H(y(x)) g(x) dx && \frac{dy}{dx} = H(y(x)) g(x) \\ &= \int g(x) dx. \end{aligned}$$

EXAMPLE 1 Solve the differential equation

$$\frac{dy}{dx} = (1 + y)e^x, \quad y > -1.$$

Solution Since $1 + y$ is never zero for $y > -1$, we can solve the equation by separating the variables.

$$\begin{aligned} \frac{dy}{dx} &= (1 + y)e^x \\ dy &= (1 + y)e^x dx && \text{Treat } dy/dx \text{ as a quotient of differentials} \\ &&& \text{and multiply both sides by } dx. \\ \frac{dy}{1 + y} &= e^x dx && \text{Divide by } (1 + y). \\ \int \frac{dy}{1 + y} &= \int e^x dx && \text{Integrate both sides.} \\ \ln(1 + y) &= e^x + C && C \text{ represents the combined constants} \\ &&& \text{of integration.} \end{aligned}$$

The last equation gives y as an implicit function of x . ■

EXAMPLE 2 Solve the equation $y(x + 1) \frac{dy}{dx} = x(y^2 + 1)$.

Solution We change to differential form, separate the variables, and integrate:

$$\begin{aligned} y(x + 1) dy &= x(y^2 + 1) dx \\ \frac{y dy}{y^2 + 1} &= \frac{x dx}{x + 1} && x \neq -1 \\ \int \frac{y dy}{1 + y^2} &= \int \left(1 - \frac{1}{x + 1}\right) dx && \text{Divide } x \text{ by } x + 1. \\ \frac{1}{2} \ln(1 + y^2) &= x - \ln|x + 1| + C. \end{aligned}$$

The last equation gives the solution y as an implicit function of x . ■

The initial value problem

$$\frac{dy}{dt} = ky, \quad y(0) = y_0$$

involves a separable differential equation, and the solution $y = y_0 e^{kt}$ expresses exponential change. We now present several examples of such change.

Unlimited Population Growth

Strictly speaking, the number of individuals in a population (of people, plants, animals, or bacteria, for example) is a discontinuous function of time because it takes on discrete values. However, when the number of individuals becomes large enough, the population can be approximated by a continuous function. Differentiability of the approximating function is another reasonable hypothesis in many settings, allowing for the use of calculus to model and predict population sizes.

If we assume that the proportion of reproducing individuals remains constant and assume a constant fertility, then at any instant t the birth rate is proportional to the number $y(t)$ of individuals present. We likewise assume that the death rate of the population is stable and proportional to $y(t)$. If, further, we neglect departures and arrivals, the growth rate dy/dt is the birth rate minus the death rate, which is the difference of the two proportionalities under our assumptions. In other words, $dy/dt = ky$ so that $y = y_0 e^{kt}$, where y_0 is the size of the population at time $t = 0$. As with all kinds of growth, there may be limitations imposed by the surrounding environment, but we will not go into these here. (We treat one model imposing such limitations in Section 16.4.) When k is positive, the proportionality $dy/dt = ky$ models *unlimited population growth*. (See Figure 7.6.)

EXAMPLE 3 The biomass of a yeast culture in an experiment is initially 29 grams. After 30 minutes the mass is 37 grams. Assuming that the equation for unlimited population growth gives a good model for the growth of the yeast when the mass is below 100 grams, how long will it take for the mass to double from its initial value?

Solution Let $y(t)$ be the yeast biomass after t minutes. We use the exponential growth model $dy/dt = ky$ for unlimited population growth, with solution $y = y_0 e^{kt}$. We have $y_0 = y(0) = 29$. We are also told that

$$y(30) = 29e^{k(30)} = 37.$$

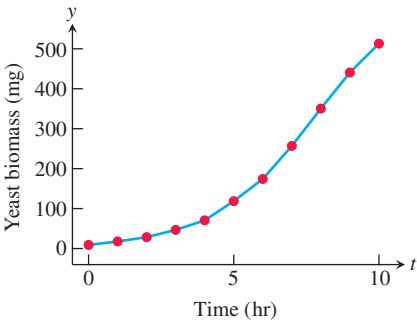


FIGURE 7.6 Graph of the growth of a yeast population over a 10-hour period, based on the data in Table 7.3.

TABLE 7.3 Population of yeast

Time (hr)	Yeast biomass (mg)
0	9.6
1	18.3
2	29.0
3	47.2
4	71.1
5	119.1
6	174.6
7	257.3
8	350.7
9	441.0
10	513.3

Solving this equation for k , we find

$$\begin{aligned} e^{k(30)} &= \frac{37}{29} \\ 30k &= \ln\left(\frac{37}{29}\right) \\ k &= \frac{1}{30} \ln\left(\frac{37}{29}\right) \approx 0.008118. \end{aligned}$$

Then the mass of the yeast in grams after t minutes is given by the equation

$$y = 29e^{(0.008118)t}.$$

To solve the problem, we find the time t for which $y(t) = 58$, which is twice the initial amount.

$$\begin{aligned} 29e^{(0.008118)t} &= 58 \\ (0.008118)t &= \ln\left(\frac{58}{29}\right) \\ t &= \frac{\ln 2}{0.008118} \approx 85.38 \end{aligned}$$

It takes about 85 minutes for the yeast population to double. ■

In the next example we model the number of people within a given population who are infected by a disease that is being eradicated from the population. Here the constant of proportionality k is negative, and the model describes an exponentially decaying number of infected individuals.

EXAMPLE 4 One model for the way diseases die out when properly treated assumes that the rate dy/dt at which the number of infected people changes is proportional to the number y . The number of people cured is proportional to the number y that are infected with the disease. Suppose that in the course of any given year, the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take to reduce the number to 1000?

Solution We use the equation $y = y_0 e^{kt}$. There are three things to find: the value of y_0 , the value of k , and the time t when $y = 1000$.

The value of y_0 . We are free to count time beginning anywhere we want. If we count from today, then $y = 10,000$ when $t = 0$, so $y_0 = 10,000$. Our equation is now

$$y = 10,000 e^{kt}. \quad (5)$$

The value of k . When $t = 1$ year, the number of cases will be 80% of its present value, or 8000. Hence,

$$\begin{aligned} 8000 &= 10,000 e^{k(1)} && \text{Eq. (5) with } t = 1 \text{ and } y = 8000 \\ e^k &= 0.8 \\ \ln(e^k) &= \ln 0.8 && \text{Take logs of both sides} \\ k &= \ln 0.8 < 0. && \ln 0.8 \approx -0.223 \end{aligned}$$

At any given time t ,

$$y = 10,000 e^{(\ln 0.8)t}. \quad (6)$$

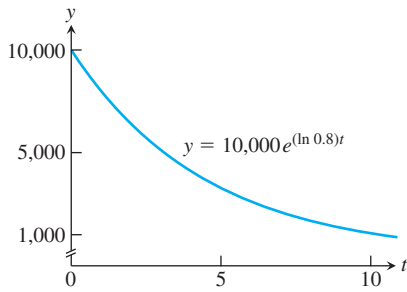


FIGURE 7.7 A graph of the number of people infected by a disease exhibits exponential decay (Example 4).

For radon-222 gas, t is measured in days and $k = 0.18$. For radium-226, which used to be painted on watch dials to make them glow at night (a dangerous practice), t is measured in years and $k = 4.3 \times 10^{-4}$.

The value of t that makes $y = 1000$. We set y equal to 1000 in Equation (6) and solve for t :

$$\begin{aligned} 1000 &= 10,000e^{(\ln 0.8)t} \\ e^{(\ln 0.8)t} &= 0.1 \\ (\ln 0.8)t &= \ln 0.1 && \text{Take logs of both sides} \\ t &= \frac{\ln 0.1}{\ln 0.8} \approx 10.32 \text{ years.} \end{aligned}$$

It will take a little more than 10 years to reduce the number of cases to 1000. (See Figure 7.7.)

Radioactivity

Some atoms are unstable and can spontaneously emit mass or radiation. This process is called **radioactive decay**, and an element whose atoms go spontaneously through this process is called **radioactive**. Sometimes when an atom emits some of its mass through this process of radioactivity, the remainder of the atom re-forms to make an atom of some new element. For example, radioactive carbon-14 decays into nitrogen; radium, through a number of intermediate radioactive steps, decays into lead.

Experiments have shown that at any given time, the rate at which a radioactive element decays (as measured by the number of nuclei that change per unit time) is approximately proportional to the number of radioactive nuclei present. Thus, the decay of a radioactive element is described by the equation $dy/dt = -ky$, $k > 0$. It is conventional to use $-k$, with $k > 0$, to emphasize that y is decreasing. If y_0 is the number of radioactive nuclei present at time zero, the number still present at any later time t will be

$$y = y_0 e^{-kt}, \quad k > 0.$$

In Section 1.5, we defined the **half-life** of a radioactive element to be the time required for half of the radioactive nuclei present in a sample to decay. It is an interesting fact that the half-life is a constant that does not depend on the number of radioactive nuclei initially present in the sample, but only on the radioactive substance. We found that the half-life is given by the following equation.

$$\text{Half-life} = \frac{\ln 2}{k} \quad (7)$$

For example, the half-life for radon-222 is

$$\text{half-life} = \frac{\ln 2}{0.18} \approx 3.9 \text{ days.}$$

Carbon-14 dating uses the half-life of 5730 years.

EXAMPLE 5 The decay of radioactive elements can sometimes be used to date events from Earth's past. In a living organism, the ratio of radioactive carbon, carbon-14, to ordinary carbon stays fairly constant during the lifetime of the organism, being approximately equal to the ratio in the organism's atmosphere at the time. After the organism's death, however, no new carbon is ingested, and the proportion of carbon-14 in the organism's remains decreases as the carbon-14 decays.

Scientists who do carbon-14 dating often use a figure of 5730 years for its half-life. Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.

Solution We use the decay equation $y = y_0 e^{-kt}$. There are two things to find: the value of k and the value of t when y is $0.9 y_0$ (90% of the radioactive nuclei are still present). That is, find t when $y_0 e^{-kt} = 0.9 y_0$, or $e^{-kt} = 0.9$.

The value of k . We use the half-life Equation (7):

$$k = \frac{\ln 2}{\text{half-life}} = \frac{\ln 2}{5730} \quad (\text{about } 1.2 \times 10^{-4}).$$

The value of t that makes $e^{-kt} = 0.9$.

$$\begin{aligned} e^{-kt} &= 0.9 \\ e^{-(\ln 2/5730)t} &= 0.9 \\ -\frac{\ln 2}{5730}t &= \ln 0.9 && \text{Take logs of both sides} \\ t &= -\frac{5730 \ln 0.9}{\ln 2} \approx 871 \text{ years} \end{aligned}$$

The sample is about 871 years old. ■

Heat Transfer: Newton's Law of Cooling

Hot soup left in a tin cup cools to the temperature of the surrounding air. A hot silver bar immersed in a large tub of water cools to the temperature of the surrounding water. In situations like these, the rate at which an object's temperature is changing at any given time is roughly proportional to the difference between its temperature and the temperature of the surrounding medium. This observation is called *Newton's Law of Cooling*, although it applies to warming as well.

If H is the temperature of the object at time t , and H_s is the constant surrounding temperature, then the differential equation is

$$\frac{dH}{dt} = -k(H - H_s). \quad (8)$$

If we substitute y for $(H - H_s)$, then

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}(H - H_s) = \frac{dH}{dt} - \frac{d}{dt}(H_s) \\ &= \frac{dH}{dt} - 0 && H_s \text{ is a constant.} \\ &= \frac{dH}{dt} \\ &= -k(H - H_s) && \text{Eq. (8)} \\ &= -ky. && H - H_s = y \end{aligned}$$

We know that the solution of the equation $dy/dt = -ky$ is $y = y_0 e^{-kt}$, where $y(0) = y_0$. Substituting $(H - H_s)$ for y , this says that

$$H - H_s = (H_0 - H_s)e^{-kt}, \quad (9)$$

where H_0 is the temperature at $t = 0$. This equation is the solution to Newton's Law of Cooling.

EXAMPLE 6 A hard-boiled egg at 98°C is put in a sink of 18°C water. After 5 min, the egg's temperature is 38°C . Assuming that the water has not warmed appreciably, how much longer will it take the egg to reach 20°C ?

Solution We find how long it would take the egg to cool from 98°C to 20°C and subtract the 5 min that have already elapsed. Using Equation (9) with $H_s = 18$ and $H_0 = 98$, the egg's temperature t min after it is put in the sink is

$$H = 18 + (98 - 18)e^{-kt} = 18 + 80e^{-kt}.$$

To find k , we use the information that $H = 38$ when $t = 5$:

$$38 = 18 + 80e^{-5k}$$

$$e^{-5k} = \frac{1}{4}$$

$$-5k = \ln \frac{1}{4} = -\ln 4$$

$$k = \frac{1}{5} \ln 4 = 0.2 \ln 4 \quad (\text{about } 0.28).$$

The egg's temperature at time t is $H = 18 + 80e^{-(0.2 \ln 4)t}$. Now find the time t when $H = 20$:

$$20 = 18 + 80e^{-(0.2 \ln 4)t}$$

$$80e^{-(0.2 \ln 4)t} = 2$$

$$e^{-(0.2 \ln 4)t} = \frac{1}{40}$$

$$-(0.2 \ln 4)t = \ln \frac{1}{40} = -\ln 40$$

$$t = \frac{\ln 40}{0.2 \ln 4} \approx 13 \text{ min.}$$

The egg's temperature will reach 20°C about 13 min after it is put in the water to cool. Since it took 5 min to reach 38°C , it will take about 8 min more to reach 20°C . ■

EXERCISES 7.2

Verifying Solutions

In Exercises 1–4, show that each function $y = f(x)$ is a solution of the accompanying differential equation.

1. $2y' + 3y = e^{-x}$

a. $y = e^{-x}$

b. $y = e^{-x} + e^{-(3/2)x}$

c. $y = e^{-x} + Ce^{-(3/2)x}$

2. $y' = y^2$

a. $y = -\frac{1}{x}$

b. $y = -\frac{1}{x+3}$

c. $y = -\frac{1}{x+C}$

3. $y = \frac{1}{x} \int_1^x \frac{e^t}{t} dt, \quad x^2 y' + xy = e^x$

4. $y = \frac{1}{\sqrt{1+x^4}} \int_1^x \sqrt{1+t^4} dt, \quad y' + \frac{2x^3}{1+x^4} y = 1$

Initial Value Problems

In Exercises 5–8, show that each function is a solution of the given initial value problem.

Differential equation	Initial equation	Solution candidate
5. $y' + y = \frac{2}{1+4e^{2x}}$	$y(-\ln 2) = \frac{\pi}{2}$	$y = e^{-x} \tan^{-1}(2e^x)$
6. $y' = e^{-x^2} - 2xy$	$y(2) = 0$	$y = (x-2)e^{-x^2}$
7. $xy' + y = -\sin x, \quad x > 0$	$y\left(\frac{\pi}{2}\right) = 0$	$y = \frac{\cos x}{x}$
8. $x^2 y' = xy - y^2, \quad x > 1$	$y(e) = e$	$y = \frac{x}{\ln x}$

Separable Differential Equations

Solve the differential equation in Exercises 9–22.

9. $2\sqrt{xy} \frac{dy}{dx} = 1, \quad x, y > 0$

10. $\frac{dy}{dx} = x^2 \sqrt{y}, \quad y > 0$

11. $\frac{dy}{dx} = e^{x-y}$

12. $\frac{dy}{dx} = 3x^2 e^{-y}$

13. $\frac{dy}{dx} = \sqrt{y} \cos^2 \sqrt{y}$

14. $\sqrt{2xy} \frac{dy}{dx} = 1$

15. $\sqrt{x} \frac{dy}{dx} = e^{y+\sqrt{x}}, \quad x > 0$

16. $(\sec x) \frac{dy}{dx} = e^{y+\sin x}$

17. $\frac{dy}{dx} = 2x\sqrt{1-y^2}, \quad -1 < y < 1$

18. $\frac{dy}{dx} = \frac{e^{2x-y}}{e^{x+y}}$

19. $y^2 \frac{dy}{dx} = 3x^2 y^3 - 6x^2$

20. $\frac{dy}{dx} = xy + 3x - 2y - 6$

21. $\frac{1}{x} \frac{dy}{dx} = ye^{x^2} + 2\sqrt{y} e^{x^2}$

22. $\frac{dy}{dx} = e^{x-y} + e^x + e^{-y} + 1$

Applications and Examples

The answers to most of the following exercises are in terms of logarithms and exponentials. A calculator can be helpful, enabling you to express the answers in decimal form.

23. Human evolution continues The analysis of tooth shrinkage by C. Loring Brace and colleagues at the University of Michigan's

Museum of Anthropology indicates that human tooth size is continuing to decrease and that the evolutionary process has not yet come to a halt. In northern Europeans, for example, tooth size reduction now has a rate of 1% per 1000 years.

- a. If t represents time in years and y represents tooth size, use the condition that $y = 0.99y_0$ when $t = 1000$ to find the value of k in the equation $y = y_0 e^{kt}$. Then use this value of k to answer the following questions.
- b. In about how many years will human teeth be 90% of their present size?
- c. What will be our descendants' tooth size 20,000 years from now (as a percentage of our present tooth size)?

- 24. Atmospheric pressure** The earth's atmospheric pressure p is often modeled by assuming that the rate dp/dh at which p changes with the altitude h above sea level is proportional to p . Suppose that the pressure at sea level is 1013 hectopascals and that the pressure at an altitude of 20 km is 90 hectopascals.

- a. Solve the initial value problem

Differential equation: $dp/dh = kp$ (k a constant)

Initial condition: $p = p_0$ when $h = 0$

to express p in terms of h . Determine the values of p_0 and k from the given altitude-pressure data.

- b. What is the atmospheric pressure at $h = 50$ km?
 - c. At what altitude does the pressure equal 900 hectopascals?
- 25. First-order chemical reactions** In some chemical reactions, the rate at which the amount of a substance changes with time is proportional to the amount present. For the change of δ -gluconolactone into gluconic acid, for example,

$$\frac{dy}{dt} = -0.6y$$

when t is measured in hours. If there are 100 grams of δ -gluconolactone present when $t = 0$, how many grams will be left after the first hour?

- 26. The inversion of sugar** The processing of raw sugar has a step called "inversion" that changes the sugar's molecular structure. Once the process has begun, the rate of change of the amount of raw sugar is proportional to the amount of raw sugar remaining. If 1000 kg of raw sugar reduces to 800 kg of raw sugar during the first 10 hours, how much raw sugar will remain after another 14 hours?
- 27. Working underwater** The intensity $L(x)$ of light x meters beneath the surface of the ocean satisfies the differential equation

$$\frac{dL}{dx} = -kL.$$

As a diver, you know from experience that diving to 6 m in the Caribbean Sea cuts the intensity in half. You cannot work without artificial light when the intensity falls below one-tenth of the surface value. About how deep can you expect to work without artificial light?

- 28. Voltage in a discharging capacitor** Suppose that electricity is draining from a capacitor at a rate that is proportional to the voltage V across its terminals and that, if t is measured in seconds,

$$\frac{dV}{dt} = -\frac{1}{40}V.$$

Solve this equation for V , using V_0 to denote the value of V when $t = 0$. How long will it take the voltage to drop to 10% of its original value?

- 29. Cholera bacteria** Suppose that the bacteria in a colony can grow unchecked, by the law of exponential change. The colony starts with 1 bacterium and doubles every half-hour. How many bacteria will the colony contain at the end of 24 hours? (Under favorable laboratory conditions, the number of cholera bacteria can double every 30 min. In an infected person, many bacteria are destroyed, but this example helps explain why a person who feels well in the morning may be dangerously ill by evening.)

- 30. Growth of bacteria** A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 hours there are 10,000 bacteria. At the end of 5 hours there are 40,000. How many bacteria were present initially?

- 31. The incidence of a disease** (Continuation of Example 4.) Suppose that in any given year the number of cases can be reduced by 25% instead of 20%.

- a. How long will it take to reduce the number of cases to 1000?
- b. How long will it take to eradicate the disease—that is, reduce the number of cases to less than 1?

- 32. Drug concentration** An antibiotic is administered intravenously into the bloodstream at a constant rate r . As the drug flows through the patient's system and acts on the infection that is present, it is removed from the bloodstream at a rate proportional to the amount in the bloodstream at that time. Since the amount of blood in the patient is constant, this means that the concentration $y = y(t)$ of the antibiotic in the bloodstream can be modeled by the differential equation

$$\frac{dy}{dt} = r - ky, \quad k > 0 \text{ and constant.}$$

- a. If $y(0) = y_0$, find the concentration $y(t)$ at any time t .
- b. Assume that $y_0 < (r/k)$ and find $\lim_{y \rightarrow \infty} y(t)$. Sketch the solution curve for the concentration.

- 33. Endangered species** Biologists consider a species of animal or plant to be endangered if it is expected to become extinct within 20 years. If a certain species of wildlife is counted to have 1147 members at the present time, and the population has been steadily declining exponentially at an annual rate averaging 39% over the past 7 years, do you think the species is endangered? Explain your answer.

- 34. The U.S. population** The U.S. Census Bureau keeps a running clock totaling the U.S. population. On April 19, 2021, the total was increasing at the rate of 1 person every 40 s. The population figure for 1:32 P.M. EST on that day was 330,215,841.

- a. Assuming exponential growth at a constant rate, find the rate constant for the population's growth (people per 365-day year).
- b. At this rate, what will the U.S. population be at 1:32 P.M. EST on April 19, 2028?

- 35. Oil depletion** Suppose the amount of oil pumped from one of the canyon wells in Whittier, California, decreases at the continuous rate of 10% per year. When will the well's output fall to one-fifth of its present value?

36. Continuous price discounting To encourage buyers to place 100-unit orders, your firm's sales department applies a continuous discount that makes the unit price a function $p(x)$ of the number of units x ordered. The discount decreases the price at the rate of \$0.01 per unit ordered. The price per unit for a 100-unit order is $p(100) = \$20.09$.

- a. Find $p(x)$ by solving the following initial value problem.

$$\text{Differential equation: } \frac{dp}{dx} = -\frac{1}{100}p$$

$$\text{Initial condition: } p(100) = 20.09$$

- b. Find the unit price $p(10)$ for a 10-unit order and the unit price $p(90)$ for a 90-unit order.
- c. The sales department has asked you to find out if it is discounting so much that the firm's revenue, $r(x) = x \cdot p(x)$, will actually be less for a 100-unit order than, say, for a 90-unit order. Reassure them by showing that r has its maximum value at $x = 100$.
- d. Graph the revenue function $r(x) = xp(x)$ for $0 \leq x \leq 200$.
- 37. Plutonium-239** The half-life of the plutonium isotope is 24,360 years. If 10 g of plutonium is released into the atmosphere by a nuclear accident, how many years will it take for 80% of the isotope to decay?
- 38. Polonium-210** The half-life of polonium is 139 days, but your sample will not be useful to you after 95% of the radioactive nuclei present on the day the sample arrives has disintegrated. For about how many days after the sample arrives will you be able to use the polonium?
- 39. The mean life of a radioactive nucleus** Physicists using the radioactivity equation $y = y_0 e^{-kt}$ call the number $1/k$ the *mean life* of a radioactive nucleus. The mean life of a radon nucleus is about $1/0.18 = 5.6$ days. The mean life of a carbon-14 nucleus is more than 8000 years. Show that 95% of the radioactive nuclei originally present in a sample will disintegrate within three mean lifetimes, i.e., by time $t = 3/k$. Thus, the mean life of a nucleus gives a quick way to estimate how long the radioactivity of a sample will last.
- 40. Californium-252** What costs \$27 million per gram and can be used to treat brain cancer, analyze coal for its sulfur content, and detect explosives in luggage? The answer is californium-252, a radioactive isotope discovered by Glenn Seaborg in 1950. Much less than 1g of it is produced each year. The half-life of the isotope is 2.645 years—long enough for a useful service life and short enough to have a high radioactivity per unit mass. One microgram of the isotope releases 170 million neutrons per minute.
- a. What is the value of k in the decay equation for this isotope?
- b. What is the isotope's mean life? (See Exercise 39.)
- c. How long will it take 95% of a sample's radioactive nuclei to disintegrate?
- 41. Cooling soup** Suppose that a cup of soup cooled from 90 °C to 60 °C after 10 min in a room where the temperature was 20 °C. Use Newton's Law of Cooling to answer the following questions.
- a. How much longer would it take the soup to cool to 35 °C?
- b. Instead of being left to stand in the room, the cup of 90 °C soup is put in a freezer where the temperature is -15 °C. How long will it take the soup to cool from 90 °C to 35 °C?

42. A beam of unknown temperature An aluminum beam was brought from the outside cold into a machine shop where the temperature was held at 18 °C. After 10 min, the beam warmed to 2 °C and after another 10 min, it was 10 °C. Use Newton's Law of Cooling to estimate the beam's initial temperature.

43. Surrounding medium of unknown temperature A pan of warm water (46 °C) was put in a refrigerator. Ten minutes later, the water's temperature was 39 °C; 10 min after that, it was 33 °C. Use Newton's Law of Cooling to estimate how cold the refrigerator was.

44. Silver cooling in air The temperature of an ingot of silver is 60 °C above room temperature right now. Twenty minutes ago, it was 70 °C above room temperature. How far above room temperature will the silver be

- a. 15 min from now?
- b. 2 hours from now?
- c. When will the silver be 10 °C above room temperature?

45. The age of Crater Lake The charcoal from a tree killed in the volcanic eruption that formed Crater Lake in Oregon contained 44.5% of the carbon-14 found in living matter. About how old is Crater Lake?

46. The sensitivity of carbon-14 dating to measurement To see the effect of a relatively small error in the estimate of the amount of carbon-14 in a sample being dated, consider this hypothetical situation:

- a. A bone fragment found in central Illinois in the year 2000 contains 17% of its original carbon-14 content. Estimate the year the animal died.
- b. Repeat part (a), assuming 18% instead of 17%.
- c. Repeat part (a), assuming 16% instead of 17%.

47. Carbon-14 The oldest known frozen human mummy, discovered in the Schnalstal glacier of the Italian Alps in 1991 and called *Otzi*, was found wearing straw shoes and a leather coat with goat fur, and holding a copper ax and stone dagger. It was estimated that Otzi died 5000 years before he was discovered in the melting glacier. How much of the original carbon-14 remained in Otzi at the time of his discovery?

48. Art forgery A painting attributed to Vermeer (1632–1675), which should contain no more than 96.2% of its original carbon-14, contains 99.5% instead. About how old is the forgery?

49. Lascaux Cave paintings Prehistoric cave paintings of animals were found in the Lascaux Cave in France in 1940. Scientific analysis revealed that only 15% of the original carbon-14 in the paintings remained. What is an estimate of the age of the paintings?

50. Incan mummy The frozen remains of a young Incan woman were discovered by archeologist Johan Reinhard on Mt. Ampato in Peru during an expedition in 1995.

- a. How much of the original carbon-14 was present if the estimated age of the "Ice Maiden" was 500 years?
- b. If a 1% error can occur in the carbon-14 measurement, what is the oldest possible age for the Ice Maiden?

7.3 Hyperbolic Functions

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x} . The hyperbolic functions simplify many mathematical expressions and occur frequently in mathematical and engineering applications.

Definitions and Identities

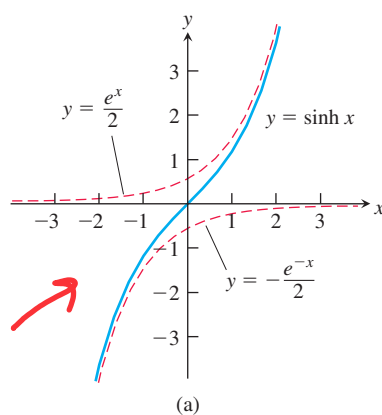
$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1 - 1}{2} = 0$$

The hyperbolic sine and hyperbolic cosine functions are defined by the equations

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

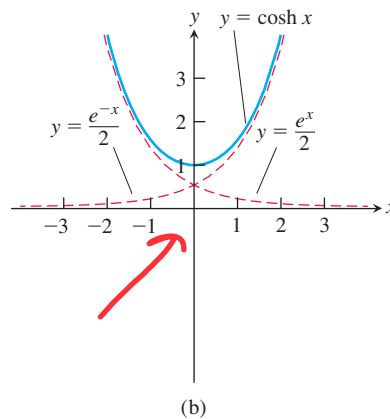
We pronounce $\sinh x$ as “cinch x ,” rhyming with “pinch x ,” and $\cosh x$ as “kosh x ,” rhyming with “gosh x .” From this basic pair, we define the hyperbolic tangent, cotangent, secant, and cosecant functions. The defining equations and graphs of these functions are shown in Table 7.4. We will see that the hyperbolic functions bear many similarities to the trigonometric functions after which they are named.

TABLE 7.4 The six basic hyperbolic functions



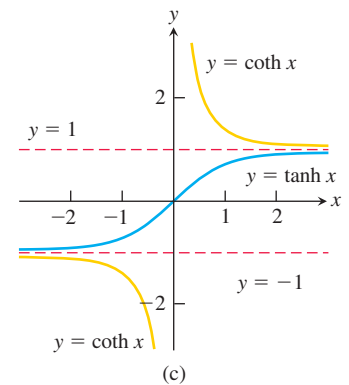
Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$



Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

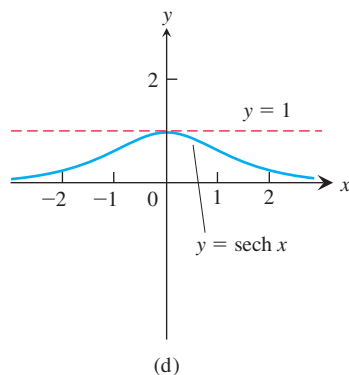


Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

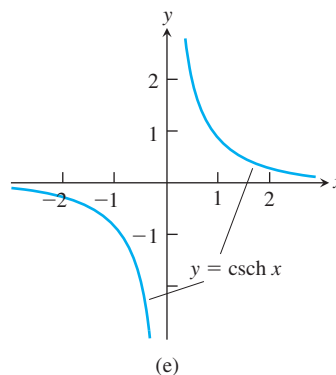
Hyperbolic cotangent:

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$



Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

TABLE 7.5 Identities for hyperbolic functions

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= 1 \\ \sinh 2x &= 2 \sinh x \cosh x \\ \cosh 2x &= \cosh^2 x + \sinh^2 x \\ \cosh^2 x &= \frac{\cosh 2x + 1}{2} \\ \sinh^2 x &= \frac{\cosh 2x - 1}{2} \\ \tanh^2 x &= 1 - \operatorname{sech}^2 x \\ \coth^2 x &= 1 + \operatorname{csch}^2 x \end{aligned}$$

Hyperbolic functions satisfy the identities in Table 7.5. Except for differences in sign, these resemble identities we know for the trigonometric functions. The identities are proved directly from the definitions, as we show here for the second one:

$$\begin{aligned} 2 \sinh x \cosh x &= 2 \left(\frac{e^x - e^{-x}}{2} \right) \left(\frac{e^x + e^{-x}}{2} \right) \\ &= \frac{e^{2x} - e^{-2x}}{2} && \text{Simplify.} \\ &= \sinh 2x. && \text{Definition of } \sinh \end{aligned}$$

The other identities are obtained similarly, by substituting in the definitions of the hyperbolic functions and using algebra.

For any real number u , we know the point with coordinates $(\cos u, \sin u)$ lies on the unit circle $x^2 + y^2 = 1$. So the trigonometric functions are sometimes called the *circular* functions. Because of the first identity

$$\cosh^2 u - \sinh^2 u = 1,$$

with u substituted for x in Table 7.5, the point having coordinates $(\cosh u, \sinh u)$ lies on the right-hand branch of the hyperbola $x^2 - y^2 = 1$. This is where the *hyperbolic* functions get their names (see Exercise 86).

Hyperbolic functions are useful in finding integrals, which we will see in Chapter 8. They play an important role in science and engineering as well. The hyperbolic cosine describes the shape of a hanging cable or wire that is strung between two points at the same height and hanging freely (see Exercise 83). The shape of the St. Louis Arch is an inverted hyperbolic cosine. The hyperbolic tangent occurs in the formula for the velocity of an ocean wave moving over water having a constant depth, and the inverse hyperbolic tangent describes how relative velocities sum according to Einstein's Law in the Special Theory of Relativity.

Derivatives and Integrals of Hyperbolic Functions

The six hyperbolic functions, being rational combinations of the differentiable functions e^x and e^{-x} , have derivatives at every point at which they are defined (Table 7.6). Again, there are similarities to trigonometric functions.

The derivative formulas are obtained from the derivative of e^x :

$$\begin{aligned} \frac{d}{dx}(\sinh x) &= \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right) && \text{Definition of } \sinh x \\ &= \frac{e^x + e^{-x}}{2} && \text{Derivative of } e^x \\ &= \cosh x. && \text{Definition of } \cosh x \end{aligned}$$

This gives the first derivative formula. From the definition, we can calculate the derivative of the hyperbolic cosecant function, as follows:

$$\begin{aligned} \frac{d}{dx}(\operatorname{csch} x) &= \frac{d}{dx} \left(\frac{1}{\sinh x} \right) && \text{Definition of } \operatorname{csch} x \\ &= -\frac{\cosh x}{\sinh^2 x} && \text{Quotient Rule for derivatives} \\ &= -\frac{1}{\sinh x} \frac{\cosh x}{\sinh x} && \text{Rearrange factors.} \\ &= -\operatorname{csch} x \coth x && \text{Definitions of } \operatorname{csch} x \text{ and } \coth x \end{aligned}$$

The other formulas in Table 7.6 are obtained similarly.

The derivative formulas lead to the integral formulas in Table 7.7.

TABLE 7.6 Derivatives of hyperbolic functions

$$\begin{aligned} \frac{d}{dx}(\sinh x) &= \cosh x \quad \checkmark \\ \frac{d}{dx}(\cosh x) &= \sinh x \quad \times \\ \frac{d}{dx}(\tanh x) &= \operatorname{sech}^2 x \quad \checkmark \\ \frac{d}{dx}(\coth x) &= -\operatorname{csch}^2 x \quad \checkmark \\ \frac{d}{dx}(\operatorname{sech} x) &= -\operatorname{sech} x \tanh x \quad \times \\ \frac{d}{dx}(\operatorname{csch} x) &= -\operatorname{csch} x \coth x \quad \checkmark \end{aligned}$$

TABLE 7.7 Integral formulas for hyperbolic functions

$$\begin{aligned} \int \sinh x \, dx &= \cosh x + C \\ \int \cosh x \, dx &= \sinh x + C \\ \int \operatorname{sech}^2 x \, dx &= \tanh x + C \\ \int \operatorname{csch}^2 x \, dx &= -\coth x + C \\ \int \operatorname{sech} x \tanh x \, dx &= -\operatorname{sech} x + C \\ \int \operatorname{csch} x \coth x \, dx &= -\operatorname{csch} x + C \end{aligned}$$

EXAMPLE 1 We illustrate the derivative and integral formulas.

$$\begin{aligned} \text{(a)} \quad \frac{d}{dt}(\tanh \sqrt{1+t^2}) &= \operatorname{sech}^2 \sqrt{1+t^2} \cdot \frac{d}{dt}(\sqrt{1+t^2}) \\ &= \frac{t}{\sqrt{1+t^2}} \operatorname{sech}^2 \sqrt{1+t^2} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int \coth 5x \, dx &= \int \frac{\cosh 5x}{\sinh 5x} \, dx = \frac{1}{5} \int \frac{du}{u} && \begin{array}{l} u = \sinh 5x, \\ du = 5 \cosh 5x \, dx \end{array} \\ &= \frac{1}{5} \ln |u| + C = \frac{1}{5} \ln |\sinh 5x| + C \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \int_0^1 \sinh^2 x \, dx &= \int_0^1 \frac{\cosh 2x - 1}{2} \, dx && \text{Table 7.5} \\ &= \frac{1}{2} \int_0^1 (\cosh 2x - 1) \, dx = \frac{1}{2} \left[\frac{\sinh 2x}{2} - x \right]_0^1 \\ &= \frac{\sinh 2}{4} - \frac{1}{2} \approx 0.40672 && \text{Evaluate with a calculator.} \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \int_0^{\ln 2} 4e^x \sinh x \, dx &= \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2} \, dx = \int_0^{\ln 2} (2e^{2x} - 2) \, dx \\ &= \left[e^{2x} - 2x \right]_0^{\ln 2} = (e^{2 \ln 2} - 2 \ln 2) - (1 - 0) \\ &= 4 - 2 \ln 2 - 1 \approx 1.6137 \end{aligned}$$

Inverse Hyperbolic Functions

The inverses of the six basic hyperbolic functions are very useful in integration (see Chapter 8). Since $d(\sinh x)/dx = \cosh x > 0$, the hyperbolic sine is an increasing function of x . We denote its inverse by

$$y = \sinh^{-1} x.$$

For every value of x in the interval $-\infty < x < \infty$, the value of $y = \sinh^{-1} x$ is the number whose hyperbolic sine is x . The graphs of $y = \sinh x$ and $y = \sinh^{-1} x$ are shown in Figure 7.8a. Other notations used for the inverse hyperbolic sine function include $\operatorname{arcsinh} x$, $\operatorname{arsinh} x$, and $\operatorname{argsinh} x$.

The function $y = \cosh x$ is not one-to-one because its graph in Table 7.4 does not pass the horizontal line test. The restricted function $y = \cosh x, x \geq 0$, however, is one-to-one and therefore has an inverse, denoted by

$$y = \cosh^{-1} x.$$

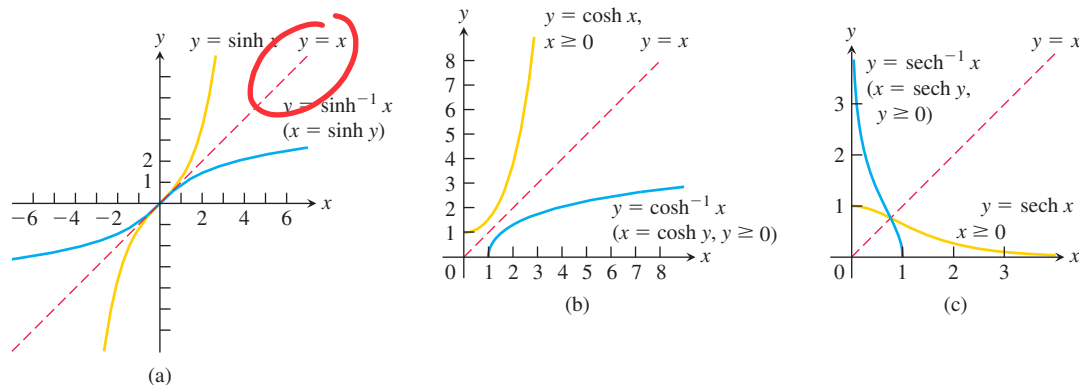


FIGURE 7.8 The graphs of the inverse hyperbolic sine, cosine, and secant of x . Notice the symmetries about the line $y = x$.

For every value of $x \geq 1$, $y = \cosh^{-1} x$ is the number in the interval $0 \leq y < \infty$ whose hyperbolic cosine is x . The graphs of $y = \cosh x$, $x \geq 0$, and $y = \cosh^{-1} x$ are shown in Figure 7.8b.

Like $y = \cosh x$, the function $y = \operatorname{sech} x = 1/\cosh x$ fails to be one-to-one, but its restriction to nonnegative values of x does have an inverse, denoted by

$$y = \operatorname{sech}^{-1} x.$$

For every value of x in the interval $(0, 1]$, $y = \operatorname{sech}^{-1} x$ is the nonnegative number whose hyperbolic secant is x . The graphs of $y = \operatorname{sech} x$, $x \geq 0$, and $y = \operatorname{sech}^{-1} x$ are shown in Figure 7.8c.

The hyperbolic tangent, cotangent, and cosecant are one-to-one on their domains and therefore have inverses, denoted by

$$y = \tanh^{-1} x, \quad y = \coth^{-1} x, \quad y = \operatorname{csch}^{-1} x.$$

These functions are graphed in Figure 7.9.

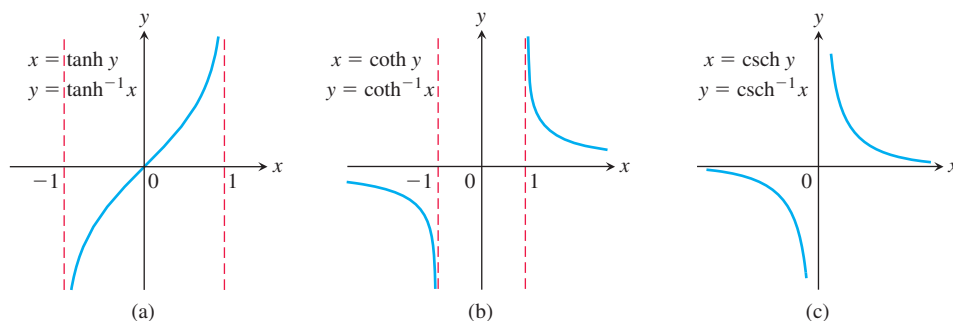


FIGURE 7.9 The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of x .

Useful Identities

TABLE 7.8 Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1} \frac{1}{x}$$

We can use the identities in Table 7.8 to express $\operatorname{sech}^{-1} x$, $\operatorname{csch}^{-1} x$, and $\coth^{-1} x$ in terms of $\cosh^{-1} x$, $\sinh^{-1} x$, and $\tanh^{-1} x$. These identities are direct consequences of the definitions. For example, if $0 < x \leq 1$, then

$$\operatorname{sech} \left(\cosh^{-1} \left(\frac{1}{x} \right) \right) = \frac{1}{\cosh \left(\cosh^{-1} \left(\frac{1}{x} \right) \right)} = \frac{1}{\left(\frac{1}{x} \right)} = x.$$

We also know that $\operatorname{sech}(\operatorname{sech}^{-1} x) = x$, so because the hyperbolic secant is one-to-one on $(0, 1]$, we have

$$\cosh^{-1} \left(\frac{1}{x} \right) = \operatorname{sech}^{-1} x.$$

Derivatives of Inverse Hyperbolic Functions

An important use of inverse hyperbolic functions lies in antiderivatives that reverse the derivative formulas in Table 7.9.

The restrictions $|x| < 1$ and $|x| > 1$ on the derivative formulas for $\tanh^{-1} x$ and $\coth^{-1} x$ come from the natural restrictions on the values of these functions. (See Figure 7.9a and b.) The distinction between $|x| < 1$ and $|x| > 1$ becomes important when we convert the derivative formulas into integral formulas.

We illustrate how the derivatives of the inverse hyperbolic functions are found in Example 2, where we calculate $d(\cosh^{-1} x)/dx$. The other derivatives are obtained by similar calculations.

TABLE 7.9 Derivatives of inverse hyperbolic functions

$\frac{d(\sinh^{-1} x)}{dx} = \frac{1}{\sqrt{1+x^2}}$	$\Rightarrow \int \frac{1}{\sqrt{1+x^2}} dx = \sinh^{-1} x + C$
$\frac{d(\cosh^{-1} x)}{dx} = \frac{1}{\sqrt{x^2-1}}, \quad x > 1$	
$\frac{d(\tanh^{-1} x)}{dx} = \frac{1}{1-x^2}, \quad x < 1$	
$\frac{d(\coth^{-1} x)}{dx} = \frac{1}{1-x^2}, \quad x > 1$	
$\frac{d(\operatorname{sech}^{-1} x)}{dx} = -\frac{1}{x\sqrt{1-x^2}}, \quad 0 < x < 1$	
$\frac{d(\operatorname{csch}^{-1} x)}{dx} = -\frac{1}{ x \sqrt{1+x^2}}, \quad x \neq 0$	

EXAMPLE 2 Show that if x is greater than 1, then

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}.$$

Solution We find the derivative of $y = \cosh^{-1} x$ for $x > 1$ by applying Theorem 3 of Section 3.8 with $f(x) = \cosh x$ and $f^{-1}(x) = \cosh^{-1} x$. Theorem 3 can be applied because the derivative of $\cosh x$ is positive when $x > 0$.

$$\begin{aligned}
 (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3, Section 3.8} \\
 &= \frac{1}{\sinh(\cosh^{-1} x)} && f'(x) = \sinh x \\
 &= \frac{1}{\sqrt{\cosh^2(\cosh^{-1} x) - 1}} && \cosh^2 x - \sinh^2 x = 1, (x \geq 0), \\
 &&& \sinh x = \sqrt{\cosh^2 x - 1} \\
 &= \frac{1}{\sqrt{x^2 - 1}} && \cosh(\cosh^{-1} x) = x
 \end{aligned}$$

HISTORICAL BIOGRAPHY

Sonya Kovalevsky
(1850–1891)

Kovalevsky, a Russian mathematician, primarily worked on the theory of partial differential equations, and a central result on the existence of solutions still bears her name. She published numerous papers on partial differential equations, eventually gaining recognition as the first woman to be elected a member of the Russian Imperial Academy of Sciences in 1889.

To know more, visit the companion Website.

With appropriate substitutions, the derivative formulas in Table 7.9 lead to the integration formulas in Table 7.10. Each of the formulas in Table 7.10 can be verified by differentiating the expression on the right-hand side.

EXAMPLE 3 Evaluate

$$\int_0^1 \frac{2 \, dx}{\sqrt{3+4x^2}}.$$

TABLE 7.10 Integrals leading to inverse hyperbolic functions

1.	$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C,$	$a > 0$
2.	$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C,$	$x > a > 0$
3.	$\int \frac{dx}{a^2 - x^2} = \begin{cases} \frac{1}{a} \tanh^{-1}\left(\frac{x}{a}\right) + C, & x^2 < a^2 \\ \frac{1}{a} \coth^{-1}\left(\frac{x}{a}\right) + C, & x^2 > a^2 \end{cases}$	
4.	$\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a} \operatorname{sech}^{-1}\left(\frac{x}{a}\right) + C,$	$0 < x < a$
5.	$\int \frac{dx}{x\sqrt{a^2 + x^2}} = -\frac{1}{a} \operatorname{csch}^{-1}\left \frac{x}{a}\right + C,$	$x \neq 0 \text{ and } a > 0$

Solution The indefinite integral is

$$\begin{aligned}
 \int \frac{2 \, dx}{\sqrt{3 + 4x^2}} &= \int \frac{du}{\sqrt{3 + u^2}} && u = 2x, \, du = 2 \, dx \\
 &= \sinh^{-1}\left(\frac{u}{\sqrt{3}}\right) + C && \text{Formula 1 from Table 7.10 with } a^2 = 3 \\
 &= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \int_0^1 \frac{2 \, dx}{\sqrt{3 + 4x^2}} &= \left[\sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) \right]_0^1 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0) \\
 &= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - 0 \approx 0.98665.
 \end{aligned}$$

EXERCISES 7.3

Values and Identities

Each of Exercises 1–4 gives a value of $\sinh x$ or $\cosh x$. Use the definitions and the identity $\cosh^2 x - \sinh^2 x = 1$ to find the values of the remaining five hyperbolic functions.

1. $\sinh x = -\frac{3}{4}$	2. $\sinh x = \frac{4}{3}$
3. $\cosh x = \frac{17}{15}, \, x > 0$	4. $\cosh x = \frac{13}{5}, \, x > 0$

Rewrite the expressions in Exercises 5–10 in terms of exponentials and simplify the results as much as you can.

5. $2 \cosh(\ln x)$	6. $\sinh(2 \ln x)$
7. $\cosh 5x + \sinh 5x$	8. $\cosh 3x - \sinh 3x$

9. $(\sinh x + \cosh x)^4$

10. $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$

11. Prove the identities

$$\begin{aligned}
 \sinh(x + y) &= \sinh x \cosh y + \cosh x \sinh y, \\
 \cosh(x + y) &= \cosh x \cosh y + \sinh x \sinh y.
 \end{aligned}$$

Then use them to show that

a. $\sinh 2x = 2 \sinh x \cosh x.$

b. $\cosh 2x = \cosh^2 x + \sinh^2 x.$

12. Use the definitions of $\cosh x$ and $\sinh x$ to show that

$$\cosh^2 x - \sinh^2 x = 1.$$

Finding Derivatives

In Exercises 13–24, find the derivative of y with respect to the appropriate variable.

13. $y = 6 \sinh \frac{x}{3}$ 14. $y = \frac{1}{2} \sinh(2x + 1)$
 15. $y = 2\sqrt{t} \tanh \sqrt{t}$ 16. $y = t^2 \tanh \frac{1}{t}$
 17. $y = \ln(\sinh z)$ 18. $y = \ln(\cosh z)$
 19. $y = (\operatorname{sech} \theta)(1 - \ln \operatorname{sech} \theta)$
 20. $y = (\operatorname{csch} \theta)(1 - \ln \operatorname{csch} \theta)$
 21. $y = \ln \cosh v - \frac{1}{2} \tanh^2 v$ 22. $y = \ln \sinh v - \frac{1}{2} \coth^2 v$
 23. $y = (x^2 + 1) \operatorname{sech}(\ln x)$

(Hint: Before differentiating, express in terms of exponentials and simplify.)

24. $y = (4x^2 - 1) \operatorname{csch}(\ln 2x)$

In Exercises 25–36, find the derivative of y with respect to the appropriate variable.

25. $y = \sinh^{-1} \sqrt{x}$ 26. $y = \cosh^{-1} 2\sqrt{x+1}$
 27. $y = (1 - \theta) \tanh^{-1} \theta$ 28. $y = (\theta^2 + 2\theta) \tanh^{-1}(\theta + 1)$
 29. $y = (1 - t) \coth^{-1} \sqrt{t}$ 30. $y = (1 - t^2) \coth^{-1} t$
 31. $y = \cos^{-1} x - x \operatorname{sech}^{-1} x$ 32. $y = \ln x + \sqrt{1 - x^2} \operatorname{sech}^{-1} x$
 33. $y = \operatorname{csch}^{-1} \left(\frac{1}{2} \right)^\theta$ 34. $y = \operatorname{csch}^{-1} 2^\theta$
 35. $y = \sinh^{-1}(\tan x)$
 36. $y = \cosh^{-1}(\sec x)$, $0 < x < \pi/2$

Integration Formulas

Verify the integration formulas in Exercises 37–40.

37. a. $\int \operatorname{sech} x \, dx = \tan^{-1}(\sinh x) + C$
 b. $\int \operatorname{sech} x \, dx = \sin^{-1}(\tanh x) + C$
 38. $\int x \operatorname{sech}^{-1} x \, dx = \frac{x^2}{2} \operatorname{sech}^{-1} x - \frac{1}{2} \sqrt{1 - x^2} + C$
 39. $\int x \coth^{-1} x \, dx = \frac{x^2 - 1}{2} \coth^{-1} x + \frac{x}{2} + C$
 40. $\int \tanh^{-1} x \, dx = x \tanh^{-1} x + \frac{1}{2} \ln(1 - x^2) + C$

Evaluating Integrals

Evaluate the integrals in Exercises 41–60.

41. $\int \sinh 2x \, dx$ 42. $\int \sinh \frac{x}{5} \, dx$
 43. $\int 6 \cosh \left(\frac{x}{2} - \ln 3 \right) dx$ 44. $\int 4 \cosh(3x - \ln 2) \, dx$
 45. $\int \tanh \frac{x}{7} \, dx$ 46. $\int \coth \frac{\theta}{\sqrt{3}} \, d\theta$
 47. $\int \operatorname{sech}^2 \left(x - \frac{1}{2} \right) dx$ 48. $\int \operatorname{csch}^2(5 - x) \, dx$

49. $\int \frac{\operatorname{sech} \sqrt{t} \tanh \sqrt{t} \, dt}{\sqrt{t}}$ 50. $\int \frac{\operatorname{csch}(\ln t) \coth(\ln t) \, dt}{t}$
 51. $\int_{\ln 2}^{\ln 4} \coth x \, dx$ 52. $\int_0^{\ln 2} \tanh 2x \, dx$
 53. $\int_{-\ln 4}^{-\ln 2} 2e^\theta \cosh \theta \, d\theta$ 54. $\int_0^{\ln 2} 4e^{-\theta} \sinh \theta \, d\theta$
 55. $\int_{-\pi/4}^{\pi/4} \cosh(\tan \theta) \sec^2 \theta \, d\theta$ 56. $\int_0^{\pi/2} 2 \sinh(\sin \theta) \cos \theta \, d\theta$
 57. $\int_1^2 \frac{\cosh(\ln t)}{t} \, dt$ 58. $\int_1^4 \frac{8 \cosh \sqrt{x}}{\sqrt{x}} \, dx$
 59. $\int_{-\ln 2}^0 \cosh^2 \left(\frac{x}{2} \right) dx$ 60. $\int_0^{\ln 10} 4 \sinh^2 \left(\frac{x}{2} \right) dx$

Inverse Hyperbolic Functions and Integrals

Since the hyperbolic functions can be expressed in terms of exponential functions, it is possible to express the inverse hyperbolic functions in terms of logarithms, as shown in the following table.

$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}),$	$-\infty < x < \infty$
$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}),$	$x \geq 1$
$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x},$	$ x < 1$
$\operatorname{sech}^{-1} x = \ln \left(\frac{1 + \sqrt{1 - x^2}}{x} \right),$	$0 < x \leq 1$
$\operatorname{csch}^{-1} x = \ln \left(\frac{1}{x} + \frac{\sqrt{1 + x^2}}{ x } \right),$	$x \neq 0$
$\coth^{-1} x = \frac{1}{2} \ln \frac{x+1}{x-1},$	$ x > 1$

Use these formulas to express the numbers in Exercises 61–66 in terms of natural logarithms.

61. $\sinh^{-1}(-5/12)$ 62. $\cosh^{-1}(5/3)$
 63. $\tanh^{-1}(-1/2)$ 64. $\coth^{-1}(5/4)$
 65. $\operatorname{sech}^{-1}(3/5)$ 66. $\operatorname{csch}^{-1}(-1/\sqrt{3})$

Evaluate the integrals in Exercises 67–74 in terms of

- a. inverse hyperbolic functions.
 b. natural logarithms.

67. $\int_0^{2\sqrt{3}} \frac{dx}{\sqrt{4 + x^2}}$ 68. $\int_0^{1/3} \frac{6 \, dx}{\sqrt{1 + 9x^2}}$
 69. $\int_{5/4}^2 \frac{dx}{1 - x^2}$ 70. $\int_0^{1/2} \frac{dx}{1 - x^2}$
 71. $\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1 - 16x^2}}$ 72. $\int_1^2 \frac{dx}{x\sqrt{4 + x^2}}$
 73. $\int_0^\pi \frac{\cos x \, dx}{\sqrt{1 + \sin^2 x}}$ 74. $\int_1^e \frac{dx}{x\sqrt{1 + (\ln x)^2}}$

Applications and Examples

75. Show that if a function f is defined on an interval symmetric about the origin (so that f is defined at $-x$ whenever it is defined at x), then

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}.$$

Then show that $(f(x) + f(-x))/2$ is even and that $(f(x) - f(-x))/2$ is odd.

76. Derive the formula $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$ for all real x . Explain in your derivation why the plus sign is used with the square root instead of the minus sign.

77. **Skydiving** If a body of mass m falling from rest under the action of gravity encounters an air resistance proportional to the square of the velocity, then the body's velocity t s into the fall satisfies the differential equation

$$m \frac{dv}{dt} = mg - kv^2,$$

where k is a constant that depends on the body's aerodynamic properties and the density of the air. (We assume that the fall is short enough so that the variation in the air's density will not affect the outcome significantly.)

- a. Show that

$$v = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{gk}{m}} t\right)$$

satisfies the differential equation and the initial condition that $v = 0$ when $t = 0$.

- b. Find the body's *limiting velocity*, $\lim_{t \rightarrow \infty} v$.

- c. For a 75-kg skydiver ($mg = 735$ N), with time in seconds and distance in meters, a typical value for k is 0.235. What is the diver's limiting velocity?

78. **Accelerations whose magnitudes are proportional to displacement** Suppose that the position of a body moving along a coordinate line at time t is

$$\text{a. } s = a \cos kt + b \sin kt. \quad \text{b. } s = a \cosh kt + b \sinh kt.$$

Show in both cases that the acceleration d^2s/dt^2 is proportional to s but that in the first case it is directed toward the origin, whereas in the second case it is directed away from the origin.

79. **Volume** A region in the first quadrant is bounded above by the curve $y = \cosh x$, below by the curve $y = \sinh x$, and on the left and right by the y -axis and the line $x = 2$, respectively. Find the volume of the solid generated by revolving the region about the x -axis.

80. **Volume** The region enclosed by the curve $y = \operatorname{sech} x$, the x -axis, and the lines $x = \pm \ln \sqrt{3}$ is revolved about the x -axis to generate a solid. Find the volume of the solid.

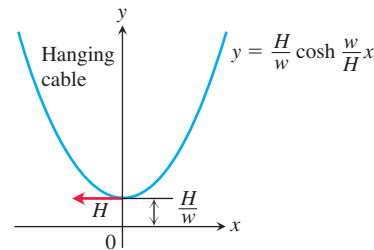
81. **Arc length** Find the length of the graph of $y = (1/2)\cosh 2x$ from $x = 0$ to $x = \ln \sqrt{5}$.

82. Use the definitions of the hyperbolic functions to find each of the following limits.

a. $\lim_{x \rightarrow \infty} \tanh x$	b. $\lim_{x \rightarrow -\infty} \tanh x$
c. $\lim_{x \rightarrow \infty} \sinh x$	d. $\lim_{x \rightarrow -\infty} \sinh x$
e. $\lim_{x \rightarrow \infty} \operatorname{sech} x$	f. $\lim_{x \rightarrow \infty} \coth x$
g. $\lim_{x \rightarrow 0^+} \coth x$	h. $\lim_{x \rightarrow 0^-} \coth x$
i. $\lim_{x \rightarrow -\infty} \operatorname{csch} x$	

83. **Hanging cables** Imagine a cable, like a telephone line or TV cable, strung from one support to another and hanging freely. The cable's weight per unit length is a constant w , and the horizontal tension at its lowest point is a *vector* of length H . If we choose a coordinate system for the plane of the cable in which the x -axis is horizontal, the force of gravity is straight down, the positive y -axis points straight up, and the lowest point of the cable lies at the point $y = H/w$ on the y -axis (see accompanying figure), then it can be shown that the cable lies along the graph of the hyperbolic cosine

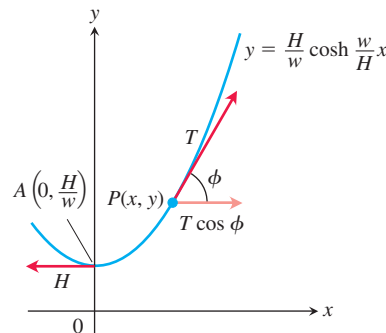
$$y = \frac{H}{w} \cosh \frac{w}{H} x.$$



Such a curve is sometimes called a **chain curve** or a **catenary**, the latter deriving from the Latin *catena*, meaning “chain.”

- a. Let $P(x, y)$ denote an arbitrary point on the cable. The next accompanying figure displays the tension at P as a vector of length (magnitude) T , as well as the tension H at the lowest point A . Show that the cable's slope at P is

$$\tan \phi = \frac{dy}{dx} = \sinh \frac{w}{H} x.$$

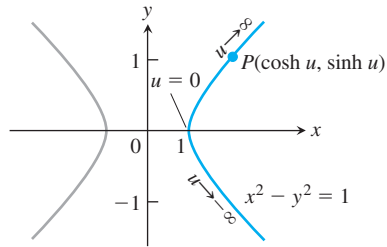


- b. Using the result from part (a) and the fact that the horizontal tension at P must equal H (the cable is not moving), show that $T = wy$. Hence, the magnitude of the tension at $P(x, y)$ is exactly equal to the weight of y units of cable.
84. (Continuation of Exercise 83.) The length of arc AP in the Exercise 83 figure is $s = (1/a)\sinh ax$, where $a = w/H$. Show that the coordinates of P may be expressed in terms of s as

$$x = \frac{1}{a} \sinh^{-1} as, \quad y = \sqrt{s^2 + \frac{1}{a^2}}.$$

85. **Area** Show that the area of the region in the first quadrant enclosed by the curve $y = (1/a)\cosh ax$, the coordinate axes, and the line $x = b$ is the same as the area of a rectangle of height $1/a$ and length s , where s is the length of the curve from $x = 0$ to $x = b$. Draw a figure illustrating this result.

86. The hyperbolic in hyperbolic functions Just as $x = \cos u$ and $y = \sin u$ are identified with points (x, y) on the unit circle, the functions $x = \cosh u$ and $y = \sinh u$ are identified with points (x, y) on the right-hand branch of the unit hyperbola, $x^2 - y^2 = 1$.



Since $\cosh^2 u - \sinh^2 u = 1$, the point $(\cosh u, \sinh u)$ lies on the right-hand branch of the hyperbola $x^2 - y^2 = 1$ for every value of u .

Another analogy between hyperbolic and circular functions is that the variable u in the coordinates $(\cosh u, \sinh u)$ for the points of the right-hand branch of the hyperbola $x^2 - y^2 = 1$ is twice the area of the sector AOP pictured in the figure following part (c). To see why this is so, carry out the following steps.

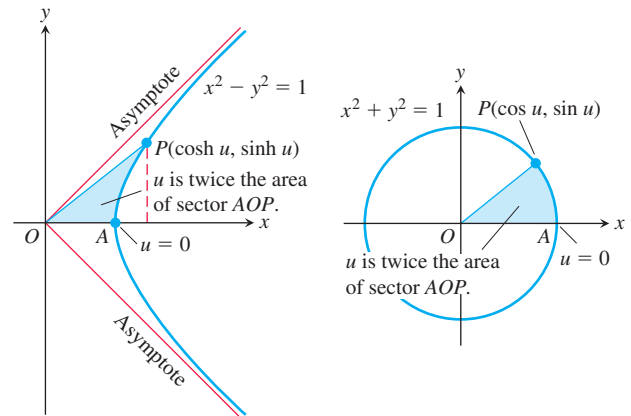
a. Show that the area $A(u)$ of sector AOP is

$$A(u) = \frac{1}{2} \cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2 - 1} \, dx.$$

b. Differentiate both sides of the equation in part (a) with respect to u to show that

$$A'(u) = \frac{1}{2}.$$

c. Solve this last equation for $A(u)$. What is the value of $A(0)$? What is the value of the constant of integration C in your solution? With C determined, what does your solution say about the relationship of u to $A(u)$?



One of the analogies between hyperbolic and circular functions is revealed by these two diagrams (Exercise 86).

7.4 Relative Rates of Growth

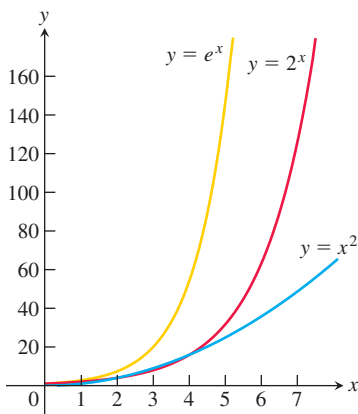


FIGURE 7.10 The graphs of e^x , 2^x , and x^2 .

It is often important in mathematics, computer science, and engineering to compare the rates at which functions of x grow as x becomes large. Exponential functions are important in these comparisons because of their very fast growth, and logarithmic functions because of their very slow growth. In this section we introduce the *little-oh* and *big-oh* notation used to describe the results of these comparisons. We restrict our attention to functions whose values eventually become and remain positive as $x \rightarrow \infty$.

Growth Rates of Functions

You may have noticed that exponential functions like 2^x and e^x seem to grow more rapidly as x gets large than do polynomials and rational functions. These exponentials certainly grow more rapidly than x itself, and you can see 2^x outgrowing x^2 as x increases in Figure 7.10. In fact, as $x \rightarrow \infty$, the functions 2^x and e^x grow faster than any power of x , even $x^{1,000,000}$ (Exercise 19). In contrast, logarithmic functions like $y = \log_2 x$ and $y = \ln x$ grow more slowly as $x \rightarrow \infty$ than any positive power of x (Exercise 21).

To get a feeling for how rapidly the values of $y = e^x$ grow with increasing x , think of graphing the function on a large blackboard, with the axes scaled in centimeters. At $x = 1$ cm, the graph is $e^1 \approx 3$ cm above the x -axis. At $x = 6$ cm, the graph is $e^6 \approx 403$ cm ≈ 4 m high (it is about to go through the ceiling if it hasn't done so already). At $x = 10$ cm, the graph is $e^{10} \approx 22,026$ cm ≈ 220 m high, higher than most buildings. At $x = 24$ cm, the graph is more than halfway to the moon, and at $x = 43$ cm

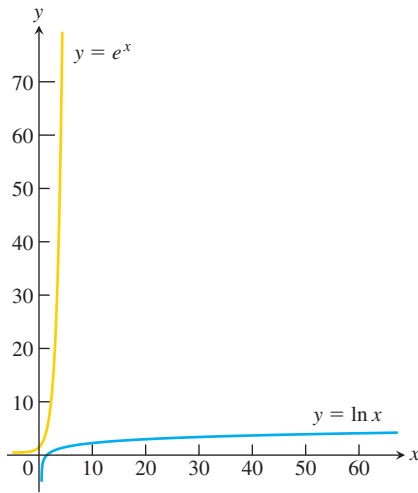


FIGURE 7.11 Scale drawings of the graphs of e^x and $\ln x$.

from the origin, the graph is high enough to reach past the sun's closest stellar neighbor, the red dwarf star Proxima Centauri. By contrast, with axes scaled in centimeters, you have to go nearly 5 light-years out on the x -axis to find a point where the graph of $y = \ln x$ is even $y = 43$ cm high. See Figure 7.11.

These important comparisons of exponential, polynomial, and logarithmic functions can be made precise by defining what it means for a function $f(x)$ to grow faster than another function $g(x)$ as $x \rightarrow \infty$.

DEFINITION Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. f grows faster than g as $x \rightarrow \infty$ if

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \infty$$

or, equivalently, if

$$\lim_{x \rightarrow \infty} \frac{g(x)}{f(x)} = 0.$$

We also say that g grows slower than f as $x \rightarrow \infty$.

2. f and g grow at the same rate as $x \rightarrow \infty$ if

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L$$

where L is finite and positive.

According to these definitions, $y = 2x$ does not grow faster than $y = x$. The two functions grow at the same rate because

$$\lim_{x \rightarrow \infty} \frac{2x}{x} = \lim_{x \rightarrow \infty} 2 = 2,$$

which is a finite, positive limit. The reason for this departure from more colloquial usage is that we want “ f grows faster than g ” to mean that for large x -values g is negligible when compared with f .

EXAMPLE 1 We compare the growth rates of several common functions.

(a) e^x grows faster than x^2 as $x \rightarrow \infty$ because

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty. \quad \text{Using l'Hôpital's Rule twice}$$

(b) 3^x grows faster than 2^x as $x \rightarrow \infty$ because

$$\lim_{x \rightarrow \infty} \frac{3^x}{2^x} = \lim_{x \rightarrow \infty} \left(\frac{3}{2} \right)^x = \infty.$$

(c) x^2 grows faster than $\ln x$ as $x \rightarrow \infty$ because

$$\lim_{x \rightarrow \infty} \frac{x^2}{\ln x} = \lim_{x \rightarrow \infty} \frac{2x}{1/x} = \lim_{x \rightarrow \infty} 2x^2 = \infty. \quad \text{l'Hôpital's Rule}$$

(d) $\ln x$ grows slower than $x^{1/n}$ as $x \rightarrow \infty$ for any positive integer n because

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\ln x}{x^{1/n}} &= \lim_{x \rightarrow \infty} \frac{1/x}{(1/n)x^{(1/n)-1}} && \text{L'Hôpital's Rule} \\ &= \lim_{x \rightarrow \infty} \frac{n}{x^{1/n}} = 0. && n \text{ is constant.}\end{aligned}$$

(e) As part (b) suggests, exponential functions with different bases never grow at the same rate as $x \rightarrow \infty$. If $a > b > 0$, then a^x grows faster than b^x . Since $(a/b) > 1$,

$$\lim_{x \rightarrow \infty} \frac{a^x}{b^x} = \lim_{x \rightarrow \infty} \left(\frac{a}{b}\right)^x = \infty.$$

(f) In contrast to exponential functions, logarithmic functions with different bases $a > 1$ and $b > 1$ always grow at the same rate as $x \rightarrow \infty$:

$$\lim_{x \rightarrow \infty} \frac{\log_a x}{\log_b x} = \lim_{x \rightarrow \infty} \frac{\ln x / \ln a}{\ln x / \ln b} = \frac{\ln b}{\ln a}.$$

The limiting ratio is always finite and never zero. ■

If f grows at the same rate as g as $x \rightarrow \infty$, and g grows at the same rate as h as $x \rightarrow \infty$, then f grows at the same rate as h as $x \rightarrow \infty$. The reason is that

$$\lim_{x \rightarrow \infty} \frac{f}{g} = L_1 \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{g}{h} = L_2$$

together imply

$$\lim_{x \rightarrow \infty} \frac{f}{h} = \lim_{x \rightarrow \infty} \frac{f}{g} \cdot \frac{g}{h} = L_1 L_2.$$

If L_1 and L_2 are finite and nonzero, then so is $L_1 L_2$.

EXAMPLE 2 Show that $\sqrt{x^2 + 5}$ and $(2\sqrt{x} - 1)^2$ grow at the same rate as $x \rightarrow \infty$.

Solution We show that the functions grow at the same rate by showing that they both grow at the same rate as the function $g(x) = x$:

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 5}}{x} &= \lim_{x \rightarrow \infty} \sqrt{1 + \frac{5}{x^2}} = 1, \\ \lim_{x \rightarrow \infty} \frac{(2\sqrt{x} - 1)^2}{x} &= \lim_{x \rightarrow \infty} \left(\frac{2\sqrt{x} - 1}{\sqrt{x}}\right)^2 = \lim_{x \rightarrow \infty} \left(2 - \frac{1}{\sqrt{x}}\right)^2 = 4.\end{aligned}$$
■

Order and Oh-Notation

The “little-oh” and “big-oh” notation was invented by number theorists over a hundred years ago and is now commonplace in mathematical analysis and computer science. According to this definition, saying $f = o(g)$ as $x \rightarrow \infty$ is another way to say that f grows slower than g as $x \rightarrow \infty$.

DEFINITION A function f is **of smaller order than** g as $x \rightarrow \infty$ if

$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$. We indicate this by writing $f = o(g)$ (“ f is little-oh of g ”).

EXAMPLE 3 Here we use little-oh notation.

(a) $\ln x = o(x)$ as $x \rightarrow \infty$ because $\lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0$

(b) $x^2 = o(x^3 + 1)$ as $x \rightarrow \infty$ because $\lim_{x \rightarrow \infty} \frac{x^2}{x^3 + 1} = 0$ ■

DEFINITION Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is **of at most the order of** g as $x \rightarrow \infty$ if there is a positive integer M for which

$$\frac{f(x)}{g(x)} \leq M,$$

for x sufficiently large. We indicate this by writing $f = O(g)$ (“ f is big-oh of g ”).

EXAMPLE 4 Here we use big-oh notation.

(a) $x + \sin x = O(x)$ as $x \rightarrow \infty$ because $\frac{x + \sin x}{x} \leq 2$ for x sufficiently large.

(b) $e^x + x^2 = O(e^x)$ as $x \rightarrow \infty$ because $\frac{e^x + x^2}{e^x} \rightarrow 1$ as $x \rightarrow \infty$.

(c) $x = O(e^x)$ as $x \rightarrow \infty$ because $\frac{x}{e^x} \rightarrow 0$ as $x \rightarrow \infty$. ■

If you look at the definitions again, you will see that $f = o(g)$ implies $f = O(g)$ for functions that are positive for all sufficiently large x . Also, if f and g grow at the same rate, then $f = O(g)$ and $g = O(f)$ (Exercise 11).

Sequential vs. Binary Search

Computer scientists often measure the efficiency of an algorithm by counting the number of steps a computer must take to execute the algorithm. There can be significant differences in how efficiently algorithms perform, even if they are designed to accomplish the same task. These differences are often described using big-oh notation. Here is an example.

One edition of *Webster's International Dictionary* lists about 26,000 words that begin with the letter *a*. One way to look up a word, or to learn if it is not there, is to read through the list one word at a time until you either find the word or determine that it is not there. This method, called **sequential search**, makes no particular use of the words' alphabetical arrangement in the list. You are sure to get an answer, but it might take 26,000 steps.

Another way to find the word or to learn it is not there is to go straight to the middle of the list (give or take a few words). If you do not find the word, then go to the middle of the half that contains it and forget about the half that does not. (You know which half contains it because you know the list is ordered alphabetically.) This method, called a **binary search**, eliminates roughly 13,000 words in a single step. If you do not find the word on the second try, then jump to the middle of the half that contains it. Continue this way until you have either found the word or divided the list in half so many times there are no words left. How many times do you have to divide the list to find the word or learn that it is not there? At most 15, because

$$(26,000/2^{15}) < 1.$$

That certainly beats a possible 26,000 steps.

For a list of length n , a sequential search algorithm takes on the order of n steps to find a word or determine that it is not in the list. A binary search, as the second algorithm is called, takes on the order of $\log_2 n$ steps. The reason is that if $2^{m-1} < n \leq 2^m$, then

$m - 1 < \log_2 n \leq m$, and the number of bisections required to narrow the list to one word will be at most $m = \lceil \log_2 n \rceil$, the integer ceiling of the number $\log_2 n$.

Big-oh notation provides a compact way to say all this. The number of steps in a sequential search of an ordered list is $O(n)$; the number of steps in a binary search is $O(\log_2 n)$. In our example, there is a big difference between the two (26,000 versus 15), and the difference can only increase with n because n grows faster than $\log_2 n$ as $n \rightarrow \infty$.

EXERCISES 7.4

Comparisons with the Exponential e^x

- Which of the following functions grow faster than e^x as $x \rightarrow \infty$? Which grow at the same rate as e^x ? Which grow slower?
 - $x - 3$
 - $x^3 + \sin^2 x$
 - $\sqrt{x}$
 - 4^x
 - $(3/2)^x$
 - $e^{x/2}$
 - $e^x/2$
 - $\log_{10} x$
- Which of the following functions grow faster than e^x as $x \rightarrow \infty$? Which grow at the same rate as e^x ? Which grow slower?
 - $10x^4 + 30x + 1$
 - $x \ln x - x$
 - $\sqrt{1 + x^4}$
 - $(5/2)^x$
 - e^{-x}
 - xe^x
 - $e^{\cos x}$
 - e^{x-1}

Comparisons with the Power x^2

- Which of the following functions grow faster than x^2 as $x \rightarrow \infty$? Which grow at the same rate as x^2 ? Which grow slower?
 - $x^2 + 4x$
 - $x^5 - x^2$
 - $\sqrt{x^4 + x^3}$
 - $(x + 3)^2$
 - $x \ln x$
 - 2^x
 - $x^3 e^{-x}$
 - $8x^2$
- Which of the following functions grow faster than x^2 as $x \rightarrow \infty$? Which grow at the same rate as x^2 ? Which grow slower?
 - $x^2 + \sqrt{x}$
 - $10x^2$
 - $x^2 e^{-x}$
 - $\log_{10}(x^2)$
 - $x^3 - x^2$
 - $(1/10)^x$
 - $(1.1)^x$
 - $x^2 + 100x$

Comparisons with the Logarithm $\ln x$

- Which of the following functions grow faster than $\ln x$ as $x \rightarrow \infty$? Which grow at the same rate as $\ln x$? Which grow slower?
 - $\log_3 x$
 - $\ln 2x$
 - $\ln \sqrt{x}$
 - $\sqrt{x}$
 - x
 - $5 \ln x$
 - $1/x$
 - e^x
- Which of the following functions grow faster than $\ln x$ as $x \rightarrow \infty$? Which grow at the same rate as $\ln x$? Which grow slower?
 - $\log_2(x^2)$
 - $\log_{10} 10x$
 - $1/\sqrt{x}$
 - $1/x^2$
 - $x - 2 \ln x$
 - e^{-x}
 - $\ln(\ln x)$
 - $\ln(2x + 5)$

Ordering Functions by Growth Rates

- Order the following functions from slowest growing to fastest growing as $x \rightarrow \infty$.
 - e^x
 - x^x
 - $(\ln x)^x$
 - $e^{x/2}$
- Order the following functions from slowest growing to fastest growing as $x \rightarrow \infty$.
 - 2^x
 - x^2
 - $(\ln 2)^x$
 - e^x

Big-oh and Little-oh; Order

- True, or false? As $x \rightarrow \infty$,
 - $x = o(x)$
 - $x = o(x + 5)$
 - $x = O(x + 5)$
 - $x = O(2x)$
 - $e^x = o(e^{2x})$
 - $x + \ln x = O(x)$
 - $\ln x = o(\ln 2x)$
 - $\sqrt{x^2 + 5} = O(x)$
- True, or false? As $x \rightarrow \infty$,
 - $\frac{1}{x+3} = O\left(\frac{1}{x}\right)$
 - $\frac{1}{x} + \frac{1}{x^2} = O\left(\frac{1}{x}\right)$
 - $\frac{1}{x} - \frac{1}{x^2} = o\left(\frac{1}{x}\right)$
 - $2 + \cos x = O(2)$
 - $e^x + x = O(e^x)$
 - $x \ln x = o(x^2)$
 - $\ln(\ln x) = O(\ln x)$
 - $\ln(x) = o(\ln(x^2 + 1))$
- Show that if positive functions $f(x)$ and $g(x)$ grow at the same rate as $x \rightarrow \infty$, then $f = O(g)$ and $g = O(f)$.
- When is a polynomial $f(x)$ of smaller order than a polynomial $g(x)$ as $x \rightarrow \infty$? Give reasons for your answer.
- When is a polynomial $f(x)$ of at most the order of a polynomial $g(x)$ as $x \rightarrow \infty$? Give reasons for your answer.
- What do the conclusions we drew in Section 2.8 about the limits of rational functions tell us about the relative growth of polynomials as $x \rightarrow \infty$?

Other Comparisons

- T** 15. Investigate

$$\lim_{x \rightarrow \infty} \frac{\ln(x+1)}{\ln x} \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{\ln(x+999)}{\ln x}.$$

Then use l'Hôpital's Rule to explain what you find.

16. (Continuation of Exercise 15.) Show that the value of

$$\lim_{x \rightarrow \infty} \frac{\ln(x+a)}{\ln x}$$

is the same no matter what value you assign to the constant a . What does this say about the relative rates at which the functions $f(x) = \ln(x+a)$ and $g(x) = \ln x$ grow?

17. Show that $\sqrt{10x+1}$ and $\sqrt{x+1}$ grow at the same rate as $x \rightarrow \infty$ by showing that they both grow at the same rate as $\sqrt{x}$ as $x \rightarrow \infty$.
18. Show that $\sqrt{x^4+x}$ and $\sqrt{x^4-x^3}$ grow at the same rate as $x \rightarrow \infty$ by showing that they both grow at the same rate as x^2 as $x \rightarrow \infty$.
19. Show that e^x grows faster as $x \rightarrow \infty$ than x^n for any positive integer n , even $x^{1,000,000}$. (Hint: What is the n th derivative of x^n ?)
20. **The function e^x outgrows any polynomial** Show that e^x grows faster as $x \rightarrow \infty$ than any polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

21. a. Show that $\ln x$ grows slower as $x \rightarrow \infty$ than $x^{1/n}$ for any positive integer n , even $x^{1/1,000,000}$.

T b. Although the values of $x^{1/1,000,000}$ eventually overtake the values of $\ln x$, you have to go way out on the x -axis before this happens. Find a value of x greater than 1 for which $x^{1/1,000,000} > \ln x$. You might start by observing that when $x > 1$ the equation $\ln x = x^{1/1,000,000}$ is equivalent to the equation $\ln(\ln x) = (\ln x)/1,000,000$.

T c. Even $x^{1/10}$ takes a long time to overtake $\ln x$. Experiment with a calculator to find the value of $x > 10$ at which the graphs of $x^{1/10}$ and $\ln x$ cross, or, equivalently, at which $\ln x = 10 \ln(\ln x)$. Bracket the crossing point between powers of 10 and then close in by successive halving.

T d. (Continuation of part (c).) The value of x at which $\ln x = 10 \ln(\ln x)$ is too far out for some graphers and root finders to identify. Try it on the equipment available to you and see what happens.

22. **The function $\ln x$ grows slower than any polynomial** Show that $\ln x$ grows slower as $x \rightarrow \infty$ than any nonconstant polynomial.

Algorithms and Searches

23. a. Suppose you have three different algorithms for solving the same problem and each algorithm takes a number of steps that is of the order of one of the functions listed here:

$$n \log_2 n, \quad n^{3/2}, \quad n(\log_2 n)^2.$$

Which of the algorithms is the most efficient in the long run? Give reasons for your answer.

- T** b. Graph the functions in part (a) together to get a sense of how rapidly each one grows.

24. Repeat Exercise 23 for the functions

$$n, \quad \sqrt{n} \log_2 n, \quad (\log_2 n)^2.$$

T 25. Suppose you are looking for an item in an ordered list one million items long. How many steps might it take to find that item with a sequential search? A binary search?

T 26. You are looking for an item in an ordered list 450,000 items long (the length of *Webster's Third New International Dictionary*). How many steps might it take to find the item with a sequential search? A binary search?

CHAPTER 7 Questions to Guide Your Review

- How is the natural logarithm function defined as an integral? What are its domain, range, and derivative? What arithmetic properties does it have? Comment on its graph.
- What integrals lead to logarithms? Give examples.
- What are the integrals of $\tan x$ and $\cot x$? Of $\sec x$ and $\csc x$?
- How is the exponential function e^x defined? What are its domain, range, and derivative? What laws of exponents does it obey? Comment on its graph.
- How are the functions a^x and $\log_a x$ defined? Are there any restrictions on a ? How is the graph of $\log_a x$ related to the graph of $\ln x$? What truth is there in the statement that there are really only one exponential function and one logarithmic function?
- How do you solve separable first-order differential equations?
- What is the law of exponential change? How can it be derived from an initial value problem? What are some of the applications of the law?
- What are the six basic hyperbolic functions? Comment on their domains, ranges, and graphs. What are some of the identities relating them?
- What are the derivatives of the six basic hyperbolic functions? What are the corresponding integral formulas? What similarities do you see here to the six basic trigonometric functions?
- How are the inverse hyperbolic functions defined? Comment on their domains, ranges, and graphs. How can you find values of $\operatorname{sech}^{-1} x$, $\operatorname{csch}^{-1} x$, and $\operatorname{coth}^{-1} x$ using a calculator's keys for $\cosh^{-1} x$, $\sinh^{-1} x$, and $\tanh^{-1} x$?
- What integrals lead naturally to inverse hyperbolic functions?
- How do you compare the growth rates of positive functions as $x \rightarrow \infty$?
- What roles do the functions e^x and $\ln x$ play in growth comparisons?
- Describe big-oh and little-oh notation. Give examples.
- Which is more efficient—a sequential search or a binary search? Explain.

CHAPTER 7 Practice Exercises

Integration

Evaluate the integrals in Exercises 1–12.

1. $\int e^x \sin(e^x) dx$
2. $\int e^t \cos(3e^t - 2) dt$
3. $\int_0^\pi \tan \frac{x}{3} dx$
4. $\int_{1/6}^{1/4} 2 \cot \pi x dx$
5. $\int_{-\pi/2}^{\pi/6} \frac{\cos t}{1 - \sin t} dt$
6. $\int e^x \sec e^x dx$
7. $\int \frac{\ln(x-5)}{x-5} dx$
8. $\int \frac{\cos(1 - \ln v)}{v} dv$
9. $\int_1^7 \frac{3}{x} dx$
10. $\int_1^{32} \frac{1}{5x} dx$
11. $\int_e^{e^2} \frac{1}{x\sqrt{\ln x}} dx$
12. $\int_2^4 (1 + \ln t) t \ln t dt$

Solving Equations with Logarithmic or Exponential Terms

In Exercises 13–18, solve for y .

13. $3^y = 2^{y+1}$
14. $4^{-y} = 3^{y+2}$
15. $9e^{2y} = x^2, x > 0$
16. $3^y = 3 \ln x$
17. $\ln(y-1) = x + \ln y$
18. $\ln(10 \ln y) = \ln 5x$

Comparing Growth Rates of Functions

19. Does f grow faster, slower, or at the same rate as g as $x \rightarrow \infty$? Give reasons for your answers.

- a. $f(x) = \log_2 x, g(x) = \log_3 x$
- b. $f(x) = x, g(x) = x + \frac{1}{x}$
- c. $f(x) = x/100, g(x) = xe^{-x}$
- d. $f(x) = x, g(x) = \arctan x$
- e. $f(x) = \operatorname{arccsc} x, g(x) = 1/x$
- f. $f(x) = \sinh x, g(x) = e^x$

20. Does f grow faster, slower, or at the same rate as g as $x \rightarrow \infty$? Give reasons for your answers.

- a. $f(x) = 3^{-x}, g(x) = 2^{-x}$
- b. $f(x) = \ln 2x, g(x) = \ln x^2$
- c. $f(x) = 10x^3 + 2x^2, g(x) = e^x$
- d. $f(x) = \arctan(1/x), g(x) = 1/x$
- e. $f(x) = \arcsin(1/x), g(x) = 1/x^2$
- f. $f(x) = \operatorname{sech} x, g(x) = e^{-x}$

21. True, or false? Give reasons for your answers.

- a. $\frac{1}{x^2} + \frac{1}{x^4} = O\left(\frac{1}{x^2}\right)$
- b. $\frac{1}{x^2} + \frac{1}{x^4} = O\left(\frac{1}{x^4}\right)$
- c. $x = o(x + \ln x)$
- d. $\ln(\ln x) = o(\ln x)$
- e. $\arctan x = O(1)$
- f. $\cosh x = O(e^x)$

22. True, or false? Give reasons for your answers.

- a. $\frac{1}{x^4} = O\left(\frac{1}{x^2} + \frac{1}{x^4}\right)$
- b. $\frac{1}{x^4} = O\left(\frac{1}{x^2} + \frac{1}{x^4}\right)$
- c. $\ln x = o(x + 1)$
- d. $\ln 2x = O(\ln x)$
- e. $\sec^{-1} x = O(1)$
- f. $\sinh x = O(e^x)$

Theory and Applications

23. The function $f(x) = e^x + x$, being differentiable and one-to-one, has a differentiable inverse $f^{-1}(x)$. Find the value of df^{-1}/dx at the point $f(\ln 2)$.

24. Find the inverse of the function $f(x) = 1 + (1/x), x \neq 0$. Then show that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$ and that

$$\left. \frac{df^{-1}}{dx} \right|_{f(x)} = \frac{1}{f'(x)}.$$

25. A particle is traveling upward and to the right along the curve $y = \ln x$. Its x -coordinate is increasing at the rate $(dx/dt) = \sqrt{x}$ m/s. At what rate is the y -coordinate changing at the point $(e^2, 2)$?

26. A girl is sliding down a slide shaped like the curve $y = 3e^{-x/3}$. Her y -coordinate is changing at the rate

$$dy/dt = (-1/4)\sqrt{3-y} \text{ m/s}.$$

At approximately what rate is her x -coordinate changing when she reaches the bottom of the slide at $x = 3$ m?

27. The functions $f(x) = \ln 5x$ and $g(x) = \ln 3x$ differ by a constant. What constant? Give reasons for your answer.

28. a. If $(\ln x)/x = (\ln 2)/2$, must $x = 2$?

- b. If $(\ln x)/x = -2 \ln 2$, must $x = 1/2$?

Give reasons for your answers.

29. The quotient $(\log_4 x)/(\log_2 x)$ has a constant value. What value? Give reasons for your answer.

- T** 30. **$\log_x(2)$ vs. $\log_2(x)$** How does $f(x) = \log_x(2)$ compare with $g(x) = \log_2(x)$? Here is one way to find out.

- a. Use the equation $\log_a b = (\ln b)/(\ln a)$ to express $f(x)$ and $g(x)$ in terms of natural logarithms.

- b. Graph f and g together. Comment on the behavior of f in relation to the signs and values of g .

In Exercises 31–34, solve the differential equation.

31. $\frac{dy}{dx} = \sqrt{y} \cos^2 \sqrt{y}$

32. $y' = \frac{3y(x+1)^2}{y-1}$

33. $yy' = \sec y^2 \sec^2 x$

34. $y \cos^2 x dy + \sin x dx = 0$

In Exercises 35–38, solve the initial value problem.

35. $\frac{dy}{dx} = e^{-x-y-2}, y(0) = -2$

36. $\frac{dy}{dx} = \frac{y \ln y}{1+x^2}, y(0) = e^2$

37. $x dy - (y + \sqrt{y}) dx = 0, y(1) = 1$

38. $y^{-2} \frac{dx}{dy} = \frac{e^x}{e^{2x} + 1}, y(0) = 1$

39. What is the age of a sample of charcoal in which 90% of the carbon-14 originally present has decayed?

40. **Cooling a pie** A deep-dish apple pie, whose internal temperature was 104°C when removed from the oven, was set out on a breezy 5°C porch to cool. Fifteen minutes later, the pie's internal temperature was 82°C . How long did it take the pie to cool from there to 21°C ?

41. Find the length of the curve $y = \ln(e^x - 1) - \ln(e^x + 1)$, $\ln 2 \leq x \leq \ln 3$.
42. In 1934, the Austrian biologist Ludwig von Bertalanffy derived and published the von Bertalanffy growth equation, which continues to be widely used and is especially important in fisheries studies. Let $L(t)$ denote the length of a fish at time t and assume $L(0) = L_0$. The von Bertalanffy equation is

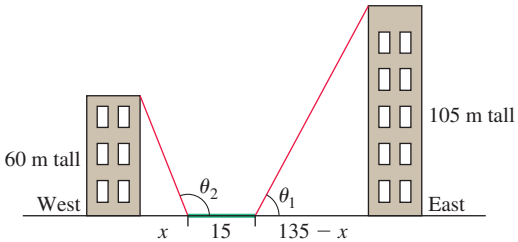
$$\frac{dL}{dt} = k(A - L),$$

where $A = \lim_{t \rightarrow \infty} L(t)$ is the asymptotic length of the fish and k is a proportionality constant.

Assume that $L(t)$ is the length in meters of a shark of age t years. In addition, assume $A = 3$, $L(0) = 0.5$ m, and $L(5) = 1.75$ m.

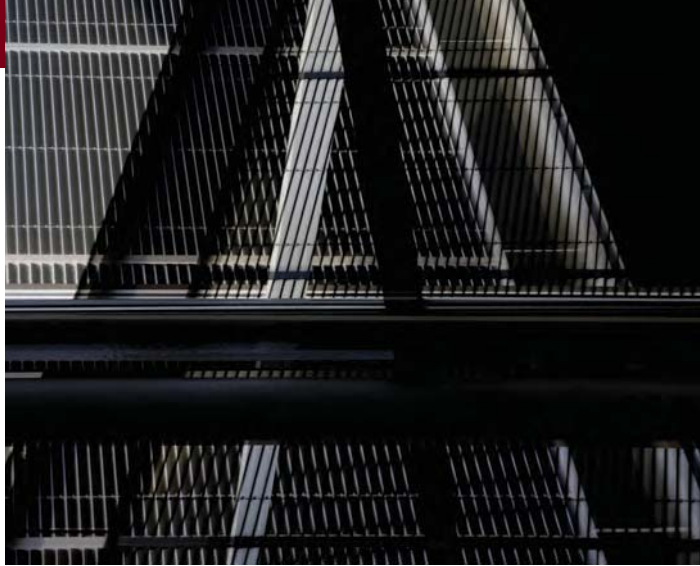
- Solve the von Bertalanffy differential equation.
- What is $L(10)$?
- When does $L = 2.5$ m?

CHAPTER 7 Additional and Advanced Exercises

- Let $A(t)$ be the area of the region in the first quadrant enclosed by the coordinate axes, the curve $y = e^{-x}$, and the vertical line $x = t$, $t > 0$. Let $V(t)$ be the volume of the solid generated by revolving the region about the x -axis. Find the following limits.
 - $\lim_{t \rightarrow \infty} A(t)$
 - $\lim_{t \rightarrow \infty} V(t)/A(t)$
 - $\lim_{t \rightarrow 0^+} V(t)/A(t)$
- Varying a logarithm's base**
 - Find $\lim_{a \rightarrow 0^+, 1^-, 1^+, \text{ and } \infty} \log_a 2$.
 - T** Graph $y = \log_a 2$ as a function of a over the interval $0 < a \leq 4$.
- T** Graph $f(x) = \tan^{-1} x + \tan^{-1}(1/x)$ for $-5 \leq x \leq 5$. Then use calculus to explain what you see. How would you expect f to behave beyond the interval $[-5, 5]$? Give reasons for your answer.
- T** Graph $f(x) = (\sin x)^{\sin x}$ over $[0, 3\pi]$. Explain what you see.
- Even-odd decompositions**
 - Suppose that g is an even function of x and h is an odd function of x . Show that if $g(x) + h(x) = 0$ for all x , then $g(x) = 0$ for all x and $h(x) = 0$ for all x .
 - Use the result in part (a) to show that if $f(x) = f_E(x) + f_O(x)$ is the sum of an even function $f_E(x)$ and an odd function $f_O(x)$, then $f_E(x) = (f(x) + f(-x))/2$ and $f_O(x) = (f(x) - f(-x))/2$.
 - What is the significance of the result in part (b)?
- Let g be a function that is differentiable throughout an open interval containing the origin. Suppose g has the following properties.
 - $g(x + y) = \frac{g(x) + g(y)}{1 - g(x)g(y)}$ for all real numbers x, y , and $x + y$ in the domain of g .
 - $\lim_{h \rightarrow 0} g(h) = 0$
 - $\lim_{h \rightarrow 0} \frac{g(h)}{h} = 1$
 - Show that $g(0) = 0$.
 - Show that $g'(x) = 1 + [g(x)]^2$.
 - Find $g(x)$ by solving the differential equation in part (b).
- Center of mass** Find the center of mass of a thin plate of constant density covering the region in the first and fourth quadrants enclosed by the curves $y = 1/(1 + x^2)$ and $y = -1/(1 + x^2)$ and by the lines $x = 0$ and $x = 1$.
- Solid of revolution** The region between the curve $y = 1/(2\sqrt{x})$ and the x -axis from $x = 1/4$ to $x = 4$ is revolved about the x -axis to generate a solid.
 - Find the volume of the solid.
 - Find the centroid of the region.
- The Rule of 70** If you use the approximation $\ln 2 \approx 0.70$ (in place of $0.69314 \dots$), you can derive a rule of thumb that says, "To estimate how many years it will take an amount of money to double when invested at r percent compounded continuously, divide r into 70." For instance, an amount of money invested at 5% will double in about $70/5 = 14$ years. If you want it to double in 10 years instead, you have to invest it at $70/10 = 7\%$. Show how the Rule of 70 is derived. (A similar "Rule of 72" uses 72 instead of 70, because 72 has more integer factors.)
- T** **Urban gardening** A vegetable garden 15 m wide is to be grown between two buildings, which are 150 m apart along an east–west line. If the buildings are 60 m and 105 m tall, where should the garden be placed in order to receive the maximum number of hours of sunlight exposure? (*Hint:* Determine the value of x in the accompanying figure that maximizes sunlight exposure for the garden.)
 

8

Techniques of Integration



OVERVIEW The Fundamental Theorem tells us how to evaluate a definite integral once we have an antiderivative for the integrand function. However, finding antiderivatives (or indefinite integrals) is not as straightforward as finding derivatives. In this chapter we study a number of important techniques that apply to finding integrals for specialized classes of functions such as trigonometric functions, products of certain functions, and rational functions. Since we cannot always find an antiderivative, we develop numerical methods for calculating definite integrals. We also study integrals for which the domain or range is infinite, called *improper integrals*.

8.1 Using Basic Integration Formulas

Table 8.1 summarizes the indefinite integrals of many of the functions we have studied so far, and the substitution method helps us use the table to evaluate more complicated functions involving these basic ones. In this section we combine the Substitution Rules (studied in Chapter 5) with algebraic methods and trigonometric identities to help us use Table 8.1. A more extensive Table of Integrals is given at the back of the chapter, and we discuss its use in Section 8.6.

Sometimes we have to rewrite an integral to match it to a standard form of the type displayed in Table 8.1. We start with an example of this procedure.

EXAMPLE 1 Evaluate the integral

$$\int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}} dx.$$

Solution We rewrite the integral and apply the Substitution Rule for Definite Integrals presented in Section 5.6, to find

$$\begin{aligned} \int_3^5 \frac{2x - 3}{\sqrt{x^2 - 3x + 1}} dx &= \int_1^{11} \frac{du}{\sqrt{u}} \quad \begin{array}{l} u = x^2 - 3x + 1, du = (2x - 3) dx; \\ u = 1 \text{ when } x = 3, u = 11 \text{ when } x = 5 \end{array} \\ &= \int_1^{11} u^{-1/2} du \\ &= 2\sqrt{u} \Big|_1^{11} = 2(\sqrt{11} - 1) \approx 4.63. \quad \text{Table 8.1, Formula 2} \end{aligned}$$

TABLE 8.1 Basic integration formulas

1. $\int k \, dx = kx + C$ (any number k)	12. $\int \tan x \, dx = \ln \sec x + C$
2. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$ ($n \neq -1$)	13. $\int \cot x \, dx = \ln \sin x + C$
3. $\int \frac{dx}{x} = \ln x + C$	14. $\int \sec x \, dx = \ln \sec x + \tan x + C$
4. $\int e^x \, dx = e^x + C$	15. $\int \csc x \, dx = -\ln \csc x + \cot x + C$
5. $\int a^x \, dx = \frac{a^x}{\ln a} + C$ ($a > 0, a \neq 1$)	16. $\int \sinh x \, dx = \cosh x + C$
6. $\int \sin x \, dx = -\cos x + C$	17. $\int \cosh x \, dx = \sinh x + C$
7. $\int \cos x \, dx = \sin x + C$	18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
8. $\int \sec^2 x \, dx = \tan x + C$	19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$
9. $\int \csc^2 x \, dx = -\cot x + C$	20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec}\left \frac{x}{a}\right + C$
10. $\int \sec x \tan x \, dx = \sec x + C$	21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$ ($a > 0$)
11. $\int \csc x \cot x \, dx = -\csc x + C$	22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$ ($x > a > 0$)

EXAMPLE 2 Complete the square to evaluate

$$\int \frac{dx}{\sqrt{8x - x^2}}.$$

Solution We complete the square to simplify the denominator:

$$\begin{aligned} 8x - x^2 &= -(x^2 - 8x) = -(x^2 - 8x + 16 - 16) \\ &= -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2. \end{aligned}$$

Then

$$\begin{aligned} \int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x - 4)^2}} \\ &= \int \frac{du}{\sqrt{a^2 - u^2}} && a = 4, u = (x - 4), \\ &&& du = dx \\ &= \arcsin\left(\frac{u}{a}\right) + C && \text{Table 8.1, Formula 18} \\ &= \arcsin\left(\frac{x - 4}{4}\right) + C. \end{aligned}$$

EXAMPLE 3 Evaluate the integral

$$\int (\cos x \sin 2x + \sin x \cos 2x) dx.$$

Solution We can replace the integrand with an equivalent trigonometric expression using the Sine Addition Formula to obtain a simple substitution:

$$\begin{aligned} \int (\cos x \sin 2x + \sin x \cos 2x) dx &= \int \sin(x + 2x) dx \\ &= \int \sin 3x dx \\ &= \int \frac{1}{3} \sin u du \quad u = 3x, du = 3 dx \\ &= -\frac{1}{3} \cos 3x + C. \quad \text{Table 8.1, Formula 6} \end{aligned}$$

In Section 5.5 we found the indefinite integral of the secant function by multiplying it by a fractional form equal to one, and then integrating the equivalent result. We can use that same procedure in other instances as well, as we illustrate next.

EXAMPLE 4 Find $\int_0^{\pi/4} \frac{dx}{1 - \sin x}$.

Solution We multiply the numerator and denominator of the integrand by $1 + \sin x$. This procedure transforms the integral into one we can evaluate:

$$\begin{aligned} \int_0^{\pi/4} \frac{dx}{1 - \sin x} &= \int_0^{\pi/4} \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} dx && \text{Multiply and divide by conjugate.} \\ &= \int_0^{\pi/4} \frac{1 + \sin x}{1 - \sin^2 x} dx && \text{Simplify.} \\ &= \int_0^{\pi/4} \frac{1 + \sin x}{\cos^2 x} dx && 1 - \sin^2 x = \cos^2 x \\ &= \int_0^{\pi/4} (\sec^2 x + \sec x \tan x) dx && \text{Use Table 8.1, Formulas 8 and 10} \\ &= \left[\tan x + \sec x \right]_0^{\pi/4} = (1 + \sqrt{2} - (0 + 1)) = \sqrt{2}. \end{aligned}$$

EXAMPLE 5 Evaluate

$$\int \frac{3x^2 - 7x}{3x + 2} dx.$$

Solution The integrand is an improper fraction since the degree of the numerator is greater than the degree of the denominator. To integrate it, we perform long division to obtain a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

$$\begin{array}{r} x - 3 \\ 3x + 2 \overline{) 3x^2 - 7x} \\ \underline{3x^2 + 2x} \\ -9x \\ \underline{-9x - 6} \\ +6 \end{array}$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2} dx = \int \left(x - 3 + \frac{6}{3x + 2} \right) dx = \frac{x^2}{2} - 3x + 2 \ln|3x + 2| + C. \quad \blacksquare$$

Reducing an improper fraction by long division (Example 5) does not always lead to an expression we can integrate directly. We see what to do about that in Section 8.5.

EXAMPLE 6 Evaluate

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} dx.$$

Solution We first separate the integrand to get

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} dx = 3 \int \frac{x dx}{\sqrt{1 - x^2}} + 2 \int \frac{dx}{\sqrt{1 - x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1 - x^2, \quad du = -2x dx, \quad \text{so} \quad x dx = -\frac{1}{2} du.$$

Then we obtain

$$\begin{aligned} 3 \int \frac{x dx}{\sqrt{1 - x^2}} &= 3 \int \frac{(-1/2) du}{\sqrt{u}} = -\frac{3}{2} \int u^{-1/2} du \\ &= -\frac{3}{2} \cdot \frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1 - x^2} + C_1. \end{aligned}$$

The second of the new integrals is a standard form,

$$2 \int \frac{dx}{\sqrt{1 - x^2}} = 2 \arcsin x + C_2. \quad \text{Table 8.1, Formula 18}$$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} dx = -3\sqrt{1 - x^2} + 2 \arcsin x + C. \quad \blacksquare$$

The question of what to substitute for in an integrand is not always quite so clear. Sometimes we simply proceed by trial-and-error, and if nothing works out, we then try another method altogether. The next several sections of the text present some of these new methods, but substitution works in the following example.

EXAMPLE 7 Evaluate

$$\int \frac{dx}{(1 + \sqrt{x})^3}.$$

Solution We might try substituting for the term $\sqrt{x}$, but the derivative factor $1/\sqrt{x}$ is missing from the integrand, so this substitution will not help. The other possibility is to substitute for $(1 + \sqrt{x})$, and it turns out this works:

$$\begin{aligned} \int \frac{dx}{(1 + \sqrt{x})^3} &= \int \frac{2(u - 1) du}{u^3} & u = 1 + \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx; \\ & & dx = 2\sqrt{x} du = 2(u - 1) du \\ &= \int \left(\frac{2}{u^2} - \frac{2}{u^3} \right) du \end{aligned}$$

$$\begin{aligned}
&= -\frac{2}{u} + \frac{1}{u^2} + C \\
&= \frac{1-2u}{u^2} + C \\
&= \frac{1-2(1+\sqrt{x})}{(1+\sqrt{x})^2} + C \\
&= C - \frac{1+2\sqrt{x}}{(1+\sqrt{x})^2}.
\end{aligned}$$

When evaluating definite integrals, a property of the integrand may help us in calculating the result.

EXAMPLE 8 Evaluate

$$\int_{-\pi/2}^{\pi/2} x^3 \cos x \, dx.$$

Solution No substitution or algebraic manipulation is clearly helpful here. But we observe that the interval of integration is the symmetric interval $[-\pi/2, \pi/2]$. Moreover, the factor x^3 is an odd function, and $\cos x$ is an even function, so their product is odd. Therefore,

$$\int_{-\pi/2}^{\pi/2} x^3 \cos x \, dx = 0. \quad \text{Theorem 8, Section 5.6}$$

EXERCISES 8.1

Assorted Integrations

The integrals in Exercises 1–44 are in no particular order. Evaluate each integral using any algebraic method, trigonometric identity, or substitution you think is appropriate.

1. $\int_0^1 \frac{16x}{8x^2 + 2} \, dx$	2. $\int \frac{x^2}{x^2 + 1} \, dx$	17. $\int \frac{\ln y}{y + 4y \ln^2 y} \, dy$	18. $\int \frac{2\sqrt{y}}{2\sqrt{y}} \, dy$
3. $\int (\sec x - \tan x)^2 \, dx$	4. $\int_{\pi/4}^{\pi/3} \frac{dx}{\cos^2 x \tan x}$	19. $\int \frac{d\theta}{\sec \theta + \tan \theta}$	20. $\int \frac{dt}{t\sqrt{3} + t^2}$
5. $\int \frac{1-x}{\sqrt{1-x^2}} \, dx$	6. $\int \frac{dx}{x - \sqrt{x}}$	21. $\int \frac{4t^3 - t^2 + 16t}{t^2 + 4} \, dt$	22. $\int \frac{x + 2\sqrt{x-1}}{2x\sqrt{x-1}} \, dx$
7. $\int \frac{e^{-\cot z}}{\sin^2 z} \, dz$	8. $\int \frac{2 \ln z^3}{16z} \, dz$	23. $\int_0^{\pi/2} \sqrt{1 - \cos \theta} \, d\theta$	24. $\int (\sec t + \cot t)^2 \, dt$
9. $\int \frac{dz}{e^z + e^{-z}}$	10. $\int_1^2 \frac{8 \, dx}{x^2 - 2x + 2}$	25. $\int \frac{dy}{\sqrt{e^{2y} - 1}}$	26. $\int \frac{6 \, dy}{\sqrt{y}(1+y)}$
11. $\int_{-1}^0 \frac{4 \, dx}{1 + (2x+1)^2}$	12. $\int_{-1}^3 \frac{4x^2 - 7}{2x + 3} \, dx$	27. $\int \frac{2 \, dx}{x\sqrt{1 - 4 \ln^2 x}}$	28. $\int \frac{dx}{(x-2)\sqrt{x^2 - 4x + 3}}$
13. $\int \frac{dt}{1 - \sec t}$	14. $\int \csc t \sin 3t \, dt$	29. $\int (\csc x - \sec x)(\sin x + \cos x) \, dx$	
15. $\int_0^{\pi/4} \frac{1 + \sin \theta}{\cos^2 \theta} \, d\theta$	16. $\int \frac{d\theta}{\sqrt{2\theta - \theta^2}}$	30. $\int 3 \sinh\left(\frac{x}{2} + \ln 5\right) \, dx$	
		31. $\int_{\sqrt{2}}^3 \frac{2x^3}{x^2 - 1} \, dx$	32. $\int_{-1}^1 \sqrt{1+x^2} \sin x \, dx$
		33. $\int_{-1}^0 \sqrt{\frac{1+y}{1-y}} \, dy$	34. $\int e^{z+e^z} \, dz$

35. $\int \frac{7 dx}{(x-1)\sqrt{x^2-2x-48}}$ 36. $\int \frac{dx}{(2x+1)\sqrt{4x+4x^2}}$
37. $\int \frac{2\theta^3 - 7\theta^2 + 7\theta}{2\theta - 5} d\theta$ 38. $\int \frac{d\theta}{\cos \theta - 1}$
39. $\int \frac{dx}{1+e^x}$ 40. $\int \frac{\sqrt{x}}{1+x^3} dx$
Hint: Use long division. *Hint: Let $u = x^{3/2}$.*
41. $\int \frac{e^{3x}}{e^x + 1} dx$ 42. $\int \frac{2^x - 1}{3^x} dx$
43. $\int \frac{1}{\sqrt{x}(1+x)} dx$ 44. $\int \frac{\tan \theta + 3}{\sin \theta} d\theta$

Theory and Examples

45. **Area** Find the area of the region bounded above by $y = 2 \cos x$ and below by $y = \sec x$, $-\pi/4 \leq x \leq \pi/4$.
46. **Volume** Find the volume of the solid generated by revolving the region in Exercise 45 about the x -axis.
47. **Arc length** Find the length of the curve $y = \ln(\cos x)$, $0 \leq x \leq \pi/3$.
48. **Arc length** Find the length of the curve $y = \ln(\sec x)$, $0 \leq x \leq \pi/4$.
49. **Centroid** Find the centroid of the region bounded by the x -axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.
50. **Centroid** Find the centroid of the region bounded by the x -axis, the curve $y = \csc x$, and the lines $x = \pi/6$, $x = 5\pi/6$.

51. The functions $y = e^{x^3}$ and $y = x^3 e^{x^3}$ do not have elementary antiderivatives, but $y = (1 + 3x^3)e^{x^3}$ does. Evaluate

$$\int (1 + 3x^3) e^{x^3} dx.$$

52. Use the substitution $u = \tan x$ to evaluate the integral

$$\int \frac{dx}{1 + \sin^2 x}.$$

53. Use the substitution $u = x^4 + 1$ to evaluate the integral

$$\int x^7 \sqrt{x^4 + 1} dx.$$

54. **Using different substitutions** Show that the integral

$$\int ((x^2 - 1)(x + 1))^{-2/3} dx$$

can be evaluated with any of the following substitutions.

- a. $u = 1/(x + 1)$
- b. $u = ((x - 1)/(x + 1))^k$ for $k = 1, 1/2, 1/3, -1/3, -2/3$, and -1
- c. $u = \arctan x$
- d. $u = \tan^{-1} \sqrt{x}$
- e. $u = \tan^{-1}((x - 1)/2)$
- f. $u = \arccos x$
- g. $u = \cosh^{-1} x$

What is the value of the integral?

8.2 Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int u(x)v'(x) dx.$$

It is useful when u can be differentiated repeatedly and v' can be integrated repeatedly without difficulty. The integrals

$$\int x \cos x dx \quad \text{and} \quad \int x^2 e^x dx$$

are such integrals because $u(x) = x$ or $u(x) = x^2$ can be differentiated repeatedly, and $v'(x) = \cos x$ or $v'(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int \ln x dx \quad \text{and} \quad \int e^x \cos x dx.$$

In the first case, the integrand $\ln x$ can be rewritten as $(\ln x)(1)$, and $u(x) = \ln x$ is easy to differentiate while $v'(x) = 1$ easily integrates to x . In the second case, each part of the integrand appears again after repeated differentiation or integration.

Product Rule in Integral Form

If u and v are differentiable functions of x , the Product Rule says that

$$\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[u(x)v(x)] dx = \int [u'(x)v(x) + u(x)v'(x)] dx$$

or

$$\int \frac{d}{dx}[u(x)v(x)] dx = \int u'(x)v(x) dx + \int u(x)v'(x) dx.$$

Rearranging the terms of this last equation, we get

$$\int u(x)v'(x) dx = \int \frac{d}{dx}[u(x)v(x)] dx - \int v(x)u'(x) dx,$$

leading to the following **integration by parts** formula.

Integration by Parts Formula

$$\int \underline{u(x)} \underline{v'(x)} dx = u(x) \cdot v(x) - \int v(x) u'(x) dx \quad (1)$$

This formula allows us to exchange the problem of computing the integral $\int u(x)v'(x) dx$ for the problem of computing a different integral, $\int v(x)u'(x) dx$. In many cases, we can choose the functions u and v so that the second integral is easier to compute than the first. There can be many choices for u and v , and it is not always clear which choice works best, so sometimes we need to try several.

The formula is often given in differential form. With $v'(x) dx = dv$ and $u'(x) dx = du$, the integration by parts formula becomes

Integration by Parts Formula—Differential Version

$$\int \underline{u} \underline{dv} = uv - \int v du \quad (2)$$

The next examples illustrate the technique.

EXAMPLE 1 Find

$$\int x \cos x dx.$$

Solution There is no obvious antiderivative of $x \cos x$, so we use the integration by parts formula

$$\int u(x)v'(x) dx = u(x)v(x) - \int v(x)u'(x) dx$$

to change this expression to one that is easier to integrate. We first decide how to choose the functions $u(x)$ and $v(x)$. There is more than one way to do this, but here we choose to factor the expression $x \cos x$ into

$$u(x) = x \quad \text{and} \quad v'(x) = \cos x.$$

Next we differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$$u'(x) = 1 \quad \text{and} \quad v(x) = \sin x.$$

When finding an antiderivative for $v'(x)$, we have a choice of how to pick a constant of integration C . We choose the constant $C = 0$, since that makes this antiderivative as simple as possible. We now apply the integration by parts formula:

$$\begin{aligned} \int \underbrace{x}_{u(x)} \underbrace{\cos x}_{v'(x)} dx &= \underbrace{x}_{u(x)} \underbrace{\sin x}_{v(x)} - \int \underbrace{\sin x}_{v(x)} \underbrace{(1)}_{u'(x)} dx && \text{Integration by parts formula} \\ &= x \sin x + \cos x + C && \text{Integrate and simplify.} \end{aligned}$$

and we have found the integral of the original function. ■

There are at least four apparent choices available for $u(x)$ and $v'(x)$ in Example 1:

1. Let $u(x) = 1$ and $v'(x) = x \cos x$.
2. Let $u(x) = x$ and $v'(x) = \cos x$.
3. Let $u(x) = x \cos x$ and $v'(x) = 1$.
4. Let $u(x) = \cos x$ and $v'(x) = x$.

We used choice 2 in Example 1. The other three choices lead to integrals that we do not know how to evaluate. For instance, Choice 3, with $u'(x) = \cos x - x \sin x$, leads to the integral

$$\int (x \cos x - x^2 \sin x) dx.$$

The goal of integration by parts is to go from an integral $\int u(x)v'(x) dx$ that we don't see how to evaluate to an integral $\int v(x)u'(x) dx$ that we can evaluate. Generally, we choose $v'(x)$ first to be as much of the integrand as we can readily integrate; then we let $u(x)$ be the leftover part. When finding $v(x)$ from $v'(x)$, any antiderivative will work, and we usually pick the simplest one. In particular, no arbitrary constant of integration is needed in $v(x)$ because it would simply cancel out of the right-hand side of Equation (2).

EXAMPLE 2 Find $\int \ln x \, dx$.

Solution We have not yet seen how to find an antiderivative for $\ln x$. If we set $u(x) = \ln x$, then $u'(x)$ is the simpler function $1/x$. It may not appear that a second function $v'(x)$ is multiplying $u(x) = \ln x$, but we can choose $v'(x)$ to be the constant function $v'(x) = 1$. We use the integration by parts formula given in Equation (1), with

$$u(x) = \ln x \quad \text{and} \quad v'(x) = 1.$$

We differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$$u'(x) = \frac{1}{x} \quad \text{and} \quad v(x) = x.$$

Then

$$\begin{aligned} \int \underbrace{\ln x}_{u(x)} \cdot \underbrace{1}_{v'(x)} dx &= \underbrace{(\ln x)}_{u(x)} \underbrace{x}_{v(x)} - \int \underbrace{x}_{v(x)} \underbrace{\frac{1}{x}}_{u'(x)} dx && \text{Integration by parts formula} \\ &= x \ln x - \int 1 \, dx \\ &= x \ln x - x + C && \text{Simplify and integrate.} \end{aligned}$$

In the following examples we use the differential form to indicate the process of integration by parts. The computations are the same, with du and dv providing shorter expressions for $u'(x) dx$ and $v'(x) dx$.

Sometimes we have to use integration by parts more than once, as in the next example.

EXAMPLE 3 Evaluate

$$\int x^2 e^x \, dx.$$

Solution We use the integration by parts formula given in Equation (1), with

$$u(x) = x^2 \quad \text{and} \quad v'(x) = e^x.$$

We differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$$u'(x) = 2x \quad \text{and} \quad v(x) = e^x.$$

We summarize this choice by setting $du = u'(x) dx$ and $dv = v'(x) dx$, so

$$du = 2x dx \quad \text{and} \quad dv = e^x dx.$$

We then have

$$\int \underbrace{x^2}_{u} \underbrace{e^x dx}_{dv} = \underbrace{x^2}_{u} \underbrace{e^x}_{v} - \int \underbrace{e^x}_{v} \underbrace{2x dx}_{du} = x^2 e^x - 2 \int x e^x dx \quad \text{Integration by parts formula}$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x dx$. Then $du = dx$, $v = e^x$, and

$$\int \underbrace{x}_{u} \underbrace{e^x dx}_{dv} = \underbrace{x}_{u} \underbrace{e^x}_{v} - \int \underbrace{e^x}_{v} \underbrace{dx}_{du} = x e^x - e^x + C. \quad \begin{array}{l} \text{Integration by parts Equation (2)} \\ u = x, \quad dv = e^x dx \\ v = e^x, \quad du = dx \end{array}$$

Using this last evaluation, we then obtain

$$\begin{aligned} \int x^2 e^x dx &= x^2 e^x - 2 \int x e^x dx \\ &= x^2 e^x - 2x e^x + 2e^x + C, \end{aligned}$$

where the constant of integration is renamed after substituting for the integral on the right. ■

The technique of Example 3 works for any integral $\int x^n e^x dx$ in which n is a positive integer, because differentiating x^n will eventually lead to a constant, and repeatedly integrating e^x is easy.

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 4 Evaluate

$$\int e^x \cos x dx.$$

Solution Let $u = e^x$ and $dv = \cos x dx$. Then $du = e^x dx$, $v = \sin x$, and

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \quad dv = \sin x dx, \quad v = -\cos x, \quad du = e^x dx.$$

Then

$$\begin{aligned} \int e^x \cos x dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x dx) \right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x dx. \end{aligned}$$

The unknown integral now appears on both sides of the equation, but with opposite signs. Adding the integral to both sides and adding the constant of integration gives

$$2 \int e^x \cos x dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration then gives

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C. \quad \blacksquare$$

EXAMPLE 5 Obtain a formula that expresses the integral

$$\int \cos^n x \, dx$$

in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \quad \text{and} \quad dv = \cos x \, dx,$$

so that

$$du = (n-1)(\cos^{n-2} x)(-\sin x \, dx) \quad \text{and} \quad v = \sin x.$$

Integration by parts then gives

$$\begin{aligned} \int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x \, dx \\ &= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx. \end{aligned}$$

If we add

$$(n-1) \int \cos^n x \, dx$$

to both sides of this equation, we obtain

$$n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx.$$

We then divide through by n , and the final result is

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx. \quad \blacksquare$$

The formula found in Example 5 is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate. For example, the result in Example 5 tells us that

$$\begin{aligned} \int \cos^3 x \, dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3} \int \cos x \, dx \\ &= \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x + C. \end{aligned}$$

Evaluating Definite Integrals by Parts

The integration by parts formula in Equation (1) can be combined with Part 2 of the Fundamental Theorem in order to evaluate definite integrals by parts. Assuming that both u' and v' are continuous over the interval $[a, b]$, Part 2 of the Fundamental Theorem gives

Integration by Parts Formula for Definite Integrals

$$\int_a^b u(x) v'(x) \, dx = u(x) v(x) \Big|_a^b - \int_a^b v(x) u'(x) \, dx \quad (3)$$

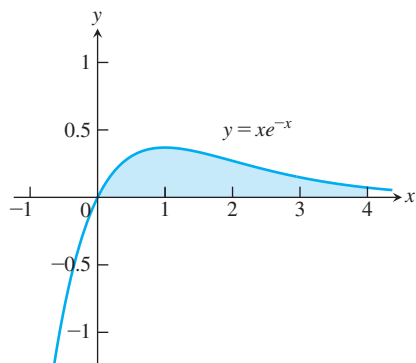


FIGURE 8.1 The region in Example 6.

EXAMPLE 6 Find the area of the region bounded by the curve $y = xe^{-x}$ and the x -axis from $x = 0$ to $x = 4$.

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x} dx.$$

Let $u = x$, $dv = e^{-x} dx$, $v = -e^{-x}$, and $du = dx$. Then

$$\begin{aligned} \int_0^4 xe^{-x} dx &= -xe^{-x} \Big|_0^4 - \int_0^4 (-e^{-x}) dx && \text{Integration by parts Formula (3)} \\ &= [-4e^{-4} - (-0e^{-0})] + \int_0^4 e^{-x} dx \\ &= -4e^{-4} - e^{-x} \Big|_0^4 \\ &= -4e^{-4} - (e^{-4} - e^{-0}) = 1 - 5e^{-4} \approx 0.91. \end{aligned}$$

EXERCISES 8.2

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2} dx$
3. $\int t^2 \cos t dt$
5. $\int_1^2 x \ln x dx$
7. $\int x e^x dx$
9. $\int x^2 e^{-x} dx$
11. $\int \tan^{-1} y dy$
13. $\int x \sec^2 x dx$
15. $\int x^3 e^x dx$
17. $\int (x^2 - 5x) e^x dx$
19. $\int x^5 e^x dx$
21. $\int e^\theta \sin \theta d\theta$
23. $\int e^{2x} \cos 3x dx$
2. $\int \theta \cos \pi \theta d\theta$
4. $\int x^2 \sin x dx$
6. $\int_1^e x^3 \ln x dx$
8. $\int x e^{3x} dx$
10. $\int (x^2 - 2x + 1) e^{2x} dx$
12. $\int \arcsin y dy$
14. $\int 4x \sec^2 2x dx$
16. $\int p^4 e^{-p} dp$
18. $\int (r^2 + r + 1) e^r dr$
20. $\int t^2 e^{4t} dt$
22. $\int e^{-y} \cos y dy$
24. $\int e^{-2x} \sin 2x dx$

Using Substitution

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}} ds$
27. $\int_0^{\pi/3} x \tan^2 x dx$
26. $\int_0^1 x \sqrt{1-x} dx$
28. $\int \ln(x + x^2) dx$

$$29. \int \sin(\ln x) dx$$

$$30. \int z(\ln z)^2 dz$$

Evaluating Integrals

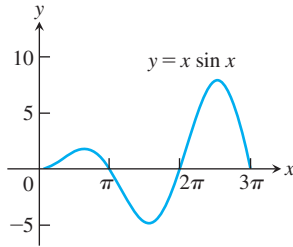
Evaluate the integrals in Exercises 31–56. Some integrals do not require integration by parts.

31. $\int x \sec x^2 dx$
33. $\int x (\ln x)^2 dx$
35. $\int \frac{\ln x}{x^2} dx$
37. $\int x^3 e^{x^4} dx$
39. $\int x^3 \sqrt{x^2 + 1} dx$
41. $\int \sin 3x \cos 2x dx$
43. $\int \sqrt{x} \ln x dx$
45. $\int \cos \sqrt{x} dx$
47. $\int_0^{\pi/2} \theta^2 \sin 2\theta d\theta$
49. $\int_{2/\sqrt{3}}^2 t \sec^{-1} t dt$
51. $\int x \arctan x dx$
53. $\int (1 + 2x^2) e^{x^2} dx$
55. $\int \sqrt{x} (\arcsin \sqrt{x}) dx$
32. $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$
34. $\int \frac{1}{x(\ln x)^2} dx$
36. $\int \frac{(\ln x)^3}{x} dx$
38. $\int x^5 e^{x^3} dx$
40. $\int x^2 \sin x^3 dx$
42. $\int \sin 2x \cos 4x dx$
44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$
46. $\int \sqrt{x} e^{\sqrt{x}} dx$
48. $\int_0^{\pi/2} x^3 \cos 2x dx$
50. $\int_0^{1/\sqrt{2}} 2x \arcsin(x^2) dx$
52. $\int x^2 \tan^{-1} \frac{x}{2} dx$
54. $\int \frac{x e^x}{(x+1)^2} dx$
56. $\int \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}} dx$

Theory and Examples

57. Finding area Find the area of the region enclosed by the curve $y = x \sin x$ and the x -axis (see the accompanying figure) for

- $0 \leq x \leq \pi$.
- $\pi \leq x \leq 2\pi$.
- $2\pi \leq x \leq 3\pi$.
- What pattern do you see here? What is the area between the curve and the x -axis for $n\pi \leq x \leq (n+1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.

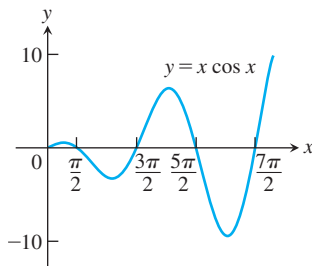


58. Finding area Find the area of the region enclosed by the curve $y = x \cos x$ and the x -axis (see the accompanying figure) for

- $\pi/2 \leq x \leq 3\pi/2$.
- $3\pi/2 \leq x \leq 5\pi/2$.
- $5\pi/2 \leq x \leq 7\pi/2$.
- What pattern do you see? What is the area between the curve and the x -axis for

$$\left(\frac{2n-1}{2}\right)\pi \leq x \leq \left(\frac{2n+1}{2}\right)\pi,$$

n an arbitrary positive integer? Give reasons for your answer.



59. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

60. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$

- about the y -axis.
- about the line $x = 1$.

61. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \leq x \leq \pi/2$, about

- the y -axis.
- the line $x = \pi/2$.

62. Finding volume Find the volume of the solid generated by revolving the region bounded by the x -axis and the curve $y = x \sin x$, $0 \leq x \leq \pi$, about

- the y -axis.
- the line $x = \pi$.

(See Exercise 57 for a graph.)

63. Consider the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$.

- Find the area of the region.
- Find the volume of the solid formed by revolving this region about the x -axis.
- Find the volume of the solid formed by revolving this region about the line $x = -2$.
- Find the centroid of the region.

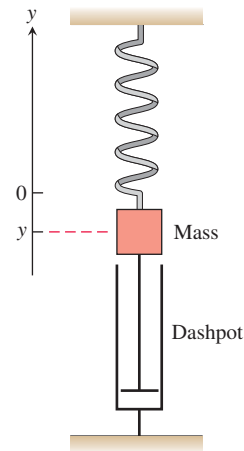
64. Consider the region bounded by the graphs of $y = \arctan x$, $y = 0$, and $x = 1$.

- Find the area of the region.
- Find the volume of the solid formed by revolving this region about the y -axis.

65. Average value A retarding force, symbolized by the dashpot in the accompanying figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t} \cos t, \quad t \geq 0.$$

Find the average value of y over the interval $0 \leq t \leq 2\pi$.



66. Average value In a mass-spring-dashpot system like the one in Exercise 65, the mass's position at time t is

$$y = 4e^{-t}(\sin t - \cos t), \quad t \geq 0.$$

Find the average value of y over the interval $0 \leq t \leq 2\pi$.

Reduction Formulas

In Exercises 67–73, use integration by parts to establish the reduction formula.

$$67. \int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx$$

$$68. \int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx$$

$$69. \int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx, \quad a \neq 0$$

$$70. \int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx$$

$$71. \int x^m (\ln x)^n dx = \frac{x^{m+1}}{m+1} (\ln x)^n - \frac{n}{m+1} \int x^m (\ln x)^{n-1} dx, \quad m \neq -1$$

$$72. \int x^n \sqrt{x+1} dx = \frac{2x^n}{2n+3} (x+1)^{3/2} - \frac{2n}{2n+3} \int x^{n-1} \sqrt{x+1} dx$$

$$73. \int \frac{x^n}{\sqrt{x+1}} dx = \frac{2x^n}{2n+1} \sqrt{x+1} - \frac{2n}{2n+1} \int \frac{x^{n-1}}{\sqrt{x+1}} dx$$

74. Use Example 5 to show that

$$\begin{aligned} \int_0^{\pi/2} \sin^n x dx &= \int_0^{\pi/2} \cos^n x dx \\ &= \begin{cases} \left(\frac{\pi}{2}\right) \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n}, & n \text{ even} \\ \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{1 \cdot 3 \cdot 5 \cdots n}, & n \text{ odd} \end{cases} \end{aligned}$$

75. Show that

$$\int_a^b \left(\int_x^b f(t) dt \right) dx = \int_a^b (x-a) f(x) dx.$$

76. Use integration by parts to obtain the formula

$$\int \sqrt{1-x^2} dx = \frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} dx.$$

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\begin{aligned} \int f^{-1}(x) dx &= \int y f'(y) dy && y = f^{-1}(x), \quad x = f(y) \\ &= y f(y) - \int f(y) dy && dx = f'(y) dy \\ &= x f^{-1}(x) - \int f(y) dy && \text{Integration by parts with } u = y, dv = f'(y) dy \end{aligned}$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\begin{aligned} \int \ln x dx &= \int y e^y dy && y = \ln x, \quad x = e^y \\ &= y e^y - e^y + C && dx = e^y dy \\ &= x \ln x - x + C. \end{aligned}$$

For the integral of $\arccos x$, we get

$$\begin{aligned} \int \arccos x dx &= x \arccos x - \int \cos y dy && y = \arccos x \\ &= x \arccos x - \sin y + C \\ &= x \arccos x - \sin(\arccos x) + C. \end{aligned}$$

Use the formula

$$\int f^{-1}(x) dx = x f^{-1}(x) - \int f(y) dy \quad y = f^{-1}(x) \quad (4)$$

to evaluate the integrals in Exercises 77–80. Express your answers in terms of x .

$$77. \int \operatorname{arcsec} x dx$$

$$78. \int \arctan x dx$$

$$79. \int \sec^{-1} x dx$$

$$80. \int \log_2 x dx$$

Another way to integrate $f^{-1}(x)$ (when f^{-1} is integrable) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of f^{-1} as

$$\int f^{-1}(x) dx = x f^{-1}(x) - \int x \left(\frac{d}{dx} f^{-1}(x) \right) dx. \quad (5)$$

Exercises 81 and 82 compare the results of using Equations (4) and (5).

81. Equations (4) and (5) give different formulas for the integral of $\arccos x$:

$$\text{a. } \int \arccos x dx = x \arccos x - \sin(\arccos x) + C \quad \text{Eq. (4)}$$

$$\text{b. } \int \arccos x dx = x \arccos x - \sqrt{1-x^2} + C \quad \text{Eq. (5)}$$

Can both integrations be correct? Explain.

82. Equations (4) and (5) lead to different formulas for the integral of $\arctan x$:

$$\text{a. } \int \arctan x dx = x \arctan x - \ln \sec(\arctan x) + C \quad \text{Eq. (4)}$$

$$\text{b. } \int \arctan x dx = x \arctan x - \ln \sqrt{1+x^2} + C \quad \text{Eq. (5)}$$

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 83 and 84 with (a) Eq. (4) and (b) Eq. (5). In each case, check your work by differentiating your answer with respect to x .

$$83. \int \sinh^{-1} x dx$$

$$84. \int \tanh^{-1} x dx$$

8.3 Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, as in the integral

$$\int \sec^2 x \, dx = \tan x + C.$$

The general idea is to use identities to transform the integrals we must find into integrals that are easier to work with.

Products of Powers of Sines and Cosines

We begin with integrals of the form

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to m and n being odd or even.

Case 1 If m is odd in $\int \sin^m x \cos^n x \, dx$, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

Ex 1 $\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \quad (1)$

Then we substitute $u = \cos x$ and $du = -\sin x \, dx$.

Case 2 If n is odd in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

Ex 2 $\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$

We then substitute $u = \sin x$ and $du = \cos x \, dx$.

Case 3 If both m and n are even in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2} \quad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Here are some examples illustrating each case.

EXAMPLE 1 Evaluate $\int \sin^3 x \cos^2 x \, dx$.

Solution This is an example of Case 1.

$$\begin{aligned} \int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx && m \text{ is odd.} \\ &= \int (1 - \cos^2 x)(\cos^2 x) \sin x \, dx && \sin^2 x = 1 - \cos^2 x \\ &= \int (1 - u^2)(u^2)(-du) && u = \cos x, \, du = -\sin x \, dx \\ &= \int (u^4 - u^2) \, du && \text{Distribute.} \\ &= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C \end{aligned}$$



8 = 2 x 4
6 = 2 x 3

EXAMPLE 2 Evaluate

$$\int \cos^5 x \, dx.$$

Solution This is an example of Case 2, where $m = 0$ is even and $n = 5$ is odd.

$$\begin{aligned} \int \cos^5 x \, dx &= \int \cos^4 x \cos x \, dx \\ &= \int (1 - \sin^2 x)^2 \cos x \, dx \\ &= \int (1 - u^2)^2 \, du && u = \sin x, \, du = \cos x \, dx \\ &= \int (1 - 2u^2 + u^4) \, du && \text{Square } 1 - u^2. \\ &= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C \end{aligned}$$

EXAMPLE 3 Evaluate

$$\int \sin^2 x \cos^4 x \, dx.$$

Solution This is an example of Case 3.

$$\begin{aligned} \int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right)^2 dx && m \text{ and } n \text{ both even} \\ &= \frac{1}{8} \int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) \, dx \\ &= \frac{1}{8} \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) \, dx \\ &= \frac{1}{8} \left[x + \frac{1}{2} \sin 2x - \int (\cos^2 2x + \cos^3 2x) \, dx \right] \end{aligned}$$

For the term involving $\cos^2 2x$, we use

$$\begin{aligned} \int \cos^2 2x \, dx &= \frac{1}{2} \int (1 + \cos 4x) \, dx && \text{Use the identity} \\ &= \frac{1}{2} \left(x + \frac{1}{4} \sin 4x \right) + C_1. && \cos^2 \theta = (1 + \cos 2\theta)/2, \\ & && \text{with } \theta = 2x. \end{aligned}$$

For the $\cos^3 2x$ term, we have

$$\begin{aligned} \int \cos^3 2x \, dx &= \int (1 - \sin^2 2x) \cos 2x \, dx && u = \sin 2x, \, du = 2 \cos 2x \, dx \\ &= \frac{1}{2} \int (1 - u^2) \, du = \frac{1}{2} \left(\sin 2x - \frac{1}{3} \sin^3 2x \right) + C_2. \end{aligned}$$

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16} \left(x - \frac{1}{4} \sin 4x + \frac{1}{3} \sin^3 2x \right) + C.$$

Eliminating Square Roots

In the next example, we use the identity $\cos^2 \theta = (1 + \cos 2\theta)/2$ to eliminate a square root.

EXAMPLE 4 Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x} \, dx.$$

Solution To eliminate the square root, we use the identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \quad \text{or} \quad 1 + \cos 2\theta = 2 \cos^2 \theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2 \cos^2 2x.$$

Therefore,

$$\begin{aligned} \int_0^{\pi/4} \sqrt{1 + \cos 4x} \, dx &= \int_0^{\pi/4} \sqrt{2 \cos^2 2x} \, dx = \int_0^{\pi/4} \sqrt{2} \sqrt{\cos^2 2x} \, dx \\ &= \sqrt{2} \int_0^{\pi/4} |\cos 2x| \, dx = \sqrt{2} \int_0^{\pi/4} \cos 2x \, dx \quad \begin{array}{l} \cos 2x \geq 0 \\ \text{on } [0, \pi/4] \end{array} \\ &= \sqrt{2} \left[\frac{\sin 2x}{2} \right]_0^{\pi/4} = \frac{\sqrt{2}}{2} [1 - 0] = \frac{\sqrt{2}}{2}. \end{aligned}$$

$\int \cos^n x \sec^n x \, dx \leftarrow$

$\Rightarrow \int \tan^m x \sec^n x \, dx$

Integrals of Powers of $\tan x$ and $\sec x$

We know how to integrate the tangent and secant functions and their squares. To integrate higher powers, we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

EXAMPLE 5 Evaluate

$$\int \tan^4 x \, dx.$$

$\int \sin^n x \cos^n x \, dx$

Solution

$$\begin{aligned} \int \tan^4 x \, dx &= \int \tan^2 x \cdot \tan^2 x \, dx \\ &= \int \tan^2 x \cdot (\sec^2 x - 1) \, dx \quad \tan^2 x = \sec^2 x - 1 \\ &= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\ &= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx \quad \tan^2 x = \sec^2 x - 1 \\ &= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx + \int dx \end{aligned}$$

In the first integral, we let

$$u = \tan x, \quad du = \sec^2 x \, dx$$

and have

$$\int u^2 du = \frac{1}{3}u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x dx = \frac{1}{3}\tan^3 x - \tan x + x + C. \quad \blacksquare$$

EXAMPLE 6 Evaluate

$$\int \sec^3 x dx.$$

Solution We integrate by parts using

$$u = \sec x, \quad dv = \sec^2 x dx, \quad v = \tan x, \quad du = \sec x \tan x dx.$$

Then

$$\begin{aligned} \int \sec^3 x dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x) dx && \text{Integrate by parts.} \\ &= \sec x \tan x - \int (\sec^2 x - 1)\sec x dx && \tan^2 x = \sec^2 x - 1 \\ &= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \end{aligned}$$

Combining the two secant-cubed integrals gives

$$2 \int \sec^3 x dx = \sec x \tan x + \int \sec x dx$$

and therefore

$$\int \sec^3 x dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C. \quad \blacksquare$$

EXAMPLE 7 Evaluate

$$\int \tan^4 x \sec^4 x dx.$$

Solution

$$\begin{aligned} \int (\tan^4 x)(\sec^4 x) dx &= \int (\tan^4 x)(1 + \tan^2 x)(\sec^2 x) dx && \sec^2 x = 1 + \tan^2 x \\ &= \int (\tan^4 x + \tan^6 x)(\sec^2 x) dx && \text{Distribute.} \\ &= \int (u^4 + u^6) du = \frac{u^5}{5} + \frac{u^7}{7} + C && u = \tan x, \\ &= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C && du = \sec^2 x dx \end{aligned} \quad \blacksquare$$

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx \, dx, \quad \int \sin mx \cos nx \, dx, \quad \text{and} \quad \int \cos mx \cos nx \, dx$$

arise in many applications involving periodic functions. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the following identities.

$$\sin mx \sin nx = \frac{1}{2} [\cos(m-n)x - \cos(m+n)x] \quad (3)$$

$$\sin mx \cos nx = \frac{1}{2} [\sin(m-n)x + \sin(m+n)x] \quad (4)$$

$$\cos mx \cos nx = \frac{1}{2} [\cos(m-n)x + \cos(m+n)x] \quad (5)$$

These identities come from the angle sum formulas for the sine and cosine functions (Section 1.3). They give functions whose antiderivatives are easily found.

EXAMPLE 8 Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned} \int \sin 3x \cos 5x \, dx &= \frac{1}{2} \int [\sin(-2x) + \sin 8x] \, dx \\ &= \frac{1}{2} \int (\sin 8x - \sin 2x) \, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C. \end{aligned}$$

EXERCISES 8.3

Powers of Sines and Cosines

Evaluate the integrals in Exercises 1–22.

1. $\int \cos 2x \, dx$

2. $\int_0^{\pi} 3 \sin \frac{x}{3} \, dx$

3. $\int \cos^3 x \sin x \, dx$

4. $\int \sin^4 2x \cos 2x \, dx$

5. $\int \sin^3 x \, dx$

6. $\int \cos^3 4x \, dx$

7. $\int \sin^5 x \, dx$

8. $\int_0^{\pi} \sin^5 \frac{x}{2} \, dx$

9. $\int \cos^3 x \, dx$

10. $\int_0^{\pi/6} 3 \cos^5 3x \, dx$

11. $\int \sin^3 x \cos^3 x \, dx$

13. $\int \cos^2 x \, dx$

15. $\int_0^{\pi/2} \sin^7 y \, dy$

17. $\int_0^{\pi} 8 \sin^4 x \, dx$

19. $\int 16 \sin^2 x \cos^2 x \, dx$

21. $\int 8 \cos^3 2\theta \sin 2\theta \, d\theta$

12. $\int \cos^3 2x \sin^5 2x \, dx$

14. $\int_0^{\pi/2} \sin^2 x \, dx$

16. $\int 7 \cos^7 t \, dt$

18. $\int 8 \cos^4 2\pi x \, dx$

20. $\int_0^{\pi} 8 \sin^4 y \cos^2 y \, dy$

22. $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta$

Integrating Square Roots

Evaluate the integrals in Exercises 23–32.

23. $\int_0^{2\pi} \sqrt{\frac{1 - \cos x}{2}} dx$

24. $\int_0^\pi \sqrt{1 - \cos 2x} dx$

25. $\int_0^\pi \sqrt{1 - \sin^2 t} dt$

26. $\int_0^\pi \sqrt{1 - \cos^2 \theta} d\theta$

27. $\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1 - \cos x}} dx$

28. $\int_0^{\pi/6} \sqrt{1 + \sin x} dx$

(Hint: Multiply by $\sqrt{\frac{1 - \sin x}{1 - \sin x}}$.)

29. $\int_{5\pi/6}^\pi \frac{\cos^4 x}{\sqrt{1 - \sin x}} dx$

30. $\int_{\pi/2}^{3\pi/4} \sqrt{1 - \sin 2x} dx$

31. $\int_0^{\pi/2} \theta \sqrt{1 - \cos 2\theta} d\theta$

32. $\int_{-\pi}^\pi (1 - \cos^2 t)^{3/2} dt$

Powers of Tangents and Secants

Evaluate the integrals in Exercises 33–52.

33. $\int \sec^2 x \tan x dx$

34. $\int \sec x \tan^2 x dx$

35. $\int \sec^3 x \tan x dx$

36. $\int \sec^3 x \tan^3 x dx$

37. $\int \sec^2 x \tan^2 x dx$

38. $\int \sec^4 x \tan^2 x dx$

39. $\int_{-\pi/3}^0 2 \sec^3 x dx$

40. $\int e^x \sec^3 e^x dx$

41. $\int \sec^4 \theta d\theta$

42. $\int \tan^4 x \sec^3 x dx$

43. $\int_{\pi/4}^{\pi/2} \csc^4 \theta d\theta$

44. $\int \sec^6 x dx$

45. $\int 4 \tan^3 x dx$

46. $\int_{-\pi/4}^{\pi/4} 6 \tan^4 x dx$

47. $\int \tan^5 x dx$

48. $\int \cot^6 2x dx$

49. $\int_{\pi/6}^{\pi/3} \cot^3 x dx$

50. $\int 8 \cot^4 t dt$

51. $\int_{\pi/4}^{\pi/3} \tan^5 \theta \sec^4 \theta d\theta$

52. $\int \cot^3 t \csc^4 t dt$

Products of Sines and Cosines

Evaluate the integrals in Exercises 53–58.

53. $\int \sin 3x \cos 2x dx$

54. $\int \sin 2x \cos 3x dx$

55. $\int_{-\pi}^\pi \sin 3x \sin 3x dx$

56. $\int_0^{\pi/2} \sin x \cos x dx$

57. $\int \cos 3x \cos 4x dx$

58. $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x dx$

Exercises 59–64 require the use of various trigonometric identities before you evaluate the integrals.

59. $\int \sin^2 \theta \cos 3\theta d\theta$

60. $\int \cos^2 2\theta \sin \theta d\theta$

61. $\int \cos^3 \theta \sin 2\theta d\theta$

62. $\int \sin^3 \theta \cos 2\theta d\theta$

63. $\int \sin \theta \cos \theta \cos 3\theta d\theta$

64. $\int \sin \theta \sin 2\theta \sin 3\theta d\theta$

Assorted Integrations

Use any method to evaluate the integrals in Exercises 65–70.

65. $\int \frac{\sec^3 x}{\tan x} dx$

66. $\int \frac{\sin^3 x}{\cos^4 x} dx$

67. $\int \frac{\tan^2 x}{\csc x} dx$

68. $\int \frac{\cot x}{\cos^2 x} dx$

69. $\int x \sin^2 x dx$

70. $\int x \cos^3 x dx$

Applications**71. Arc length** Find the length of the curve

$$y = \ln(\sin x), \quad \frac{\pi}{6} \leq x \leq \frac{\pi}{2}$$

72. Center of gravity Find the center of gravity of the region bounded by the x -axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.**73. Volume** Find the volume generated by revolving one arch of the curve $y = \sin x$ about the x -axis.**74. Area** Find the area between the x -axis and the curve $y = \sqrt{1 + \cos 4x}$, $0 \leq x \leq \pi$.**75. Centroid** Find the centroid of the region bounded by the graphs of $y = x + \cos x$ and $y = 0$ for $0 \leq x \leq 2\pi$.**76. Volume** Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sin x + \sec x$, $y = 0$, $x = 0$, and $x = \pi/3$ about the x -axis.**77. Volume** Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \arctan x$, $x = 0$, and $y = \pi/4$ about the y -axis.**78. Average Value** Find the average value of the function

$$f(x) = \frac{1}{1 - \sin \theta} \text{ on } [0, \pi/6].$$

8.4 Trigonometric Substitutions

Trigonometric substitutions occur when we replace the variable of integration by a trigonometric function. The most common substitutions are $x = a \tan \theta$, $x = a \sin \theta$, and $x = a \sec \theta$. These substitutions are effective in transforming integrals involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, and $\sqrt{x^2 - a^2}$ into integrals with respect to θ , since they come from the reference right triangles in Figure 8.2.

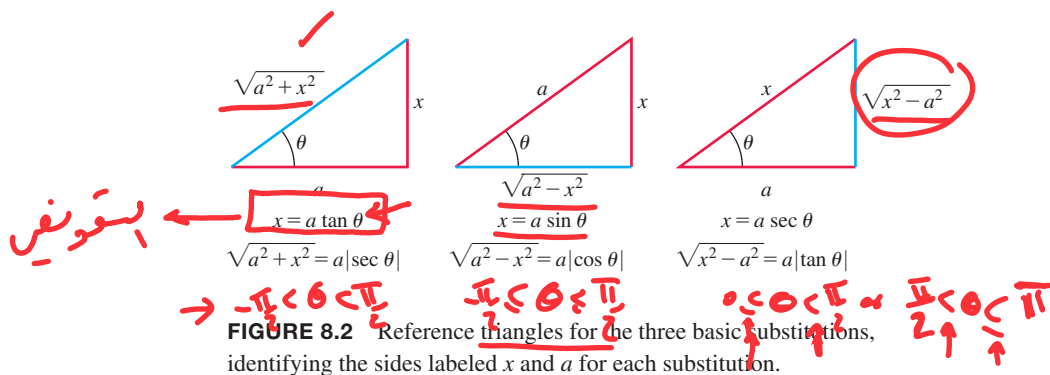


FIGURE 8.2 Reference triangles for the three basic substitutions, identifying the sides labeled x and a for each substitution.

With $x = a \tan \theta$,

$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta.$$

With $x = a \sin \theta$,

$$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta.$$

With $x = a \sec \theta$,

$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2(\sec^2 \theta - 1) = a^2 \tan^2 \theta.$$

We want any substitution we use in an integration to be reversible so that we can change back to the original variable afterward. For example, if $x = a \tan \theta$, we want to be able to set $\theta = \arctan(x/a)$ after the integration takes place. If $x = a \sin \theta$, we want to be able to set $\theta = \arcsin(x/a)$ when we're done, and similarly for $x = a \sec \theta$.

As we know from Section 1.5, the functions in these substitutions have inverses only for selected values of θ (Figure 8.3). For reversibility,

$$x = a \tan \theta \text{ requires } \theta = \arctan\left(\frac{x}{a}\right) \text{ with } -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a \sin \theta \text{ requires } \theta = \arcsin\left(\frac{x}{a}\right) \text{ with } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

$$x = a \sec \theta \text{ requires } \theta = \operatorname{arcsec}\left(\frac{x}{a}\right) \text{ with } \begin{cases} 0 \leq \theta < \frac{\pi}{2} & \text{if } \frac{x}{a} \geq 1, \\ \frac{\pi}{2} < \theta \leq \pi & \text{if } \frac{x}{a} \leq -1. \end{cases}$$

To simplify calculations with the substitution $x = a \sec \theta$, we will restrict its use to integrals in which $x/a \geq 1$. This will place θ in $[0, \pi/2)$ and make $\tan \theta \geq 0$. We will then have $\sqrt{x^2 - a^2} = \sqrt{a^2 \tan^2 \theta} = |a \tan \theta| = a \tan \theta$, free of absolute values, provided $a > 0$.

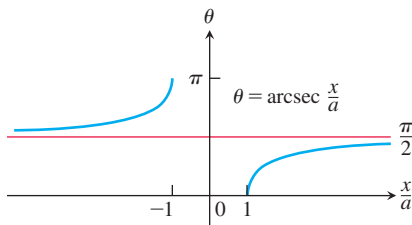
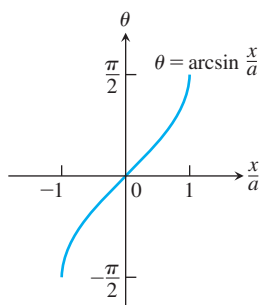
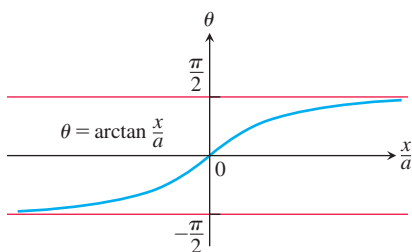


FIGURE 8.3 The arctangent, arcsine, and arcsecant of x/a , graphed as functions of x/a .

Procedure for a Trigonometric Substitution

1. Write down the substitution for x , calculate the differential dx , and specify the selected values of θ for the substitution.
2. Substitute the trigonometric expression and the calculated differential into the integrand, and then simplify the results algebraically.
3. Evaluate the trigonometric integral, keeping in mind the restrictions on the angle θ for reversibility.
4. Draw an appropriate reference triangle to reverse the substitution in the integration result and convert it back to the original variable x .

EXAMPLE 1 Evaluate

$$\int \frac{dx}{\sqrt{4+x^2}}.$$

Solution We set

$$\begin{aligned} x &= 2 \tan \theta, & dx &= 2 \sec^2 \theta d\theta, & -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \\ 4 + x^2 &= 4 + 4 \tan^2 \theta = 4(1 + \tan^2 \theta) = 4 \sec^2 \theta. \end{aligned}$$

Then

$$\begin{aligned} \int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2 \sec^2 \theta d\theta}{\sqrt{4 \sec^2 \theta}} = \int \frac{\sec^2 \theta d\theta}{|\sec \theta|} && \sqrt{\sec^2 \theta} = |\sec \theta| \\ &= \int \sec \theta d\theta && \sec \theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ &= \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{\sqrt{4+x^2}}{2} + \frac{x}{2} \right| + C. && \text{From Fig. 8.4} \end{aligned}$$

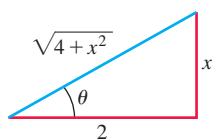


FIGURE 8.4 Reference triangle for $x = 2 \tan \theta$ (Example 1):

$$\tan \theta = \frac{x}{2}$$

and

$$\sec \theta = \frac{\sqrt{4+x^2}}{2}.$$

Notice how we expressed $\ln |\sec \theta + \tan \theta|$ in terms of x : We drew a reference triangle for the original substitution $x = 2 \tan \theta$ (Figure 8.4) and read the ratios from the triangle. ■

EXAMPLE 2 Here we find an expression for the inverse hyperbolic sine function in terms of the natural logarithm. Following the same procedure as in Example 1, we find that

$$\begin{aligned} \int \frac{dx}{\sqrt{a^2+x^2}} &= \int \sec \theta d\theta && x = a \tan \theta, dx = a \sec^2 \theta d\theta \\ &= \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a} \right| + C && \text{Fig. 8.2} \end{aligned}$$

From Table 7.9, $\sinh^{-1}(x/a)$ is also an antiderivative of $1/\sqrt{a^2+x^2}$, so the two antiderivatives differ by a constant, giving

$$\sinh^{-1} \frac{x}{a} = \ln \left| \frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a} \right| + C.$$

Setting $x = 0$ in this last equation, we find that $0 = \ln|1| + C$, so $C = 0$. Since $\sqrt{a^2 + x^2} > |x|$, we conclude that $\frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a} > 0$, and therefore

$$\sinh^{-1} \frac{x}{a} = \ln \left(\frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a} \right)$$

(See also Exercise 76 in Section 7.3.)

EXAMPLE 3 Evaluate

$$\int \frac{x^2 dx}{\sqrt{9 - x^2}}.$$

Solution We set

$$\begin{aligned} x &= 3 \sin \theta, & dx &= 3 \cos \theta d\theta, & -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ 9 - x^2 &= 9 - 9 \sin^2 \theta = 9(1 - \sin^2 \theta) = 9 \cos^2 \theta. \end{aligned}$$

Then

$$\begin{aligned} \int \frac{x^2 dx}{\sqrt{9 - x^2}} &= \int \frac{9 \sin^2 \theta \cdot 3 \cos \theta d\theta}{|3 \cos \theta|} \\ &= 9 \int \sin^2 \theta d\theta && \cos \theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ &= 9 \int \frac{1 - \cos 2\theta}{2} d\theta && \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \\ &= \frac{9}{2} \left(\theta - \frac{\sin 2\theta}{2} \right) + C \\ &= \frac{9}{2} (\theta - \sin \theta \cos \theta) + C && \sin 2\theta = 2 \sin \theta \cos \theta \\ &= \frac{9}{2} \left(\arcsin \frac{x}{3} - \frac{x}{3} \cdot \frac{\sqrt{9 - x^2}}{3} \right) + C && \text{From Fig. 8.5} \\ &= \frac{9}{2} \arcsin \frac{x}{3} - \frac{x}{2} \sqrt{9 - x^2} + C. \end{aligned}$$

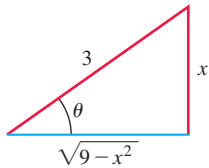


FIGURE 8.5 Reference triangle for $x = 3 \sin \theta$ (Example 3):

$$\sin \theta = \frac{x}{3}$$

and

$$\cos \theta = \frac{\sqrt{9 - x^2}}{3}.$$

EXAMPLE 4 Evaluate

$$\int \frac{dx}{\sqrt{25x^2 - 4}}, \quad x > \frac{2}{5}.$$

Solution We first rewrite the radical as

$$\begin{aligned} \sqrt{25x^2 - 4} &= \sqrt{25 \left(x^2 - \frac{4}{25} \right)} \\ &= 5 \sqrt{x^2 - \left(\frac{2}{5} \right)^2} && \sqrt{x^2 - a^2} \text{ with } a = \frac{2}{5} \end{aligned}$$

to put the radicand in the form $x^2 - a^2$. We then substitute

$$x = \frac{2}{5} \sec \theta, \quad dx = \frac{2}{5} \sec \theta \tan \theta \, d\theta, \quad 0 < \theta < \frac{\pi}{2}.$$

We then get

$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25} \sec^2 \theta - \frac{4}{25} = \frac{4}{25} (\sec^2 \theta - 1) = \frac{4}{25} \tan^2 \theta$$

and

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5} |\tan \theta| = \frac{2}{5} \tan \theta. \quad \tan \theta > 0 \text{ for } 0 < \theta < \pi/2$$

With these substitutions, we have

$$\begin{aligned} \int \frac{dx}{\sqrt{25x^2 - 4}} &= \int \frac{dx}{5\sqrt{x^2 - (4/25)}} = \int \frac{(2/5) \sec \theta \tan \theta \, d\theta}{5 \cdot (2/5) \tan \theta} \\ &= \frac{1}{5} \int \sec \theta \, d\theta = \frac{1}{5} \ln |\sec \theta + \tan \theta| + C \\ &= \frac{1}{5} \ln \left| \frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2} \right| + C. \quad \text{From Fig. 8.6} \end{aligned}$$

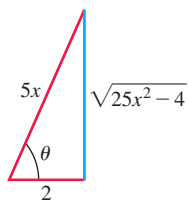


FIGURE 8.6 If $x = (2/5) \sec \theta$, $0 < \theta < \pi/2$, then $\theta = \operatorname{arcsec}(5x/2)$, and we can read the values of the other trigonometric functions of θ from this right triangle (Example 4).

EXERCISES 8.4

Using Trigonometric Substitutions

Evaluate the integrals in Exercises 1–14.

1. $\int \frac{dx}{\sqrt{9 + x^2}}$
2. $\int \frac{3 \, dx}{\sqrt{1 + 9x^2}}$
3. $\int_{-2}^2 \frac{dx}{4 + x^2}$
4. $\int_0^2 \frac{dx}{8 + 2x^2}$
5. $\int_0^{3/2} \frac{dx}{\sqrt{9 - x^2}}$
6. $\int_0^{1/2\sqrt{2}} \frac{2 \, dx}{\sqrt{1 - 4x^2}}$
7. $\int \sqrt{25 - t^2} \, dt$
8. $\int \sqrt{1 - 9t^2} \, dt$
9. $\int \frac{dx}{\sqrt{4x^2 - 49}}, \quad x > \frac{7}{2}$
10. $\int \frac{5 \, dx}{\sqrt{25x^2 - 9}}, \quad x > \frac{3}{5}$
11. $\int \frac{\sqrt{y^2 - 49}}{y} \, dy, \quad y > 7$
12. $\int \frac{\sqrt{y^2 - 25}}{y^3} \, dy, \quad y > 5$
13. $\int \frac{dx}{x^2 \sqrt{x^2 - 1}}, \quad x > 1$
14. $\int \frac{2 \, dx}{x^3 \sqrt{x^2 - 1}}, \quad x > 1$

Assorted Integrations

Use any method to evaluate the integrals in Exercises 15–38. Most will require trigonometric substitutions, but some can be evaluated by other methods.

15. $\int \frac{dx}{x \sqrt{x^2 - 1}}$
16. $\int \frac{dx}{1 + x^2}$
17. $\int \frac{x \, dx}{\sqrt{x^2 - 1}}$
18. $\int \frac{dx}{\sqrt{1 - x^2}}$
19. $\int \frac{x}{\sqrt{9 - x^2}} \, dx$
20. $\int \frac{x^2}{4 + x^2} \, dx$
21. $\int \frac{x^3 \, dx}{\sqrt{x^2 + 4}}$
22. $\int \frac{dx}{x^2 \sqrt{x^2 + 1}}$
23. $\int \frac{8 \, dw}{w^2 \sqrt{4 - w^2}}$
24. $\int \frac{\sqrt{9 - w^2}}{w^2} \, dw$
25. $\int \sqrt{\frac{x+1}{1-x}} \, dx$
26. $\int x \sqrt{x^2 - 4} \, dx$
27. $\int_0^{\sqrt{3}/2} \frac{4x^2 \, dx}{(1 - x^2)^{3/2}}$
28. $\int_0^1 \frac{dx}{(4 - x^2)^{3/2}}$
29. $\int \frac{dx}{(x^2 - 1)^{3/2}}, \quad x > 1$
30. $\int \frac{x^2 \, dx}{(x^2 - 1)^{5/2}}, \quad x > 1$
31. $\int \frac{(1 - x^2)^{3/2}}{x^6} \, dx$
32. $\int \frac{(1 - x^2)^{1/2}}{x^4} \, dx$
33. $\int \frac{8 \, dx}{(4x^2 + 1)^2}$
34. $\int \frac{6 \, dt}{(9t^2 + 1)^2}$

35. $\int \frac{x^3 dx}{x^2 - 1}$

36. $\int \frac{x dx}{25 + 4x^2}$

37. $\int \frac{v^2 dv}{(1 - v^2)^{5/2}}$

38. $\int \frac{(1 - r^2)^{5/2}}{r^8} dr$

In Exercises 39–48, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

39. $\int_0^{\ln 4} \frac{e^t dt}{\sqrt{e^{2t} + 9}}$

40. $\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t dt}{(1 + e^{2t})^{3/2}}$

41. $\int_{1/12}^{1/4} \frac{2 dt}{\sqrt{t} + 4t\sqrt{t}}$

42. $\int_1^e \frac{dy}{y\sqrt{1 + (\ln y)^2}}$

43. $\int \frac{x dx}{\sqrt{1 + x^4}}$

44. $\int \frac{\sqrt{1 - (\ln x)^2}}{x \ln x} dx$

45. $\int \sqrt{\frac{4 - x}{x}} dx$
 (Hint: Let $x = u^2$.)

46. $\int \sqrt{\frac{x}{1 - x^3}} dx$
 (Hint: Let $u = x^{3/2}$.)

47. $\int \sqrt{x} \sqrt{1 - x} dx$

48. $\int \frac{\sqrt{x - 2}}{\sqrt{x - 1}} dx$

Complete the Square Before Using Trigonometric Substitutions

For Exercises 49–52, complete the square before using an appropriate trigonometric substitution.

49. $\int \sqrt{8 - 2x - x^2} dx$

50. $\int \frac{1}{\sqrt{x^2 - 2x + 5}} dx$

51. $\int \frac{\sqrt{x^2 + 4x + 3}}{x + 2} dx$

52. $\int \frac{\sqrt{x^2 + 2x + 2}}{x^2 + 2x + 1} dx$

Initial Value Problems

Solve the initial value problems in Exercises 53–56 for y as a function of x .

53. $x \frac{dy}{dx} = \sqrt{x^2 - 4}, \quad x \geq 2, \quad y(2) = 0$

54. $\sqrt{x^2 - 9} \frac{dy}{dx} = 1, \quad x > 3, \quad y(5) = \ln 3$

55. $(x^2 + 4) \frac{dy}{dx} = 3, \quad y(2) = 0$

56. $(x^2 + 1)^2 \frac{dy}{dx} = \sqrt{x^2 + 1}, \quad y(0) = 1$

Applications and Examples

57. Area Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \sqrt{9 - x^2}/3$.

58. Area Find the area enclosed by the ellipse

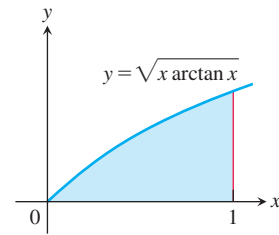
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

59. Consider the region bounded by the graphs of $y = \sin^{-1} x$, $y = 0$, and $x = 1/2$.

a. Find the area of the region.

b. Find the centroid of the region.

60. Consider the region bounded by the graphs of $y = \sqrt{x} \arctan x$ and $y = 0$ for $0 \leq x \leq 1$. Find the volume of the solid formed by revolving this region about the x -axis (see accompanying figure).



61. Evaluate $\int x^3 \sqrt{1 - x^2} dx$ using

a. integration by parts.

b. a u -substitution.

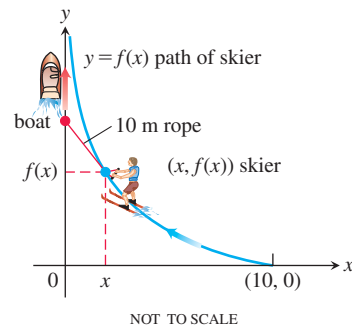
c. a trigonometric substitution.

62. Path of a water skier Suppose that a boat is positioned at the origin with a water skier tethered to the boat at the point $(10, 0)$ on a rope 10 m long. As the boat travels along the positive y -axis, the skier is pulled behind the boat along an unknown path $y = f(x)$, as shown in the accompanying figure.

a. Show that $f'(x) = \frac{-\sqrt{100 - x^2}}{x}$.

(Hint: Assume that the skier is always pointed directly at the boat and the rope is on a line tangent to the path $y = f(x)$.)

b. Solve the equation in part (a) for $f(x)$, using $f(10) = 0$.



63. Find the average value of $f(x) = \frac{\sqrt{x+1}}{\sqrt{x}}$ on the interval $[1, 3]$.

64. Find the length of the curve $y = 1 - e^{-x}$, $0 \leq x \leq 1$.

8.5 Integration of Rational Functions by Partial Fractions

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called *partial fractions*, which are more easily integrated. For instance, the rational function $(5x - 3)/(x^2 - 2x - 3)$ can be rewritten as

$$f(x) = \frac{p(x)}{q(x)}$$

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 1} + \frac{3}{x - 3} = 2\ln|x+1| + 3\ln|x-3| + C$$

You can verify this equation algebraically by placing the fractions on the right side over a common denominator $(x + 1)(x - 3)$. The skill acquired in writing rational functions as such a sum is useful in other settings as well (for instance, when using certain transform methods to solve differential equations). To integrate the rational function $(5x - 3)/(x^2 - 2x - 3)$ on the left side of the expression we are considering, we simply sum the integrals of the fractions on the right side:

$$\begin{aligned} \int \frac{5x - 3}{(x + 1)(x - 3)} dx &= \int \frac{2}{x + 1} dx + \int \frac{3}{x - 3} dx \\ &= 2 \ln|x + 1| + 3 \ln|x - 3| + C. \end{aligned}$$

The method for rewriting rational functions as a sum of simpler fractions is called **the method of partial fractions**. In the case of our example, it consists of finding constants A and B such that

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}. \quad (1)$$

(Pretend for a moment that we do not know that $A = 2$ and $B = 3$ will work.) We call the fractions $A/(x + 1)$ and $B/(x - 3)$ **partial fractions** because their denominators are only part of the original denominator $x^2 - 2x - 3$. We call A and B **undetermined coefficients** until suitable values for them have been found.

To find A and B , we first clear Equation (1) of fractions and regroup in powers of x , obtaining

$$5x - 3 = A(x - 3) + B(x + 1) = (A + B)x - 3A + B.$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

$$A + B = 5, \quad -3A + B = -3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$.

General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on three things:

- The degree of $f(x)$ must be less than the degree of $g(x)$. That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term. Example 3 of this section illustrates such a case.
- We must know the factors of $g(x)$. In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.
- The values of the undetermined coefficients form a system of n linear equations in n unknowns. For large n , solving such systems may require linear algebra methods (such as Gaussian Elimination).

Here is how we find the partial fractions of a proper fraction $f(x)/g(x)$ when the factors of g are known. A quadratic polynomial (or factor) is **irreducible** if it cannot be written as the product of two linear factors with real coefficients. That is, the polynomial has no real roots.

Method of Partial Fractions When $f(x)/g(x)$ Is Proper

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$. In this case, $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{(x^2 + px + q)} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions.
4. Find the values of the undetermined coefficients.

There are often multiple ways to find the values of the undetermined coefficients in Step 4. To find the values of the coefficients that satisfy Equation (1), we equated coefficients of like powers of x . In the next example, we instead will assign convenient values of x , leading to simple equations that we can solve for the undetermined coefficients.

EXAMPLE 1 Use partial fractions to evaluate

$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} dx.$$

Solution Note that each of the factors $(x - 1)$, $(x + 1)$, and $(x + 3)$ is raised only to the first power. Therefore, the partial fraction decomposition has the form

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}.$$

To find the values of the undetermined coefficients A , B , and C , we clear fractions and get

$$x^2 + 4x + 1 = A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1).$$

On the right side, we notice that a factor $(x - 1)$ is present in all terms except for the one containing A . Therefore, letting $x = 1$ allows us to solve for A .

$$x = 1: \quad 1^2 + 4(1) + 1 = A(2)(4) + B(0) + C(0)$$

$$6 = 8A$$

$$A = \frac{3}{4}$$

In a similar manner, we can let x equal -1 to find B or -3 to find C .

$$\begin{aligned} x = -1: \quad & (-1)^2 + 4(-1) + 1 = A(0) + B(-2)(2) + C(0) \\ & -2 = -4B \end{aligned}$$

$$B = \frac{1}{2}$$

$$\begin{aligned} x = -3: \quad & (-3)^2 + 4(-3) + 1 = A(0) + B(0) + C(-4)(-2) \\ & -2 = 8C \end{aligned}$$

$$C = -\frac{1}{4}$$

Hence we have

$$\begin{aligned} \int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)} dx &= \int \left[\frac{3}{4} \frac{1}{x-1} + \frac{1}{2} \frac{1}{x+1} - \frac{1}{4} \frac{1}{x+3} \right] dx \\ &= \frac{3}{4} \ln|x-1| + \frac{1}{2} \ln|x+1| - \frac{1}{4} \ln|x+3| + K, \end{aligned}$$

where K is the arbitrary constant of integration (we call it K here to avoid confusion with the undetermined coefficient we labeled as C). ■

You can solve for the undetermined coefficients (A, B , etc.) by equating coefficients of like powers of x or by assigning convenient values to x . You should choose the method that is most convenient for the problem at hand.

EXAMPLE 2 Use partial fractions to evaluate

$$\int \frac{6x + 7}{(x + 2)^2} dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2} \quad \text{Two terms because } (x + 2) \text{ is squared}$$

$$6x + 7 = A(x + 2) + B \quad \text{Multiply both sides by } (x + 2)^2.$$

$$= Ax + (2A + B)$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\begin{aligned} \int \frac{6x + 7}{(x + 2)^2} dx &= \int \left(\frac{6}{x + 2} - \frac{5}{(x + 2)^2} \right) dx \\ &= 6 \int \frac{dx}{x + 2} - 5 \int (x + 2)^{-2} dx \\ &= 6 \ln|x + 2| + 5(x + 2)^{-1} + C. \end{aligned} \quad \text{■}$$

The next example shows how to handle the case when $f(x)/g(x)$ is an improper fraction. It is a case where the degree of f is larger than the degree of g .

EXAMPLE 3 Use partial fractions to evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx.$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r} 2x \\ x^2 - 2x - 3 \overline{) 2x^3 - 4x^2 - x - 3} \\ \underline{2x^3 - 4x^2 - 6x} \\ 5x - 3 \end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\begin{aligned} \int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx &= \int 2x dx + \int \frac{5x - 3}{x^2 - 2x - 3} dx \\ &= \int 2x dx + \int \frac{2}{x + 1} dx + \int \frac{3}{x - 3} dx \\ &= x^2 + 2 \ln|x + 1| + 3 \ln|x - 3| + C. \end{aligned}$$

EXAMPLE 4 Use partial fractions to evaluate

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx.$$

Solution The denominator has an irreducible quadratic factor $x^2 + 1$ as well as a repeated linear factor $(x - 1)^2$, so we write

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}. \quad (2)$$

Clearing the equation of fractions gives

$$\begin{aligned} -2x + 4 &= (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1) \\ &= (A + C)x^3 + (-2A + B - C + D)x^2 \\ &\quad + (A - 2B + C)x + (B - C + D). \end{aligned}$$

Equating coefficients of like terms gives

$$\begin{aligned} \text{Coefficients of } x^3: \quad 0 &= A + C \\ \text{Coefficients of } x^2: \quad 0 &= -2A + B - C + D \\ \text{Coefficients of } x^1: \quad -2 &= A - 2B + C \\ \text{Coefficients of } x^0: \quad 4 &= B - C + D \end{aligned}$$

We solve these equations simultaneously to find the values of A , B , C , and D .

$$\begin{aligned} -4 &= -2A, & A &= 2 & \text{Subtract fourth equation from second.} \\ C &= -A = -2 & & & \text{From the first equation} \\ B &= (A + C + 2)/2 = 1 & & & \text{From the third equation and } C = -A \\ D &= 4 - B + C = 1. & & & \text{From the fourth equation} \end{aligned}$$

We substitute these values into Equation (2), obtaining

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}.$$

Finally, using the expansion above, we can integrate:

$$\begin{aligned}\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx &= \int \left(\frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \right) dx \\ &= \int \left(\frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \right) dx \\ &= \ln(x^2 + 1) + \tan^{-1} x - 2 \ln|x - 1| - \frac{1}{x - 1} + K.\end{aligned}$$

We use the letter K instead of C to represent an arbitrary constant here because we have already used C to represent a variable in the partial fraction representation. ■

EXAMPLE 5 Use partial fractions to evaluate

$$\int \frac{dx}{x(x^2 + 1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}.$$

Multiplying by $x(x^2 + 1)^2$, we have

$$\begin{aligned}1 &= A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x \\ &= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\ &= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A.\end{aligned}$$

If we equate coefficients, we get the system

$$A + B = 0, \quad C = 0, \quad 2A + B + D = 0, \quad C + E = 0, \quad A = 1.$$

Solving this system gives $A = 1$, $B = -1$, $C = 0$, $D = -1$, and $E = 0$. Thus,

$$\begin{aligned}\int \frac{dx}{x(x^2 + 1)^2} &= \int \left[\frac{1}{x} + \frac{-x}{x^2 + 1} + \frac{-x}{(x^2 + 1)^2} \right] dx \\ &= \int \frac{dx}{x} - \int \frac{x dx}{x^2 + 1} - \int \frac{x dx}{(x^2 + 1)^2} \\ &= \int \frac{dx}{x} - \frac{1}{2} \int \frac{du}{u} - \frac{1}{2} \int \frac{du}{u^2} \quad \begin{array}{l} u = x^2 + 1, \\ du = 2x dx \end{array} \\ &= \ln|x| - \frac{1}{2} \ln|u| + \frac{1}{2u} + K \\ &= \ln|x| - \frac{1}{2} \ln(x^2 + 1) + \frac{1}{2(x^2 + 1)} + K \\ &= \ln \frac{|x|}{\sqrt{x^2 + 1}} + \frac{1}{2(x^2 + 1)} + K. \quad \blacksquare\end{aligned}$$

HISTORICAL BIOGRAPHY

Oliver Heaviside
(1850–1925)

Heaviside studied electricity and languages on his own. He was able to simplify Maxwell's 20 equations into the two we now call Maxwell's equations. Heaviside's contributions in mathematics are in the areas of vector algebra and vector calculus.

To know more, visit the companion Website.

Determining Coefficients by Differentiating

Another way to determine the constants that appear in partial fractions is to differentiate, as in the next example.

EXAMPLE 6 Find A , B , and C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

Solution We first clear fractions:

$$x-1 = A(x+1)^2 + B(x+1) + C.$$

Substituting $x = -1$ shows $C = -2$. We then differentiate both sides with respect to x , obtaining

$$1 = 2A(x+1) + B.$$

Substituting $x = -1$ shows $B = 1$. We differentiate again to get $0 = 2A$, which shows $A = 0$. Hence,

$$\frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}.$$

EXERCISES 8.5

Expanding Quotients into Partial Fractions

Expand the quotients in Exercises 1–8 by partial fractions.

1. $\frac{5x-13}{(x-3)(x-2)}$

2. $\frac{5x-7}{x^2-3x+2}$

3. $\frac{x+4}{(x+1)^2}$

4. $\frac{2x+2}{x^2-2x+1}$

5. $\frac{z+1}{z^2(z-1)}$

6. $\frac{z}{z^3-z^2-6z}$

7. $\frac{t^2+8}{t^2-5t+6}$

8. $\frac{t^4+9}{t^4+9t^2}$

Nonrepeated Linear Factors

In Exercises 9–16, express the integrand as a sum of partial fractions and evaluate the integrals.

9. $\int \frac{dx}{1-x^2}$

10. $\int \frac{dx}{x^2+2x}$

11. $\int \frac{x+4}{x^2+5x-6} dx$

12. $\int \frac{2x+1}{x^2-7x+12} dx$

13. $\int_4^8 \frac{y dy}{y^2-2y-3}$

14. $\int_{1/2}^1 \frac{y+4}{y^2+y} dy$

15. $\int \frac{dt}{t^3+t^2-2t}$

16. $\int \frac{x+3}{2x^3-8x} dx$

Repeated Linear Factors

In Exercises 17–20, express the integrand as a sum of partial fractions and evaluate the integrals.

17. $\int_0^1 \frac{x^3 dx}{x^2+2x+1}$

18. $\int_{-1}^0 \frac{x^3 dx}{x^2-2x+1}$

19. $\int \frac{dx}{(x^2-1)^2}$

20. $\int \frac{x^2 dx}{(x-1)(x^2+2x+1)}$

Irreducible Quadratic Factors

In Exercises 21–32, express the integrand as a sum of partial fractions and evaluate the integrals.

21. $\int_0^1 \frac{dx}{(x+1)(x^2+1)}$

22. $\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t} dt$

23. $\int \frac{y^2+2y+1}{(y^2+1)^2} dy$

24. $\int \frac{8x^2+8x+2}{(4x^2+1)^2} dx$

25. $\int \frac{2s+2}{(s^2+1)(s-1)^3} ds$

26. $\int \frac{s^4+81}{s(s^2+9)^2} ds$

27. $\int \frac{x^2-x+2}{x^3-1} dx$

28. $\int \frac{1}{x^4+x} dx$

29. $\int \frac{x^2}{x^4-1} dx$

30. $\int \frac{x^2+x}{x^4-3x^2-4} dx$

31. $\int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2} d\theta$

32. $\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3} d\theta$

Improper Fractions

In Exercises 33–38, perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

33. $\int \frac{2x^3 - 2x^2 + 1}{x^2 - x} dx$

34. $\int \frac{x^4}{x^2 - 1} dx$

35. $\int \frac{9x^3 - 3x + 1}{x^3 - x^2} dx$

36. $\int \frac{16x^3}{4x^2 - 4x + 1} dx$

37. $\int \frac{y^4 + y^2 - 1}{y^3 + y} dy$

38. $\int \frac{2y^4}{y^3 - y^2 + y - 1} dy$

Evaluating Integrals

Evaluate the integrals in Exercises 39–54.

39. $\int \frac{e^t dt}{e^{2t} + 3e^t + 2}$

40. $\int \frac{e^{4t} + 2e^{2t} - e^t}{e^{2t} + 1} dt$

41. $\int \frac{\cos y dy}{\sin^2 y + \sin y - 6}$

42. $\int \frac{\sin \theta d\theta}{\cos^2 \theta + \cos \theta - 2}$

43. $\int \frac{(x-2)^2 \tan^{-1}(2x) - 12x^3 - 3x}{(4x^2 + 1)(x-2)^2} dx$

44. $\int \frac{(x+1)^2 \tan^{-1}(3x) + 9x^3 + x}{(9x^2 + 1)(x+1)^2} dx$

45. $\int \frac{1}{x^{3/2} - \sqrt{x}} dx$

46. $\int \frac{1}{(x^{1/3} - 1)\sqrt{x}} dx$
(Hint: Let $x = u^6$.)

47. $\int \frac{\sqrt{x+1}}{x} dx$
(Hint: Let $x+1 = u^2$.)

48. $\int \frac{1}{x\sqrt{x+9}} dx$

49. $\int \frac{1}{x(x^4 + 1)} dx$
(Hint: Multiply by $\frac{x^3}{x^3}$.)

50. $\int \frac{1}{x^6(x^5 + 4)} dx$

51. $\int \frac{1}{\cos 2\theta \sin \theta} d\theta$

52. $\int \frac{1}{\cos \theta + \sin 2\theta} d\theta$

53. $\int \frac{\sqrt{1+\sqrt{x}}}{x} dx$

54. $\int \frac{\sqrt{x}}{\sqrt{2-\sqrt{x}} + \sqrt{x}} dx$

Use any method to evaluate the integrals in Exercises 55–66.

55. $\int \frac{x^3 - 2x^2 - 3x}{x + 2} dx$

56. $\int \frac{x + 2}{x^3 - 2x^2 - 3x} dx$

57. $\int \frac{2^x - 2^{-x}}{2^x + 2^{-x}} dx$

58. $\int \frac{2^x}{2^{2x} + 2^x - 2} dx$

59. $\int \frac{1}{x^4 - 1} dx$

60. $\int \frac{x^4 - 1}{x^5 - 5x + 1} dx$

61. $\int \frac{\ln x + 2}{x(\ln x + 1)(\ln x + 3)} dx$

62. $\int \frac{2}{x(\ln x - 2)^3} dx$

63. $\int \frac{1}{\sqrt{x^2 - 1}} dx$

64. $\int \frac{x}{x + \sqrt{x^2 + 2}} dx$

65. $\int x^5 \sqrt{x^3 + 1} dx$

66. $\int x^2 \sqrt{1 - x^2} dx$

Initial Value Problems

Solve the initial value problems in Exercises 67–70 for x as a function of t .

67. $(t^2 - 3t + 2) \frac{dx}{dt} = 1 \quad (t > 2), \quad x(3) = 0$

68. $(3t^4 + 4t^2 + 1) \frac{dx}{dt} = 2\sqrt{3}, \quad x(1) = -\pi\sqrt{3}/4$

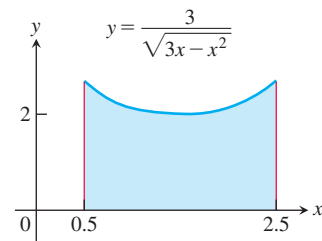
69. $(t^2 + 2t) \frac{dx}{dt} = 2x + 2 \quad (t, x > 0), \quad x(1) = 1$

70. $(t + 1) \frac{dx}{dt} = x^2 + 1 \quad (t > -1), \quad x(0) = 0$

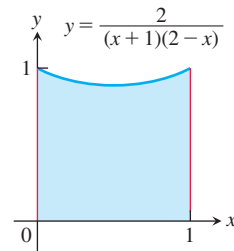
Applications and Examples

In Exercises 71 and 72, find the volume of the solid generated by revolving the shaded region about the indicated axis.

71. The x -axis



72. The y -axis



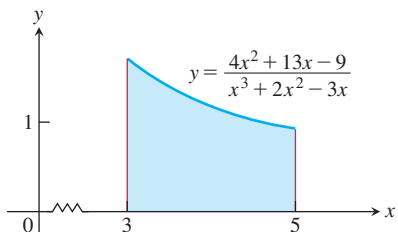
73. Find the length of the curve $y = \ln(1 - x^2)$, $0 \leq x \leq \frac{1}{2}$.

74. Evaluate $\int \sec \theta d\theta$ by

a. multiplying by $\frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta}$ and then using a u -substitution.

b. writing the integral as $\int \frac{1}{\cos \theta} d\theta$. Then multiply by $\frac{\cos \theta}{\cos \theta}$, use a trigonometric identity and a u -substitution, and finally integrate using partial fractions.

- T 75.** Find, to two decimal places, the x -coordinate of the centroid of the region in the first quadrant bounded by the x -axis, the curve $y = \arctan x$, and the line $x = \sqrt{3}$.
- T 76.** Find the x -coordinate of the centroid of this region to two decimal places.



- T 77. Social diffusion** Sociologists sometimes use the phrase “social diffusion” to describe the way information spreads through a population. The information might be a rumor, a cultural fad, or news about a technical innovation. In a sufficiently large population, the number of people x who have the information is treated as a differentiable function of time t , and the rate of diffusion, dx/dt , is assumed to be proportional to the number of people who have the information times the number of people who do not. This leads to the equation

$$\frac{dx}{dt} = kx(N - x),$$

where N is the number of people in the population.

Suppose t is in days, $k = 1/250$, and two people start a rumor at time $t = 0$ in a population of $N = 1000$ people.

- a.** Find x as a function of t .
- b.** When will half the population have heard the rumor? (This is when the rumor will be spreading the fastest.)

- T 78. Second-order chemical reactions** Many chemical reactions are the result of the interaction of two molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the amount of substance B at time $t = 0$, and if x is the amount of product at time t , then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt} = k(a - x)(b - x),$$

or

$$\frac{1}{(a - x)(b - x)} \frac{dx}{dt} = k,$$

where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t **(a)** if $a = b$, and **(b)** if $a \neq b$. Assume in each case that $x = 0$ when $t = 0$.

8.6 Integral Tables and Computer Algebra Systems

In this section we discuss how to use tables and computer algebra systems (CAS) to evaluate integrals.

Integral Tables

A Brief Table of Integrals is provided at the back of the text, after the index. (More extensive tables appear in compilations such as *CRC Mathematical Tables*, which contain thousands of integrals.) The integration formulas are stated in terms of constants a, b, c, m, n , and so on. These constants can usually assume any real value and need not be integers. Occasional limitations on their values are stated with the formulas. Formula 21 requires $n \neq -1$, for example, and Formula 27 requires $n \neq -2$.

The formulas also assume that the constants do not take on values that require dividing by zero or taking even roots of negative numbers. For example, Formula 24 assumes that $a \neq 0$, and Formulas 29a and 29b cannot be used unless b is positive.

EXAMPLE 1 Find

$$\int x(2x + 5)^{-1} dx.$$

Solution We use Formula 24 at the back of the text (not 22, which requires $n \neq -1$):

$$\int x(ax + b)^{-1} dx = \frac{x}{a} - \frac{b}{a^2} \ln |ax + b| + C.$$

With $a = 2$ and $b = 5$, we have

$$\int x(2x + 5)^{-1} dx = \frac{x}{2} - \frac{5}{4} \ln |2x + 5| + C. \quad \blacksquare$$

EXAMPLE 2 Find

$$\int \frac{dx}{x\sqrt{2x-4}}.$$

Solution We use Formula 29b:

$$\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}} \arctan \sqrt{\frac{ax-b}{b}} + C.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x-4}} = \frac{2}{\sqrt{4}} \arctan \sqrt{\frac{2x-4}{4}} + C = \arctan \sqrt{\frac{x-2}{2}} + C. \quad \blacksquare$$

EXAMPLE 3 Find

$$\int x \arcsin x \, dx.$$

Solution We begin by using Formula 106:

$$\int x^n \arcsin ax \, dx = \frac{x^{n+1}}{n+1} \arcsin ax - \frac{a}{n+1} \int \frac{x^{n+1} dx}{\sqrt{1-a^2x^2}}, \quad n \neq -1.$$

With $n = 1$ and $a = 1$, we have

$$\int x \arcsin x \, dx = \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{x^2 dx}{\sqrt{1-x^2}}.$$

Next we use Formula 49 to find the integral on the right:

$$\int \frac{x^2}{\sqrt{a^2-x^2}} dx = \frac{a^2}{2} \arcsin\left(\frac{x}{a}\right) - \frac{1}{2} x \sqrt{a^2-x^2} + C.$$

With $a = 1$,

$$\int \frac{x^2}{\sqrt{1-x^2}} dx = \frac{1}{2} \arcsin x - \frac{1}{2} x \sqrt{1-x^2} + C.$$

The combined result is

$$\begin{aligned} \int x \arcsin x \, dx &= \frac{x^2}{2} \arcsin x - \frac{1}{2} \left(\frac{1}{2} \arcsin x - \frac{1}{2} x \sqrt{1-x^2} + C \right) \\ &= \left(\frac{x^2}{2} - \frac{1}{4} \right) \arcsin x + \frac{1}{4} x \sqrt{1-x^2} + C'. \quad \blacksquare \end{aligned}$$

Reduction Formulas

The time required for repeated integrations by parts can sometimes be shortened by applying reduction formulas like the following.

$$\int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx \quad (1)$$

$$\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx \quad (2)$$

$$\int \sin^n x \cos^m x \, dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n} \int \sin^{n-2} x \cos^m x \, dx \quad (n \neq -m). \quad (3)$$

By applying such a formula repeatedly, we can eventually express the original integral in terms of a power low enough to be evaluated directly. The next example illustrates this procedure.

EXAMPLE 4 Find

$$\int \tan^5 x \, dx.$$

Solution We apply Equation (1) with $n = 5$ to get

$$\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \int \tan^3 x \, dx.$$

We then apply Equation (1) again, with $n = 3$, to evaluate the remaining integral:

$$\int \tan^3 x \, dx = \frac{1}{2} \tan^2 x - \int \tan x \, dx = \frac{1}{2} \tan^2 x + \ln |\cos x| + C_1.$$

The combined result is

$$\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \ln |\cos x| + C. \quad \blacksquare$$

As their form suggests, reduction formulas are derived using integration by parts. (See Example 5 in Section 8.3.)

Integration with a CAS

A powerful capability of computer algebra systems is their ability to integrate symbolically. This is performed with the **integrate command** specified by the particular system (for example, **int** in Maple, **Integrate** in Mathematica).

EXAMPLE 5 Suppose that you want to evaluate the indefinite integral of the function

$$f(x) = x^2 \sqrt{a^2 + x^2}.$$

Using Maple, you first define or name the function:

$$> f := x^2 * \text{sqrt}(a^2 + x^2);$$

Then you use the integrate command on f , identifying the variable of integration:

$$> \text{int}(f, x);$$

Maple returns the answer

$$\frac{x(a^2 + x^2)^{3/2}}{4} - \frac{a^2 x \sqrt{a^2 + x^2}}{8} - \frac{a^4 \ln(x + \sqrt{a^2 + x^2})}{8}$$

If you want to see whether the answer can be simplified, enter

$$> \text{simplify}(\%);$$

Maple returns

$$-\frac{a^4 \ln(x + \sqrt{a^2 + x^2})}{8} + \frac{x \sqrt{a^2 + x^2} (a^2 + 2x^2)}{8}$$

If you want the definite integral for $0 \leq x \leq \pi/2$, you can use the format

$$> \text{int}(f, x = 0..Pi/2);$$

Maple will return the expression

$$\frac{a^4 \ln(a^2)}{16} + \frac{\pi(\pi^2 + 4a^2)^{3/2}}{64} - \frac{a^2 \pi \sqrt{\pi^2 + 4a^2}}{32} + \frac{a^4 \ln(2)}{8} - \frac{a^4 \ln(\pi + \sqrt{\pi^2 + 4a^2})}{8}$$

You can also find the definite integral for a particular value of the constant a :

```
> a:= 1;
> int(f, x = 0..1);
```

Maple returns the numerical answer

$$\frac{3}{8}\sqrt{2} + \frac{1}{8}\ln(\sqrt{2} - 1).$$

EXAMPLE 6 Use a CAS to find

$$\int \sin^2 x \cos^3 x \, dx.$$

Solution With Maple, we have the entry

```
> int((sin^2)(x)*(cos^3)(x),x);
```

with the immediate return

$$-\frac{1}{5}\sin(x)\cos(x)^4 + \frac{1}{15}\cos(x)^2\sin(x) + \frac{2}{15}\sin(x).$$

Computer algebra systems vary in how they process integrations. We used Maple in Examples 5 and 6. Mathematica would have returned somewhat different results:

1. In Example 5, given

```
In[1]:= Integrate[x^2*Sqrt[a^2 + x^2], x]
```

Mathematica returns

$$Out[1] = \frac{1}{8}\sqrt{a^2 + x^2} \left(a^2x + 2x^3 - \frac{a^3 \sinh^{-1}\left(\frac{x}{a}\right)}{\sqrt{1 + \frac{x^2}{a^2}}} \right)$$

without having to simplify an intermediate result. The answer is different from, but equivalent to, Formula 36 in the integral tables.

2. The Mathematica answer to the integral

```
In[2]:= Integrate[Sin[x]^2*Cos[x]^3, x]
```

in Example 6 is

$$Out[2] = \frac{\sin[x]}{8} - \frac{1}{48}\sin[3x] - \frac{1}{80}\sin[5x]$$

differing from the Maple answer. Both answers are correct.

Although a CAS is very powerful and can aid us in solving difficult problems, each CAS has its own limitations. There are even situations where a CAS may further complicate a problem (in the sense of producing an answer that is extremely difficult to use or interpret). Note, too, that neither Maple nor Mathematica returns an arbitrary constant $+C$. On the other hand, a little mathematical thinking on your part may reduce the problem to one that is quite easy to handle. We provide an example in Exercise 67.

Nonelementary Integrals

Many functions have antiderivatives that cannot be expressed using the standard functions that we have encountered, such as polynomials, trigonometric functions, and exponential functions. Integrals of functions that do not have elementary antiderivatives are called **nonelementary** integrals. These integrals can sometimes be expressed with infinite series (Chapter 9) or approximated using numerical methods (Section 8.7). Examples of nonelementary integrals include the error function (which measures the probability of random errors)

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

and integrals such as

$$\int \sin x^2 dx \quad \text{and} \quad \int \sqrt{1+x^4} dx$$

that arise in engineering and physics. These and a number of others, such as

$$\int \frac{e^x}{x} dx, \quad \int e^{(e^x)} dx, \quad \int \frac{1}{\ln x} dx, \quad \int \ln(\ln x) dx, \quad \int \frac{\sin x}{x} dx,$$

$$\int \sqrt{1-k^2 \sin^2 x} dx, \quad 0 < k < 1,$$

look so easy they tempt us to try them just to see how they turn out. It can be proved, however, that there is no way to express any of these integrals as finite combinations of elementary functions. The same applies to integrals that can be changed into these by substitution. The functions in these integrals all have antiderivatives, as a consequence of the Fundamental Theorem of Calculus, Part 1, because they are continuous. However, none of the antiderivatives are elementary. The integrals you are asked to evaluate in this chapter have elementary antiderivatives.

EXERCISES 8.6

Using Integral Tables

Use the table of integrals at the back of the text to evaluate the integrals in Exercises 1–26.

1. $\int \frac{dx}{x\sqrt{x-3}}$

2. $\int \frac{dx}{x\sqrt{x+4}}$

9. $\int x\sqrt{4x-x^2} dx$

10. $\int \frac{\sqrt{x-x^2}}{x} dx$

3. $\int \frac{x dx}{\sqrt{x-2}}$

4. $\int \frac{x dx}{(2x+3)^{3/2}}$

11. $\int \frac{dx}{x\sqrt{7+x^2}}$

12. $\int \frac{dx}{x\sqrt{7-x^2}}$

13. $\int \frac{\sqrt{4-x^2}}{x} dx$

14. $\int \frac{\sqrt{x^2-4}}{x} dx$

5. $\int x\sqrt{2x-3} dx$

6. $\int x(7x+5)^{3/2} dx$

15. $\int e^{2t} \cos 3t dt$

16. $\int e^{-3t} \sin 4t dt$

7. $\int \frac{\sqrt{9-4x}}{x^2} dx$

8. $\int \frac{dx}{x^2\sqrt{4x-9}}$

17. $\int x \arccos x dx$

18. $\int x \arctan x dx$

19. $\int x^2 \arctan x \, dx$ 20. $\int \frac{\tan^{-1} x}{x^2} \, dx$
21. $\int \sin 3x \cos 2x \, dx$ 22. $\int \sin 2x \cos 3x \, dx$
23. $\int 8 \sin 4t \sin \frac{t}{2} \, dt$ 24. $\int \sin \frac{t}{3} \sin \frac{t}{6} \, dt$
25. $\int \cos \frac{\theta}{3} \cos \frac{\theta}{4} \, d\theta$ 26. $\int \cos \frac{\theta}{2} \cos 7\theta \, d\theta$

Substitution and Integral Tables

In Exercises 27–40, use a substitution to change the integral into one you can find in the table. Then evaluate the integral.

27. $\int \frac{x^3 + x + 1}{(x^2 + 1)^2} \, dx$ 28. $\int \frac{x^2 + 6x}{(x^2 + 3)^2} \, dx$
29. $\int \arcsin \sqrt{x} \, dx$ 30. $\int \frac{\cos^{-1} \sqrt{x}}{\sqrt{x}} \, dx$
31. $\int \frac{\sqrt{x}}{\sqrt{1-x}} \, dx$ 32. $\int \frac{\sqrt{2-x}}{\sqrt{x}} \, dx$
33. $\int \cot t \sqrt{1 - \sin^2 t} \, dt, \quad 0 < t < \pi/2$
34. $\int \frac{dt}{\tan t \sqrt{4 - \sin^2 t}}$
35. $\int \frac{dy}{y\sqrt{3 + (\ln y)^2}}$ 36. $\int \tan^{-1} \sqrt{y} \, dy$
37. $\int \frac{1}{\sqrt{x^2 + 2x + 5}} \, dx$ 38. $\int \frac{x^2}{\sqrt{x^2 - 4x + 5}} \, dx$
(Hint: Complete the square.)
39. $\int \sqrt{5 - 4x - x^2} \, dx$ 40. $\int x^2 \sqrt{2x - x^2} \, dx$

Using Reduction Formulas

Use reduction formulas to evaluate the integrals in Exercises 41–50.

41. $\int \sin^5 2x \, dx$ 42. $\int 8 \cos^4 2\pi t \, dt$
43. $\int \sin^2 2\theta \cos^3 2\theta \, d\theta$ 44. $\int 2 \sin^2 t \sec^4 t \, dt$
45. $\int 4 \tan^3 2x \, dx$ 46. $\int 8 \cot^4 t \, dt$
47. $\int 2 \sec^3 \pi x \, dx$ 48. $\int 3 \sec^4 3x \, dx$
49. $\int \csc^5 x \, dx$ 50. $\int 16x^3 (\ln x)^2 \, dx$

Evaluate the integrals in Exercises 51–56 by making a substitution (possibly trigonometric) and then applying a reduction formula.

51. $\int e^t \sec^3(e^t - 1) \, dt$ 52. $\int \frac{\csc^3 \sqrt{\theta}}{\sqrt{\theta}} \, d\theta$
53. $\int_0^1 2\sqrt{x^2 + 1} \, dx$ 54. $\int_0^{\sqrt{3}/2} \frac{dy}{(1 - y^2)^{5/2}}$
55. $\int_1^2 \frac{(r^2 - 1)^{3/2}}{r} \, dr$ 56. $\int_0^{1/\sqrt{3}} \frac{dt}{(t^2 + 1)^{7/2}}$

Applications

57. Surface area Find the area of the surface generated by revolving the curve $y = \sqrt{x^2 + 2}$, $0 \leq x \leq \sqrt{2}$, about the x -axis.

58. Arc length Find the length of the curve $y = x^2$, $0 \leq x \leq \sqrt{3}/2$.

59. Centroid Find the centroid of the region cut from the first quadrant by the curve $y = 1/\sqrt{x+1}$ and the line $x = 3$.

60. Moment about y-axis A thin plate of constant density $\delta = 1$ occupies the region enclosed by the curve $y = 36/(2x + 3)$ and the line $x = 3$ in the first quadrant. Find the moment of the plate about the y -axis.

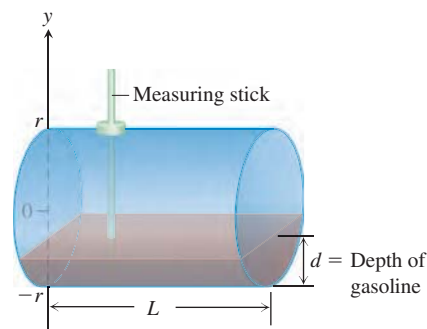
T 61. Use the integral table and a calculator to find, to two decimal places, the area of the surface generated by revolving the curve $y = x^2$, $-1 \leq x \leq 1$, about the x -axis.

62. Volume The head of your firm's accounting department has asked you to find a formula she can use in a computer program to calculate the year-end inventory of gasoline in the company's tanks. A typical tank is shaped like a right circular cylinder of radius r and length L , mounted horizontally, as shown in the accompanying figure. The data come to the accounting office as depth measurements taken with a vertical measuring stick marked in centimeters.

a. Show, in the notation of the figure, that the volume of gasoline that fills the tank to a depth d is

$$V = 2L \int_{-r}^{-r+d} \sqrt{r^2 - y^2} \, dy.$$

b. Evaluate the integral.



63. What is the largest value that

$$\int_a^b \sqrt{x - x^2} \, dx$$

can have for any a and b ? Give reasons for your answer.

64. What is the largest value that

$$\int_a^b x\sqrt{2x - x^2} dx$$

can have for any a and b ? Give reasons for your answer.

COMPUTER EXPLORATIONS

In Exercises 65 and 66, use a CAS to perform the integrations.

65. Evaluate the integrals

a. $\int x \ln x dx$ b. $\int x^2 \ln x dx$ c. $\int x^3 \ln x dx$.

d. What pattern do you see? Predict the formula for $\int x^4 \ln x dx$ and then see if you are correct by evaluating it with a CAS.

e. What is the formula for $\int x^n \ln x dx$, $n \geq 1$? Check your answer using a CAS.

66. Evaluate the integrals

a. $\int \frac{\ln x}{x^2} dx$ b. $\int \frac{\ln x}{x^3} dx$ c. $\int \frac{\ln x}{x^4} dx$.

d. What pattern do you see? Predict the formula for

$$\int \frac{\ln x}{x^5} dx$$

and then see if you are correct by evaluating it with a CAS.

e. What is the formula for

$$\int \frac{\ln x}{x^n} dx, \quad n \geq 2?$$

Check your answer using a CAS.

67. a. Use a CAS to evaluate

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx,$$

where n is an arbitrary positive integer. Does your CAS find the result?

b. In succession, find the integral when $n = 1, 2, 3, 5$, and 7 . Comment on the complexity of the results.

c. Now substitute $x = (\pi/2) - u$ and add the new and old integrals. What is the value of

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx?$$

This exercise illustrates how a little mathematical ingenuity can sometimes solve a problem not immediately amenable to solution by a CAS.

8.7 Numerical Integration

The antiderivatives of some functions, like $\sin(x^2)$, $1/\ln x$, and $\sqrt{1+x^4}$, have no elementary formulas. When we cannot find a workable antiderivative for a function f that we have to integrate, we can partition the interval of integration, replace f by a closely fitting polynomial on each subinterval, integrate the polynomials, and add the results to approximate the definite integral of f . This procedure is an example of numerical integration. In this section we start by revisiting the Midpoint Rule, which we studied in Section 5.2. We then study two new methods, the *Trapezoidal Rule* and *Simpson's Rule*. A key goal in our analysis is to control the possible error that is introduced when computing an approximation to an integral.

Approximating Integrals with the Midpoint Rule

In Section 5.2 we introduced the Midpoint Rule to approximate a definite integral over an interval $[a, b]$. The rule is based on subdividing $[a, b]$ into n equal subintervals,

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$$

each of width

$$\Delta x = \frac{b-a}{n}.$$

We then approximate the integral using n rectangles, where the height of the k th rectangle is the value of f at the midpoint $c_k = (x_{k-1} + x_k)/2$ of the k th subinterval $[x_{k-1}, x_k]$.

Midpoint Rule for Approximating a Definite Integral

$$\int_a^b f(x) dx \approx \sum_{k=1}^n f(c_k) \left(\frac{b-a}{n} \right) = [f(c_1) + f(c_2) + \dots + f(c_n)] \left(\frac{b-a}{n} \right)$$

$$\text{with } c_k = \frac{x_{k-1} + x_k}{2} \text{ and } x_k = a + k \left(\frac{b-a}{n} \right).$$

Trapezoidal Approximations

The Trapezoidal Rule for the value of a definite integral is based on approximating the region between a curve and the x -axis with trapezoids instead of rectangles, as in Figure 8.7. It is not necessary for the subdivision points $x_0, x_1, x_2, \dots, x_n$ in the figure to be evenly spaced, but the resulting formula is simpler if they are. We therefore assume that the length of each subinterval is

$$\Delta x = \frac{b - a}{n}.$$

The length $\Delta x = (b - a)/n$ is called the **step size** or **mesh size**. The area of the trapezoid that lies above the i th subinterval is

$$\Delta x \left(\frac{y_{i-1} + y_i}{2} \right) = \frac{\Delta x}{2} (y_{i-1} + y_i),$$

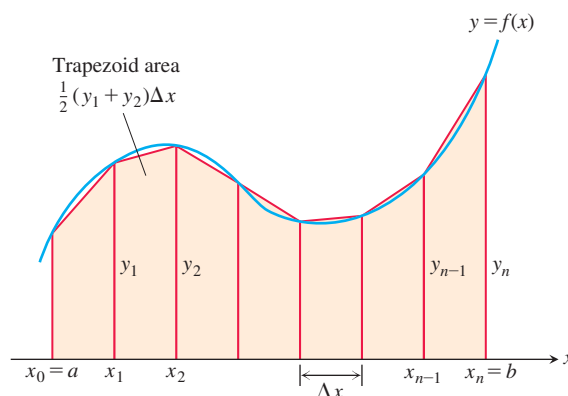


FIGURE 8.7 The Trapezoidal Rule approximates short stretches of the curve $y = f(x)$ with line segments. To approximate the integral of f from a to b , we add the areas of the trapezoids made by vertically joining the ends of the segments to the x -axis.

where $y_{i-1} = f(x_{i-1})$ and $y_i = f(x_i)$. (See Figure 8.7.) The area below the curve $y = f(x)$ and above the x -axis is then approximated by adding the areas of all the trapezoids:

$$\begin{aligned} T &= \frac{1}{2}(y_0 + y_1)\Delta x + \frac{1}{2}(y_1 + y_2)\Delta x + \cdots \\ &\quad + \frac{1}{2}(y_{n-2} + y_{n-1})\Delta x + \frac{1}{2}(y_{n-1} + y_n)\Delta x \\ &= \Delta x \left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n \right) \\ &= \frac{\Delta x}{2} (y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n), \end{aligned}$$

where

$$y_0 = f(a), \quad y_1 = f(x_1), \quad \dots, \quad y_{n-1} = f(x_{n-1}), \quad y_n = f(b).$$

The Trapezoidal Rule says: Use T to estimate the integral of f from a to b .

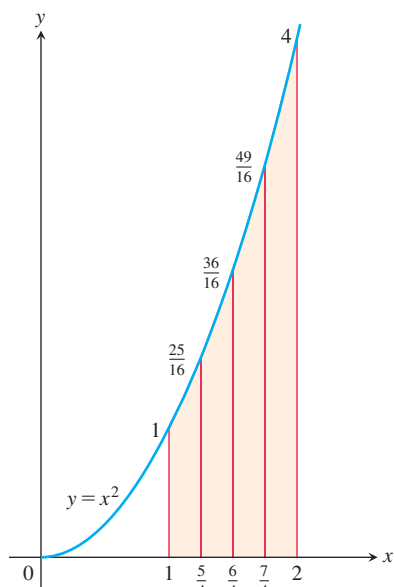


FIGURE 8.8 The trapezoidal approximation of the area under the graph of $y = x^2$ from $x = 1$ to $x = 2$ is a slight overestimate (Example 1).

TABLE 8.2

x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4

The Trapezoidal Rule

To approximate $\int_a^b f(x) dx$, use

$$T = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n).$$

The y 's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \dots, x_{n-1} = a + (n-1)\Delta x, x_n = b,$$

where $\Delta x = (b - a)/n$.

EXAMPLE 1 Use the Trapezoidal Rule with $n = 4$ to estimate $\int_1^2 x^2 dx$. Compare the estimate with the exact value.

Solution Partition $[1, 2]$ into four subintervals of equal length (Figure 8.8). Then evaluate $y = x^2$ at each partition point (Table 8.2).

Using these y -values, $n = 4$, and $\Delta x = (2 - 1)/4 = 1/4$ in the Trapezoidal Rule, we have

$$\begin{aligned} T &= \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4) \\ &= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right) \\ &= \frac{75}{32} = 2.34375. \end{aligned}$$

Since the parabola is concave *up*, the approximating segments lie above the curve, giving each trapezoid slightly more area than the corresponding strip under the curve. The exact value of the integral is

$$\int_1^2 x^2 dx = \left.\frac{x^3}{3}\right|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

The T approximation overestimates the integral by about half a percent of its true value of $7/3$. The percentage error is $(2.34375 - 7/3)/(7/3) \approx 0.00446$, or 0.446%. ■

Simpson's Rule: Approximations Using Parabolas

Another rule for approximating the definite integral of a continuous function results from using parabolas instead of the straight-line segments that produced trapezoids. As before, we partition the interval $[a, b]$ into n subintervals of equal length $h = \Delta x = (b - a)/n$, but this time we require that n be an *even* number. On each consecutive pair of intervals we approximate the curve $y = f(x) \geq 0$ by a parabola, as shown in Figure 8.9. A typical parabola passes through three consecutive points (x_{i-1}, y_{i-1}) , (x_i, y_i) , and (x_{i+1}, y_{i+1}) on the curve.

Let's calculate the shaded area beneath a parabola passing through three consecutive points. To simplify our calculations, we first take the case where $x_0 = -h$, $x_1 = 0$, and $x_2 = h$ (Figure 8.10), where $h = \Delta x = (b - a)/n$. The area under the parabola will be the same if we shift the y -axis to the left or right. The parabola has an equation of the form

$$y = Ax^2 + Bx + C,$$

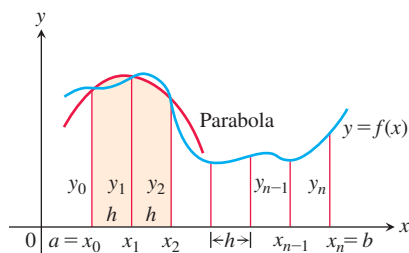


FIGURE 8.9 Simpson's Rule approximates short stretches of the curve with parabolas.

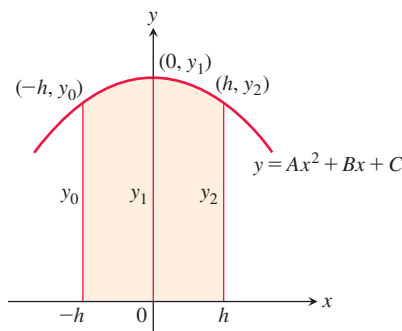


FIGURE 8.10 By integrating from $-h$ to h , we find the shaded area to be

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

so the area under it from $x = -h$ to $x = h$ is

$$\begin{aligned} A_p &= \int_{-h}^h (Ax^2 + Bx + C) dx \\ &= \left[\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx \right]_{-h}^h \\ &= \frac{2Ah^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6C). \end{aligned}$$

Since the curve passes through the three points $(-h, y_0)$, $(0, y_1)$, and (h, y_2) , we also have

$$y_0 = Ah^2 - Bh + C, \quad y_1 = C, \quad y_2 = Ah^2 + Bh + C.$$

After some algebraic manipulation, we find that

$$A_p = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Now shifting the parabola horizontally to its shaded position in Figure 8.9 does not change the area under it. Thus the area under the parabola through (x_0, y_0) , (x_1, y_1) , and (x_2, y_2) in Figure 8.9 is still

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

Similarly, the area under the parabola through the points (x_2, y_2) , (x_3, y_3) , and (x_4, y_4) is

$$\frac{h}{3}(y_2 + 4y_3 + y_4).$$

Computing the areas under all the parabolas and adding the results give the approximation

$$\begin{aligned} \int_a^b f(x) dx &\approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots \\ &\quad + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n) \\ &= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n). \end{aligned}$$

HISTORICAL BIOGRAPHY

Thomas Simpson
(1720–1761)

Simpson was a successful text writer and did most of his research on probability. Simpson's rule to approximate definite integrals was developed before he was born. It is another of history's beautiful quirks that one of the ablest mathematicians of the 18th century is remembered not for his own work but for a rule that was never his, that he never claimed, and that bears his name only because he happened to mention it in one of his books.

To know more, visit the companion Website.

The result is known as Simpson's Rule. The function need not be positive, as in our derivation, but the number n of subintervals must be even for us to apply the rule because each parabolic arc uses two subintervals.

Simpson's Rule

To approximate $\int_a^b f(x) dx$, use

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

The y 's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \dots, x_{n-1} = a + (n-1)\Delta x, x_n = b.$$

The number n is even, and $\Delta x = (b - a)/n$.

Note the pattern of the coefficients in the above rule: 1, 4, 2, 4, 2, 4, 2, ..., 4, 1.

EXAMPLE 2 Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4 dx$.

TABLE 8.3

x	$y = 5x^4$
0	0
$\frac{1}{2}$	$\frac{5}{16}$
1	5
$\frac{3}{2}$	$\frac{405}{16}$
2	80

Solution Partition $[0, 2]$ into four subintervals and evaluate $y = 5x^4$ at the partition points (Table 8.3). Then apply Simpson's Rule with $n = 4$ and $\Delta x = 1/2$:

$$\begin{aligned}
 S &= \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) \\
 &= \frac{1}{6}\left(0 + 4\left(\frac{5}{16}\right) + 2(5) + 4\left(\frac{405}{16}\right) + 80\right) \\
 &= 32\frac{1}{12}.
 \end{aligned}$$

This estimate differs from the exact value (32) by only $1/12$, a percentage error of less than three-tenths of one percent, and this was with just four subintervals. ■

Error Analysis

Whenever we use an approximation technique, we must consider how accurate the approximation might be. The following theorem gives formulas for estimating the errors when using the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule. The **error** is the difference between the approximation obtained by using the rule and the actual value of the definite integral $\int_a^b f(x) dx$.

THEOREM 1—Error Estimates in the Midpoint, Trapezoidal, and Simpson's Rules

If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then the error E_M in the Midpoint Rule approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_M| \leq \frac{M(b-a)^3}{24n^2}. \quad \text{Midpoint Rule}$$

If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then the error E_T in the Trapezoidal Rule approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}. \quad \text{Trapezoidal Rule}$$

If $f^{(4)}$ is continuous and M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then the error E_S in the Simpson's Rule approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}. \quad \text{Simpson's Rule}$$

To give an idea of why Theorem 1 is true in the case of the Trapezoidal Rule, we begin with a result which says that if f'' is continuous on the interval $[a, b]$, then

$$\int_a^b f(x) dx = T - \frac{b-a}{12} \cdot f''(c)(\Delta x)^2$$

for some number c between a and b . This result follows from Taylor's Remainder Theorem, which we discuss in Theorem 24 of Section 9.9. It follows that as Δx approaches zero, the error defined by

$$E_T = -\frac{b-a}{12} \cdot f''(c)(\Delta x)^2$$

approaches zero at the rate of the square of Δx .

The inequality

$$|E_T| \leq \frac{b-a}{12} \max |f''(x)| (\Delta x)^2,$$

where “max” refers to the maximum of $|f''(x)|$ over the interval $[a, b]$, gives an upper bound for the magnitude of the error. In practice, we usually cannot find the exact value of $\max |f''(x)|$ and have to estimate an upper bound or “worst case” value for it instead. If M is any upper bound for the values of $|f''(x)|$ on $[a, b]$, so that $|f''(x)| \leq M$ for every x in $[a, b]$, then

$$|E_T| \leq \frac{b-a}{12} M (\Delta x)^2.$$

If we substitute $(b-a)/n$ for Δx , we get

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}.$$

The upper bound for the error in the Midpoint Rule,

$$|E_M| \leq \frac{M(b-a)^3}{24n^2},$$

is based on a similar argument. Taylor’s Remainder Theorem implies that the Midpoint Rule approximation and the definite integral differ by at most

$$\frac{|f''(c)| (b-a)^3}{24n^2}$$

for some $c \in [a, b]$. Taking M to be at least as large as the maximum of $|f''(c)|$ on $[a, b]$, we obtain the error bound in Theorem 1. Note that this is half as large as the error bound for the Trapezoidal Rule. This does not mean that the Midpoint Rule is always more accurate than the Trapezoidal Rule, but rather that the largest possible error that might occur is only half as large as the largest possible error for the Trapezoidal Rule.

To estimate the error in Simpson’s Rule, we start with a result, again following from Taylor’s Remainder Theorem, that says that if the fourth derivative $f^{(4)}$ is continuous, then

$$\int_a^b f(x) dx = S - \frac{b-a}{180} \cdot f^{(4)}(c) (\Delta x)^4$$

for some point c between a and b . Thus, as Δx approaches zero, the error,

$$E_S = -\frac{b-a}{180} \cdot f^{(4)}(c) (\Delta x)^4,$$

approaches zero as the *fourth power* of Δx . (This helps to explain why Simpson’s Rule is likely to give better results than the Trapezoidal Rule.)

The inequality

$$|E_S| \leq \frac{b-a}{180} \max |f^{(4)}(x)| (\Delta x)^4,$$

where “max” refers to the maximum of $|f^{(4)}(x)|$ over the interval $[a, b]$, gives an upper bound for the magnitude of the error. As with $\max |f''|$ in the error formula for the Trapezoidal Rule, we usually cannot find the exact value of $\max |f^{(4)}(x)|$ and have to replace it with an upper bound. If M is any upper bound for the values of $|f^{(4)}(x)|$ on $[a, b]$, then

$$|E_S| \leq \frac{b-a}{180} M (\Delta x)^4.$$

Substituting $(b-a)/n$ for Δx in this last expression gives

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}.$$

You might wonder why we don't just take M to be the maximum value of $|f''(x)|$ on $[a, b]$ for the first two rules, or the maximum value of $|f^{(4)}(x)|$ for Simpson's Rule. The reason is that sometimes this maximum is hard to compute, while a less accurate upper bound is easily found. We can certainly set M to the maximum value if we can compute it.

EXAMPLE 3 Find an upper bound for the error in estimating $\int_0^2 5x^4 dx$ using Simpson's Rule with $n = 4$ (Example 2).

Solution To estimate the error, we first find an upper bound M for the magnitude of the fourth derivative of $f(x) = 5x^4$ on the interval $0 \leq x \leq 2$. Since the fourth derivative has the constant value $f^{(4)}(x) = 120$, we take $M = 120$. With $b - a = 2$ and $n = 4$, the error estimate for Simpson's Rule gives

$$|E_s| \leq \frac{M(b-a)^5}{180n^4} = \frac{120(2)^5}{180 \cdot 4^4} = \frac{1}{12}.$$

This estimate is consistent with the result of Example 2. ■

Theorem 1 can also be used to estimate the number of subintervals required when using the Trapezoidal or Simpson's Rule if we specify a certain tolerance for the error.

EXAMPLE 4 Estimate the minimum number of subintervals needed to approximate the integral in Example 3 using Simpson's Rule with an error of magnitude less than 10^{-4} .

Solution Using the inequality in Theorem 1, if we choose the number of subintervals n to satisfy

$$\frac{M(b-a)^5}{180n^4} < 10^{-4},$$

then the error E_s in Simpson's Rule satisfies $|E_s| < 10^{-4}$, as required.

From the solution in Example 3, we have $M = 120$ and $b - a = 2$, so we want n to satisfy

$$\frac{120(2)^5}{180n^4} < \frac{1}{10^4},$$

or, equivalently,

$$n^4 > \frac{64 \cdot 10^4}{3}.$$

It follows that

$$n > 10 \left(\frac{64}{3} \right)^{1/4} \approx 21.5.$$

Since n must be even in Simpson's Rule, we estimate the minimum number of subintervals required for the error tolerance to be $n = 22$. ■

EXAMPLE 5 As we saw in Chapter 7, the value of $\ln 2$ can be calculated from the integral

$$\ln 2 = \int_1^2 \frac{1}{x} dx.$$

Table 8.4 shows values of T and S for approximations of $\int_1^2 (1/x) dx$ using various values of n . Notice how Simpson's Rule dramatically improves over the Trapezoidal Rule.

TABLE 8.4 Trapezoidal Rule approximations (T_n) and Simpson's Rule approximations (S_n) of $\ln 2 = \int_1^2 (1/x) dx$

n	T_n	Error less than ...	S_n	Error less than ...
10	0.6937714032	0.0006242227	0.6931502307	0.0000030502
20	0.6933033818	0.0001562013	0.6931473747	0.0000001942
30	0.6932166154	0.0000694349	0.6931472190	0.0000000385
40	0.6931862400	0.0000390595	0.6931471927	0.0000000122
50	0.6931721793	0.0000249988	0.6931471856	0.0000000050
100	0.6931534305	0.0000062500	0.6931471809	0.0000000004

In particular, notice that when we double the value of n (thereby halving the value of $h = \Delta x$), the T error is divided by 2 *squared*, whereas the S error is divided by 2 *to the fourth*.

This has a dramatic effect as $\Delta x = (2 - 1)/n$ gets very small. The Simpson approximation for $n = 50$ rounds accurately to seven places and for $n = 100$ is accurate to nine decimal places (billionths)! ■

If $f(x)$ is a polynomial of degree less than 4, then its fourth derivative is zero, and

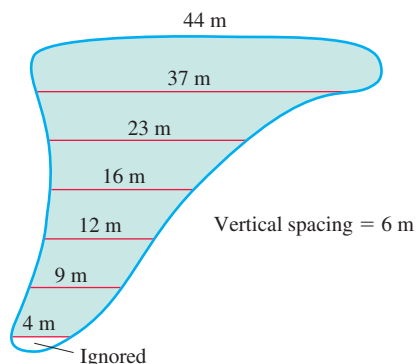
$$E_S = -\frac{b-a}{180} f^{(4)}(c) (\Delta x)^4 = -\frac{b-a}{180} (0) (\Delta x)^4 = 0.$$

Thus, there will be no error in the Simpson approximation of any integral of f . In other words, if f is a constant, a linear function, or a quadratic or cubic polynomial, Simpson's Rule will give the value of any integral of f exactly, whatever the number of subdivisions. Similarly, if f is a constant or a linear function, then its second derivative is zero, and

$$E_T = -\frac{b-a}{12} f''(c) (\Delta x)^2 = -\frac{b-a}{12} (0) (\Delta x)^2 = 0.$$

The Trapezoidal Rule will therefore give the exact value of any integral of f . This is no surprise, for the trapezoids fit the graph perfectly.

Although decreasing the step size Δx reduces the error in the Simpson and Trapezoidal approximations in theory, it may fail to do so in practice. When Δx is very small, say $\Delta x = 10^{-8}$, computer or calculator round-off errors in the arithmetic required to evaluate S and T may accumulate to such an extent that the error formulas no longer describe what is going on. Shrinking Δx below a certain size can actually make things worse. You should consult a text on numerical analysis for more sophisticated methods if you are having problems with round-off error using the rules discussed in this section.

**FIGURE 8.11** The dimensions of the swamp in Example 6.

EXAMPLE 6 A town wants to drain and fill a polluted swamp (Figure 8.11). The swamp averages 1.5 m deep. About how many cubic meters of dirt will it take to fill the area after the swamp is drained?

Solution To calculate the volume of the swamp, we estimate the surface area and multiply by 1.5. To estimate the area, we use Simpson's Rule with $\Delta x = 6$ m, and the y 's equal to the distances measured across the swamp, as shown in Figure 8.11.

$$\begin{aligned} S &= \frac{\Delta x}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6) \\ &= \frac{6}{3} (44 + 148 + 46 + 64 + 24 + 36 + 4) = 732 \end{aligned}$$

The volume is about $(732)(1.5) = 1098 \text{ m}^3$. ■

EXERCISES
8.7

For some exercises, a calculator may be helpful for expressing answers in decimal form.

Estimating Definite Integrals

The instructions for the integrals in Exercises 1–10 have three parts, one for the Midpoint Rule, one for the Trapezoidal Rule, and one for Simpson’s Rule.

I. Using the Midpoint Rule

- a. Estimate the integral with $n = 4$ steps and find an upper bound for $|E_M|$.
- b. Evaluate the integral directly and find $|E_M|$.
- c. Use the formula $(|E_M|/(\text{true value})) \times 100$ to express $|E_M|$ as a percentage of the integral’s true value.

II. Using the Trapezoidal Rule

- a. Estimate the integral with $n = 4$ steps and find an upper bound for $|E_T|$.
- b. Evaluate the integral directly and find $|E_T|$.
- c. Use the formula $(|E_T|/(\text{true value})) \times 100$ to express $|E_T|$ as a percentage of the integral’s true value.

III. Using Simpson’s Rule

- a. Estimate the integral with $n = 4$ steps and find an upper bound for $|E_S|$.
- b. Evaluate the integral directly and find $|E_S|$.
- c. Use the formula $(|E_S|/(\text{true value})) \times 100$ to express $|E_S|$ as a percentage of the integral’s true value.

1. $\int_1^2 x \, dx$
2. $\int_1^3 (2x - 1) \, dx$
3. $\int_{-1}^1 (x^2 + 1) \, dx$
4. $\int_{-2}^0 (x^2 - 1) \, dx$
5. $\int_0^2 (t^3 + t) \, dt$
6. $\int_{-1}^1 (t^3 + 1) \, dt$
7. $\int_1^2 \frac{1}{s^2} \, ds$
8. $\int_2^4 \frac{1}{(s - 1)^2} \, ds$
9. $\int_0^\pi \sin t \, dt$
10. $\int_0^1 \sin \pi t \, dt$

Estimating the Number of Subintervals

In Exercises 11–22, estimate the minimum number of subintervals needed to approximate the integrals with an error of magnitude less than 10^{-4} by (a) the Trapezoidal Rule and (b) Simpson’s Rule. (The integrals in Exercises 11–18 are the integrals from Exercises 1–8.)

11. $\int_1^2 x \, dx$
12. $\int_1^3 (2x - 1) \, dx$
13. $\int_{-1}^1 (x^2 + 1) \, dx$
14. $\int_{-2}^0 (x^2 - 1) \, dx$
15. $\int_0^2 (t^3 + t) \, dt$
16. $\int_{-1}^1 (t^3 + 1) \, dt$

17. $\int_1^2 \frac{1}{s^2} \, ds$
18. $\int_2^4 \frac{1}{(s - 1)^2} \, ds$
19. $\int_0^3 \sqrt{x + 1} \, dx$
20. $\int_0^3 \frac{1}{\sqrt{x + 1}} \, dx$
21. $\int_0^2 \sin(x + 1) \, dx$
22. $\int_{-1}^1 \cos(x + \pi) \, dx$

Estimates with Numerical Data

23. **Volume of water in a swimming pool** A rectangular swimming pool is 5 m wide and 10 m long. The accompanying table shows the depth $h(x)$ of the water at 1-m intervals from one end of the pool to the other. Estimate the volume of water in the pool using the Trapezoidal Rule with $n = 10$ applied to the integral

$$V = \int_0^{10} 5 \cdot h(x) \, dx.$$

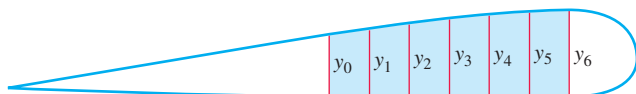
Position (m) x	Depth (m) $h(x)$	Position (m) x	Depth (m) $h(x)$
0	1.20	6	2.30
1	1.64	7	2.38
2	1.82	8	2.46
3	1.98	9	2.54
4	2.10	10	2.60
5	2.20		

24. **Distance traveled** The accompanying table shows time-to-speed data for a car accelerating from rest to 130 km/h. How far had the car traveled by the time it reached this speed? (Use trapezoids to estimate the area under the velocity curve, but be careful: The time intervals vary in length.)

Speed change	Time (s)
Zero to 30 km/h	2.2
40 km/h	3.2
50 km/h	4.5
60 km/h	5.9
70 km/h	7.8
80 km/h	10.2
90 km/h	12.7
100 km/h	16.0
110 km/h	20.6
120 km/h	26.2
130 km/h	37.1

25. **Wing design** The design of a new airplane requires a gasoline tank of constant cross-sectional area in each wing. A scale drawing of a cross-section is shown here. The tank must hold 2000 kg

of gasoline, which has a density of 673 kg/m^3 . Estimate the length of the tank by Simpson's Rule.



$$y_0 = 0.5 \text{ m}, \quad y_1 = 0.55 \text{ m}, \quad y_2 = 0.6 \text{ m}, \quad y_3 = 0.65 \text{ m}, \\ y_4 = 0.7 \text{ m}, \quad y_5 = y_6 = 0.75 \text{ m} \quad \text{Horizontal spacing} = 0.3 \text{ m}$$

- 26. Oil consumption on Pathfinder Island** A diesel generator runs continuously, consuming oil at a gradually increasing rate until it must be temporarily shut down to have the filters replaced. Use the Trapezoidal Rule to estimate the amount of oil consumed by the generator during that week.

Day	Oil consumption rate (liters/hour)
Sun	0.019
Mon	0.020
Tue	0.021
Wed	0.023
Thu	0.025
Fri	0.028
Sat	0.031
Sun	0.035

Theory and Examples

- 27. Usable values of the sine-integral function** The sine-integral function,

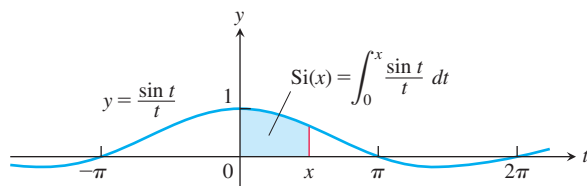
$$\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt, \quad \text{"Sine integral of } x\text{"}$$

is one of the many functions in engineering whose formulas cannot be simplified. There is no elementary formula for the antiderivative of $(\sin t)/t$. The values of $\text{Si}(x)$, however, are readily estimated by numerical integration.

Although the notation does not show it explicitly, the function being integrated is

$$f(t) = \begin{cases} \frac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0, \end{cases}$$

the continuous extension of $(\sin t)/t$ to the interval $[0, x]$. The function has derivatives of all orders at every point of its domain. Its graph is smooth, and you can expect good results from Simpson's Rule.



- a. Use the fact that $|f^{(4)}| \leq 1$ on $[0, \pi/2]$ to give an upper bound for the error that will occur if

$$\text{Si}\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{\sin t}{t} dt$$

is estimated by Simpson's Rule with $n = 4$.

- b. Estimate $\text{Si}(\pi/2)$ by Simpson's Rule with $n = 4$.
c. Express the error bound you found in part (a) as a percentage of the value you found in part (b).
- 28. The error function** The error function,

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

which is important in probability and in the theories of heat flow and signal transmission, must be evaluated numerically because there is no elementary expression for the antiderivative of e^{-t^2} .

- a. Use Simpson's Rule with $n = 10$ to estimate $\text{erf}(1)$.
b. In $[0, 1]$,

$$\left| \frac{d^4}{dt^4}(e^{-t^2}) \right| \leq 12.$$

Give an upper bound for the magnitude of the error of the estimate in part (a).

- 29.** Prove that the sum T in the Trapezoidal Rule for $\int_a^b f(x) dx$ is a Riemann sum for f continuous on $[a, b]$. (Hint: Use the Intermediate Value Theorem to show the existence of c_k in the subinterval $[x_{k-1}, x_k]$ satisfying $f(c_k) = (f(x_{k-1}) + f(x_k))/2$.)
30. Prove that the sum S in Simpson's Rule for $\int_a^b f(x) dx$ is a Riemann sum for f continuous on $[a, b]$. (See Exercise 29.)
31. Elliptic integrals The length of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

turns out to be

$$\text{Length} = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \cos^2 t} dt,$$

where $e = \sqrt{a^2 - b^2}/a$ is the ellipse's eccentricity. The integral in this formula, called an *elliptic integral*, is nonelementary except when $e = 0$ or 1 .

- a. Use the Trapezoidal Rule with $n = 10$ to estimate the length of the ellipse when $a = 1$ and $e = 1/2$.
b. Use the fact that the absolute value of the second derivative of $f(t) = \sqrt{1 - e^2 \cos^2 t}$ is less than 1 to find an upper bound for the error in the estimate you obtained in part (a).

Applications

- 32.** The length of one arch of the curve $y = \sin x$ is given by

$$L = \int_0^\pi \sqrt{1 + \cos^2 x} dx.$$

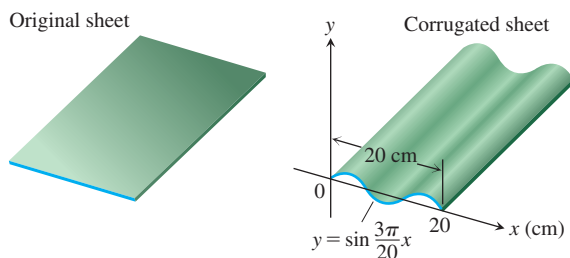
Estimate L by Simpson's Rule with $n = 8$.

T When solving Exercises 33–40, you may need to use a calculator or a computer.

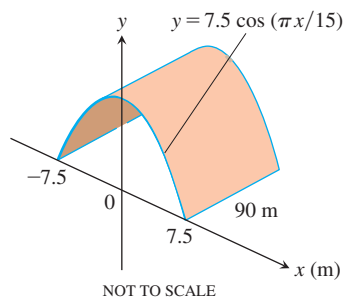
33. Your metal fabrication company is bidding for a contract to make sheets of corrugated iron roofing like the one shown here. The cross-sections of the corrugated sheets are to conform to the curve

$$y = \sin \frac{3\pi}{20}x, \quad 0 \leq x \leq 20 \text{ cm.}$$

If the roofing is to be stamped from flat sheets by a process that does not stretch the material, how wide should the original material be? To find out, use numerical integration to approximate the length of the sine curve to two decimal places.



34. Your engineering firm is bidding for the contract to construct the tunnel shown here. The tunnel is 90 m long and 15 m wide at the base. The cross-section is shaped like one arch of the curve $y = 7.5 \cos(\pi x/15)$. Upon completion, the tunnel's inside surface (excluding the roadway) will be treated with a waterproof sealer that costs \$26.11 per square meter to apply. How much will it cost to apply the sealer? (*Hint*: Use numerical integration to find the length of the cosine curve.)



Find, to two decimal places, the areas of the surfaces generated by revolving the curves in Exercises 35 and 36 about the x -axis.

35. $y = \sin x, \quad 0 \leq x \leq \pi$

36. $y = x^2/4, \quad 0 \leq x \leq 2$

37. Use numerical integration to estimate the value of

$$\arcsin 0.6 = \int_0^{0.6} \frac{dx}{\sqrt{1-x^2}}.$$

For reference, $\arcsin 0.6 = 0.64350$ to five decimal places.

38. Use numerical integration to estimate the value of

$$\pi = 4 \int_0^1 \frac{1}{1+x^2} dx.$$

39. **Drug assimilation** An average adult under age 60 years assimilates a 12-hour cold medicine into his or her system at a rate modeled by

$$\frac{dy}{dt} = 6 - \ln(2t^2 - 3t + 3),$$

where y is measured in milligrams and t is the time in hours since the medication was taken. What amount of medicine is absorbed into a person's system over a 12-hour period?

40. **Effects of an antihistamine** The concentration of an antihistamine in the bloodstream of a healthy adult is modeled by

$$C = 12.5 - 4 \ln(t^2 - 3t + 4),$$

where C is measured in grams per liter and t is the time in hours since the medication was taken. What is the average level of concentration in the bloodstream over a 6-hour period?

8.8 Improper Integrals

Up to now, we have required definite integrals to satisfy two properties. First, the domain of integration $[a, b]$ must be finite. Second, the range of the integrand must be finite on this domain. In practice, we may encounter problems that fail to meet one or both of these conditions. The integral for the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ is an example for which the domain is infinite (Figure 8.12a). The integral for the area under the curve of $y = 1/\sqrt{x}$ between $x = 0$ and $x = 1$ is an example for which the range of the integrand is infinite (Figure 8.12b). In either case, the integrals are said to be *improper* and are calculated as limits. We will see in Chapter 9 that improper integrals are useful for investigating the convergence of certain infinite series.

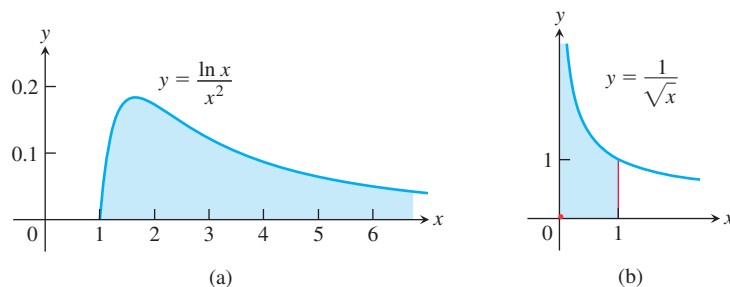


FIGURE 8.12 Are the areas under these infinite curves finite? We will see that the answer is yes for both curves.

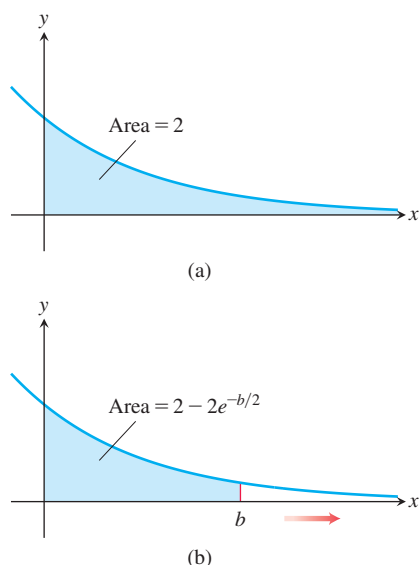


FIGURE 8.13 (a) The area in the first quadrant under the curve $y = e^{-x/2}$. (b) The area is an improper integral of the first type.

Infinite Limits of Integration

Consider the infinite region (unbounded on the right) that lies under the curve $y = e^{-x/2}$ in the first quadrant (Figure 8.13a). You might think this region has infinite area, but we will see that the value is finite. We assign a value to the area in the following way. First find the area $A(b)$ of the portion of the region that is bounded on the right by $x = b$ (Figure 8.13b).

$$A(b) = \int_0^b e^{-x/2} dx = -2e^{-x/2} \Big|_0^b = -2e^{-b/2} + 2 = 2 - 2e^{-b/2},$$

which is a little less than 2. Then find the limit of $A(b)$ as $b \rightarrow \infty$.

$$\lim_{b \rightarrow \infty} A(b) = \lim_{b \rightarrow \infty} (2 - 2e^{-b/2}) = 2$$

Therefore, the value we assign to the area under the curve from 0 to ∞ is

$$\int_0^{\infty} e^{-x/2} dx = \lim_{b \rightarrow \infty} \int_0^b e^{-x/2} dx = 2.$$

DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx,$$

where c is any real number.

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

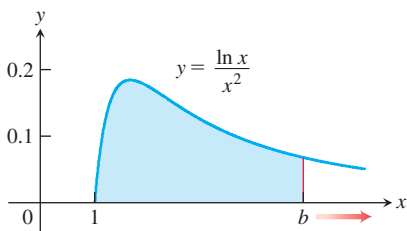


FIGURE 8.14 The area under this curve is an improper integral (Example 1).

The choice of c in Part 3 of the definition is unimportant. We can evaluate or determine the convergence or divergence of $\int_{-\infty}^{\infty} f(x) dx$ with any convenient choice.

Any of the integrals in the above definition can be interpreted as an area if $f \geq 0$ on the interval of integration. For instance, we interpreted the improper integral in Figure 8.13 as an area. In that case, the area has the finite value 2. If $f \geq 0$ and the improper integral diverges, we say the area under the curve is **infinite**.

EXAMPLE 1 Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?

Solution We find the area under the curve from $x = 1$ to $x = b$ and examine the limit as $b \rightarrow \infty$. If the limit is finite, we take it to be the area under the curve (Figure 8.14). The area from 1 to b is

$$\begin{aligned} \int_1^b \frac{\ln x}{x^2} dx &= \left[(\ln x) \left(-\frac{1}{x} \right) \right]_1^b - \int_1^b \left(-\frac{1}{x} \right) \left(\frac{1}{x} \right) dx && \text{Integration by parts with} \\ &= -\frac{\ln b}{b} - \left[\frac{1}{x} \right]_1^b && u = \ln x, dv = dx/x^2, \\ &= -\frac{\ln b}{b} - \frac{1}{b} + 1. && du = dx/x, v = -1/x \end{aligned}$$

The limit of the area as $b \rightarrow \infty$ is

$$\begin{aligned} \int_1^{\infty} \frac{\ln x}{x^2} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x^2} dx \\ &= \lim_{b \rightarrow \infty} \left[-\frac{\ln b}{b} - \frac{1}{b} + 1 \right] \\ &= - \left[\lim_{b \rightarrow \infty} \frac{\ln b}{b} \right] - 0 + 1 \\ &= - \left[\lim_{b \rightarrow \infty} \frac{1/b}{1} \right] + 1 = 0 + 1 = 1. && \text{L'Hôpital's Rule} \end{aligned}$$

Thus, the improper integral converges and the area has finite value 1. ■

EXAMPLE 2 Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

Solution According to Part 3 of the definition, we can choose $c = 0$ and write

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

$$\begin{aligned} \int_{-\infty}^0 \frac{dx}{1+x^2} &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{1+x^2} \\ &= \lim_{a \rightarrow -\infty} \left[\tan^{-1} x \right]_a^0 \\ &= \lim_{a \rightarrow -\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2} \right) = \frac{\pi}{2} \end{aligned}$$

HISTORICAL BIOGRAPHY

Lejeune Dirichlet
(1805–1859)

Dirichlet, a German mathematician, investigated the solution and equilibrium of systems of differential equations and discovered many results on the convergence of series. In 1855, Dirichlet succeeded Gauss as the professor of mathematics at Göttingen.

To know more, visit the companion Website.

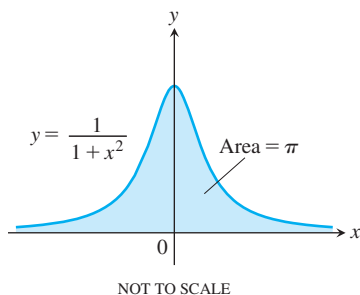


FIGURE 8.15 The area under this curve is finite (Example 2).

$$\begin{aligned}\int_0^\infty \frac{dx}{1+x^2} &= \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{1+x^2} \\ &= \lim_{b \rightarrow \infty} \tan^{-1} x \Big|_0^b \\ &= \lim_{b \rightarrow \infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}\end{aligned}$$

Thus,

$$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Since $1/(1+x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the x -axis (Figure 8.15). ■

The Integral $\int_1^\infty \frac{dx}{x^p}$

The function $y = 1/x$ is the boundary between the convergent and divergent improper integrals with integrands of the form $y = 1/x^p$. As the next example shows, the improper integral converges if $p > 1$ and diverges if $p \leq 1$.

EXAMPLE 3 For what values of p does the integral $\int_1^\infty dx/x^p$ converge? When the integral does converge, what is its value?

Solution If $p \neq 1$, then

$$\int_1^b \frac{dx}{x^p} = \left. \frac{x^{-p+1}}{-p+1} \right|_1^b = \left(\frac{1}{1-p} \right) (b^{-p+1} - 1) = \left(\frac{1}{1-p} \right) \left(\frac{1}{b^{p-1}} - 1 \right).$$

Thus,

$$\begin{aligned}\int_1^\infty \frac{dx}{x^p} &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^p} \\ &= \lim_{b \rightarrow \infty} \left[\left(\frac{1}{1-p} \right) \left(\frac{1}{b^{p-1}} - 1 \right) \right] = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}\end{aligned}$$

because

$$\lim_{b \rightarrow \infty} \frac{1}{b^{p-1}} = \begin{cases} 0, & p > 1 \\ \infty, & p < 1. \end{cases}$$

Therefore, the integral converges to the value $1/(p-1)$ if $p > 1$, and it diverges if $p < 1$.

If $p = 1$, the integral also diverges:

$$\begin{aligned}\int_1^\infty \frac{dx}{x^p} &= \int_1^\infty \frac{dx}{x} \\ &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x} \\ &= \lim_{b \rightarrow \infty} \left[\ln |x| \right]_1^b \\ &= \lim_{b \rightarrow \infty} (\ln b - \ln 1) = \infty. \quad \blacksquare\end{aligned}$$

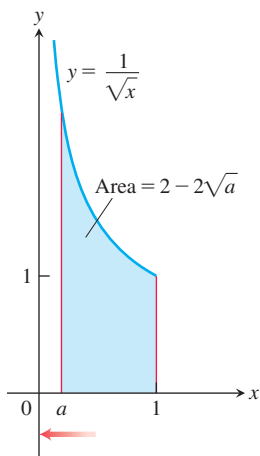


FIGURE 8.16 The area under this curve is an example of an improper integral of the second kind.

Integrands with Vertical Asymptotes

Another type of improper integral arises when the integrand has a vertical asymptote—an infinite discontinuity—at a limit of integration or at some point between the limits of integration. If the integrand f is positive over the interval of integration, we can again interpret the improper integral as the area under the graph of f and above the x -axis between the limits of integration.

Consider the region in the first quadrant that lies under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ (Figure 8.12b). First we find the area of the portion from a to 1 (Figure 8.16):

$$\int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x} \Big|_a^1 = 2 - 2\sqrt{a}.$$

Then we find the limit of this area as $a \rightarrow 0^+$:

$$\lim_{a \rightarrow 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a \rightarrow 0^+} (2 - 2\sqrt{a}) = 2.$$

Therefore, the area under the curve from 0 to 1 is finite and is defined to be

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a \rightarrow 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = 2.$$

DEFINITION Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx.$$

3. If $f(x)$ is discontinuous at c , where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

In Part 3 of the definition, the integral on the left side of the equation converges if *both* integrals on the right side converge; otherwise, it diverges.

EXAMPLE 4 Investigate the convergence of

$$\int_0^1 \frac{1}{1-x} dx.$$

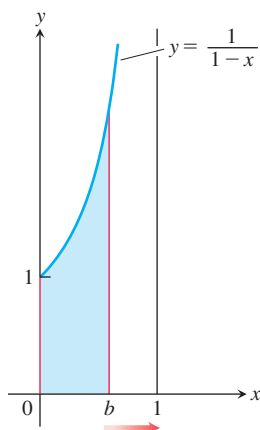


FIGURE 8.17 The area beneath the curve and above the x -axis for $[0, 1)$ is not a real number (Example 4).

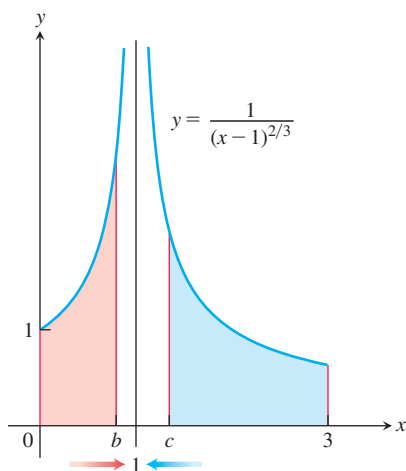


FIGURE 8.18 Example 5 shows that the area under the curve exists (so it is a real number).

Solution The integrand $f(x) = 1/(1-x)$ is continuous on $[0, 1)$ but is discontinuous at $x = 1$ and becomes infinite as $x \rightarrow 1^-$ (Figure 8.17). We evaluate the integral as

$$\begin{aligned}\lim_{b \rightarrow 1^-} \int_0^b \frac{1}{1-x} dx &= \lim_{b \rightarrow 1^-} \left[-\ln|1-x| \right]_0^b \\ &= \lim_{b \rightarrow 1^-} [-\ln(1-b) + 0] = \infty.\end{aligned}$$

The limit is infinite, so the integral diverges. ■

EXAMPLE 5 Evaluate

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}.$$

Solution The integrand has a vertical asymptote at $x = 1$ and is continuous on $[0, 1)$ and $(1, 3]$ (Figure 8.18). Thus, by Part 3 of the definition above,

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}.$$

Next, we evaluate each improper integral on the right-hand side of this equation.

$$\begin{aligned}\int_0^1 \frac{dx}{(x-1)^{2/3}} &= \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{b \rightarrow 1^-} 3(x-1)^{1/3} \Big|_0^b \\ &= \lim_{b \rightarrow 1^-} [3(b-1)^{1/3} + 3] = 3 \\ \int_1^3 \frac{dx}{(x-1)^{2/3}} &= \lim_{c \rightarrow 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{c \rightarrow 1^+} 3(x-1)^{1/3} \Big|_c^3 \\ &= \lim_{c \rightarrow 1^+} [3(3-1)^{1/3} - 3(c-1)^{1/3}] = 3\sqrt[3]{2}\end{aligned}$$

We conclude that

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}. \quad \text{■}$$

Improper Integrals with a CAS

Computer algebra systems can evaluate many convergent improper integrals. To evaluate the integral

$$\int_2^\infty \frac{x+3}{(x-1)(x^2+1)} dx$$

(which converges) using Maple, enter

$$> f := (x+3)/((x-1)*(x^2+1));$$

Then use the integration command

$$> \text{int}(f, x = 2..infinity);$$

Maple returns the answer

$$\ln(5) + \arctan(2) - \frac{\pi}{2}$$

To obtain a numerical result, use the evaluation command **evalf** and specify the number of digits as follows:

```
> evalf(%, 6);
```

The symbol % instructs the computer to evaluate the last expression on the screen, in this case $(-1/2)\pi + \ln(5) + \arctan(2)$. Maple returns 1.14579.

If you are using Mathematica, entering

```
In[1]:= Integrate[(x + 3)/((x - 1)(x^2 + 1)), {x, 2, Infinity}]
```

returns

$$\text{Out}[1] = -\frac{\pi}{2} + \text{ArcTan}[2] + \text{Log}[5].$$

To obtain a numerical result with six digits, use the command “N[%, 6]”; it also yields 1.14579.

Tests for Convergence and Divergence

When we cannot evaluate an improper integral directly, we try to determine whether it converges or diverges. If the integral diverges, that’s the end of the story. If it converges, we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

THEOREM 2—Direct Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. if $\int_a^\infty g(x) dx$ converges, then $\int_a^\infty f(x) dx$ also converges.
2. if $\int_a^\infty f(x) dx$ diverges, then $\int_a^\infty g(x) dx$ also diverges.

HISTORICAL BIOGRAPHY

Karl Weierstrass
(1815–1897)

Weierstrass attended the University of Bonn to learn public administration, but he found that his passion was for mathematics. In his Berlin lectures in the 1860s, he also proved several theorems for continuous and complex functions. The standards of rigor that he set greatly affected the future of mathematics.

To know more, visit the companion Website.

Outline of a Proof Assume that $\int_a^\infty g(x) dx$ converges. For each number $b \geq a$, let $I(b) = \int_a^b f(x) dx$. Since $f(x)$ is nonnegative, we know that $I(b)$ is an increasing function of b . Using the completeness property of the real numbers (Appendix A.7), it can be shown that since this integral increases with b , it either has a limit or it diverges to infinity as $b \rightarrow \infty$. We will not give a proof of this fact here. Since $f(x) \leq g(x)$ for every x ,

$$I(b) = \int_a^b f(x) dx \leq \int_a^b g(x) dx \leq \int_a^\infty g(x) dx.$$

Thus, the finite number $M = \int_a^\infty g(x) dx$ is an upper bound to the values of $I(b)$. Therefore $I(b)$ cannot diverge to infinity, so it must converge to a finite value as $b \rightarrow \infty$. Hence $\int_a^\infty f(x) dx$ converges. This establishes that statement 1 holds. Since statement 2 is the contrapositive form of statement 1, it must hold as well. ■

Although the theorem is stated for Type I improper integrals, a similar result is true for integrals of Type II as well.

EXAMPLE 6 These examples illustrate how we use Theorem 2.

(a) $\int_1^\infty e^{-x^2} dx$ converges because $0 < e^{-x^2} < e^{-x}$ for every $x \geq 1$ (Figure 8.19) and

$$\begin{aligned} \int_1^\infty e^{-x} dx &= \lim_{b \rightarrow \infty} \int_1^b e^{-x} dx \\ &= \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_1^b \\ &= \lim_{b \rightarrow \infty} (-e^{-b} + e^{-1}) \\ &= \frac{1}{e} \end{aligned}$$

converges.

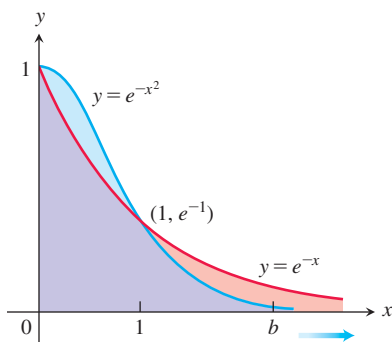


FIGURE 8.19 The graph of e^{-x^2} lies below the graph of e^{-x} for $x > 1$ (Example 6a).

(b) $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ converges because

$$0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2} \quad \text{on } [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x^2} dx \quad \text{converges.} \quad \text{Example 3}$$

(c) $\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}} dx$ diverges because

$$\frac{1}{\sqrt{x^2 - 0.1}} \geq \frac{1}{x} \quad \text{on } [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x} dx \quad \text{diverges.} \quad \text{Example 3}$$

(d) $\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}} dx$ converges because

$$0 \leq \frac{\cos x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}} \quad \text{on} \quad \left[0, \frac{\pi}{2}\right], \quad 0 \leq \cos x \leq 1 \text{ on } \left[0, \frac{\pi}{2}\right]$$

and

$$\begin{aligned} \int_0^{\pi/2} \frac{dx}{\sqrt{x}} &= \lim_{a \rightarrow 0^+} \int_a^{\pi/2} \frac{dx}{\sqrt{x}} \\ &= \lim_{a \rightarrow 0^+} \left. \sqrt{4x} \right|_a^{\pi/2} \quad 2\sqrt{x} = \sqrt{4x} \\ &= \lim_{a \rightarrow 0^+} (\sqrt{2\pi} - \sqrt{4a}) = \sqrt{2\pi} \quad \text{converges.} \quad \blacksquare \end{aligned}$$

Although we have shown that the integrals in parts (a), (b), and (d) of Example 6 converge, we do not know the exact values of these integrals. For example, our computations in part (d) imply that

$$0 \leq \int_0^{\pi/2} \frac{\cos x}{\sqrt{x}} dx \leq \int_0^{\pi/2} \frac{dx}{\sqrt{x}} = \sqrt{2\pi},$$

but unless we do further calculations the most we can say is that the integral is some real number between 0 and $\sqrt{2\pi}$.

THEOREM 3—Limit Comparison Test

If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty,$$

then

$$\int_a^\infty f(x) dx \quad \text{and} \quad \int_a^\infty g(x) dx$$

either both converge or both diverge.

We omit the proof of Theorem 3, which is similar to that of Theorem 2.

If two functions $f(x)$ and $g(x)$ satisfy the hypotheses of Theorem 3 and their improper integrals both converge, it need not be the case that these two integrals have the same value. This is illustrated in the next example.

EXAMPLE 7 Show that

$$\int_1^\infty \frac{dx}{1+x^2}$$

converges by comparison with $\int_1^\infty (1/x^2) dx$. Find and compare the values of the two integrals.

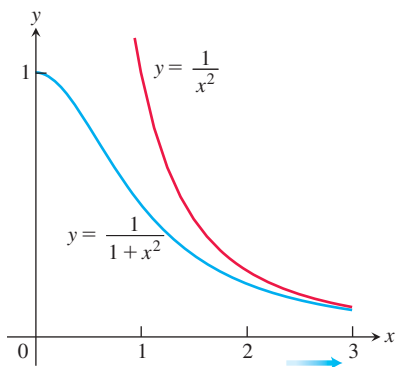


FIGURE 8.20 The functions in Example 7.

Solution The functions $f(x) = 1/x^2$ and $g(x) = 1/(1+x^2)$ are positive and continuous on $[1, \infty)$. Also,

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow \infty} \frac{1/x^2}{1/(1+x^2)} = \lim_{x \rightarrow \infty} \frac{1+x^2}{x^2} \\ &= \lim_{x \rightarrow \infty} \left(\frac{1}{x^2} + 1 \right) = 0 + 1 = 1,\end{aligned}$$

which is a positive finite limit (Figure 8.20). Therefore, $\int_1^\infty \frac{dx}{1+x^2}$ converges because $\int_1^\infty \frac{dx}{x^2}$ converges.

The integrals converge to different values, however:

$$\int_1^\infty \frac{dx}{x^2} = \frac{1}{2-1} = 1 \quad \text{Example 3}$$

and

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{1+x^2} = \lim_{b \rightarrow \infty} [\tan^{-1} b - \tan^{-1} 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}. \quad \blacksquare$$

TABLE 8.5

b	$\int_1^b \frac{1-e^{-x}}{x} dx$
2	0.5226637569
5	1.3912002736
10	2.0832053156
100	4.3857862516
1000	6.6883713446
10000	8.9909564376
100000	11.2935415306

EXAMPLE 8 Investigate the convergence of $\int_1^\infty \frac{1-e^{-x}}{x} dx$.

Solution The integrand suggests a comparison of $f(x) = (1-e^{-x})/x$ with $g(x) = 1/x$. However, we cannot use the Direct Comparison Test because $f(x) \leq g(x)$ and the integral of $g(x)$ diverges. On the other hand, using the Limit Comparison Test, we find that

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \left(\frac{1-e^{-x}}{x} \right) \left(\frac{x}{1} \right) = \lim_{x \rightarrow \infty} (1-e^{-x}) = 1,$$

which is a positive finite limit. Therefore, $\int_1^\infty \frac{1-e^{-x}}{x} dx$ diverges because $\int_1^\infty \frac{dx}{x}$ diverges.

Approximations to the improper integral are given in Table 8.5. Note that the values of these approximations do not appear to approach a fixed finite limit as $b \rightarrow \infty$. ■

EXERCISES 8.8

Evaluating Improper Integrals

The integrals in Exercises 1–34 converge. Evaluate the integrals without using tables.

1. $\int_0^\infty \frac{dx}{x^2+1}$

2. $\int_1^\infty \frac{dx}{x^{1.001}}$

13. $\int_{-\infty}^\infty \frac{2x dx}{(x^2+1)^2}$

14. $\int_{-\infty}^\infty \frac{x dx}{(x^2+4)^{3/2}}$

3. $\int_0^1 \frac{dx}{\sqrt{x}}$

4. $\int_0^4 \frac{dx}{\sqrt{4-x}}$

15. $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}} d\theta$

16. $\int_0^2 \frac{s+1}{\sqrt{4-s^2}} ds$

5. $\int_{-1}^1 \frac{dx}{x^{2/3}}$

6. $\int_{-8}^1 \frac{dx}{x^{1/3}}$

17. $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$

18. $\int_1^\infty \frac{1}{x\sqrt{x^2-1}} dx$

7. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

8. $\int_0^1 \frac{dr}{r^{0.999}}$

19. $\int_0^\infty \frac{dv}{(1+v^2)(1+\tan^{-1}v)}$

20. $\int_0^\infty \frac{16 \tan^{-1} x}{1+x^2} dx$

9. $\int_{-\infty}^{-2} \frac{2 dx}{x^2-1}$

10. $\int_{-\infty}^2 \frac{2 dx}{x^2+4}$

23. $\int_{-\infty}^0 e^{-|x|} dx$

24. $\int_{-\infty}^\infty 2x e^{-x^2} dx$

11. $\int_2^\infty \frac{2}{v^2-v} dv$

12. $\int_2^\infty \frac{2 dt}{t^2-1}$

25. $\int_0^1 x \ln x dx$

26. $\int_0^1 (-\ln x) dx$

27. $\int_0^2 \frac{ds}{\sqrt{4-s^2}}$

28. $\int_0^1 \frac{4r \, dr}{\sqrt{1-r^4}}$

29. $\int_1^2 \frac{ds}{s\sqrt{s^2-1}}$

30. $\int_2^4 \frac{dt}{t\sqrt{t^2-4}}$

31. $\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$

32. $\int_0^2 \frac{dx}{\sqrt{|x-1|}}$

33. $\int_{-1}^{\infty} \frac{d\theta}{\theta^2 + 5\theta + 6}$

34. $\int_0^{\infty} \frac{dx}{(x+1)(x^2+1)}$

Testing for Convergence

In Exercises 35–68, use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. If more than one method applies, use whatever method you prefer.

35. $\int_{1/2}^2 \frac{dx}{x \ln x}$

36. $\int_{-1}^1 \frac{d\theta}{\theta^2 - 2\theta}$

37. $\int_{1/2}^{\infty} \frac{dx}{x(\ln x)^3}$

38. $\int_0^{\infty} \frac{d\theta}{\theta^2 - 1}$

39. $\int_0^{\pi/2} \tan \theta \, d\theta$

40. $\int_0^{\pi/2} \cot \theta \, d\theta$

41. $\int_0^1 \frac{\ln x}{x^2} dx$

42. $\int_1^2 \frac{dx}{x \ln x}$

43. $\int_0^{\ln 2} x^{-2} e^{-1/x} dx$

44. $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$

45. $\int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t}$

46. $\int_0^1 \frac{dt}{t - \sin t}$
(Hint: $t \geq \sin t$ for $t \geq 0$)

47. $\int_0^2 \frac{dx}{1-x^2}$

48. $\int_0^2 \frac{dx}{1-x}$

49. $\int_{-1}^1 \ln |x| dx$

50. $\int_{-1}^1 -x \ln |x| dx$

51. $\int_1^{\infty} \frac{dx}{x^3 + 1}$

52. $\int_4^{\infty} \frac{dx}{\sqrt{x} - 1}$

53. $\int_2^{\infty} \frac{dv}{\sqrt{v} - 1}$

54. $\int_0^{\infty} \frac{d\theta}{1 + e^{\theta}}$

55. $\int_0^{\infty} \frac{dx}{\sqrt{x^6 + 1}}$

56. $\int_2^{\infty} \frac{dx}{\sqrt{x^2 - 1}}$

57. $\int_1^{\infty} \frac{\sqrt{x+1}}{x^2} dx$

58. $\int_2^{\infty} \frac{x \, dx}{\sqrt{x^4 - 1}}$

59. $\int_{\pi}^{\infty} \frac{2 + \cos x}{x} dx$

60. $\int_{\pi}^{\infty} \frac{1 + \sin x}{x^2} dx$

61. $\int_4^{\infty} \frac{2 \, dt}{t^{3/2} - 1}$

62. $\int_2^{\infty} \frac{1}{\ln x} dx$

63. $\int_1^{\infty} \frac{e^x}{x} dx$

64. $\int_{e^e}^{\infty} \ln(\ln x) dx$

65. $\int_1^{\infty} \frac{1}{\sqrt{e^x - x}} dx$

66. $\int_1^{\infty} \frac{1}{e^x - 2^x} dx$

67. $\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4 + 1}}$

68. $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}}$

In Exercises 69–80, determine whether the improper integral converges or diverges. If it converges, evaluate the integral.

69. $\int_0^1 \frac{1}{x\sqrt{x}} dx$

70. $\int_2^{\infty} \frac{1}{x\sqrt{x}} dx$

71. $\int_0^{32} \frac{1}{\sqrt[3]{x}} dx$

72. $\int_1^{\infty} \frac{1}{\sqrt[3]{x}} dx$

73. $\int_3^{\infty} \frac{1}{x^4} dx$

74. $\int_{-2}^1 \frac{1}{x^4} dx$

75. $\int_0^{\infty} x^2 e^{x^3} dx$

76. $\int_{-\infty}^0 x^2 e^{x^3} dx$

77. $\int_{-3}^0 \frac{1}{x^2 + 3x} dx$

78. $\int_1^{\infty} \frac{1}{x^2 + 3x} dx$

79. $\int_{-\infty}^4 \frac{x}{(x^2 + 9)^{5/2}} dx$

80. $\int_{-\infty}^4 \frac{x}{(x^2 + 9)^{2/5}} dx$

Theory and Examples

81. Find the values of p for which each integral converges.

a. $\int_1^2 \frac{dx}{x(\ln x)^p}$

b. $\int_2^{\infty} \frac{dx}{x(\ln x)^p}$

82. $\int_{-\infty}^{\infty} f(x) dx$ may not equal $\lim_{b \rightarrow \infty} \int_{-b}^b f(x) dx$. Show that

$$\int_0^{\infty} \frac{2x \, dx}{x^2 + 1}$$

diverges and hence that

$$\int_{-\infty}^{\infty} \frac{2x \, dx}{x^2 + 1}$$

diverges. Then show that

$$\lim_{b \rightarrow \infty} \int_{-b}^b \frac{2x \, dx}{x^2 + 1} = 0.$$

Exercises 83–86 are about the infinite region in the first quadrant between the curve $y = e^{-x}$ and the x -axis.

83. Find the area of the region.

84. Find the centroid of the region.

85. Find the volume of the solid generated by revolving the region about the y -axis.

86. Find the volume of the solid generated by revolving the region about the x -axis.

87. Find the area of the region that lies between the curves $y = \sec x$ and $y = \tan x$ from $x = 0$ to $x = \pi/2$.

88. The region in Exercise 87 is revolved about the x -axis to generate a solid.

a. Find the volume of the solid.

b. Show that the inner and outer surfaces of the solid have infinite area.

89. Consider the infinite region in the first quadrant bounded by the graphs of $y = \frac{1}{x^2}$, $y = 0$, and $x = 1$.

- Find the area of the region.
- Find the volume of the solid formed by revolving the region (i) about the x -axis; (ii) about the y -axis.

90. Consider the infinite region in the first quadrant bounded by the graphs of $y = \frac{1}{\sqrt{x}}$, $y = 0$, $x = 0$, and $x = 1$.

- Find the area of the region.
- Find the volume of the solid formed by revolving the region (i) about the x -axis; (ii) about the y -axis.

91. Evaluate the integrals.

$$\text{a. } \int_0^1 \frac{dt}{\sqrt{t}(1+t)} \qquad \text{b. } \int_0^\infty \frac{dt}{\sqrt{t}(1+t)}$$

92. Evaluate $\int_3^\infty \frac{dx}{x\sqrt{x^2-9}}$.

93. Estimating the value of a convergent improper integral whose domain is infinite

- a. Show that

$$\int_3^\infty e^{-3x} dx = \frac{1}{3}e^{-9} < 0.000042,$$

and hence that $\int_3^\infty e^{-x^2} dx < 0.000042$. Explain why this means that $\int_0^\infty e^{-x^2} dx$ can be replaced by $\int_0^3 e^{-x^2} dx$ without introducing an error of magnitude greater than 0.000042.

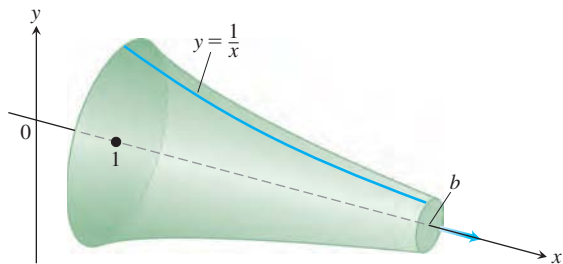
- T** b. Evaluate $\int_0^3 e^{-x^2} dx$ numerically.

94. **The infinite paint can or Gabriel's horn** As Example 3 shows, the integral $\int_1^\infty (dx/x)$ diverges. This means that the integral

$$\int_1^\infty 2\pi \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx,$$

which measures the *surface area* of the solid of revolution traced out by revolving the curve $y = 1/x$, $1 \leq x$, about the x -axis, diverges also. By comparing the two integrals, we see that, for every finite value $b > 1$,

$$\int_1^b 2\pi \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx > 2\pi \int_1^b \frac{1}{x} dx.$$



However, the integral

$$\int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx$$

for the *volume* of the solid converges.

- Calculate it.
- This solid of revolution is sometimes described as a can that does not hold enough paint to cover its own interior. Think about that for a moment. It is common sense that a finite

amount of paint cannot cover an infinite surface. But if we fill the horn with paint (a finite amount), then we *will* have covered an infinite surface. Explain the apparent contradiction.

95. **Sine-integral function** The integral

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt,$$

called the *sine-integral function*, has important applications in optics.

- T** a. Plot the integrand $(\sin t)/t$ for $t > 0$. Is the sine-integral function everywhere increasing or decreasing? Do you think $\text{Si}(x) = 0$ for $x \geq 0$? Check your answers by graphing the function $\text{Si}(x)$ for $0 \leq x \leq 25$.

- b. Explore the convergence of

$$\int_0^\infty \frac{\sin t}{t} dt.$$

If it converges, what is its value?

96. **Error function** The function

$$\text{erf}(x) = \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}} dt,$$

called the *error function*, has important applications in probability and statistics.

- T** a. Plot the error function for $0 \leq x \leq 25$.

- b. Explore the convergence of

$$\int_0^\infty \frac{2e^{-t^2}}{\sqrt{\pi}} dt.$$

If it converges, what appears to be its value? You will see how to confirm your estimate in Section 14.4, Exercise 41.

97. **Normal probability distribution** The function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

is called the *normal probability density function* with mean μ and standard deviation σ . The number μ tells where the distribution is centered, and σ measures the “scatter” around the mean.

From the theory of probability, it is known that

$$\int_{-\infty}^\infty f(x) dx = 1.$$

In what follows, let $\mu = 0$ and $\sigma = 1$.

- T** a. Draw the graph of f . Find the intervals on which f is increasing, the intervals on which f is decreasing, and any local extreme values and where they occur.

- b. Evaluate

$$\int_{-n}^n f(x) dx$$

for $n = 1, 2$, and 3 .

- c. Give a convincing argument that

$$\int_{-\infty}^\infty f(x) dx = 1.$$

(Hint: Show that $0 < f(x) < e^{-x^2/2}$ for $x > 1$, and for $b > 1$,

$$\int_b^\infty e^{-x^2/2} dx \rightarrow 0 \quad \text{as } b \rightarrow \infty.)$$

98. Show that if $f(x)$ is integrable on every interval of real numbers, and if a and b are real numbers with $a < b$, then

a. $\int_{-\infty}^a f(x) dx$ and $\int_a^{\infty} f(x) dx$ both converge if and only if $\int_{-\infty}^b f(x) dx$ and $\int_b^{\infty} f(x) dx$ both converge.

b. $\int_{-\infty}^a f(x) dx + \int_a^{\infty} f(x) dx = \int_{-\infty}^b f(x) dx + \int_b^{\infty} f(x) dx$ when the integrals involved converge.

COMPUTER EXPLORATIONS

In Exercises 99–102, use a CAS to explore the integrals for various values of p (include noninteger values). For what values of p does the

integral converge? What is the value of the integral when it does converge? Plot the integrand for various values of p .

99. $\int_0^e x^p \ln x dx$

100. $\int_e^{\infty} x^p \ln x dx$

101. $\int_0^{\infty} x^p \ln x dx$

102. $\int_{-\infty}^{\infty} x^p \ln |x| dx$

Use a CAS to evaluate the integrals.

103. $\int_0^{2/\pi} \sin \frac{1}{x} dx$

104. $\int_0^{2/\pi} x \sin \frac{1}{x} dx$

CHAPTER 8 Questions to Guide Your Review

- What is the formula for integration by parts? Where does it come from? Why might you want to use it?
- When applying the formula for integration by parts, how do you choose the u and dv ? How can you apply integration by parts to an integral of the form $\int f(x) dx$?
- If an integrand is a product of the form $\sin^n x \cos^m x$, where m and n are nonnegative integers, how do you evaluate the integral? Give a specific example of each case.
- What substitutions are made to evaluate integrals of $\sin mx \sin nx$, $\sin mx \cos nx$, and $\cos mx \cos nx$? Give an example of each case.
- What substitutions are sometimes used to transform integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$ into integrals that can be evaluated directly? Give an example of each case.
- What restrictions can you place on the variables involved in the three basic trigonometric substitutions to make sure the substitutions are reversible (have inverses)?
- What is the goal of the method of partial fractions?
- When the degree of a polynomial $f(x)$ is less than the degree of a polynomial $g(x)$, how do you write $f(x)/g(x)$ as a sum of partial fractions if $g(x)$
 - is a product of distinct linear factors?
 - consists of a repeated linear factor?
 - contains an irreducible quadratic factor?

What do you do if the degree of f is *not* less than the degree of g ?
- How are integral tables typically used? What do you do if a particular integral you want to evaluate is not listed in the table?
- What is a reduction formula? How are reduction formulas used? Give an example.
- How would you compare the relative merits of the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule?
- What is an improper integral of Type I? Type II? How are the values of various types of improper integrals defined? Give examples.
- What tests are available for determining the convergence and divergence of improper integrals that cannot be evaluated directly? Give examples of their use.
- What is a random variable? What is a continuous random variable? Give some specific examples.
- What is a probability density function? What is the probability that a continuous random variable has a value in the interval $[c, d]$?
- What is an exponentially decreasing probability density function? What are some typical events that might be modeled by this distribution? What do we mean when we say such distributions are *memoryless*?
- What is the expected value of a continuous random variable? What is the expected value of an exponentially distributed random variable?
- What is the median of a continuous random variable? What is the median of an exponential distribution?
- What does the variance of a random variable measure? What is the standard deviation of a continuous random variable X ?
- What probability density function describes the normal distribution? What are some examples typically modeled by a normal distribution? How do we usually calculate probabilities for a normal distribution?
- In a normal distribution, what percentage of the population lies within 1 standard deviation of the mean? Within 2 standard deviations?

CHAPTER 8 Practice Exercises

Integration by Parts

Evaluate the integrals in Exercises 1–8 using integration by parts.

1. $\int \ln(x+1) dx$
2. $\int x^2 \ln x dx$
3. $\int \arctan 3x dx$
4. $\int \cos^{-1}\left(\frac{x}{2}\right) dx$
5. $\int (x+1)^2 e^x dx$
6. $\int x^2 \sin(1-x) dx$
7. $\int e^x \cos 2x dx$
8. $\int x \sin x \cos x dx$

Partial Fractions

Evaluate the integrals in Exercises 9–28. It may be necessary to use a substitution first.

9. $\int \frac{x dx}{x^2 - 3x + 2}$
10. $\int \frac{x dx}{x^2 + 4x + 3}$
11. $\int \frac{dx}{x(x+1)^2}$
12. $\int \frac{x+1}{x^2(x-1)} dx$
13. $\int \frac{\sin \theta d\theta}{\cos^2 \theta + \cos \theta - 2}$
14. $\int \frac{\cos \theta d\theta}{\sin^2 \theta + \sin \theta - 6}$
15. $\int \frac{3x^2 + 4x + 4}{x^3 + x} dx$
16. $\int \frac{4x dx}{x^3 + 4x}$
17. $\int \frac{v+3}{2v^3 - 8v} dv$
18. $\int \frac{(3v-7) dv}{(v-1)(v-2)(v-3)}$
19. $\int \frac{dt}{t^4 + 4t^2 + 3}$
20. $\int \frac{t dt}{t^4 - t^2 - 2}$
21. $\int \frac{x^3 + x^2}{x^2 + x - 2} dx$
22. $\int \frac{x^3 + 1}{x^3 - x} dx$
23. $\int \frac{x^3 + 4x^2}{x^2 + 4x + 3} dx$
24. $\int \frac{2x^3 + x^2 - 21x + 24}{x^2 + 2x - 8} dx$
25. $\int \frac{dx}{x(3\sqrt{x} + 1)}$
26. $\int \frac{dx}{x(1 + \sqrt[3]{x})}$
27. $\int \frac{ds}{e^s - 1}$
28. $\int \frac{ds}{\sqrt{e^s + 1}}$

Trigonometric Substitutions

Evaluate the integrals in Exercises 29–32 (a) without using a trigonometric substitution, (b) using a trigonometric substitution.

29. $\int \frac{y dy}{\sqrt{16 - y^2}}$
30. $\int \frac{x dx}{\sqrt{4 + x^2}}$
31. $\int \frac{x dx}{4 - x^2}$
32. $\int \frac{t dt}{\sqrt{4t^2 - 1}}$

Evaluate the integrals in Exercises 33–36.

33. $\int \frac{x dx}{9 - x^2}$
34. $\int \frac{dx}{x(9 - x^2)}$
35. $\int \frac{dx}{9 - x^2}$
36. $\int \frac{dx}{\sqrt{9 - x^2}}$

Trigonometric Integrals

Evaluate the integrals in Exercises 37–44.

37. $\int \sin^3 x \cos^4 x dx$
38. $\int \cos^5 x \sin^5 x dx$
39. $\int \tan^4 x \sec^2 x dx$
40. $\int \tan^3 x \sec^3 x dx$
41. $\int \sin 5\theta \cos 6\theta d\theta$
42. $\int \sec^2 \theta \sin^3 \theta d\theta$
43. $\int \sqrt{1 + \cos(t/2)} dt$
44. $\int e^t \sqrt{\tan^2 e^t + 1} dt$

Numerical Integration

45. According to the error-bound formula for Simpson's Rule, how many subintervals should you use to be sure of estimating the value of

$$\ln 3 = \int_1^3 \frac{1}{x} dx$$

by Simpson's Rule with an error of no more than 10^{-4} in absolute value? (Remember that for Simpson's Rule, the number of subintervals has to be even.)46. A brief calculation shows that if $0 \leq x \leq 1$, then the second derivative of $f(x) = \sqrt{1+x^4}$ lies between 0 and 8. Based on this, about how many subdivisions would you need to estimate the integral of f from 0 to 1 with an error no greater than 10^{-3} in absolute value using the Trapezoidal Rule?

47. A direct calculation shows that

$$\int_0^\pi 2 \sin^2 x dx = \pi.$$

How close do you come to this value by using the Trapezoidal Rule with $n = 6$? The Midpoint Rule with $n = 6$? Simpson's Rule with $n = 6$? Try them and find out.

48. You are planning to use Simpson's Rule to estimate the value of the integral

$$\int_1^2 f(x) dx$$

with an error magnitude less than 10^{-5} . You have determined that $|f^{(4)}(x)| \leq 3$ throughout the interval of integration. How many subintervals should you use to ensure the required accuracy? (Remember that for Simpson's Rule, the number has to be even.)**T** 49. **Mean temperature** Compute the average value of the temperature function

$$f(x) = 20 \sin\left(\frac{2\pi}{365}(x - 101)\right) - 4$$

for a 365-day year. This is one way to estimate the annual mean air temperature in Fairbanks, Alaska. The National Weather Service's official figure, a numerical average of the daily normal mean air temperatures for the year, is -3.5°C , which is slightly higher than the average value of $f(x)$.50. **Heat capacity of a gas** Heat capacity C_v is the amount of heat required to raise the temperature of a given mass of gas with constant volume by 1°C , measured in units of cal/deg-mol (calories per degree gram molecular weight). The heat capacity of oxygen depends on its temperature T and satisfies the formula

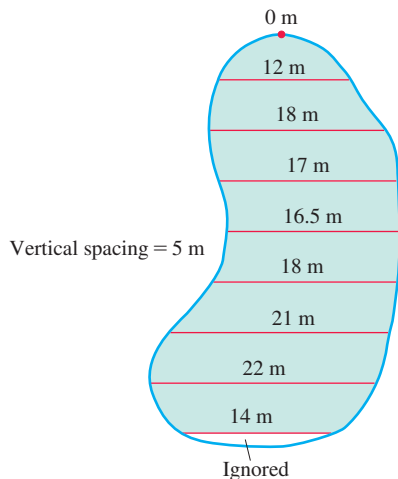
$$C_v = 8.27 + 10^{-5}(26T - 1.87T^2).$$

Use Simpson's Rule to find the average value of C_v and the temperature at which it is attained for $20^\circ\text{C} \leq T \leq 675^\circ\text{C}$.

- 51. Fuel efficiency** An automobile computer gives a digital readout of fuel consumption in liters per hour. During a trip, a passenger recorded the fuel consumption every 5 min for a full hour of travel.

Time	L/h	Time	L/h
0	2.5	35	2.5
5	2.4	40	2.4
10	2.3	45	2.3
15	2.4	50	2.4
20	2.4	55	2.4
25	2.5	60	2.3
30	2.6		

- a. Use the Trapezoidal Rule to approximate the total fuel consumption during the hour.
- b. If the automobile covered 60 km in the hour, what was its fuel efficiency (in kilometers per liter) for that portion of the trip?
- 52. A new parking lot** To meet the demand for parking, your town has allocated the area shown here. As the town engineer, you have been asked by the town council to find out if the lot can be built for \$11,000. The cost to clear the land will be \$1.00 a square meter, and the lot will cost \$20.00 a square meter to pave. Use Simpson's Rule to find out if the job can be done for \$11,000.



Improper Integrals

Evaluate the improper integrals in Exercises 53–62.

53. $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$
54. $\int_0^1 \ln x \, dx$
55. $\int_0^2 \frac{dy}{(y-1)^{2/3}}$
56. $\int_{-2}^0 \frac{d\theta}{(\theta+1)^{3/5}}$
57. $\int_3^\infty \frac{2 \, du}{u^2 - 2u}$
58. $\int_1^\infty \frac{3v-1}{4v^3 - v^2} \, dv$
59. $\int_0^\infty x^2 e^{-x} \, dx$
60. $\int_{-\infty}^0 x e^{3x} \, dx$
61. $\int_{-\infty}^\infty \frac{dx}{4x^2 + 9}$
62. $\int_{-\infty}^\infty \frac{4 \, dx}{x^2 + 16}$

Which of the improper integrals in Exercises 63–68 converge and which diverge?

63. $\int_6^\infty \frac{d\theta}{\sqrt{\theta^2 + 1}}$
64. $\int_0^\infty e^{-u} \cos u \, du$
65. $\int_1^\infty \frac{\ln z}{z} \, dz$
66. $\int_1^\infty \frac{e^{-t}}{\sqrt{t}} \, dt$
67. $\int_{-\infty}^\infty \frac{2 \, dx}{e^x + e^{-x}}$
68. $\int_{-\infty}^\infty \frac{dx}{x^2(1+e^x)}$

Assorted Integrations

Evaluate the integrals in Exercises 69–134. The integrals are listed in random order so you need to decide which integration technique to use.

69. $\int x e^{2x} \, dx$
70. $\int_0^1 x^2 e^{x^3} \, dx$
71. $\int (\tan^2 x + \sec^2 x) \, dx$
72. $\int_0^{\pi/4} \cos^2 2x \, dx$
73. $\int x \sec^2 x \, dx$
74. $\int x \sec^2(x^2) \, dx$
75. $\int \sin x \cos^2 x \, dx$
76. $\int \sin 2x \sin(\cos 2x) \, dx$
77. $\int_{-1}^0 \frac{e^x}{x^4 + e^{-x}} \, dx$
78. $\int (e^{2x} + e^{-x})^2 \, dx$
79. $\int \frac{x+1}{x^4 - x^3} \, dx$
80. $\int \frac{e^x + 1}{e^x(e^{2x} - 4)} \, dx$
81. $\int \frac{e^x + e^{3x}}{e^{2x}} \, dx$
82. $\int (e^x - e^{-x})(e^x + e^{-x})^3 \, dx$
83. $\int_0^{\pi/3} \tan^3 x \sec^2 x \, dx$
84. $\int \tan^4 x \sec^4 x \, dx$
85. $\int_0^3 (x+2)\sqrt{x+1} \, dx$
86. $\int (x+1)\sqrt{x^2 + 2x} \, dx$
87. $\int \cot x \csc^3 x \, dx$
88. $\int \sin x (\tan x - \cot x)^2 \, dx$
89. $\int \frac{x \, dx}{1 + \sqrt{x}}$
90. $\int \frac{x^3 + 2}{4 - x^2} \, dx$
91. $\int \sqrt{2x - x^2} \, dx$
92. $\int \frac{dx}{\sqrt{-2x - x^2}}$
93. $\int \frac{2 - \cos x + \sin x}{\sin^2 x} \, dx$
94. $\int \sin^2 \theta \cos^5 \theta \, d\theta$
95. $\int \frac{9 \, dv}{81 - v^4}$
96. $\int_2^\infty \frac{dx}{(x-1)^2}$
97. $\int \theta \cos(2\theta + 1) \, d\theta$
98. $\int \frac{x^3 \, dx}{x^2 - 2x + 1}$
99. $\int \frac{\sin 2\theta \, d\theta}{(1 + \cos 2\theta)^2}$
100. $\int_{\pi/4}^{\pi/2} \sqrt{1 + \cos 4x} \, dx$
101. $\int \frac{x \, dx}{\sqrt{2-x}}$
102. $\int \frac{\sqrt{1-v^2}}{v^2} \, dv$

103. $\int \frac{dy}{y^2 - 2y + 2}$

104. $\int \frac{x \, dx}{\sqrt{8 - 2x^2 - x^4}}$

105. $\int \frac{z + 1}{z^2(z^2 + 4)} \, dz$

106. $\int x^2(x - 1)^{1/3} \, dx$

107. $\int \frac{t \, dt}{\sqrt{9 - 4t^2}}$

108. $\int \frac{\arctan x}{x^2} \, dx$

109. $\int \frac{e^t \, dt}{e^{2t} + 3e^t + 2}$

110. $\int \tan^3 t \, dt$

111. $\int_1^\infty \frac{\ln y}{y^3} \, dy$

112. $\int y^{3/2}(\ln y)^2 \, dy$

113. $\int e^{\ln \sqrt{x}} \, dx$

114. $\int e^\theta \sqrt{3 + 4e^\theta} \, d\theta$

115. $\int \frac{\sin 5t \, dt}{1 + (\cos 5t)^2}$

116. $\int \frac{dv}{\sqrt{e^{2v} - 1}}$

117. $\int \frac{dr}{1 + \sqrt{r}}$

118. $\int \frac{4x^3 - 20x}{x^4 - 10x^2 + 9} \, dx$

119. $\int \frac{x^3}{1 + x^2} \, dx$

120. $\int \frac{x^2}{1 + x^3} \, dx$

121. $\int \frac{1 + x^2}{1 + x^3} \, dx$

122. $\int \frac{1 + x^2}{(1 + x)^3} \, dx$

123. $\int \sqrt{x} \cdot \sqrt{1 + \sqrt{x}} \, dx$

124. $\int \sqrt{1 + \sqrt{1 + x}} \, dx$

125. $\int \frac{1}{\sqrt{x} \cdot \sqrt{1 + x}} \, dx$

126. $\int_0^{1/2} \sqrt{1 + \sqrt{1 - x^2}} \, dx$

127. $\int \frac{\ln x}{x + x \ln x} \, dx$

128. $\int \frac{1}{x \cdot \ln x \cdot \ln(\ln x)} \, dx$

129. $\int \frac{x^{\ln x} \ln x}{x} \, dx$

130. $\int (\ln x)^{\ln x} \left[\frac{1}{x} + \frac{\ln(\ln x)}{x} \right] \, dx$

131. $\int \frac{1}{x\sqrt{1 - x^4}} \, dx$

132. $\int \frac{\sqrt{1 - x}}{x} \, dx$

133. $\int \frac{\sin^2 x}{1 + \sin^2 x} \, dx$

134. $\int \frac{1 - \cos x}{1 + \cos x} \, dx$

135. Evaluate $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} \, dx$ in two ways:

a. By evaluating $\int \frac{\sin x}{\sin x + \cos x} \, dx$, then using the Evaluation Theorem.

b. By showing that $\int_0^a f(x) \, dx = \int_0^a f(a - x) \, dx$, then using this result.

CHAPTER 8 Additional and Advanced Exercises

Evaluating Integrals

Evaluate the integrals in Exercises 1–6.

1. $\int (\arcsin x)^2 \, dx$

2. $\int \frac{dx}{x(x+1)(x+2)\cdots(x+m)}$

3. $\int x \arcsin x \, dx$

4. $\int \sin^{-1} \sqrt{y} \, dy$

5. $\int \frac{dt}{t - \sqrt{1 - t^2}}$

6. $\int \frac{dx}{x^4 + 4}$

Evaluate the limits in Exercise 7 and 8.

7. $\lim_{x \rightarrow \infty} \int_{-x}^x \sin t \, dt$

8. $\lim_{x \rightarrow 0^+} x \int_x^1 \frac{\cos t}{t^2} \, dt$

Evaluate the limits in Exercise 9 and 10 by identifying them with definite integrals and evaluating the integrals.

9. $\lim_{n \rightarrow \infty} \sum_{k=1}^n \ln \sqrt[n]{1 + \frac{k}{n}}$

10. $\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \frac{1}{\sqrt{n^2 - k^2}}$

Applications

11. **Finding arc length** Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t} \, dt, \quad 0 \leq x \leq \pi/4.$$

12. **Finding arc length** Find the length of the graph of the function $y = \ln(1 - x^2)$, $0 \leq x \leq 1/2$.

13. **Finding volume** The region in the first quadrant that is enclosed by the x -axis and the curve $y = 3x\sqrt{1 - x}$ is revolved about the y -axis to generate a solid. Find the volume of the solid.

14. **Finding volume** The region in the first quadrant that is enclosed by the x -axis, the curve $y = 5/(x\sqrt{5 - x})$, and the lines $x = 1$ and $x = 4$ is revolved about the x -axis to generate a solid. Find the volume of the solid.

15. **Finding volume** The region in the first quadrant enclosed by the coordinate axes, the curve $y = e^x$, and the line $x = 1$ is revolved about the y -axis to generate a solid. Find the volume of the solid.

16. **Finding volume** The region in the first quadrant that is bounded above by the curve $y = e^x - 1$, below by the x -axis, and on the right by the line $x = \ln 2$ is revolved about the line $x = \ln 2$ to generate a solid. Find the volume of the solid.

17. **Finding volume** Let R be the “triangular” region in the first quadrant that is bounded above by the line $y = 1$, below by the curve $y = \ln x$, and on the left by the line $x = 1$. Find the volume of the solid generated by revolving R about

- a. the x -axis. b. the line $y = 1$.

18. **Finding volume** (Continuation of Exercise 17.) Find the volume of the solid generated by revolving the region R about

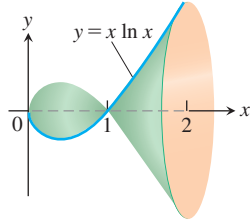
- a. the y -axis. b. the line $x = 1$.

19. **Finding volume** The region between the x -axis and the curve

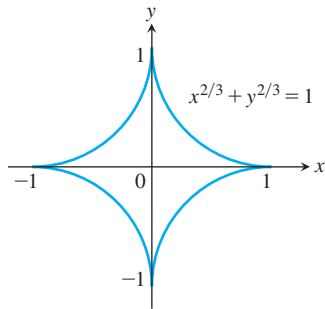
$$y = f(x) = \begin{cases} 0, & x = 0 \\ x \ln x, & 0 < x \leq 2 \end{cases}$$

is revolved about the x -axis to generate the solid shown here.

- Show that f is continuous at $x = 0$.
- Find the volume of the solid.



- Finding volume** The infinite region bounded by the coordinate axes and the curve $y = -\ln x$ in the first quadrant is revolved about the x -axis to generate a solid. Find the volume of the solid.
- Centroid of a region** Find the centroid of the region in the first quadrant that is bounded below by the x -axis, above by the curve $y = \ln x$, and on the right by the line $x = e$.
- Centroid of a region** Find the centroid of the region in the plane enclosed by the curves $y = \pm(1 - x^2)^{-1/2}$ and the lines $x = 0$ and $x = 1$.
- Length of a curve** Find the length of the curve $y = \ln x$ from $x = 1$ to $x = e$.
- Finding surface area** Find the area of the surface generated by revolving the curve in Exercise 23 about the y -axis.
- The surface generated by an astroid** The graph of the equation $x^{2/3} + y^{2/3} = 1$ is an *astroid* (see accompanying figure). Find the area of the surface generated by revolving the curve about the x -axis.



- Length of a curve** Find the length of the curve $y = \int_1^x \sqrt{t-1} dt$, $1 \leq x \leq 16$.

- For what value or values of a does

$$\int_1^\infty \left(\frac{ax}{x^2+1} - \frac{1}{2x} \right) dx$$

converge? Evaluate the corresponding integral(s).

- For each $x > 0$, let $G(x) = \int_0^\infty e^{-xt} dt$. Prove that $xG(x) = 1$ for each $x > 0$.
- Infinite area and finite volume** What values of p have the following property? The area of the region between the curve $y = x^{-p}$, $1 \leq x < \infty$, and the x -axis is infinite but the volume of the solid generated by revolving the region about the x -axis is finite.
- Infinite area and finite volume** What values of p have the following property? The area of the region in the first quadrant enclosed by the curve $y = x^{-p}$, the y -axis, the line $x = 1$, and the interval $[0, 1]$ on the x -axis is infinite, but the volume of the

solid generated by revolving the region about one of the coordinate axes is finite.

- Integrating the square of the derivative** If f is continuously differentiable on $[0, 1]$, and $f(1) = f(0) = -1/6$, prove that

$$\int_0^1 (f'(x))^2 dx \geq 2 \int_0^1 f(x) dx + \frac{1}{4}.$$

Hint: Consider the inequality $0 \leq \int_0^1 \left(f'(x) + x - \frac{1}{2} \right)^2 dx$.

Source: *Mathematics Magazine*, vol. 84, no. 4, Oct. 2011.

- (Continuation of Exercise 31.) If f is continuously differentiable on $[0, a]$ for $a > 0$, and $f(a) = f(0) = b$, prove that

$$\int_0^a (f'(x))^2 dx \geq 2 \int_0^a f(x) dx - \left(2ab + \frac{a^3}{12} \right).$$

Hint: Consider the inequality $0 \leq \int_0^a \left(f'(x) + x - \frac{a}{2} \right)^2 dx$.

Source: *Mathematics Magazine*, vol. 84, no. 4, Oct. 2011.

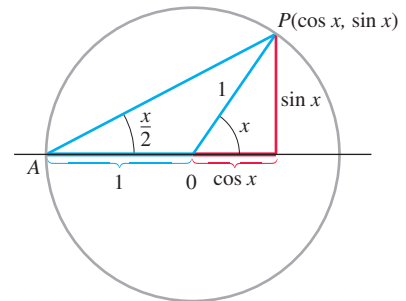
The Substitution $z = \tan(x/2)$

The substitution

$$z = \tan \frac{x}{2} \quad (1)$$

reduces the problem of integrating a rational expression in $\sin x$ and $\cos x$ to a problem of integrating a rational function of z . This in turn can be integrated by partial fractions.

From the accompanying figure



we can read the relation

$$\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x}.$$

To see the effect of the substitution, we calculate

$$\begin{aligned} \cos x &= 2 \cos^2\left(\frac{x}{2}\right) - 1 = \frac{2}{\sec^2(x/2)} - 1 \\ &= \frac{2}{1 + \tan^2(x/2)} - 1 = \frac{2}{1 + z^2} - 1 \\ \cos x &= \frac{1 - z^2}{1 + z^2}, \end{aligned} \quad (2)$$

and

$$\begin{aligned} \sin x &= 2 \sin \frac{x}{2} \cos \frac{x}{2} = 2 \frac{\sin(x/2)}{\cos(x/2)} \cdot \cos^2\left(\frac{x}{2}\right) \\ &= 2 \tan \frac{x}{2} \cdot \frac{1}{\sec^2(x/2)} = \frac{2 \tan(x/2)}{1 + \tan^2(x/2)} \\ \sin x &= \frac{2z}{1 + z^2}. \end{aligned} \quad (3)$$

Finally, $x = 2 \arctan z$, so

$$dx = \frac{2 dz}{1 + z^2}. \quad (4)$$

Examples

$$\begin{aligned} \text{a. } \int \frac{1}{1 + \cos x} dx &= \int \frac{1 + z^2}{2} \frac{2 dz}{1 + z^2} \\ &= \int dz = z + C \\ &= \tan\left(\frac{x}{2}\right) + C \end{aligned}$$

$$\begin{aligned} \text{b. } \int \frac{1}{2 + \sin x} dx &= \int \frac{1 + z^2}{2 + 2z + 2z^2} \frac{2 dz}{1 + z^2} \\ &= \int \frac{dz}{z^2 + z + 1} = \int \frac{dz}{(z + (1/2))^2 + 3/4} \\ &= \int \frac{du}{u^2 + a^2} \\ &= \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C \\ &= \frac{2}{\sqrt{3}} \arctan \frac{2z + 1}{\sqrt{3}} + C \\ &= \frac{2}{\sqrt{3}} \arctan \frac{1 + 2 \tan(x/2)}{\sqrt{3}} + C \end{aligned}$$

Use the substitutions in Equations (1)–(4) to evaluate the integrals in Exercises 33–40. Integrals like these arise in calculating the average angular velocity of the output shaft of a universal joint when the input and output shafts are not aligned.

$$33. \int \frac{dx}{1 - \sin x}$$

$$34. \int \frac{dx}{1 + \sin x + \cos x}$$

$$35. \int_0^{\pi/2} \frac{dx}{1 + \sin x}$$

$$36. \int_{\pi/3}^{\pi/2} \frac{dx}{1 - \cos x}$$

$$37. \int_0^{\pi/2} \frac{d\theta}{2 + \cos \theta}$$

$$38. \int_{\pi/2}^{2\pi/3} \frac{\cos \theta d\theta}{\sin \theta \cos \theta + \sin \theta}$$

$$39. \int \frac{dt}{\sin t - \cos t}$$

$$40. \int \frac{\cos t dt}{1 - \cos t}$$

Use the substitution $z = \tan(\theta/2)$ to evaluate the integrals in Exercises 41 and 42.

$$41. \int \sec \theta d\theta$$

$$42. \int \csc \theta d\theta$$

The Gamma Function and Stirling's Formula

Euler's gamma function $\Gamma(x)$ ("gamma of x "; Γ is a Greek capital g) uses an integral to extend the factorial function from the nonnegative integers to other real values. The formula is

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0.$$

For each positive x , the number $\Gamma(x)$ is the integral of $t^{x-1}e^{-t}$ with respect to t from 0 to ∞ . Figure 8.21 shows the graph of Γ near the origin. You will see how to calculate $\Gamma(1/2)$ if you do Additional Exercise 23 in Chapter 14.

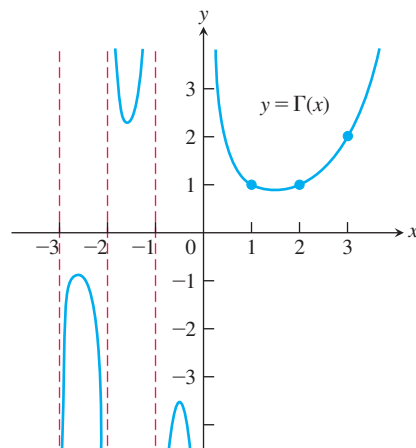


FIGURE 8.21 Euler's gamma function $\Gamma(x)$ is a continuous function of x whose value at each positive integer $n + 1$ is $n!$. The defining integral formula for Γ is valid only for $x > 0$, but we can extend Γ to negative noninteger values of x with the formula $\Gamma(x) = (\Gamma(x + 1))/x$, which is the subject of Exercise 43.

43. If n is a nonnegative integer, then $\Gamma(n + 1) = n!$

a. Show that $\Gamma(1) = 1$.

b. Then apply integration by parts to the integral for $\Gamma(x + 1)$ to show that $\Gamma(x + 1) = x\Gamma(x)$. This gives

$$\Gamma(2) = 1\Gamma(1) = 1$$

$$\Gamma(3) = 2\Gamma(2) = 2$$

$$\Gamma(4) = 3\Gamma(3) = 6$$

$$\vdots$$

$$\Gamma(n + 1) = n\Gamma(n) = n! \quad (1)$$

c. Use mathematical induction to verify Equation (1) for every nonnegative integer n .

44. Stirling's formula Scottish mathematician James Stirling (1692–1770) showed that

$$\lim_{x \rightarrow \infty} \left(\frac{e}{x}\right)^x \sqrt{\frac{x}{2\pi}} \Gamma(x) = 1,$$

so, for large x ,

$$\Gamma(x) = \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} (1 + \varepsilon(x)) \quad \varepsilon(x) \rightarrow 0 \text{ as } x \rightarrow \infty. \quad (2)$$

Dropping $\varepsilon(x)$ leads to the approximation

$$\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \quad (\text{Stirling's formula}). \quad (3)$$

a. Stirling's approximation for $n!$ Use Equation (3) and the fact that $n! = n\Gamma(n)$ to show that

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2n\pi} \quad (\text{Stirling's approximation}). \quad (4)$$

As you will see if you do Exercise 114 in Section 9.1, Equation (4) leads to the approximation

$$\sqrt[n]{n!} \approx \frac{n}{e}. \quad (5)$$

- T** b. Compare your calculator's value for $n!$ with the value given by Stirling's approximation for $n = 10, 20, 30, \dots$, as far as your calculator can go.

- T** c. A refinement of Equation (2) gives

$$\Gamma(x) = \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} e^{1/(12x)} (1 + \varepsilon(x))$$

or

$$\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} e^{1/(12x)},$$

which tells us that

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2n\pi} e^{1/(12n)}. \quad (6)$$

Compare the values given for $10!$ by your calculator, Stirling's approximation, and Equation (6).

CHAPTER 8 Technology Application Projects

Mathematica/Maple Projects

Projects can be found within MyLab Math.

- **Riemann, Trapezoidal, and Simpson Approximations**

Part I: Visualize the error involved in using Riemann sums to approximate the area under a curve.

Part II: Build a table of values and compute the relative magnitude of the error as a function of the step size Δx .

Part III: Investigate the effect of the derivative function on the error.

Parts IV and V: Trapezoidal Rule approximations.

Part VI: Simpson's Rule approximations.

- **Games of Chance: Exploring the Monte Carlo Probabilistic Technique for Numerical Integration**

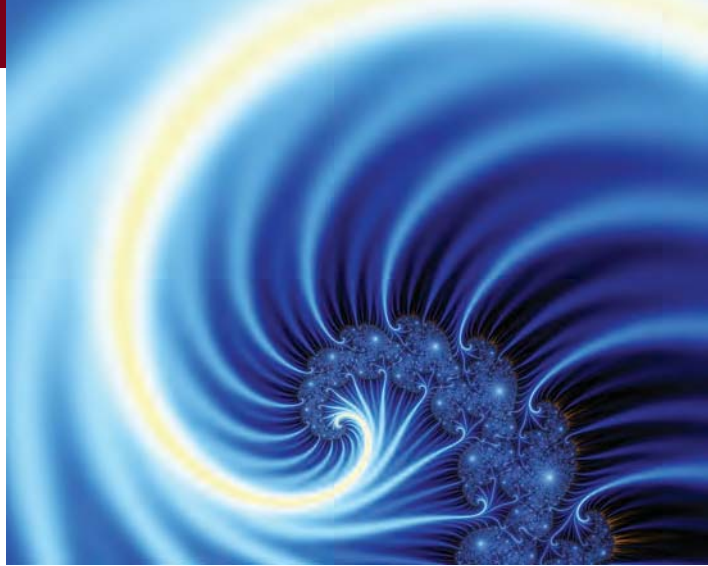
Graphically explore the Monte Carlo method for approximating definite integrals.

- **Computing Probabilities with Improper Integrals**

More explorations of the Monte Carlo method for approximating definite integrals.

9

Infinite Sequences and Series



OVERVIEW In this chapter we introduce the topic of *infinite series*. Such series give us precise ways to express many numbers and functions, both familiar and new, as arithmetic sums with infinitely many terms. For example, we will learn that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

and

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40,320} - \cdots.$$

We need to develop a method to make sense of such expressions. Everyone knows how to add two numbers together, or even several. But how do you add together infinitely many numbers? Or, when adding together functions, how do you add infinitely many powers of x ? In this chapter we answer these questions, which are part of the theory of infinite sequences and series. As with the differential and integral calculus, limits play a major role in the development of infinite series.

One common and important application of series occurs in making computations with complicated functions. A hard-to-compute function is replaced by an expression that looks like an “infinite degree polynomial,” an infinite series in powers of x , as we see with the cosine function given above. Using the first few terms of this infinite series can allow for highly accurate approximations of functions by polynomials, enabling us to work with more general functions than those we have encountered before. These new functions are commonly obtained as solutions to differential equations arising in important applications of mathematics to science and engineering.

The terms “sequence” and “series” are sometimes used interchangeably in spoken language. In mathematics, however, each has a distinct meaning. A sequence is a type of infinite list, whereas a series is an infinite sum. To understand the infinite sums described by series, we first must understand infinite sequences.

9.1 Sequences

HISTORICAL ESSAY

Sequences and Series

To read this essay, visit the companion Website.

Sequences are fundamental to the study of infinite series and to many aspects of mathematics. We saw one example of a sequence when we studied Newton’s Method in Section 4.7. Newton’s Method produces a sequence of approximations x_n that become closer and closer to the root of a differentiable function. Now we will explore general sequences of numbers and the conditions under which they converge to a finite number.

Representing Sequences

A sequence is a list of numbers

$$a_1, a_2, a_3, \dots, a_n, \dots$$

in a given order. Each of a_1, a_2, a_3 , and so on represents a number. These are the **terms** of the sequence. For example, the sequence

$$2, 4, 6, 8, 10, 12, \dots, 2n, \dots$$

has first term $a_1 = 2$, second term $a_2 = 4$, and n th term $a_n = 2n$. The integer n is called the **index** of a_n and indicates where a_n occurs in the list. Order is important. The sequence $2, 4, 6, 8, \dots$ is not the same as the sequence $4, 2, 6, 8, \dots$

We can think of the sequence

$$a_1, a_2, a_3, \dots, a_n, \dots$$

as a function that sends 1 to a_1 , 2 to a_2 , 3 to a_3 , and in general sends the positive integer n to the n th term a_n . More precisely, an **infinite sequence** of numbers is a function whose domain is the set of positive integers. For example, the function associated with the sequence

$$2, 4, 6, 8, 10, 12, \dots, 2n, \dots$$

sends 1 to $a_1 = 2$, 2 to $a_2 = 4$, and so on. The general behavior of this sequence is described by the formula $a_n = 2n$.

We can change the index to start at any given number n . For example, the sequence

$$12, 14, 16, 18, 20, 22, \dots$$

is described by the formula $a_n = 10 + 2n$, if we start with $n = 1$. It can also be described by the simpler formula $b_n = 2n$, where the index n starts at 6 and increases. To allow such simpler formulas, we let the first index of the sequence be any appropriate integer. In the sequence above, $\{a_n\}$ starts with a_1 while $\{b_n\}$ starts with b_6 .

Sequences can be described by writing rules that specify their terms, such as

$$a_n = \sqrt{n}, \quad b_n = (-1)^{n+1} \frac{1}{n}, \quad c_n = \frac{n-1}{n}, \quad d_n = (-1)^{n+1},$$

or by listing terms:

$$\begin{aligned} \{a_n\} &= \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots\} \\ \{b_n\} &= \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, (-1)^{n+1} \frac{1}{n}, \dots\right\} \\ \{c_n\} &= \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n-1}{n}, \dots\right\} \\ \{d_n\} &= \{1, -1, 1, -1, 1, -1, \dots, (-1)^{n+1}, \dots\}. \end{aligned}$$

We also sometimes write a sequence using its rule, as with

$$\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}$$

and

$$\{b_n\} = \left\{(-1)^{n+1} \frac{1}{n}\right\}_{n=1}^{\infty}.$$

Figure 9.1 shows two ways to represent sequences graphically. The first marks the first few points from $a_1, a_2, a_3, \dots, a_n, \dots$ on the real axis. The second method shows the graph of the function defining the sequence. The function is defined only on integer inputs, and the graph consists of some points in the xy -plane located at $(1, a_1), (2, a_2), \dots, (n, a_n), \dots$

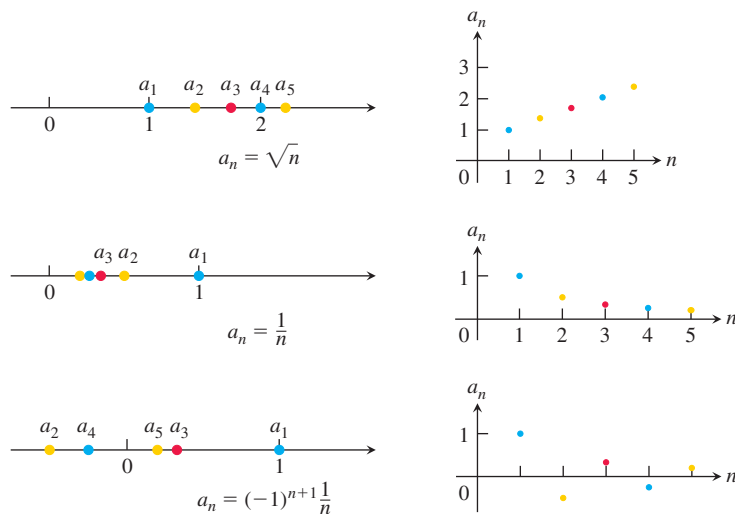


FIGURE 9.1 Sequences can be represented as points on the real line or as points in the plane where the horizontal axis n is the index number of the term and the vertical axis a_n is its value.

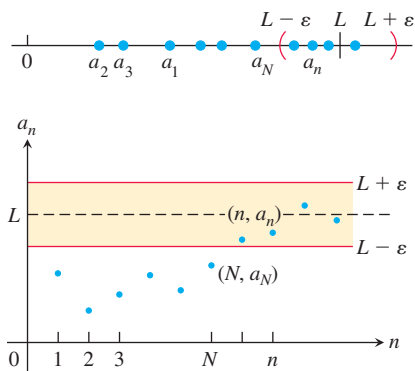


FIGURE 9.2 In the representation of a sequence as points in the plane, $a_n \rightarrow L$ if $y = L$ is a horizontal asymptote of the sequence of points $\{(n, a_n)\}$. In this figure, all the a_n 's after a_N lie within ε of L .

Convergence and Divergence

Sometimes the numbers in a sequence approach a single value as the index n increases. This happens in the sequence

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\right\},$$

whose terms approach 0 as n gets large, and in the sequence

$$\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, 1 - \frac{1}{n}, \dots\right\},$$

whose terms approach 1. On the other hand, sequences like

$$\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots\}$$

have terms that get larger than any number as n increases, and sequences like

$$\{1, -1, 1, -1, 1, -1, \dots, (-1)^{n+1}, \dots\}$$

bounce back and forth between 1 and -1 , never converging to a single value. The following definition captures the meaning of having a sequence converge to a limiting value. It says that if we specify any number $\varepsilon > 0$, then by going far enough out in the sequence, taking the index n to be larger than some value N , the difference between a_n and the limit of the sequence becomes less than ε .

HISTORICAL BIOGRAPHY

Nicole Oresme
(ca. 1320–1382)

Frenchman Oresme went to the University of Paris in the 1340s, studying theology and liberal arts. Later he was a faculty member and administrator at the same university. His work entitled *De configurationibus* (1350s) contained results in geometry and was the first to present graphs of velocities. The argument we use to show the divergence of the harmonic series was devised by Oresme in this publication.

To know more, visit the companion Website.

DEFINITIONS The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there corresponds an integer N such that

$$|a_n - L| < \varepsilon \quad \text{whenever} \quad n > N.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L , we write $\lim_{n \rightarrow \infty} a_n = L$, or simply $a_n \rightarrow L$, and call L the **limit** of the sequence (Figure 9.2).

The definition is very similar to the definition of $\lim_{x \rightarrow \infty} f(x)$, the limit of a function $f(x)$ as x tends to ∞ , discussed in Section 2.5. We will exploit this connection to calculate limits of sequences.

EXAMPLE 1 Show that

$$(a) \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \quad (b) \lim_{n \rightarrow \infty} k = k \quad (\text{where } k \text{ is a constant})$$

Solution

(a) Let $\varepsilon > 0$ be given. We must show that there exists an integer N such that

$$\left| \frac{1}{n} - 0 \right| < \varepsilon \quad \text{whenever} \quad n > N.$$

The inequality $|1/n - 0| < \varepsilon$ will hold if $1/n < \varepsilon$ or $n > 1/\varepsilon$. If N is any integer greater than $1/\varepsilon$, the inequality will hold for all $n > N$. This proves that $\lim_{n \rightarrow \infty} 1/n = 0$.

(b) Let k be a constant, and let $\varepsilon > 0$ be given. We must show that there exists an integer N such that

$$|k - k| < \varepsilon \quad \text{whenever} \quad n > N.$$

Since $k - k = 0$, we can use any positive integer for N and the inequality $|k - k| < \varepsilon$ will hold. This proves that $\lim_{n \rightarrow \infty} k = k$. ■

EXAMPLE 2 Show that the sequence $\{1, -1, 1, -1, 1, -1, \dots, (-1)^{n+1}, \dots\}$ diverges.

Solution Since the terms in this sequence alternate between 1 and -1 , our intuition is that they cannot approach arbitrarily close to some single limit L . To prove this, suppose that there did exist a number L that is the limit of this sequence. The definition of convergence then tells us that for every number $\varepsilon > 0$, there is an integer N such that $|a_n - L| < \varepsilon$ for all $n > N$. No matter what positive number ε we consider, such an integer N must exist. In particular, for the number $\varepsilon = 1/2$, there must exist some integer N such that $|a_n - L| < 1/2$ for all $n > N$. If $n > N$ is odd, then $a_n = 1$, so we have $|1 - L| < 1/2$, or

$$\frac{1}{2} < L < \frac{3}{2}.$$

On the other hand, if $n > N$ is even, then $a_n = -1$, and so we have $|-1 - L| < 1/2$, or

$$-\frac{3}{2} < L < -\frac{1}{2}.$$

However, these two conditions on L cannot both hold, so we have reached a contradiction. Therefore there is no number L to which this sequence converges.

We chose $\varepsilon = 1/2$ in this argument simply for convenience. The same argument would work if we had instead chosen $\varepsilon = 1/4$, or any value less than 1. ■

The sequence $\{\sqrt{n}\}$ also diverges, but for a different reason. As n increases, its terms become larger than any fixed number. We describe the behavior of this sequence by writing

$$\lim_{n \rightarrow \infty} \sqrt{n} = \infty.$$

In writing infinity as the limit of a sequence, we are not saying that the differences between the terms a_n and ∞ become small as n increases, nor are we asserting that there is some number infinity that the sequence approaches. We are merely using a notation that captures the idea that a_n eventually gets and stays larger than any fixed number as n gets large (see Figure 9.3a). The terms of a sequence might also decrease to negative infinity, as in Figure 9.3b.

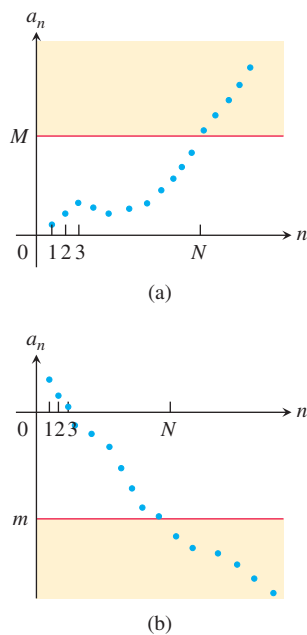


FIGURE 9.3 (a) The sequence diverges to ∞ because no matter what number M is chosen, the terms of the sequence after some index N all lie in the yellow band above M . (b) The sequence diverges to $-\infty$ because all terms after some index N lie below any chosen number m .

DEFINITION The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N , $a_n > M$. If this condition holds, then we write

$$\lim_{n \rightarrow \infty} a_n = \infty \quad \text{or} \quad a_n \rightarrow \infty.$$

Similarly, if for every number m there is an integer N such that for all $n > N$, we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n \rightarrow \infty} a_n = -\infty \quad \text{or} \quad a_n \rightarrow -\infty.$$

A sequence may diverge without diverging to infinity or negative infinity, as we saw in Example 2. The sequences $\{1, -2, 3, -4, 5, -6, 7, -8, \dots\}$ and $\{1, 0, 2, 0, 3, 0, \dots\}$ are also sequences that diverge but do not diverge to infinity or negative infinity.

The convergence or divergence of a sequence is not affected by the values of any number of its initial terms (whether we omit or change the first 10, the first 1000, or even the first million terms does not matter). From Figure 9.2, we can see that only the part of the sequence that remains after discarding some initial number of terms determines whether the sequence has a limit and the value of that limit when it does exist.

Calculating Limits of Sequences

Since sequences are functions with domain restricted to the positive integers, it is not surprising that the theorems on limits of functions given in Chapter 2 have versions for sequences.

THEOREM 1 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$.

1. *Sum Rule:* $\lim_{n \rightarrow \infty} (a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n \rightarrow \infty} (a_n - b_n) = A - B$
3. *Constant Multiple Rule:* $\lim_{n \rightarrow \infty} (k \cdot b_n) = k \cdot B$ (any number k)
4. *Product Rule:* $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = A \cdot B$
5. *Quotient Rule:* $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

The proof is similar to that of Theorem 1 in Section 2.2 and is omitted.

EXAMPLE 3 By combining Theorem 1 with the limits of Example 1, we have:

- (a) $\lim_{n \rightarrow \infty} \left(-\frac{1}{n}\right) = -1 \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = -1 \cdot 0 = 0$ Constant Multiple Rule
and Example 1a
- (b) $\lim_{n \rightarrow \infty} \left(\frac{n-1}{n}\right) = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = \lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{1}{n} = 1 - 0 = 1$ Difference Rule
and Example 1a
- (c) $\lim_{n \rightarrow \infty} \frac{5}{n^2} = 5 \cdot \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \lim_{n \rightarrow \infty} \frac{1}{n} = 5 \cdot 0 \cdot 0 = 0$ Product Rule
- (d) $\lim_{n \rightarrow \infty} \frac{4 - 7n^6}{n^6 + 3} = \lim_{n \rightarrow \infty} \frac{(4/n^6) - 7}{1 + (3/n^6)} = \frac{0 - 7}{1 + 0} = -7.$ Divide numerator and denominator
by n^6 and use the Sum and
Quotient Rules.

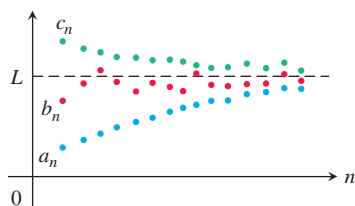


FIGURE 9.4 The terms of sequence $\{b_n\}$ are sandwiched between those of $\{a_n\}$ and $\{c_n\}$, forcing them to the same common limit L .

Be cautious in applying Theorem 1. It does not say, for example, that both of the sequences $\{a_n\}$ and $\{b_n\}$ have limits if their sum $\{a_n + b_n\}$ has a limit. For instance, $\{a_n\} = \{1, 2, 3, \dots\}$ and $\{b_n\} = \{-1, -2, -3, \dots\}$ both diverge, but their sum $\{a_n + b_n\} = \{0, 0, 0, \dots\}$ clearly converges to 0.

One consequence of Theorem 1 is that every nonzero multiple of a divergent sequence $\{a_n\}$ diverges. Suppose, to the contrary, that $\{ca_n\}$ converges for some number $c \neq 0$. Then, by taking $k = 1/c$ in the Constant Multiple Rule in Theorem 1, we see that the sequence

$$\left\{\frac{1}{c} \cdot ca_n\right\} = \{a_n\}$$

converges. Thus, $\{ca_n\}$ cannot converge unless $\{a_n\}$ also converges. If $\{a_n\}$ does not converge, then $\{ca_n\}$ does not converge.

The next theorem is the sequence version of the Sandwich Theorem in Section 2.2. You are asked to prove the theorem in Exercise 119. (See Figure 9.4.)

THEOREM 2—The Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N , and if $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$ also.

An immediate consequence of Theorem 2 is that if $|b_n| \leq c_n$ and $c_n \rightarrow 0$, then $b_n \rightarrow 0$ because $-c_n \leq b_n \leq c_n$. We use this fact in the next example.

EXAMPLE 4 Since $1/n \rightarrow 0$, we know that

- (a) $\frac{\cos n}{n} \rightarrow 0$ because $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$;
- (b) $\frac{1}{2^n} \rightarrow 0$ because $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$;
- (c) $(-1)^n \frac{1}{n} \rightarrow 0$ because $-\frac{1}{n} \leq (-1)^n \frac{1}{n} \leq \frac{1}{n}$.
- (d) If $|a_n| \rightarrow 0$, then $a_n \rightarrow 0$ because $-|a_n| \leq a_n \leq |a_n|$. ■

The application of Theorems 1 and 2 is broadened by a theorem stating that applying a continuous function to a convergent sequence produces a convergent sequence. We state the theorem, leaving the proof as an exercise (Exercise 120).

THEOREM 3—The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \rightarrow L$ and if f is a function that is continuous at L and defined at all a_n , then $f(a_n) \rightarrow f(L)$.

EXAMPLE 5 Show that $\sqrt{(n+1)/n} \rightarrow 1$.

Solution We know that $(n+1)/n \rightarrow 1$. Taking $f(x) = \sqrt{x}$ and $L = 1$ in Theorem 3 gives $\sqrt{(n+1)/n} \rightarrow \sqrt{1} = 1$. ■

EXAMPLE 6 The sequence $\{1/n\}$ converges to 0. By taking $a_n = 1/n$, $f(x) = 2^x$, and $L = 0$ in Theorem 3, we see that $2^{1/n} = f(1/n) \rightarrow f(L) = 2^0 = 1$. The sequence $\{2^{1/n}\}$ converges to 1 (Figure 9.5). ■

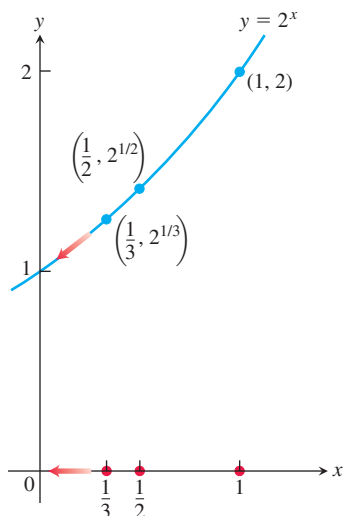


FIGURE 9.5 As $n \rightarrow \infty$, $1/n \rightarrow 0$ and $2^{1/n} \rightarrow 2^0$ (Example 6). The terms of $\{1/n\}$ are shown on the x -axis; the terms of $\{2^{1/n}\}$ are shown as the y -values on the graph of $f(x) = 2^x$.

Using L'Hôpital's Rule

The next theorem formalizes the connection between $\lim_{n \rightarrow \infty} a_n$ and $\lim_{x \rightarrow \infty} f(x)$. It enables us to use l'Hôpital's Rule to find the limits of some sequences.

THEOREM 4 Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{whenever} \quad \lim_{x \rightarrow \infty} f(x) = L.$$

Proof Suppose that $\lim_{x \rightarrow \infty} f(x) = L$. Then for each positive number ε , there is a number M such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x > M.$$

Let N be an integer that is both greater than M and greater than or equal to n_0 . Since $a_n = f(n)$, it follows that for all $n > N$, we have

$$|a_n - L| = |f(n) - L| < \varepsilon. \quad \blacksquare$$

EXAMPLE 7 Show that

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0.$$

Solution The function $(\ln x)/x$ is defined for all $x \geq 1$ and agrees with the given sequence at positive integers. Therefore, by Theorem 4, $\lim_{n \rightarrow \infty} (\ln n)/n$ will equal $\lim_{x \rightarrow \infty} (\ln x)/x$ if the latter exists. A single application of l'Hôpital's Rule shows that

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = \frac{0}{1} = 0.$$

We conclude that $\lim_{n \rightarrow \infty} (\ln n)/n = 0$. ■

EXAMPLE 8 Does the sequence whose n th term is

$$a_n = \left(\frac{n+1}{n-1}\right)^n, \quad n \geq 2$$

converge? If so, find $\lim_{n \rightarrow \infty} a_n$.

Solution The function $f(x) = \left(\frac{x+1}{x-1}\right)^x$ is defined for all real numbers $x \geq 2$ and agrees with a_n at all integers $n \geq 2$. If we can show that $\lim_{x \rightarrow \infty} \left(\frac{x+1}{x-1}\right)^x = L$, then by Theorem 4, $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n-1}\right)^n = L$.

The limit $\lim_{x \rightarrow \infty} f(x)$ leads to the indeterminate form 1^∞ . To evaluate this limit, we apply l'Hôpital's Rule after taking the natural logarithm of $f(x)$:

$$\ln f(x) = \ln \left(\frac{x+1}{x-1}\right)^x = x \ln \left(\frac{x+1}{x-1}\right).$$

Then

$$\begin{aligned} \lim_{x \rightarrow \infty} \ln f(x) &= \lim_{x \rightarrow \infty} x \ln \left(\frac{x+1}{x-1}\right) && \infty \cdot 0 \text{ form} \\ &= \lim_{x \rightarrow \infty} \frac{\ln \left(\frac{x+1}{x-1}\right)}{1/x} && \frac{0}{0} \text{ form} \\ &= \lim_{x \rightarrow \infty} \frac{-2/(x^2 - 1)}{-1/x^2} && \text{Apply l'Hôpital's Rule.} \\ &= \lim_{x \rightarrow \infty} \frac{2x^2}{x^2 - 1} = 2. && \text{Simplify and evaluate.} \end{aligned}$$

Therefore, $\lim_{x \rightarrow \infty} \left(\frac{x+1}{x-1}\right)^x = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{\ln f(x)} = e^2$. Applying Theorem 4, we conclude that the sequence $\{a_n\}$ also converges to e^2 . ■

Commonly Occurring Limits

The next theorem gives some limits that arise frequently.

Factorial Notation

The notation $n!$ (“ n factorial”) means the product $1 \cdot 2 \cdot 3 \cdots n$ of the integers from 1 to n . Notice that $(n + 1)! = (n + 1) \cdot n!$. Thus, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$ and $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5 \cdot 4! = 120$. We define $0!$ to be 1. Factorials grow even faster than exponentials, as the table suggests. The values in the table are rounded.

n	e^n	$n!$
1	3	1
5	148	120
10	22,026	3,628,800
20	4.9×10^8	2.4×10^{18}

THEOREM 5 The following six sequences converge to the limits listed below.

$$1. \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$

$$2. \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$$

$$3. \lim_{n \rightarrow \infty} x^{1/n} = 1 \quad (x > 0)$$

$$4. \lim_{n \rightarrow \infty} x^n = 0 \quad (|x| < 1)$$

$$5. \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$$

$$6. \lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$$

In Formulas (3) through (6), x remains fixed as $n \rightarrow \infty$.

Proof The first limit was computed in Example 7. The next two can be proved by taking logarithms and applying Theorem 4 (Exercises 117 and 118). The remaining proofs are given in Appendix A.6. ■

EXAMPLE 9 These are examples involving the limits in Theorem 5.

$$(a) \frac{\ln(n^2)}{n} = \frac{2 \ln n}{n} \rightarrow 2 \cdot 0 = 0$$

Formula 1

$$(b) \sqrt[n]{n^2} = n^{2/n} = (n^{1/n})^2 \rightarrow (1)^2 = 1$$

Formula 2

$$(c) \sqrt[3]{3n} = 3^{1/n} (n^{1/n}) \rightarrow 1 \cdot 1 = 1$$

Formula 3 with $x = 3$ and Formula 2

$$(d) \left(-\frac{1}{2}\right)^n \rightarrow 0$$

Formula 4 with $x = -\frac{1}{2}$

$$(e) \left(\frac{n-2}{n}\right)^n = \left(1 + \frac{-2}{n}\right)^n \rightarrow e^{-2}$$

Formula 5 with $x = -2$

$$(f) \frac{100^n}{n!} \rightarrow 0$$

Formula 6 with $x = 100$ ■

Recursive Definitions

So far, we have calculated each a_n directly from the value of n . But sequences are often defined **recursively** by giving

1. The value(s) of the initial term or terms, and
2. a rule called a **recursion formula** for calculating any later term from terms that precede it.

EXAMPLE 10

- (a) The statements $a_1 = 1$ and $a_n = a_{n-1} + 1$ for $n > 1$ define the sequence $1, 2, 3, \dots, n, \dots$ of positive integers. With $a_1 = 1$, we have $a_2 = a_1 + 1 = 2$, $a_3 = a_2 + 1 = 3$, and so on.
- (b) The statements $a_1 = 1$ and $a_n = n \cdot a_{n-1}$ for $n > 1$ define the sequence $1, 2, 6, 24, \dots, n!, \dots$ of factorials. With $a_1 = 1$, we have $a_2 = 2 \cdot a_1 = 2$, $a_3 = 3 \cdot a_2 = 6$, $a_4 = 4 \cdot a_3 = 24$, and so on.
- (c) The statements $a_1 = 1, a_2 = 1$, and $a_{n+1} = a_n + a_{n-1}$ for $n > 2$ define the sequence $1, 1, 2, 3, 5, \dots$ of **Fibonacci numbers**. With $a_1 = 1$ and $a_2 = 1$, we have $a_3 = 1 + 1 = 2$, $a_4 = 2 + 1 = 3$, $a_5 = 3 + 2 = 5$, and so on.

- (d) As we can see by applying Newton's method (see Exercise 145), the statements $x_0 = 1$ and $x_{n+1} = x_n - [(\sin x_n - x_n^2)/(\cos x_n - 2x_n)]$ for $n > 0$ define a sequence that, when it converges, gives a solution to the equation $\sin x - x^2 = 0$. ■

Bounded Monotonic Sequences

Two concepts that play a key role in determining the convergence of a sequence are those of a *bounded* sequence and a *monotonic* sequence. First we define bounded sequences.

DEFINITION A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \leq M$ for all n . The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

A sequence $\{a_n\}$ is **bounded from below** if there exists a number m such that $a_n \geq m$ for all n . The number m is a **lower bound** for $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is **bounded**. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an **unbounded** sequence.

EXAMPLE 11

- (a) The sequence $1, 2, 3, \dots, n, \dots$ has no upper bound because it eventually surpasses every number M . However, it is bounded below by every real number less than or equal to 1. The number $m = 1$ is the greatest lower bound of the sequence.
- (b) The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$ is bounded above by every real number greater than or equal to 1. The upper bound $M = 1$ is the least upper bound (Exercise 137). The sequence is also bounded below by every number less than or equal to $\frac{1}{2}$, which is its greatest lower bound. ■

Convergent sequences are bounded.

Suppose that a sequence $\{a_n\}$ converges to a number L . Applying the definition of convergence with the particular value $\varepsilon = 1$, there must exist a number N such that $|a_n - L| < 1$ if $n > N$. That is,

$$L - 1 < a_n < L + 1 \quad \text{for } n > N.$$

If M is a number larger than both $L + 1$ and all of the finitely many numbers $a_1, a_2, \dots, a_N$, then for every index n we have $a_n \leq M$, and therefore $\{a_n\}$ is bounded from above. Similarly, if m is a number smaller than both $L - 1$ and all of the numbers $a_1, a_2, \dots, a_N$, then m is a lower bound for the sequence. Therefore, all convergent sequences are bounded.

Although it is true that every convergent sequence is bounded, there are bounded sequences that fail to converge. One example is the bounded sequence $\{(-1)^{n+1}\}$ discussed in Example 2. The problem here is that some bounded sequences bounce around in the band determined by any lower bound m and any upper bound M but do not converge (Figure 9.6). An important type of sequence that does not behave that way is one for which each term is at least as large, or at least as small, as its predecessor.

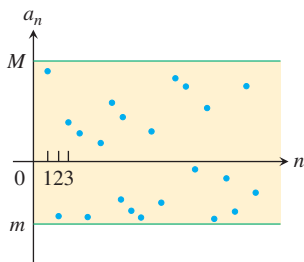


FIGURE 9.6 Some bounded sequences bounce around between their bounds and fail to converge to any limiting value.

DEFINITIONS A sequence $\{a_n\}$ is **nondecreasing** if $a_n \leq a_{n+1}$ for all n . That is, $a_1 \leq a_2 \leq a_3 \leq \dots$. The sequence is **nonincreasing** if $a_n \geq a_{n+1}$ for all n . The sequence $\{a_n\}$ is **monotonic** if it is either nondecreasing or nonincreasing.

EXAMPLE 12

- (a) The sequence $1, 2, 3, \dots, n, \dots$ is nondecreasing.
- (b) The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$ is nondecreasing.
- (c) The sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^n}, \dots$ is nonincreasing.
- (d) The constant sequence $3, 3, 3, \dots, 3, \dots$ is both nondecreasing and nonincreasing.
- (e) The sequence $1, -1, 1, -1, 1, -1, \dots$ is not monotonic. ■

A sequence that is bounded from above always has a least upper bound. Likewise, a sequence bounded from below always has a greatest lower bound. These results are based on the *completeness property* of the real numbers, discussed in Appendix A.9. We now prove that if L is the least upper bound of a nondecreasing sequence, then the sequence converges to L , and that if L is the greatest lower bound of a nonincreasing sequence, then the sequence converges to L .

THEOREM 6—The Monotonic Sequence Theorem

If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

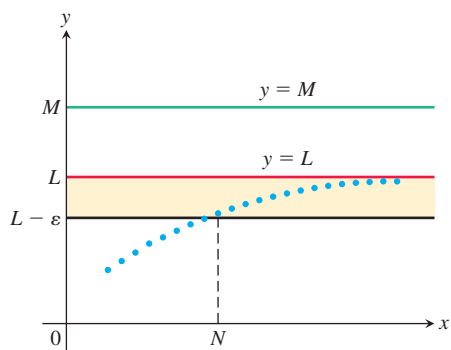


FIGURE 9.7 If the terms of a nondecreasing sequence have an upper bound M , then they have a limit $L \leq M$.

Proof Suppose $\{a_n\}$ is nondecreasing, L is its least upper bound, and we plot the points $(1, a_1), (2, a_2), \dots, (n, a_n), \dots$ in the xy -plane. If M is an upper bound of the sequence, all these points will lie on or below the line $y = M$ (Figure 9.7). The line $y = L$ is the lowest such line. None of the points (n, a_n) lies above $y = L$, but some do lie above any lower line $y = L - \varepsilon$, if ε is a positive number (because $L - \varepsilon$ is not an upper bound). That is,

- a. $a_n \leq L$ for all values of n , and
- b. given any $\varepsilon > 0$, there exists at least one integer N for which $a_N > L - \varepsilon$.

The fact that $\{a_n\}$ is nondecreasing tells us further that

$$a_n \geq a_N > L - \varepsilon \quad \text{for all } n \geq N.$$

Thus, all the numbers a_n beyond the N th number lie within ε of L . This is precisely the condition for L to be the limit of the sequence $\{a_n\}$.

The proof for nonincreasing sequences bounded from below is similar. ■

It is important to realize that Theorem 6 does not say that convergent sequences are monotonic. The sequence $\{(-1)^{n+1}/n\}$ converges and is bounded, but it is not monotonic since it alternates between positive and negative values as it tends toward zero. What the theorem does say is that a nondecreasing sequence converges when it is bounded from above, but it diverges to infinity otherwise.

EXERCISES 9.1

Finding Terms of a Sequence

Each of Exercises 1–6 gives a formula for the n th term a_n of a sequence $\{a_n\}$. Find the values of a_1, a_2, a_3 , and a_4 .

1. $a_n = \frac{1-n}{n^2}$

2. $a_n = \frac{1}{n!}$

3. $a_n = \frac{(-1)^{n+1}}{2n-1}$

4. $a_n = 2 + (-1)^n$

5. $a_n = \frac{2^n}{2^{n+1}}$

6. $a_n = \frac{2^n - 1}{2^n}$

Each of Exercises 7–12 gives the first term or two of a sequence along with a recursion formula for the remaining terms. Write out the first ten terms of the sequence.

7. $a_1 = 1, \quad a_{n+1} = a_n + (1/2^n)$

8. $a_1 = 1, \quad a_{n+1} = a_n/(n+1)$

88. $a_n = n\left(1 - \cos \frac{1}{n}\right)$ 89. $a_n = \sqrt{n} \sin \frac{1}{\sqrt{n}}$
 90. $a_n = (3^n + 5^n)^{1/n}$ 91. $a_n = \tan^{-1} n$
 92. $a_n = \frac{1}{\sqrt{n}} \arctan n$ 93. $a_n = \left(\frac{1}{3}\right)^n + \frac{1}{\sqrt{2^n}}$
 94. $a_n = \sqrt[n]{n^2 + n}$ 95. $a_n = \frac{(\ln n)^{200}}{n}$
 96. $a_n = \frac{(\ln n)^5}{\sqrt{n}}$ 97. $a_n = n - \sqrt{n^2 - n}$
 98. $a_n = \frac{1}{\sqrt{n^2 - 1} - \sqrt{n^2 + n}}$
 99. $a_n = \frac{1}{n} \int_1^n \frac{1}{x} dx$ 100. $a_n = \int_1^n \frac{1}{x^p} dx, \quad p > 1$

Recursively Defined Sequences

In Exercises 101–108, assume that each sequence converges and find its limit.

101. $a_1 = 2, \quad a_{n+1} = \frac{72}{1 + a_n}$
 102. $a_1 = -1, \quad a_{n+1} = \frac{a_n + 6}{a_n + 2}$
 103. $a_1 = -4, \quad a_{n+1} = \sqrt{8 + 2a_n}$
 104. $a_1 = 0, \quad a_{n+1} = \sqrt{8 + 2a_n}$
 105. $a_1 = 5, \quad a_{n+1} = \sqrt{5a_n}$
 106. $a_1 = 3, \quad a_{n+1} = 12 - \sqrt{a_n}$
 107. $2, 2 + \frac{1}{2}, 2 + \frac{1}{2 + \frac{1}{2}}, 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}, \dots$
 108. $\sqrt{1}, \sqrt{1 + \sqrt{1}}, \sqrt{1 + \sqrt{1 + \sqrt{1}}}, \dots$
 $\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}, \dots$

Theory and Examples

109. The first term of a sequence is $x_1 = 1$. Each succeeding term is the sum of all those that come before it:

$$x_{n+1} = x_1 + x_2 + \cdots + x_n.$$

Write out enough early terms of the sequence to deduce a general formula for x_n that holds for $n \geq 2$.

110. A sequence of rational numbers is described as follows:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \dots, \frac{a}{b}, \frac{a+2b}{a+b}, \dots$$

Here the numerators form one sequence, the denominators form a second sequence, and their ratios form a third sequence. Let x_n and y_n be, respectively, the numerator and the denominator of the n th fraction $r_n = x_n/y_n$.

- a. Verify that $x_1^2 - 2y_1^2 = -1$, that $x_2^2 - 2y_2^2 = +1$, and, more generally, that if $a^2 - 2b^2 = -1$ or $+1$, then

$$(a + 2b)^2 - 2(a + b)^2 = +1 \quad \text{or} \quad -1,$$

respectively.

- b. The fractions $r_n = x_n/y_n$ approach a limit as n increases. What is that limit? (*Hint*: Use part (a) to show that $r_n^2 - 2 = \pm(1/y_n)^2$ and that y_n is not less than n .)

111. **Newton's method** The following sequences come from the recursion formula for Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Do the sequences converge? If so, to what value? In each case, begin by identifying the function f that generates the sequence.

- a. $x_0 = 1, \quad x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}$
 b. $x_0 = 1, \quad x_{n+1} = x_n - \frac{\tan x_n - 1}{\sec^2 x_n}$
 c. $x_0 = 1, \quad x_{n+1} = x_n - 1$

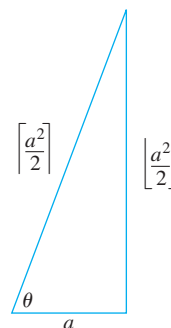
112. a. Suppose that $f(x)$ is differentiable for all x in $[0, 1]$ and that $f(0) = 0$. Define sequence $\{a_n\}$ by the rule $a_n = n f(1/n)$. Show that $\lim_{n \rightarrow \infty} a_n = f'(0)$. Use the result in part (a) to find the limits of the following sequences $\{a_n\}$.

- b. $a_n = n \tan^{-1} \frac{1}{n}$ c. $a_n = n(e^{1/n} - 1)$
 d. $a_n = n \ln\left(1 + \frac{2}{n}\right)$

113. **Pythagorean triples** A triple of positive integers a , b , and c is called a **Pythagorean triple** if $a^2 + b^2 = c^2$. Let a be an odd positive integer and let

$$b = \left\lfloor \frac{a^2}{2} \right\rfloor \quad \text{and} \quad c = \left\lceil \frac{a^2}{2} \right\rceil$$

be, respectively, the integer floor and ceiling for $a^2/2$.



- a. Show that $a^2 + b^2 = c^2$. (*Hint*: Let $a = 2n + 1$ and express b and c in terms of n .)
 b. By direct calculation, or by appealing to the accompanying figure, find

$$\lim_{a \rightarrow \infty} \frac{\left\lfloor \frac{a^2}{2} \right\rfloor}{\left\lceil \frac{a^2}{2} \right\rceil}.$$

114. The n th root of $n!$

- a. Show that $\lim_{n \rightarrow \infty} (2n\pi)^{1/(2n)} = 1$ and hence, using Stirling's approximation (Chapter 8, Additional Exercise 44a), that

$$\sqrt[n]{n!} \approx \frac{n}{e} \quad \text{for large values of } n.$$

- T** b. Test the approximation in part (a) for $n = 40, 50, 60, \dots$, as far as your calculator will allow.

- 115. a.** Assuming that $\lim_{n \rightarrow \infty} (1/n^c) = 0$ if c is any positive constant, show that

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n^c} = 0$$

if c is any positive constant.

- b. Prove that $\lim_{n \rightarrow \infty} (1/n^c) = 0$ if c is any positive constant.

(Hint: If $\varepsilon = 0.001$ and $c = 0.04$, how large should N be to ensure that $|1/n^c - 0| < \varepsilon$ if $n > N$?)

- 116. The zipper theorem** Prove the “zipper theorem” for sequences: If $\{a_n\}$ and $\{b_n\}$ both converge to L , then the sequence

$$a_1, b_1, a_2, b_2, \dots, a_n, b_n, \dots$$

converges to L .

- 117.** Prove that $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

- 118.** Prove that $\lim_{n \rightarrow \infty} x^{1/n} = 1$, ($x > 0$).

- 119.** Prove Theorem 2.

- 120.** Prove Theorem 3.

In Exercises 121–124, determine whether the sequence is monotonic and whether it is bounded.

121. $a_n = \frac{3n+1}{n+1}$

122. $a_n = \frac{(2n+3)!}{(n+1)!}$

123. $a_n = \frac{2^n 3^n}{n!}$

124. $a_n = 2 - \frac{2}{n} - \frac{1}{2^n}$

In Exercises 125–134, determine whether the sequence is monotonic, whether it is bounded, and whether it converges.

125. $a_n = 1 - \frac{1}{n}$

126. $a_n = n - \frac{1}{n}$

127. $a_n = \frac{2^n - 1}{2^n}$

128. $a_n = \frac{2^n - 1}{3^n}$

129. $a_n = ((-1)^n + 1) \left(\frac{n+1}{n} \right)$

- 130.** The first term of a sequence is $x_1 = \cos(1)$. The next terms are $x_2 = x_1$ or $\cos(2)$, whichever is larger; and $x_3 = x_2$ or $\cos(3)$, whichever is larger (farther to the right). In general,

$$x_{n+1} = \max\{x_n, \cos(n+1)\}.$$

131. $a_n = \frac{1 + \sqrt{2n}}{\sqrt{n}}$

132. $a_n = \frac{n+1}{n}$

133. $a_n = \frac{4^{n+1} + 3^n}{4^n}$

134. $a_1 = 1, \quad a_{n+1} = 2a_n - 3$

In Exercises 135–136, use the definition of convergence to prove the given limit.

135. $\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$

136. $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n^2} \right) = 1$

- 137. The sequence $\{n/(n+1)\}$ has a least upper bound of 1** Show that if M is a number less than 1, then the terms of $\{n/(n+1)\}$ eventually exceed M . That is, if $M < 1$, there is an integer N such that $n/(n+1) > M$ whenever $n > N$. Since $n/(n+1) < 1$ for every n , this proves that 1 is a least upper bound for $\{n/(n+1)\}$.

- 138. Uniqueness of least upper bounds** Show that if M_1 and M_2 are least upper bounds for the sequence $\{a_n\}$, then $M_1 = M_2$. That is, a sequence cannot have two different least upper bounds.

- 139.** Is it true that a sequence $\{a_n\}$ of positive numbers must converge if it is bounded from above? Give reasons for your answer.

- 140.** Prove that if $\{a_n\}$ is a convergent sequence, then to every positive number ε there corresponds an integer N such that

$$|a_m - a_n| < \varepsilon \quad \text{whenever } m > N \quad \text{and } n > N.$$

- 141. Uniqueness of limits** Prove that limits of sequences are unique. That is, show that if L_1 and L_2 are numbers such that $a_n \rightarrow L_1$ and $a_n \rightarrow L_2$, then $L_1 = L_2$.

- 142. Limits and subsequences** If the terms of one sequence appear in another sequence in their given order, we call the first sequence a **subsequence** of the second. Prove that if two subsequences of a sequence $\{a_n\}$ have different limits $L_1 \neq L_2$, then $\{a_n\}$ diverges.

- 143.** For a sequence $\{a_n\}$ the terms of even index are denoted by a_{2k} and the terms of odd index by a_{2k+1} . Prove that if $a_{2k} \rightarrow L$ and $a_{2k+1} \rightarrow L$, then $a_n \rightarrow L$.

- 144.** Prove that a sequence $\{a_n\}$ converges to 0 if and only if the sequence of absolute values $\{|a_n|\}$ converges to 0.

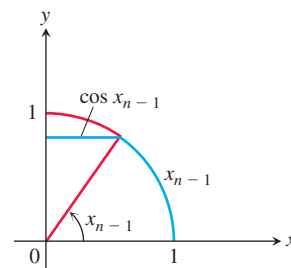
- 145. Sequences generated by Newton's method** Newton's method, applied to a differentiable function $f(x)$, begins with a starting value x_0 and constructs from it a sequence of numbers $\{x_n\}$ that under favorable circumstances converges to a zero of f . The recursion formula for the sequence is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

- a. Show that the recursion formula for $f(x) = x^2 - a$, $a > 0$, can be written as $x_{n+1} = (x_n + a/x_n)/2$.

- T** b. Starting with $x_0 = 1$ and $a = 3$, calculate successive terms of the sequence until the display begins to repeat. What number is being approximated? Explain.

- T** **146. A recursive definition of $\pi/2$** If you start with $x_1 = 1$ and if you define the subsequent terms of $\{x_n\}$ by the rule $x_n = x_{n-1} + \cos x_{n-1}$, you generate a sequence that converges rapidly to $\pi/2$. (a) Try it. (b) Use the accompanying figure to explain why the convergence is so rapid.



COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for the sequences in Exercises 147–158.

- a. Calculate and then plot the first 25 terms of the sequence. Does the sequence appear to be bounded from above or below? Does it appear to converge or diverge? If it does converge, what is the limit L ?
- b. If the sequence converges, find an integer N such that $|a_n - L| \leq 0.01$ for $n \geq N$. How far in the sequence do you have to get for the terms to lie within 0.0001 of L ?

147. $a_n = \sqrt[n]{n}$

148. $a_n = \left(1 + \frac{0.5}{n}\right)^n$

149. $a_1 = 1, \quad a_{n+1} = a_n + \frac{1}{5^n}$

150. $a_1 = 1, \quad a_{n+1} = a_n + (-2)^n$

151. $a_n = \sin n$

152. $a_n = n \sin \frac{1}{n}$

153. $a_n = \frac{\sin n}{n}$

154. $a_n = \frac{\ln n}{n}$

155. $a_n = (0.9999)^n$

156. $a_n = (123456)^{1/n}$

157. $a_n = \frac{8^n}{n!}$

158. $a_n = \frac{n^{41}}{19^n}$

9.2 Infinite Series

An *infinite series* is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots.$$

The goal of this section is to understand the meaning of such an infinite sum and to develop methods to calculate it. Since there are infinitely many terms to add in an infinite series, we cannot just keep adding to see what comes out. Instead we look at the result of summing just the first n terms of the sequence. The sum of the first n terms

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n$$

is an ordinary finite sum and can be calculated by normal addition. It is called the *n th partial sum*. As n gets larger, we expect the partial sums to get closer and closer to a limiting value in the same sense that the terms of a sequence approach a limit, as discussed in Section 9.1.

For example, to assign meaning to an expression like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots,$$

we add the terms one at a time from the beginning and look for a pattern in how these partial sums grow.

Partial sum		Value	Suggestive expression for partial sum
First:	$s_1 = 1$	1	$2 - 1$
Second:	$s_2 = 1 + \frac{1}{2}$	$\frac{3}{2}$	$2 - \frac{1}{2}$
Third:	$s_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$\frac{7}{4}$	$2 - \frac{1}{4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n th:	$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	$\frac{2^n - 1}{2^{n-1}}$	$2 - \frac{1}{2^{n-1}}$

Indeed there is a pattern. The partial sums form a sequence whose n th term is

$$s_n = 2 - \frac{1}{2^{n-1}}.$$

This sequence of partial sums converges to 2 because $\lim_{n \rightarrow \infty} (1/2^{n-1}) = 0$. We say

“the sum of the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \cdots$ is 2.”

Is the sum of any finite number of terms in this series equal to 2? No. Can we actually add an infinite number of terms one by one? No. But we can still define their sum by defining it to be the limit of the sequence of partial sums as $n \rightarrow \infty$, in this case 2 (Figure 9.8). Our knowledge of sequences and limits enables us to break away from the confines of finite sums.

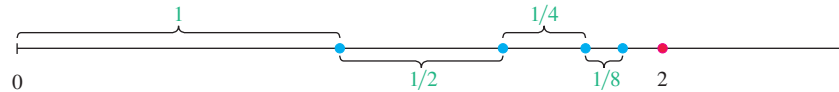


FIGURE 9.8 As the lengths $1, 1/2, 1/4, 1/8, \dots$ are added one by one, the sum approaches 2.

HISTORICAL BIOGRAPHY

Blaise Pascal

(1623–1662)

Pascal was born in France and was encouraged by his father to study science. He met Fermat and was inspired to work on applied science problems. As early as 1640, he wrote an essay on conic sections and earned praise for his work from Descartes. Despite his poor health, Pascal designed an “arithmetic machine” to perform computations for tax collecting. Pascal also contributed to the development of differential calculus.

To know more, visit the companion Website.

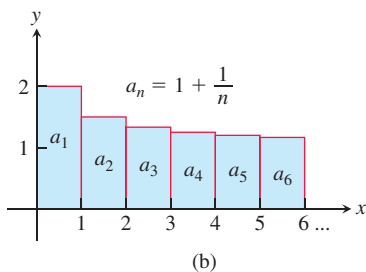
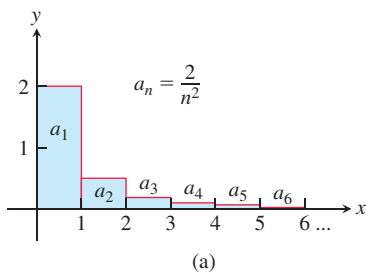


FIGURE 9.9 The sum of a series with positive terms can be interpreted as a total area of an infinite collection of rectangles. The series converges when the total area of the rectangles is finite (a) and diverges when the total area is unbounded (b). Note that the total area can be infinite even if the area of the rectangles is decreasing.

DEFINITIONS Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the **n th term** of the series. The sequence $\{s_n\}$ defined by

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$\vdots$$

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^n a_k$$

$$\vdots$$

is the **sequence of partial sums** of the series, the number s_n being the **n th partial sum**. If the sequence of partial sums converges to a limit L , we say that the series **converges** and that its **sum** is L . In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

If all the terms a_n in an infinite series are positive, then we can represent each term in the series by the area of a rectangle. The series converges if the total area is finite, and diverges otherwise. Figure 9.9a shows an example where the series converges, and Figure 9.9b shows an example where it diverges. The convergence of the total area is related to the convergence or divergence of improper integrals, as we found in Section 8.8. We make this connection explicit in the next section, where we develop an important test for convergence of series, the Integral Test.

When we begin to study a given series $a_1 + a_2 + \cdots + a_n + \cdots$, we might not know whether it converges or diverges. In either case, it is convenient to use sigma notation to write the series as

$$\sum_{n=1}^{\infty} a_n, \quad \sum_{k=1}^{\infty} a_k, \quad \text{or} \quad \sum a_n$$

A useful shorthand when summation from 1 to ∞ is understood

Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1},$$

in which a and r are fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$. The **ratio** r can be positive, as in

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots, \quad r = 1/2, a = 1$$

or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots, \quad r = -1/3, a = 1$$

If $r = 1$, the n th partial sum of the geometric series is

$$s_n = a + a \cdot 1 + a \cdot 1^2 + \cdots + a \cdot 1^{n-1} = na,$$

and the series diverges because $\lim_{n \rightarrow \infty} s_n = \pm\infty$, depending on the sign of a . If $r = -1$, the series diverges because the n th partial sums alternate between a and 0 and never approach a single limit. If $|r| \neq 1$, we can determine the convergence or divergence of the series in the following way:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} \quad \text{Write the } n\text{th partial sum.}$$

$$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n \quad \text{Multiply } s_n \text{ by } r.$$

$$s_n - rs_n = a - ar^n$$

Subtract rs_n from s_n . Most of the terms on the right cancel.

$$s_n(1 - r) = a(1 - r^n)$$

Factor.

$$s_n = \frac{a(1 - r^n)}{1 - r}, \quad (r \neq 1).$$

We can solve for s_n if $r \neq 1$.

If $|r| < 1$, then $r^n \rightarrow 0$ as $n \rightarrow \infty$ (as in Section 9.1), so $s_n \rightarrow a/(1 - r)$ in this case. On the other hand, if $|r| > 1$, then $|r^n| \rightarrow \infty$ and the series diverges. We summarize these results as follows.

Geometric Series

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1 - r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}, \quad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

The formula $a/(1 - r)$ for the sum of a geometric series applies *only* when the summation index begins with $n = 1$ in the expression $\sum_{n=1}^{\infty} ar^{n-1}$ (or with the index $n = 0$ if we write the series as $\sum_{n=0}^{\infty} ar^n$).

EXAMPLE 1 The geometric series with $a = 1/9$ and $r = 1/3$ is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9} \left(\frac{1}{3}\right)^{n-1}.$$

Since $|r| = 1/3 < 1$, the series converges to

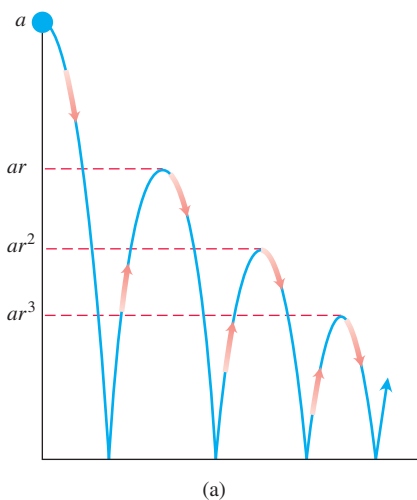
$$\frac{1/9}{1 - (1/3)} = \frac{1}{6}.$$

EXAMPLE 2 The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

is a geometric series with $a = 5$ and $r = -1/4$. Since $|r| = 1/4 < 1$, the series converges to

$$\frac{a}{1 - r} = \frac{5}{1 + (1/4)} = 4.$$



EXAMPLE 3 You drop a ball from a meters above a flat surface. Each time the ball hits the surface after falling a distance h , it rebounds a distance rh , where r is positive but less than 1. Find the total distance the ball travels up and down (Figure 9.10).

Solution The total distance is

$$s = a + \underbrace{2ar + 2ar^2 + 2ar^3 + \cdots}_{\text{This sum is } 2ar/(1-r)} = a + \frac{2ar}{1-r} = a \frac{1+r}{1-r}.$$

If $a = 6$ m and $r = 2/3$, for instance, the distance is

$$s = 6 \cdot \frac{1 + (2/3)}{1 - (2/3)} = 6 \left(\frac{5/3}{1/3} \right) = 30 \text{ m.}$$

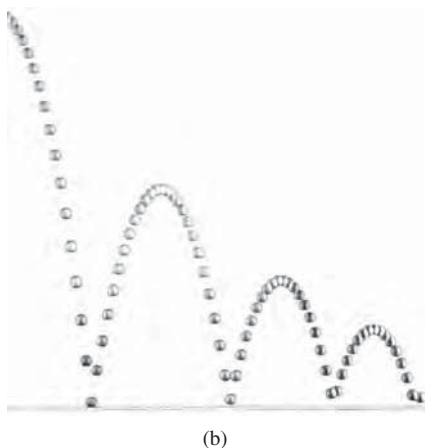


FIGURE 9.10 (a) Example 3 shows how to use a geometric series to calculate the total vertical distance traveled by a bouncing ball if the height of each rebound is reduced by the factor r . (b) A stroboscopic photo of a bouncing ball. (Source: Berenice Abbott/Science Source)

EXAMPLE 4 Express the repeating decimal $5.232323 \dots$ as the ratio of two integers.

Solution From the definition of a decimal number, we get a geometric series.

$$\begin{aligned} 5.232323 \dots &= 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \cdots \\ &= 5 + \frac{23}{100} \underbrace{\left(1 + \frac{1}{100} + \left(\frac{1}{100} \right)^2 + \cdots \right)}_{1/(1 - 1/100)} \quad \begin{array}{l} a = 1, \\ r = 1/100 \\ |r| = 1/100 < 1 \end{array} \\ &= 5 + \frac{23}{100} \left(\frac{1}{0.99} \right) = 5 + \frac{23}{99} = \frac{518}{99} \end{aligned}$$

Unfortunately, formulas like the one for the sum of a convergent geometric series are rare, and we usually have to settle for an estimate of a series' sum (more about this later). The next example, however, is another case in which we can find the sum exactly.

EXAMPLE 5 Find the sum of the “telescoping” series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Solution We look for a pattern in the sequence of partial sums that might lead to a formula for s_k . The key observation is the partial fraction decomposition

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

so

$$\sum_{n=1}^k \frac{1}{n(n+1)} = \sum_{n=1}^k \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

and

$$s_k = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1} \right).$$

Removing parentheses and canceling adjacent terms of opposite sign collapse the sum to

$$s_k = 1 - \frac{1}{k+1}.$$

We now see that $s_k \rightarrow 1$ as $k \rightarrow \infty$. The series converges, and its sum is 1:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1. \quad \blacksquare$$

The n th-Term Test for a Divergent Series

One reason why a series may fail to converge is that its terms don't become small.

EXAMPLE 6 The series

$$\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{n+1}{n} + \cdots$$

diverges because the partial sums eventually outgrow every preassigned number. Each term is greater than 1, so the sum of n terms is greater than n . ■

We now show that $\lim_{n \rightarrow \infty} a_n$ must equal zero if the series $\sum_{n=1}^{\infty} a_n$ converges. To see why, let S represent the series' sum, and $s_n = a_1 + a_2 + \cdots + a_n$ the n th partial sum. When n is large, both s_n and s_{n-1} are close to S , so their difference, a_n , is close to zero. More formally, s_n and s_{n-1} both converge to S as n increases, so

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (s_n - s_{n-1}) = \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} s_{n-1} = S - S = 0. \quad \text{Difference Rule for sequences}$$

This establishes the following theorem.

THEOREM 7 If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \rightarrow 0$.

Theorem 7 leads to a test for detecting the kind of divergence that occurred in Example 6.

The n th-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n \rightarrow \infty} a_n$ fails to exist or is different from zero.

EXAMPLE 7 The following are all examples of divergent series.

(a) $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \rightarrow \infty$. lim a_n fails to exist.

Caution

Theorem 7 does not say that $\sum_{n=1}^{\infty} a_n$ converges if $a_n \rightarrow 0$. It is possible for a series to diverge when $a_n \rightarrow 0$. (See Example 8.)

(b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \rightarrow 1$. $\lim_{n \rightarrow \infty} a_n \neq 0$

(c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n \rightarrow \infty} (-1)^{n+1}$ does not exist.

(d) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n \rightarrow \infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$. ■

EXAMPLE 8 The series

$$1 + \underbrace{\frac{1}{2} + \frac{1}{2}}_{2 \text{ terms}} + \underbrace{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}_{4 \text{ terms}} + \cdots + \underbrace{\frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n}}_{2^n \text{ terms}} + \cdots$$

diverges because the terms can be grouped into infinitely many clusters each of which adds to 1, so the partial sums increase without bound. However, the terms of the series form a sequence that converges to 0. Example 1 of Section 9.3 shows that the harmonic series $\sum 1/n$ also behaves in this manner. ■

Combining Series

Whenever we have two convergent series, we can add them term by term, subtract them term by term, or multiply them by constants to make new convergent series.

THEOREM 8 If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum ka_n = k \sum a_n = kA$ (any number k).

Proof The three rules for series follow from the analogous rules for sequences in Theorem 1, Section 9.1. To prove the Sum Rule for series, let

$$A_n = a_1 + a_2 + \cdots + a_n, \quad B_n = b_1 + b_2 + \cdots + b_n.$$

Then the partial sums of $\sum (a_n + b_n)$ are

$$\begin{aligned} s_n &= (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) \\ &= (a_1 + \cdots + a_n) + (b_1 + \cdots + b_n) \\ &= A_n + B_n. \end{aligned}$$

Since $A_n \rightarrow A$ and $B_n \rightarrow B$, we have $s_n \rightarrow A + B$ by the Sum Rule for sequences. The proof of the Difference Rule is similar.

To prove the Constant Multiple Rule for series, observe that the partial sums of $\sum ka_n$ form the sequence

$$s_n = ka_1 + ka_2 + \cdots + ka_n = k(a_1 + a_2 + \cdots + a_n) = kA_n,$$

which converges to kA by the Constant Multiple Rule for sequences. ■

As corollaries of Theorem 8, we have the following results. We omit the proofs.

1. Every nonzero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n + b_n)$ and $\sum (a_n - b_n)$ both diverge.

Caution Remember that $\sum (a_n + b_n)$ can converge *even if* both $\sum a_n$ and $\sum b_n$ diverge. For example, both $\sum a_n = 1 + 1 + 1 + \cdots$ and $\sum b_n = (-1) + (-1) + (-1) + \cdots$ diverge, whereas $\sum (a_n + b_n) = 0 + 0 + 0 + \cdots$ converges to 0. ●

EXAMPLE 9 Find the sums of the following series.

$$\begin{aligned}
 \text{(a)} \quad \sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}} &= \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}} \right) \\
 &= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} && \text{Difference Rule} \\
 &= \frac{1}{1 - (1/2)} - \frac{1}{1 - (1/6)} && \text{Geometric series with } a = 1 \text{ and } r = 1/2, 1/6 \\
 &&& \text{Both series converge since } |1/2| < 1 \text{ and } |1/6| < 1. \\
 &= 2 - \frac{6}{5} = \frac{4}{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \sum_{n=0}^{\infty} \frac{4}{2^n} &= 4 \sum_{n=0}^{\infty} \frac{1}{2^n} && \text{Constant Multiple Rule} \\
 &= 4 \left(\frac{1}{1 - (1/2)} \right) && \text{Geometric series with } a = 1 \text{ and } r = 1/2 \\
 &&& \text{Converges since } |r| = 1/2 < 1. \\
 &= 8
 \end{aligned}$$

Adding or Deleting Terms

We can add a finite number of terms to a series or delete a finite number of terms without altering the series' convergence or divergence, although in the case of convergence, this will usually change the sum. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$ and

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n.$$

Conversely, if $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$, then $\sum_{n=1}^{\infty} a_n$ converges. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \sum_{n=4}^{\infty} \frac{1}{5^n}$$

and

$$\sum_{n=4}^{\infty} \frac{1}{5^n} = \left(\sum_{n=1}^{\infty} \frac{1}{5^n} \right) - \frac{1}{5} - \frac{1}{25} - \frac{1}{125}.$$

The convergence or divergence of a series is not affected by its first few terms. Only the “tail” of the series, the part that remains when we sum beyond some finite number of initial terms, influences whether it converges or diverges.

Reindexing

As long as we preserve the order of its terms, we can reindex any series without altering its convergence. To raise the starting value of the index h units, replace the n in the formula for a_n by $n - h$:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h} = a_1 + a_2 + a_3 + \cdots.$$

To lower the starting value of the index h units, replace the n in the formula for a_n by $n + h$:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1-h}^{\infty} a_{n+h} = a_1 + a_2 + a_3 + \cdots.$$

HISTORICAL BIOGRAPHY

Richard Dedekind
(1831–1916)

Dedekind grew up in Germany and in 1850 entered the University of Göttingen. There he studied with Bernhard Riemann and Carl Gauss. Like Gauss, Dedekind preferred to study the theoretical aspects of number theory. His work on irrational numbers gave the subject a logical foundation.

To know more, visit the companion Website.

We saw this reindexing in starting a geometric series with the index $n = 0$ instead of the index $n = 1$, but we can use any other starting index value as well. We usually give preference to indexings that lead to simple expressions.

EXAMPLE 10 We can write the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

as

$$\sum_{n=0}^{\infty} \frac{1}{2^n}, \quad \sum_{n=5}^{\infty} \frac{1}{2^{n-5}}, \quad \text{or even} \quad \sum_{n=-4}^{\infty} \frac{1}{2^{n+4}}.$$

The partial sums remain the same no matter what indexing we choose to use. ■

EXERCISES 9.2

Finding n th Partial Sums

In Exercises 1–6, find a formula for the n th partial sum of each series and use it to find the series' sum if the series converges.

1. $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots$

2. $\frac{9}{100} + \frac{9}{100^2} + \frac{9}{100^3} + \cdots + \frac{9}{100^n} + \cdots$

3. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + (-1)^{n-1} \frac{1}{2^{n-1}} + \cdots$

4. $1 - 2 + 4 - 8 + \cdots + (-1)^{n-1} 2^{n-1} + \cdots$

5. $\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{(n+1)(n+2)} + \cdots$

6. $\frac{5}{1 \cdot 2} + \frac{5}{2 \cdot 3} + \frac{5}{3 \cdot 4} + \cdots + \frac{5}{n(n+1)} + \cdots$

Series with Geometric Terms

In Exercises 7–14, write out the first eight terms of each series to show how the series starts. Then find the sum of the series or show that it diverges.

7. $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$

8. $\sum_{n=2}^{\infty} \frac{1}{4^n}$

9. $\sum_{n=1}^{\infty} \left(1 - \frac{7}{4^n}\right)$

10. $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}$

11. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right)$

12. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$

13. $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right)$

14. $\sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{5^n}\right)$

In Exercises 15–22, determine whether the geometric series converges or diverges. If a series converges, find its sum.

15. $1 + \left(\frac{2}{5}\right) + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \cdots$

16. $1 + (-3) + (-3)^2 + (-3)^3 + (-3)^4 + \cdots$

17. $\left(\frac{1}{8}\right) + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \left(\frac{1}{8}\right)^4 + \left(\frac{1}{8}\right)^5 + \cdots$

18. $\left(\frac{-2}{3}\right)^2 + \left(\frac{-2}{3}\right)^3 + \left(\frac{-2}{3}\right)^4 + \left(\frac{-2}{3}\right)^5 + \left(\frac{-2}{3}\right)^6 + \cdots$

19. $1 - \left(\frac{2}{e}\right) + \left(\frac{2}{e}\right)^2 - \left(\frac{2}{e}\right)^3 + \left(\frac{2}{e}\right)^4 - \cdots$

20. $\left(\frac{1}{3}\right)^{-2} - \left(\frac{1}{3}\right)^{-1} + 1 - \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 - \cdots$

21. $1 + \left(\frac{10}{9}\right)^2 + \left(\frac{10}{9}\right)^4 + \left(\frac{10}{9}\right)^6 + \left(\frac{10}{9}\right)^8 + \cdots$

22. $\frac{9}{4} - \frac{27}{8} + \frac{81}{16} - \frac{243}{32} + \frac{729}{64} - \cdots$

Repeating Decimals

Express each of the numbers in Exercises 23–30 as the ratio of two integers.

23. $0.\overline{23} = 0.23\ 23\ 23\ \dots$

24. $0.\overline{234} = 0.234\ 234\ 234\ \dots$

25. $0.\overline{7} = 0.7777\ \dots$

26. $0.\overline{d} = 0.d\ d\ d\ d\ \dots$, where d is a digit

27. $0.0\overline{6} = 0.06666\ \dots$

28. $1.\overline{414} = 1.414\ 414\ 414\ \dots$

29. $1.24\overline{123} = 1.24\ 123\ 123\ 123\ \dots$

30. $3.\overline{142857} = 3.142857\ 142857\ \dots$

Using the n th-Term Test

In Exercises 31–38, use the n th-Term Test for divergence to show that the series is divergent, or state that the test is inconclusive.

31. $\sum_{n=1}^{\infty} \frac{n}{n+10}$

32. $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)(n+3)}$

33. $\sum_{n=0}^{\infty} \frac{1}{n+4}$

34. $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$

35. $\sum_{n=1}^{\infty} \cos \frac{1}{n}$

36. $\sum_{n=0}^{\infty} \frac{e^n}{e^n + n}$

37. $\sum_{n=1}^{\infty} \ln \frac{1}{n}$

38. $\sum_{n=0}^{\infty} \cos n\pi$

69. $\sum_{n=1}^{\infty} \ln \left(\frac{n}{n+1} \right)$

70. $\sum_{n=1}^{\infty} \ln \left(\frac{n}{2n+1} \right)$

71. $\sum_{n=0}^{\infty} \left(\frac{e}{\pi} \right)^n$

72. $\sum_{n=0}^{\infty} \frac{e^{n\pi}}{\pi^{ne}}$

73. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1} - \frac{n+2}{n+3} \right)$

74. $\sum_{n=2}^{\infty} \left(\sin \left(\frac{\pi}{n} \right) - \sin \left(\frac{\pi}{n-1} \right) \right)$

75. $\sum_{n=1}^{\infty} \left(\cos \left(\frac{\pi}{n} \right) + \sin \left(\frac{\pi}{n} \right) \right)$

76. $\sum_{n=0}^{\infty} (\ln(4e^n - 1) - \ln(2e^n + 1))$

Telescoping Series

In Exercises 39–44, find a formula for the n th partial sum of the series and use it to determine whether the series converges or diverges. If a series converges, find its sum.

39. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$

40. $\sum_{n=1}^{\infty} \left(\frac{3}{n^2} - \frac{3}{(n+1)^2} \right)$

41. $\sum_{n=1}^{\infty} (\ln \sqrt{n+1} - \ln \sqrt{n})$

42. $\sum_{n=1}^{\infty} (\tan(n) - \tan(n-1))$

43. $\sum_{n=1}^{\infty} \left(\arccos \left(\frac{1}{n+1} \right) - \arccos \left(\frac{1}{n+2} \right) \right)$

44. $\sum_{n=1}^{\infty} (\sqrt{n+4} - \sqrt{n+3})$

Find the sum of each series in Exercises 45–52.

45. $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$

46. $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$

47. $\sum_{n=1}^{\infty} \frac{40n}{(2n-1)^2(2n+1)^2}$

48. $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

49. $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$

50. $\sum_{n=1}^{\infty} \left(\frac{1}{2^{1/n}} - \frac{1}{2^{1/(n+1)}} \right)$

51. $\sum_{n=1}^{\infty} \left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)} \right)$

52. $\sum_{n=1}^{\infty} (\tan^{-1}(n) - \tan^{-1}(n+1))$

Convergence or Divergence

Which series in Exercises 53–76 converge, and which diverge? Give reasons for your answers. If a series converges, find its sum.

53. $\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}} \right)^n$

54. $\sum_{n=0}^{\infty} (\sqrt{2})^n$

55. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{2^n}$

56. $\sum_{n=1}^{\infty} (-1)^{n+1} n$

57. $\sum_{n=0}^{\infty} \cos \left(\frac{n\pi}{2} \right)$

58. $\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n}$

59. $\sum_{n=0}^{\infty} e^{-2n}$

60. $\sum_{n=1}^{\infty} \ln \frac{1}{3^n}$

61. $\sum_{n=1}^{\infty} \frac{2}{10^n}$

62. $\sum_{n=0}^{\infty} \frac{1}{x^n}, \quad |x| > 1$

63. $\sum_{n=0}^{\infty} \frac{2^n - 1}{3^n}$

64. $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n} \right)^n$

65. $\sum_{n=0}^{\infty} \frac{n!}{1000^n}$

66. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

67. $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$

68. $\sum_{n=1}^{\infty} \frac{2^n + 4^n}{3^n + 4^n}$

Geometric Series with a Variable x

In each of the geometric series in Exercises 77–80, write out the first few terms of the series to find a and r , and find the sum of the series. Then express the inequality $|r| < 1$ in terms of x and find the values of x for which the inequality holds and the series converges.

77. $\sum_{n=0}^{\infty} (-1)^n x^n$

78. $\sum_{n=0}^{\infty} (-1)^n x^{2n}$

79. $\sum_{n=0}^{\infty} 3 \left(\frac{x-1}{2} \right)^n$

80. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2} \left(\frac{1}{3 + \sin x} \right)^n$

In Exercises 81–86, find the values of x for which the given geometric series converges. Also, find the sum of the series (as a function of x) for those values of x .

81. $\sum_{n=0}^{\infty} 2^n x^n$

82. $\sum_{n=0}^{\infty} (-1)^n x^{-2n}$

83. $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$

84. $\sum_{n=0}^{\infty} \left(-\frac{1}{2} \right)^n (x-3)^n$

85. $\sum_{n=0}^{\infty} \sin^n x$

86. $\sum_{n=0}^{\infty} (\ln x)^n$

Theory and Examples

87. The series in Exercise 5 can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \quad \text{and} \quad \sum_{n=-1}^{\infty} \frac{1}{(n+3)(n+4)}.$$

Write this series as a sum beginning with (a) $n = -2$, (b) $n = 0$, (c) $n = 5$.

88. The series in Exercise 6 can also be written as

$$\sum_{n=1}^{\infty} \frac{5}{n(n+1)} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{5}{(n+1)(n+2)}.$$

Write this series as a sum beginning with (a) $n = -1$, (b) $n = 3$, (c) $n = 20$.

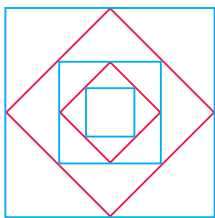
89. Make up an infinite series of nonzero terms whose sum is

a. 1 b. -3 c. 0.

90. (Continuation of Exercise 89.) Can you make an infinite series of nonzero terms that converges to any number you want? Explain.

91. Show by example that $\sum (a_n/b_n)$ may diverge even though $\sum a_n$ and $\sum b_n$ converge and no b_n equals 0.

92. Find convergent geometric series $A = \sum a_n$ and $B = \sum b_n$ that illustrate the fact that $\sum a_n b_n$ may converge without being equal to AB .
93. Show by example that $\sum (a_n/b_n)$ may converge to something other than A/B even when $A = \sum a_n$, $B = \sum b_n \neq 0$, and no b_n equals 0.
94. If $\sum a_n$ converges and $a_n > 0$ for all n , can anything be said about $\sum (1/a_n)$? Give reasons for your answer.
95. What happens if you add a finite number of terms to a divergent series or delete a finite number of terms from a divergent series? Give reasons for your answer.
96. If $\sum a_n$ converges and $\sum b_n$ diverges, can anything be said about their term-by-term sum $\sum (a_n + b_n)$? Give reasons for your answer.
97. Make up a geometric series $\sum ar^{n-1}$ that converges to the number 5 if
- $a = 2$
 - $a = 13/2$.
98. Find the value of b for which
- $$1 + e^b + e^{2b} + e^{3b} + \cdots = 9.$$
99. For what values of r does the infinite series
- $$1 + 2r + r^2 + 2r^3 + r^4 + 2r^5 + r^6 + \cdots$$
- converge? Find the sum of the series when it converges.
100. The accompanying figure shows the first five of a sequence of squares. The outermost square has an area of 4 m^2 . Each of the other squares is obtained by joining the midpoints of the sides of the squares before it. Find the sum of the areas of all the squares.



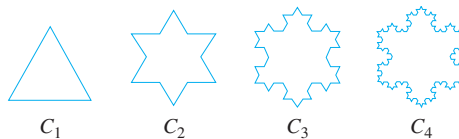
101. **Drug dosage** A patient takes a 300 mg tablet for the control of high blood pressure every morning at the same time. The concentration of the drug in the patient's system decays exponentially at a constant hourly rate of $k = 0.12$.
- How many milligrams of the drug are in the patient's system just before the second tablet is taken? Just before the third tablet is taken?
 - After the patient has taken the medication for at least six months, what quantity of drug is in the patient's body just before the next regularly scheduled morning tablet is taken?
102. Show that the error $(L - s_n)$ obtained by replacing a convergent geometric series with one of its partial sums s_n is $ar^n/(1 - r)$.

103. **The Cantor set** To construct this set, we begin with the closed interval $[0, 1]$. From that interval, we remove the middle open interval $(1/3, 2/3)$, leaving the two closed intervals $[0, 1/3]$ and $[2/3, 1]$. At the second step we remove the open middle third interval from each of those remaining. From $[0, 1/3]$ we remove the open interval $(1/9, 2/9)$, and from $[2/3, 1]$ we remove $(7/9, 8/9)$, leaving behind the four closed intervals $[0, 1/9]$, $[2/9, 1/3]$, $[2/3, 7/9]$, and $[8/9, 1]$. At the next step, we remove the open middle third interval from each closed interval left behind, so $(1/27, 2/27)$ is removed from $[0, 1/9]$, leaving the closed intervals $[0, 1/27]$ and $[2/27, 1/9]$; $(7/27, 8/27)$ is removed from $[2/9, 1/3]$, leaving behind $[2/9, 7/27]$ and $[8/27, 1/3]$, and so forth. We continue this process repeatedly without stopping, at each step removing the open third interval from every closed interval remaining behind from the preceding step. The numbers remaining in the interval $[0, 1]$, after all open middle third intervals have been removed, are the points in the Cantor set (named after Georg Cantor, 1845–1918). The set has some interesting properties.

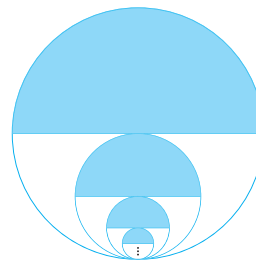
- The Cantor set contains infinitely many numbers in $[0, 1]$. List 12 numbers that belong to the Cantor set.
- Show, by summing an appropriate geometric series, that the total length of all the open middle third intervals that have been removed from $[0, 1]$ is equal to 1.

104. **Helge von Koch's snowflake curve** Helge von Koch's snowflake is a curve of infinite length that encloses a region of finite area. To see why this is so, suppose the curve is generated by starting with an equilateral triangle whose sides have length 1.

- Find the length L_n of the n th curve C_n and show that $\lim_{n \rightarrow \infty} L_n = \infty$.
- Find the area A_n of the region enclosed by C_n and show that $\lim_{n \rightarrow \infty} A_n = (8/5)A_1$.



105. The largest circle in the accompanying figure has radius 1. Consider the sequence of circles of maximum area inscribed in semicircles of diminishing size. What is the sum of the areas of all of the circles?



9.3 The Integral Test

The most basic question we can ask about a series is whether it converges. In this section we begin to study this question, starting with series that have nonnegative terms. Such a series converges if its sequence of partial sums is bounded. If we establish that a given series does converge, we generally do not have a formula available for its sum. So to get an estimate for the sum of a convergent series, we investigate the error involved when using a partial sum to approximate the total sum.

Nondecreasing Partial Sums

Suppose that $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0$ for all n . Then each partial sum is greater than or equal to its predecessor because $s_{n+1} = s_n + a_n$, so

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots$$

Since the partial sums form a nondecreasing sequence, the Monotonic Sequence Theorem (Theorem 6, Section 9.1) gives the following result.

Corollary of Theorem 6

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

EXAMPLE 1 As an application of the above corollary, consider the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

Although the n th term $1/n$ does go to zero, the series diverges because there is no upper bound for its partial sums. To see why, group the terms of the series in the following way:

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{2}{4} = \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{4}{8} = \frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right)}_{> \frac{8}{16} = \frac{1}{2}} + \cdots$$

The sum of the first two terms is 1.5. The sum of the next two terms is $1/3 + 1/4$, which is greater than $1/4 + 1/4 = 1/2$. The sum of the next four terms is $1/5 + 1/6 + 1/7 + 1/8$, which is greater than $1/8 + 1/8 + 1/8 + 1/8 = 1/2$. The sum of the next eight terms is $1/9 + 1/10 + 1/11 + 1/12 + 1/13 + 1/14 + 1/15 + 1/16$, which is greater than $8/16 = 1/2$. The sum of the next 16 terms is greater than $16/32 = 1/2$, and so on. In general, the sum of 2^m terms ending with $1/2^{m+1}$ is greater than $2^m/2^{m+1} = 1/2$. Therefore, if $n = 2^k$, then the partial sum s_n is greater than $k/2$, so the sequence of partial sums is not bounded from above. The harmonic series diverges. ■

The Integral Test

We introduce the Integral Test with a series that is related to the harmonic series, but whose n th term is $1/n^2$ instead of $1/n$.

EXAMPLE 2 Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

Caution

The series and integral need not have the same value in the convergent case. You will see in Example 6 that

$$\sum_{n=1}^{\infty} (1/n^2) \neq \int_1^{\infty} (1/x^2) dx = 1.$$

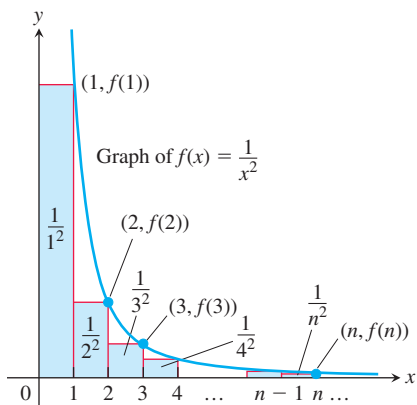


FIGURE 9.11 The sum of the areas of the rectangles under the graph of $f(x) = 1/x^2$ is less than the area under the graph (Example 2).

Solution We determine the convergence of $\sum_{n=1}^{\infty} (1/n^2)$ by comparing it with $\int_1^{\infty} (1/x^2) dx$. To carry out the comparison, we think of the terms of the series as values of the function $f(x) = 1/x^2$ and interpret these values as the areas of rectangles under the curve $y = 1/x^2$.

As Figure 9.11 shows,

$$\begin{aligned} s_n &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\ &= f(1) + f(2) + f(3) + \cdots + f(n) \\ &< f(1) + \int_1^n \frac{1}{x^2} dx \\ &< 1 + \int_1^{\infty} \frac{1}{x^2} dx \\ &= 1 + 1 = 2. \end{aligned}$$

Rectangle areas sum to less than area under graph.

$$\int_1^n (1/x^2) dx < \int_1^{\infty} (1/x^2) dx$$

As in Section 8.8, Example 3, $\int_1^{\infty} (1/x^2) dx = 1$.

Thus the partial sums of $\sum_{n=1}^{\infty} (1/n^2)$ are bounded from above (by 2), and the series converges. ■

THEOREM 9—The Integral Test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x) dx$ both converge or both diverge.

Proof We establish the test for the case $N = 1$. The proof for general N is similar.

We start with the assumption that f is a decreasing function with $f(n) = a_n$ for every n . This leads us to observe that the rectangles in Figure 9.12a, which have areas $a_1, a_2, \dots, a_n$, collectively enclose more area than that under the curve $y = f(x)$ from $x = 1$ to $x = n + 1$. That is,

$$\int_1^{n+1} f(x) dx \leq a_1 + a_2 + \cdots + a_n.$$

In Figure 9.12b the rectangles have been faced to the left instead of to the right. If we momentarily disregard the first rectangle of area a_1 , we see that

$$a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x) dx.$$

If we include a_1 , we have

$$a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^n f(x) dx.$$

Combining these results gives

$$\int_1^{n+1} f(x) dx \leq a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^n f(x) dx.$$

These inequalities hold for each n , and continue to hold as $n \rightarrow \infty$.

If $\int_1^{\infty} f(x) dx$ is finite, the right-hand inequality shows that $\sum a_n$ is finite. If $\int_1^{\infty} f(x) dx$ is infinite, the left-hand inequality shows that $\sum a_n$ is infinite. Hence the series and the integral are either both finite or both infinite. ■

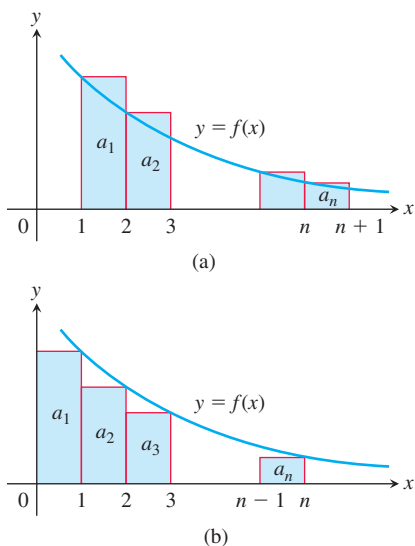


FIGURE 9.12 Subject to the conditions of the Integral Test, the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x) dx$ both converge or both diverge.

The p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

converges if $p > 1$, diverges if $p \leq 1$.

EXAMPLE 3 Show that the p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$ and diverges if $p \leq 1$.

Solution If $p > 1$, then $f(x) = 1/x^p$ is a positive, continuous, and decreasing function of x . Since

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} dx &= \int_1^{\infty} x^{-p} dx = \lim_{b \rightarrow \infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^b && \text{Evaluate the improper integral} \\ &= \frac{1}{1-p} \lim_{b \rightarrow \infty} \left(\frac{1}{b^{p-1}} - 1 \right) \\ &= \frac{1}{1-p} (0 - 1) = \frac{1}{p-1}, && \begin{array}{l} b^{p-1} \rightarrow \infty \text{ as } b \rightarrow \infty \\ \text{because } p-1 > 0. \end{array} \end{aligned}$$

the series converges by the Integral Test. We emphasize that the sum of the p -series is *not* $1/(p-1)$. The series converges, but we don't know the value it converges to.

If $p \leq 0$, the series diverges by the n th-term test. If $0 < p < 1$, then $1-p > 0$ and

$$\int_1^{\infty} \frac{1}{x^p} dx = \frac{1}{1-p} \lim_{b \rightarrow \infty} (b^{1-p} - 1) = \infty.$$

Therefore, the series diverges by the Integral Test.

If $p = 1$, we have the (divergent) harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

In summary, we have convergence for $p > 1$ but divergence for all other values of p . ■

The p -series with $p = 1$ is the **harmonic series** (Example 1). The p -Series Test shows that the harmonic series is just *barely* divergent; if we increase p to 1.000000001, for instance, the series converges!

The slowness with which the partial sums of the harmonic series approach infinity is impressive. For instance, it takes more than 178 million terms of the harmonic series to move the partial sums beyond 20. (See also Exercise 49b.)

EXAMPLE 4 The series $\sum_{n=1}^{\infty} (1/(n^2 + 1))$ is not a p -series, but we will show that it converges by the Integral Test. The function $f(x) = 1/(x^2 + 1)$ is positive, continuous, and decreasing for $x \geq 1$ since it is the reciprocal of a function that is positive, continuous, and increasing on this interval, and

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2 + 1} dx &= \lim_{b \rightarrow \infty} \left[\arctan x \right]_1^b \\ &= \lim_{b \rightarrow \infty} [\arctan b - \arctan 1] \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}. \end{aligned}$$

The Integral Test tells us that the series converges, but it does *not* say that $\pi/4$ is the sum of the series. ■

EXAMPLE 5 Determine the convergence or divergence of the series.

$$(a) \sum_{n=1}^{\infty} n e^{-n^2} \quad (b) \sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$$

Solutions

- (a) The function $f(x) = xe^{-x^2}$ is continuous and positive for $x \geq 1$. The first derivative

$$f'(x) = e^{-x^2} - 2x^2e^{-x^2} = (1 - 2x^2)e^{-x^2}$$

is negative on that interval (since $x \geq 1$ implies $1 - 2x^2 \leq 1 - 2 < 0$), so f is decreasing and we can apply the Integral Test:

$$\begin{aligned} \int_1^\infty \frac{x}{e^{x^2}} dx &= \frac{1}{2} \int_1^\infty \frac{du}{e^u} && u = x^2, du = 2x dx \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{2} e^{-u} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left(-\frac{1}{2e^b} + \frac{1}{2e} \right) = \frac{1}{2e}. \end{aligned}$$

Since the integral converges, the series also converges.

- (b) The integrand $1/(2^{\ln x})$ is a positive, continuous, and decreasing function on $(1, \infty)$, so we can apply the Integral Test.

$$\begin{aligned} \int_1^\infty \frac{dx}{2^{\ln x}} &= \int_0^\infty \frac{e^u du}{2^u} && u = \ln x, x = e^u, dx = e^u du \\ &= \int_0^\infty \left(\frac{e}{2} \right)^u du \\ &= \lim_{b \rightarrow \infty} \frac{1}{\ln\left(\frac{e}{2}\right)} \left(\left(\frac{e}{2} \right)^b - 1 \right) = \infty && (e/2) > 1 \end{aligned}$$

The improper integral diverges, so the series diverges also. ■

Error Estimation

For some convergent series, such as the geometric series or the telescoping series in Example 5 of Section 9.2, we can actually find the total sum of the series. That is, we can find the limiting value S of the sequence of partial sums. For most convergent series, however, we cannot easily find the total sum. Nevertheless, we can *estimate* the sum by adding the first n terms to get s_n , but we need to know how far off s_n is from the total sum S . An approximation to a function or to a number is more useful when it is accompanied by a bound on the size of the worst possible error that could occur. With such an error bound we can try to make an estimate or approximation that is close enough for the problem at hand. Without a bound on the error size, we are just guessing and hoping that we are close to the actual answer. We now show a way to bound the error size using integrals.

Suppose that a series $\sum a_n$ with positive terms is shown to be convergent by the Integral Test, and we want to estimate the size of the **remainder** R_n measuring the difference between the total sum S of the series and its n th partial sum s_n . That is, we wish to estimate

$$R_n = S - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

To get a lower bound for the remainder, we compare the sum of the areas of the rectangles with the area under the curve $y = f(x)$ for $x \geq n$ (see Figure 9.13a). We see that

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \geq \int_{n+1}^\infty f(x) dx.$$

Similarly, from Figure 9.13b, we find an upper bound with

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \leq \int_n^\infty f(x) dx.$$

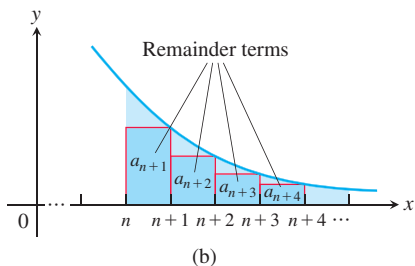
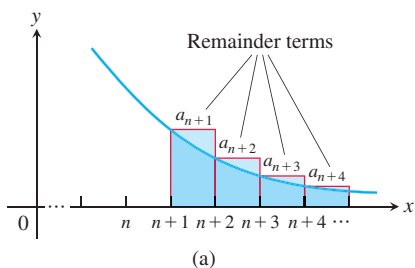


FIGURE 9.13 A geometric interpretation of Remainder Formula (1).

These comparisons prove the following result, giving bounds on the size of the remainder.

Bounds for the Remainder in the Integral Test

Suppose $\{a_k\}$ is a sequence of positive terms with $a_k = f(k)$, where f is a continuous positive decreasing function of x for all $x \geq n$, and suppose that $\sum a_n$ converges to S . Then the remainder $R_n = S - s_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx. \quad (1)$$

Since $s_n + R_n = S$, if we add the partial sum s_n to each side of the inequalities in (1), we get

$$s_n + \int_{n+1}^{\infty} f(x) dx \leq S \leq s_n + \int_n^{\infty} f(x) dx. \quad (2)$$

The inequalities in (2) are useful for estimating the error in approximating the sum of a series known to converge by the Integral Test. The error can be no larger than the length of the interval containing S , with endpoints given by (2).

EXAMPLE 6 Estimate the sum of the series $\sum (1/n^2)$ using the inequalities in (2) and $n = 10$.

Solution We have that

$$\int_n^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_n^b = \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + \frac{1}{n} \right) = \frac{1}{n}.$$

Using this result with the inequalities in (2) gives

$$s_{10} + \frac{1}{11} \leq S \leq s_{10} + \frac{1}{10}.$$

Since $s_{10} = 1 + (1/4) + (1/9) + (1/16) + \cdots + (1/100) \approx 1.54977$, these last inequalities give

$$1.64068 \leq S \leq 1.64977.$$

If we approximate the sum S by the midpoint of this interval, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx 1.6452.$$

The error in this approximation is then less than half the length of the interval, so the error is less than 0.005. Using a trigonometric *Fourier series*, we prove in Section 19.4 that S is equal to $\pi^2/6 \approx 1.64493$. ■

The p -series for $p = 2$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64493$$

EXERCISES 9.3

Applying the Integral Test

Use the Integral Test to determine whether the series in Exercises 1–12 converge or diverge. Be sure to check that the conditions of the Integral Test are satisfied.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

2. $\sum_{n=1}^{\infty} \frac{1}{n^{0.2}}$

3. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}$

4. $\sum_{n=1}^{\infty} \frac{1}{n+4}$

5. $\sum_{n=1}^{\infty} e^{-2n}$

6. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

7. $\sum_{n=1}^{\infty} \frac{n}{n^2 + 4}$

8. $\sum_{n=2}^{\infty} \frac{\ln(n^2)}{n}$

9. $\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$

10. $\sum_{n=2}^{\infty} \frac{n-4}{n^2 - 2n + 1}$

11. $\sum_{n=1}^{\infty} \frac{7}{\sqrt{n+4}}$

12. $\sum_{n=2}^{\infty} \frac{1}{5n + 10\sqrt{n}}$

Determining Convergence or Divergence

Which of the series in Exercises 13–46 converge, and which diverge? Give reasons for your answers. (When you check an answer, remember that there may be more than one way to determine the series' convergence or divergence.)

13. $\sum_{n=1}^{\infty} \frac{1}{10^n}$
14. $\sum_{n=1}^{\infty} e^{-n}$
15. $\sum_{n=1}^{\infty} \frac{n}{n+1}$
16. $\sum_{n=1}^{\infty} \frac{5}{n+1}$
17. $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$
18. $\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}}$
19. $\sum_{n=1}^{\infty} -\frac{1}{8^n}$
20. $\sum_{n=1}^{\infty} \frac{-8}{n}$
21. $\sum_{n=2}^{\infty} \frac{\ln n}{n}$
22. $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$
23. $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$
24. $\sum_{n=1}^{\infty} \frac{5^n}{4^n + 3}$
25. $\sum_{n=0}^{\infty} \frac{-2}{n+1}$
26. $\sum_{n=1}^{\infty} \frac{1}{2n-1}$
27. $\sum_{n=1}^{\infty} \frac{2^n}{n+1}$
28. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$
29. $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$
30. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$
31. $\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n}$
32. $\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n}$
33. $\sum_{n=3}^{\infty} \frac{(1/n)}{(\ln n)\sqrt{\ln^2 n - 1}}$
34. $\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln^2 n)}$
35. $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$
36. $\sum_{n=1}^{\infty} n \tan \frac{1}{n}$
37. $\sum_{n=1}^{\infty} \frac{e^n}{1 + e^{2n}}$
38. $\sum_{n=1}^{\infty} \frac{2}{1 + e^n}$
39. $\sum_{n=1}^{\infty} \frac{e^n}{10 + e^n}$
40. $\sum_{n=1}^{\infty} \frac{e^n}{(10 + e^n)^2}$
41. $\sum_{n=2}^{\infty} \frac{\sqrt{n+2} - \sqrt{n+1}}{\sqrt{n+1}\sqrt{n+2}}$
42. $\sum_{n=3}^{\infty} \frac{7}{\sqrt{n+1} \ln \sqrt{n+1}}$
43. $\sum_{n=1}^{\infty} \frac{8 \tan^{-1} n}{1 + n^2}$
44. $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$
45. $\sum_{n=1}^{\infty} \operatorname{sech} n$
46. $\sum_{n=1}^{\infty} \operatorname{sech}^2 n$

Theory and Examples

For what values of a , if any, do the series in Exercises 47 and 48 converge?

$$47. \sum_{n=1}^{\infty} \left(\frac{a}{n+2} - \frac{1}{n+4} \right) \quad 48. \sum_{n=3}^{\infty} \left(\frac{1}{n-1} - \frac{2a}{n+1} \right)$$

49. a. Draw illustrations like those in Figures 9.12a and 9.12b to show that the partial sums of the harmonic series satisfy the inequalities

$$\begin{aligned} \ln(n+1) &= \int_1^{n+1} \frac{1}{x} dx \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n} \\ &\leq 1 + \int_1^n \frac{1}{x} dx = 1 + \ln n. \end{aligned}$$

- T** b. There is absolutely no empirical evidence for the divergence of the harmonic series even though we know it diverges.

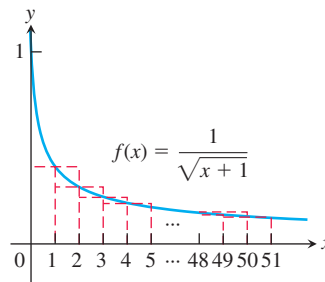
The partial sums just grow too slowly. To see what we mean, suppose you had started with $s_1 = 1$ the day the universe was formed, 13 billion years ago, and added a new term every *second*. About how large would the partial sum s_n be today, assuming a 365-day year?

50. Are there any values of x for which $\sum_{n=1}^{\infty} (1/nx)$ converges? Give reasons for your answer.
51. Is it true that if $\sum_{n=1}^{\infty} a_n$ is a divergent series of positive numbers, then there is also a divergent series $\sum_{n=1}^{\infty} b_n$ of positive numbers with $b_n < a_n$ for every n ? Is there a “smallest” divergent series of positive numbers? Give reasons for your answers.
52. (Continuation of Exercise 51.) Is there a “largest” convergent series of positive numbers? Explain.
53. $\sum_{n=1}^{\infty} (1/\sqrt{n+1})$ diverges

- a. Use the accompanying graph to show that the partial sum $s_{50} = \sum_{n=1}^{50} (1/\sqrt{n+1})$ satisfies

$$\int_1^{51} \frac{1}{\sqrt{x+1}} dx < s_{50} < \int_0^{50} \frac{1}{\sqrt{x+1}} dx.$$

Conclude that $11.5 < s_{50} < 12.3$.

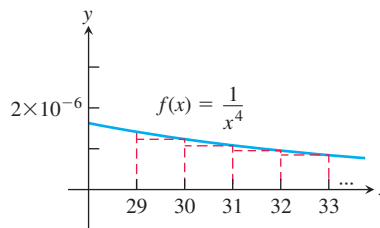


- b. What should n be in order that the partial sum

$$s_n = \sum_{i=1}^n (1/\sqrt{i+1}) \text{ satisfy } s_n > 1000?$$

54. $\sum_{n=1}^{\infty} (1/n^4)$ converges

- a. Use the accompanying graph to find an upper bound for the error if $s_{30} = \sum_{n=1}^{30} (1/n^4)$ is used to estimate the value of $\sum_{n=1}^{\infty} (1/n^4)$.



- b. Find n so that the partial sum $s_n = \sum_{i=1}^n (1/i^4)$ estimates the value of $\sum_{n=1}^{\infty} (1/n^4)$ with an error of at most 0.000001. (The exact value of this series is computed in Exercise 20 of Section 19.4.)

55. Estimate the value of $\sum_{n=1}^{\infty} (1/n^3)$ to within 0.01 of its exact value.
56. Estimate the value of $\sum_{n=2}^{\infty} (1/(n^2 + 4))$ to within 0.1 of its exact value.
57. How many terms of the convergent series $\sum_{n=1}^{\infty} (1/n^{1.1})$ should be used to estimate its value with error at most 0.00001?

58. How many terms of the convergent series $\sum_{n=4}^{\infty} 1/(n(\ln n)^3)$ should be used to estimate its value with error at most 0.01?

59. **The Cauchy condensation test** The Cauchy condensation test says: Let $\{a_n\}$ be a nonincreasing sequence ($a_n \geq a_{n+1}$ for all n) of positive terms that converges to 0. Then $\sum a_n$ converges if and only if $\sum 2^n a_{2^n}$ converges. For example, $\sum (1/n)$ diverges because $\sum 2^n \cdot (1/2^n) = \sum 1$ diverges. Show why the test works.

60. Use the Cauchy condensation test from Exercise 59 to show that

a. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges;

b. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

61. **Logarithmic p -series**

a. Show that the improper integral

$$\int_2^{\infty} \frac{dx}{x(\ln x)^p} \quad (p \text{ a positive constant})$$

converges if and only if $p > 1$.

b. What implications does the fact in part (a) have for the convergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}?$$

Give reasons for your answer.

62. (Continuation of Exercise 61.) Use the result in Exercise 61 to determine which of the following series converge and which diverge. Support your answer in each case.

a. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$

b. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1.01}}$

c. $\sum_{n=2}^{\infty} \frac{1}{n \ln(n^3)}$

d. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

63. **Euler's constant** Graphs like those in Figure 9.12 suggest that as n increases there is little change in the difference between the sum

$$1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

and the integral

$$\ln n = \int_1^n \frac{1}{x} dx.$$

To explore this idea, carry out the following steps.

a. By taking $f(x) = 1/x$ in the proof of Theorem 9, show that

$$\ln(n+1) \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq 1 + \ln n$$

or

$$0 < \ln(n+1) - \ln n \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \leq 1.$$

Thus, the sequence

$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$$

is bounded from below and from above.

b. Show that

$$\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x} dx = \ln(n+1) - \ln n,$$

and use this result to show that the sequence $\{a_n\}$ in part (a) is decreasing.

Since a decreasing sequence that is bounded from below converges, the numbers a_n defined in part (a) converge:

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \rightarrow \gamma.$$

The number γ , whose value is $0.5772\dots$, is called *Euler's constant*.

64. Use the Integral Test to show that the series

$$\sum_{n=0}^{\infty} e^{-n^2}$$

converges.

65. a. For the series $\sum (1/n^3)$, use the inequalities in Equation (2) with $n = 10$ to find an interval containing the sum S .

b. As in Example 5, use the midpoint of the interval found in part (a) to approximate the sum of the series. What is the maximum error for your approximation?

66. Repeat Exercise 65 using the series $\sum (1/n^4)$.

67. **Area** Consider the sequence $\{1/n\}_{n=1}^{\infty}$. On each subinterval $(1/(n+1), 1/n)$ within the interval $[0, 1]$, erect the rectangle with area a_n having height $1/n$ and width equal to the length of the subinterval. Find the total area $\sum a_n$ of all the rectangles. (Hint: Use the result of Example 5 in Section 9.2.)

68. **Area** Repeat Exercise 67, using trapezoids instead of rectangles. That is, on the subinterval $(1/(n+1), 1/n)$, let a_n denote the area of the trapezoid having heights $y = 1/(n+1)$ at $x = 1/(n+1)$ and $y = 1/n$ at $x = 1/n$.

9.4 Comparison Tests

We have seen how to determine the convergence of geometric series, p -series, and a few others. We can test the convergence of many more series by comparing their terms to those of a series whose convergence is already known.

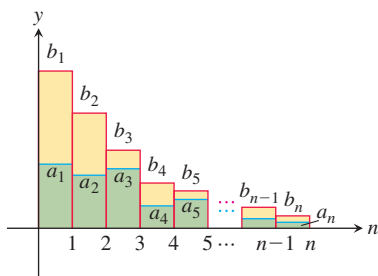


FIGURE 9.14 If the total area $\sum b_n$ of the taller b_n rectangles is finite, then so is the total area $\sum a_n$ of the shorter a_n rectangles.

THEOREM 10—Direct Comparison Test

Let $\sum a_n$ and $\sum b_n$ be two series with $0 \leq a_n \leq b_n$ for all n . Then

1. If $\sum b_n$ converges, then $\sum a_n$ also converges.
2. If $\sum a_n$ diverges, then $\sum b_n$ also diverges.

Proof The series $\sum a_n$ and $\sum b_n$ have nonnegative terms. The Corollary of Theorem 6 stated in Section 9.3 tells us that the series $\sum a_n$ and $\sum b_n$ converge if and only if their partial sums are bounded from above.

In Part (1) we assume that $\sum b_n$ converges to some number M . The partial sums $\sum_{n=1}^N a_n$ are all bounded from above by $M = \sum b_n$ because

$$s_N = a_1 + a_2 + \cdots + a_N \leq b_1 + b_2 + \cdots + b_N \leq \sum_{n=1}^{\infty} b_n = M.$$

Since the partial sums of $\sum a_n$ are bounded from above, the Corollary of Theorem 6 implies that $\sum a_n$ converges. We conclude that if $\sum b_n$ converges, then so does $\sum a_n$. Figure 9.14 illustrates this result, with each term of each series interpreted as the area of a rectangle.

In Part (2), where we assume that $\sum a_n$ diverges, the partial sums of $\sum_{n=1}^{\infty} b_n$ are not bounded from above. If they were, the partial sums for $\sum a_n$ would also be bounded from above, since

$$a_1 + a_2 + \cdots + a_N \leq b_1 + b_2 + \cdots + b_N,$$

and this would mean that $\sum a_n$ converges. We conclude that if $\sum a_n$ diverges, then so does $\sum b_n$. ■

EXAMPLE 1 We apply Theorem 10 to several series.

(a) The series

$$\sum_{n=1}^{\infty} \frac{5}{5n-1}$$

diverges because its n th term

$$\frac{5}{5n-1} = \frac{1}{n-\frac{1}{5}} > \frac{1}{n} \quad \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges and } 1/n > 0.$$

is positive and is greater than the n th term of the (positive) divergent harmonic series.

(b) The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

converges because its terms are all positive and less than or equal to the corresponding terms of

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots.$$

The geometric series on the left converges (since $|r| = 1/2 < 1$) and we have

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{1-(1/2)} = 3.$$

The fact that 3 is an upper bound for the partial sums of $\sum_{n=0}^{\infty} (1/n!)$ does not mean that the series converges to 3. As we will see in Section 9.9, the series converges to e .

HISTORICAL BIOGRAPHY

Albert of Saxony
(ca. 1316–1390)

Albert attended the University of Paris and gained recognition as a teacher at its faculty of arts. He wrote on squaring the circle and other geometric problems. He also published books on physics and mechanics, *Tractatus proportionum* being the most popular one.

To know more, visit the companion Website.

(c) The series

$$5 + \frac{2}{3} + \frac{1}{7} + 1 + \frac{1}{2 + \sqrt{1}} + \frac{1}{4 + \sqrt{2}} + \frac{1}{8 + \sqrt{3}} + \cdots + \frac{1}{2^n + \sqrt{n}} + \cdots$$

converges. To see this, we ignore the first three terms and compare the remaining terms with those of the convergent geometric series $\sum_{n=0}^{\infty} (1/2^n)$. The term $1/(2^n + \sqrt{n})$ of the truncated sequence is positive and is less than the corresponding term $1/2^n$ of the geometric series. We see that term-by-term we have the comparison of positive terms

$$1 + \frac{1}{2 + \sqrt{1}} + \frac{1}{4 + \sqrt{2}} + \frac{1}{8 + \sqrt{3}} + \cdots \leq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots.$$

So the truncated series and the original series converge by an application of the Direct Comparison Test. ■

The Limit Comparison Test

We now introduce a comparison test that is particularly useful for series in which a_n is a rational function of n .

THEOREM 11—Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Proof We will prove Part 1. Parts 2 and 3 are left as Exercises 57a and b.

We assume that $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$ where $c > 0$. Then $\varepsilon = c/2$ is a positive number, so by the definition of convergence there exists an integer N such that

$$\left| \frac{a_n}{b_n} - c \right| < \frac{c}{2} \quad \text{whenever} \quad n > N. \quad \begin{array}{l} \text{Limit definition with} \\ \varepsilon = c/2, L = c, \text{ and} \\ a_n \text{ replaced by } a_n/b_n \end{array}$$

Thus, for $n > N$,

$$\begin{aligned} -\frac{c}{2} &< \frac{a_n}{b_n} - c < \frac{c}{2}, \\ \frac{c}{2} &< \frac{a_n}{b_n} < \frac{3c}{2}, \\ \left(\frac{c}{2}\right)b_n &< a_n < \left(\frac{3c}{2}\right)b_n. \end{aligned}$$

If $\sum b_n$ converges, then $\sum (3c/2)b_n$ converges and $\sum a_n$ converges by the Direct Comparison Test. If $\sum b_n$ diverges, then $\sum (c/2)b_n$ diverges and $\sum a_n$ diverges by the Direct Comparison Test. ■

EXAMPLE 2 Which of the following series converge, and which diverge?

$$(a) \frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$$

$$(b) \frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

$$(c) \frac{1 + 2 \ln 2}{9} + \frac{1 + 3 \ln 3}{14} + \frac{1 + 4 \ln 4}{21} + \cdots = \sum_{n=2}^{\infty} \frac{1 + n \ln n}{n^2 + 5}$$

Solution We apply the Limit Comparison Test to each series.

- (a) Let $a_n = (2n + 1)/(n^2 + 2n + 1)$. For large n , we expect a_n to behave like $2n/n^2 = 2/n$ since the leading terms dominate for large n , so we let $b_n = 1/n$. The numbers a_n and b_n are positive for each n , the series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges,}$$

and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2n^2 + n}{n^2 + 2n + 1} = 2,$$

so $\sum a_n$ diverges by Part 1 of the Limit Comparison Test. We could just as well have taken $b_n = 2/n$, but $1/n$ is simpler.

- (b) Let $a_n = 1/(2^n - 1)$. For large n , we expect a_n to behave like $1/2^n$, so we let $b_n = 1/2^n$. The numbers a_n and b_n are positive for each n , the series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges,}$$

and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2^n}{2^n - 1} = \lim_{n \rightarrow \infty} \frac{1}{1 - (1/2^n)} = 1,$$

so $\sum a_n$ converges by Part 1 of the Limit Comparison Test.

- (c) Let $a_n = (1 + n \ln n)/(n^2 + 5)$. For large n , we expect a_n to behave like $(n \ln n)/n^2 = (\ln n)/n$, which is greater than $1/n$ for $n \geq 3$, so we let $b_n = 1/n$. The numbers a_n and b_n are positive for each n , the series

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n} \text{ diverges,}$$

and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n + n^2 \ln n}{n^2 + 5} = \infty,$$

so $\sum a_n$ diverges by Part 3 of the Limit Comparison Test. ■

EXAMPLE 3 Does $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ converge?

Solution First note that both $\ln n$ and $n^{3/2}$ are positive for $n \geq 3$, so the Limit Comparison Theorem can be applied. Because $\ln n$ grows more slowly than n^c for any positive constant c (Section 9.1, Exercise 115), we can compare the series to a convergent p -series. To get the p -series, we see that

$$\frac{\ln n}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}$$

for n sufficiently large. Then, taking $a_n = (\ln n)/n^{3/2}$ and $b_n = 1/n^{5/4}$, we have

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\ln n}{n^{1/4}} \\ &= \lim_{n \rightarrow \infty} \frac{1/n}{(1/4)n^{-3/4}} \quad \text{L'Hôpital's Rule} \\ &= \lim_{n \rightarrow \infty} \frac{4}{n^{1/4}} = 0.\end{aligned}$$

Since $\sum b_n = \sum (1/n^{5/4})$ is a p -series with $p > 1$, it converges. Therefore, $\sum a_n$ converges by Part 2 of the Limit Comparison Test. ■

EXERCISES 9.4

Direct Comparison Test

In Exercises 1–8, use the Direct Comparison Test to determine whether each series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 30}$
2. $\sum_{n=1}^{\infty} \frac{n-1}{n^4 + 2}$
3. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1}$
4. $\sum_{n=2}^{\infty} \frac{n+2}{n^2 - n}$
5. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$
6. $\sum_{n=1}^{\infty} \frac{1}{n3^n}$
7. $\sum_{n=1}^{\infty} \sqrt{\frac{n+4}{n^4 + 4}}$
8. $\sum_{n=1}^{\infty} \frac{\sqrt{n} + 1}{\sqrt{n^2 + 3}}$

Limit Comparison Test

In Exercises 9–16, use the Limit Comparison Test to determine whether each series converges or diverges.

9. $\sum_{n=1}^{\infty} \frac{n-2}{n^3 - n^2 + 3}$
(Hint: Limit Comparison with $\sum_{n=1}^{\infty} (1/n^2)$)
10. $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2 + 2}}$
(Hint: Limit Comparison with $\sum_{n=1}^{\infty} (1/\sqrt{n})$)
11. $\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2 + 1)(n-1)}$
12. $\sum_{n=1}^{\infty} \frac{2^n}{3 + 4^n}$
13. $\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n} 4^n}$
14. $\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4} \right)^n$
15. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$
(Hint: Limit Comparison with $\sum_{n=2}^{\infty} (1/n)$)
16. $\sum_{n=1}^{\infty} \ln \left(1 + \frac{1}{n^2} \right)$
(Hint: Limit Comparison with $\sum_{n=1}^{\infty} (1/n^2)$)
20. $\sum_{n=1}^{\infty} \frac{1 + \cos n}{n^2}$
21. $\sum_{n=1}^{\infty} \frac{2n}{3n-1}$
22. $\sum_{n=1}^{\infty} \frac{n+1}{n^2 \sqrt{n}}$
23. $\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$
24. $\sum_{n=3}^{\infty} \frac{5n^3 - 3n}{n^2(n-2)(n^2+5)}$
25. $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1} \right)^n$
26. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 2}}$
27. $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$
28. $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$
29. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} \ln n}$
30. $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$
31. $\sum_{n=1}^{\infty} \frac{1}{1 + \ln n}$
32. $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$
33. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2 - 1}}$
34. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$
35. $\sum_{n=1}^{\infty} \frac{1-n}{n2^n}$
36. $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n}$
37. $\sum_{n=1}^{\infty} \frac{1}{3^{n-1} + 1}$
38. $\sum_{n=1}^{\infty} \frac{3^{n-1} + 1}{3^n}$
39. $\sum_{n=1}^{\infty} \frac{n+1}{n^2 + 3n} \cdot \frac{1}{5n}$
40. $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n}$
41. $\sum_{n=1}^{\infty} \frac{2^n - n}{n2^n}$
42. $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n} e^n}$
43. $\sum_{n=2}^{\infty} \frac{1}{n!}$
(Hint: First show that $(1/n!) \leq (1/n(n-1))$ for $n \geq 2$.)
44. $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+2)!}$
45. $\sum_{n=1}^{\infty} \sin \frac{1}{n}$
46. $\sum_{n=1}^{\infty} \tan \frac{1}{n}$
47. $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^{1.1}}$
48. $\sum_{n=1}^{\infty} \frac{\operatorname{arcsec} n}{n^{1.3}}$
49. $\sum_{n=1}^{\infty} \frac{\coth n}{n^2}$
50. $\sum_{n=1}^{\infty} \frac{\tanh n}{n^2}$
51. $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{n}}}$
52. $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$

Determining Convergence or Divergence

Which of the series in Exercises 17–56 converge, and which diverge? Use any method, and give reasons for your answers.

17. $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}}$
18. $\sum_{n=1}^{\infty} \frac{3}{n + \sqrt{n}}$
19. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$
53. $\sum_{n=1}^{\infty} \frac{1}{1 + 2 + 3 + \cdots + n}$
54. $\sum_{n=1}^{\infty} \frac{1}{1 + 2^2 + 3^2 + \cdots + n^2}$
55. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^2}$
56. $\sum_{n=2}^{\infty} \frac{(\ln n)^2}{n}$

Theory and Examples

57. Prove (a) Part 2 and (b) Part 3 of the Limit Comparison Test.
58. If $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative numbers, can anything be said about $\sum_{n=1}^{\infty} (a_n/n)$? Explain.
59. Suppose that $a_n > 0$ and $b_n > 0$ for $n \geq N$ (N an integer). If $\lim_{n \rightarrow \infty} (a_n/b_n) = \infty$ and $\sum a_n$ converges, can anything be said about $\sum b_n$? Give reasons for your answer.
60. Prove that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ converges.
61. Suppose that $a_n > 0$ and $\lim_{n \rightarrow \infty} a_n = \infty$. Prove that $\sum a_n$ diverges.
62. Suppose that $a_n > 0$ and $\lim_{n \rightarrow \infty} n^2 a_n = 0$. Prove that $\sum a_n$ converges.
63. Show that $\sum_{n=2}^{\infty} ((\ln n)^q / n^p)$ converges for $-\infty < q < \infty$ and $p > 1$.
(Hint: Limit Comparison with $\sum_{n=2}^{\infty} 1/n^r$ for $1 < r < p$.)
64. (Continuation of Exercise 63.) Show that $\sum_{n=2}^{\infty} ((\ln n)^q / n^p)$ diverges for $-\infty < q < \infty$ and $0 < p < 1$.
(Hint: Limit Comparison with an appropriate p -series.)
65. **Decimal numbers** Any real number in the interval $[0, 1]$ can be represented by a decimal (not necessarily unique) as

$$0.d_1 d_2 d_3 d_4 \cdots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \cdots,$$

where d_i is one of the integers $0, 1, 2, 3, \dots, 9$. Prove that the series on the right-hand side always converges.

66. If $\sum a_n$ is a convergent series of positive terms, prove that $\sum \sin(a_n)$ converges.

In Exercises 67–72, use the results of Exercises 63 and 64 to determine whether each series converges or diverges.

67. $\sum_{n=2}^{\infty} \frac{(\ln n)^3}{n^4}$ 68. $\sum_{n=2}^{\infty} \sqrt{\frac{\ln n}{n}}$
69. $\sum_{n=2}^{\infty} \frac{(\ln n)^{1000}}{n^{1.001}}$ 70. $\sum_{n=2}^{\infty} \frac{(\ln n)^{1/5}}{n^{0.99}}$
71. $\sum_{n=2}^{\infty} \frac{1}{n^{1.1} (\ln n)^3}$ 72. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n \cdot \ln n}}$

COMPUTER EXPLORATIONS

73. It is not yet known whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2 n}$$

converges or diverges. Use a CAS to explore the behavior of the series by performing the following steps.

- a. Define the sequence of partial sums

$$s_k = \sum_{n=1}^k \frac{1}{n^3 \sin^2 n}.$$

What happens when you try to find the limit of s_k as $k \rightarrow \infty$? Does your CAS find a closed form answer for this limit?

- b. Plot the first 100 points (k, s_k) for the sequence of partial sums. Do they appear to converge? What would you estimate the limit to be?
- c. Next plot the first 200 points (k, s_k) . Discuss the behavior in your own words.
- d. Plot the first 400 points (k, s_k) . What happens when $k = 355$? Calculate the number $355/113$. Explain from your calculation what happened at $k = 355$. For what values of k would you guess this behavior might occur again?

74. a. Use Theorem 8 to show that

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n(n+1)} \right),$$

where $S = \sum_{n=1}^{\infty} (1/n^2)$, the sum of a convergent p -series.

- b. From Example 5, Section 9.2, show that

$$S = 1 + \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}.$$

- c. Explain why taking the first M terms in the series in part (b) gives a better approximation to S than taking the first M terms in the original series $\sum_{n=1}^{\infty} (1/n^2)$.
- d. We know the exact value of S is $\pi^2/6$. Which of these sums,

$$\sum_{n=1}^{1000000} \frac{1}{n^2} \quad \text{or} \quad 1 + \sum_{n=1}^{1000} \frac{1}{n^2(n+1)},$$

gives a better approximation to S ?

9.5 Absolute Convergence; The Ratio and Root Tests

When some of the terms of a series are positive and others are negative, the series may or may not converge. For example, the geometric series

$$5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots = \sum_{n=0}^{\infty} 5 \left(\frac{-1}{4} \right)^n \quad (1)$$

converges (since $|r| = \frac{1}{4} < 1$), whereas the different geometric series

$$1 - \frac{5}{4} + \frac{25}{16} - \frac{125}{64} + \cdots = \sum_{n=0}^{\infty} \left(\frac{-5}{4} \right)^n \quad (2)$$

diverges (since $|r| = 5/4 > 1$). In series (1), there is some cancellation in the partial sums, which may be assisting the convergence property of the series. However, if we make all of the terms positive in series (1) to form the new series

$$5 + \frac{5}{4} + \frac{5}{16} + \frac{5}{64} + \cdots = \sum_{n=0}^{\infty} \left| 5 \left(\frac{-1}{4} \right)^n \right| = \sum_{n=0}^{\infty} 5 \left(\frac{1}{4} \right)^n,$$

we see that it still converges. For a general series with both positive and negative terms, we can apply the tests for convergence that we studied before to the series of absolute values of its terms. In doing so, we are led naturally to the following concept.

DEFINITION A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

So the geometric series (1) is absolutely convergent. We observed, too, that it is also convergent. This situation is always true: An absolutely convergent series is convergent as well, which we now prove.

Caution

Be careful when using Theorem 12. A convergent series *need not* converge absolutely, as you will see in the next section.

THEOREM 12—The Absolute Convergence Test

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof For each n ,

$$-|a_n| \leq a_n \leq |a_n|, \quad \text{so} \quad 0 \leq a_n + |a_n| \leq 2|a_n|.$$

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} 2|a_n|$ converges and, by the Direct Comparison Test, the nonnegative series $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges. The equality $a_n = (a_n + |a_n|) - |a_n|$ now lets us express $\sum_{n=1}^{\infty} a_n$ as the difference of two convergent series:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n| - |a_n|) = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|.$$

Therefore, $\sum_{n=1}^{\infty} a_n$ converges. ■

EXAMPLE 1 This example gives two series that converge absolutely.

(a) For $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$, the corresponding series of absolute values is the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots.$$

The original series converges because it converges absolutely.

(b) For $\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$, which contains both positive and negative terms, the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \cdots,$$

which converges by comparison with $\sum_{n=1}^{\infty} (1/n^2)$ because $|\sin n| \leq 1$ for every n . The original series converges absolutely; therefore, it converges. ■

The Ratio Test

The Ratio Test measures the rate of growth (or decline) of a series by examining the ratio a_{n+1}/a_n . For a geometric series $\sum ar^n$, this rate is a constant $((ar^{n+1})/(ar^n) = r)$, and the series converges if and only if its ratio is less than 1 in absolute value. The Ratio Test is a powerful rule extending that result.

THEOREM 13—The Ratio Test

Let $\sum a_n$ be any series and suppose that

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho.$$

Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, and **(c)** the test is *inconclusive* if $\rho = 1$.

ρ is the Greek lowercase letter rho, which is pronounced “row.”

Proof

(a) $\rho < 1$. Let r be a number between ρ and 1. Then the number $\varepsilon = r - \rho$ is positive. Since

$$\left| \frac{a_{n+1}}{a_n} \right| \rightarrow \rho,$$

$|a_{n+1}/a_n|$ must lie within ε of ρ when n is large enough, say, for all $n \geq N$. In particular,

$$\left| \frac{a_{n+1}}{a_n} \right| < \rho + \varepsilon = r, \quad \text{when } n \geq N.$$

Hence

$$\begin{aligned} |a_{N+1}| &< r|a_N|, \\ |a_{N+2}| &< r|a_{N+1}| < r^2|a_N|, \\ |a_{N+3}| &< r|a_{N+2}| < r^3|a_N|, \\ &\vdots \\ |a_{N+m}| &< r|a_{N+m-1}| < r^m|a_N|. \end{aligned}$$

Therefore,

$$\sum_{m=N}^{\infty} |a_m| = \sum_{m=0}^{\infty} |a_{N+m}| \leq \sum_{m=0}^{\infty} |a_N| r^m = |a_N| \sum_{m=0}^{\infty} r^m.$$

The geometric series on the right-hand side converges because $0 < r < 1$, so the series of absolute values $\sum_{m=N}^{\infty} |a_m|$ converges by the Direct Comparison Test. Because adding or deleting finitely many terms in a series does not affect its convergence or divergence property, the series $\sum_{n=1}^{\infty} |a_n|$ also converges. That is, the series $\sum a_n$ is absolutely convergent.

(b) $1 < \rho \leq \infty$. From some index M on,

$$\left| \frac{a_{n+1}}{a_n} \right| > 1 \quad \text{and} \quad |a_M| < |a_{M+1}| < |a_{M+2}| < \cdots.$$

The terms of the series do not approach zero as n becomes infinite, so the series diverges by the n th-Term Test.

(c) $\rho = 1$. The two series

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

show that some other test for convergence must be used when $\rho = 1$.

$$\text{For } \sum_{n=1}^{\infty} \frac{1}{n}: \left| \frac{a_{n+1}}{a_n} \right| = \frac{1/(n+1)}{1/n} = \frac{n}{n+1} \rightarrow 1.$$

$$\text{For } \sum_{n=1}^{\infty} \frac{1}{n^2}: \left| \frac{a_{n+1}}{a_n} \right| = \frac{1/(n+1)^2}{1/n^2} = \left(\frac{n}{n+1} \right)^2 \rightarrow 1^2 = 1.$$

In both cases, $\rho = 1$, yet the first series diverges, whereas the second converges. ■

The Ratio Test is often effective when the terms of a series contain factorials of expressions involving n or expressions raised to a power involving n .

EXAMPLE 2 Investigate the convergence of the following series.

$$(a) \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \quad (b) \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!} \quad (c) \sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$$

Solution We apply the Ratio Test to each series.

(a) For the series $\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3} \cdot \frac{2^{n+1} + 5}{2^n + 5} = \frac{1}{3} \cdot \left(\frac{2 + 5 \cdot 2^{-n}}{1 + 5 \cdot 2^{-n}} \right) \rightarrow \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}.$$

The series converges absolutely (and thus converges) because $\rho = 2/3$ is less than 1. This does *not* mean that $2/3$ is the sum of the series. In fact,

$$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3} \right)^n + \sum_{n=0}^{\infty} \frac{5}{3^n} = \frac{1}{1 - (2/3)} + \frac{5}{1 - (1/3)} = \frac{21}{2}.$$

(b) If $a_n = \frac{(2n)!}{n!n!}$, then $a_{n+1} = \frac{(2n+2)!}{(n+1)!(n+1)!}$ and

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{n!n!(2n+2)(2n+1)(2n)!}{(n+1)!(n+1)!(2n)!} \\ &= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \frac{4n+2}{n+1} \rightarrow 4. \end{aligned}$$

The series diverges because $\rho = 4$ is greater than 1.

(c) If $a_n = \frac{4^n n! n!}{(2n)!}$, then

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{4^{n+1} (n+1)!(n+1)!}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{4^n n! n!} \\ &= \frac{4(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{2(n+1)}{2n+1} \rightarrow 1. \end{aligned}$$

Because the limit is $\rho = 1$, we cannot decide from the Ratio Test whether the series converges. However, when we notice that $a_{n+1}/a_n = (2n+2)/(2n+1)$, we conclude that a_{n+1} is always greater than a_n because $(2n+2)/(2n+1)$ is always greater than 1. Therefore, all terms are greater than or equal to $a_1 = 2$, and the n th term does not approach zero as $n \rightarrow \infty$. The series diverges. ■

The Root Test

The convergence tests for $\sum a_n$ that we have studied so far work best when the formula for a_n is relatively simple. However, consider the series with the terms

$$a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even.} \end{cases}$$

To investigate convergence we write out several terms of the series:

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \frac{1}{2^1} + \frac{1}{2^2} + \frac{3}{2^3} + \frac{1}{2^4} + \frac{5}{2^5} + \frac{1}{2^6} + \frac{7}{2^7} + \cdots \\ &= \frac{1}{2} + \frac{1}{4} + \frac{3}{8} + \frac{1}{16} + \frac{5}{32} + \frac{1}{64} + \frac{7}{128} + \cdots. \end{aligned}$$

Clearly, this is not a geometric series. The n th term approaches zero as $n \rightarrow \infty$, so the n th-Term Test does not tell us whether the series diverges. The Integral Test does not look promising. The Ratio Test produces

$$\left| \frac{a_{n+1}}{a_n} \right| = \begin{cases} \frac{1}{2n}, & n \text{ odd} \\ \frac{n+1}{2}, & n \text{ even.} \end{cases}$$

As $n \rightarrow \infty$, the ratio is alternately small and large and therefore has no limit. However, we will see that the following test establishes that the series converges.

THEOREM 14—The Root Test

Let $\sum a_n$ be any series and suppose that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \rho.$$

Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, and **(c)** the test is *inconclusive* if $\rho = 1$.

Proof

(a) $\rho < 1$. Choose an $\varepsilon > 0$ so small that $\rho + \varepsilon < 1$. Since $\sqrt[n]{|a_n|} \rightarrow \rho$, the terms $\sqrt[n]{|a_n|}$ eventually get to within ε of ρ . So there exists an index M such that

$$\sqrt[n]{|a_n|} < \rho + \varepsilon \quad \text{when } n \geq M.$$

Then it is also true that

$$|a_n| < (\rho + \varepsilon)^n \quad \text{for } n \geq M.$$

Now, $\sum_{n=M}^{\infty} (\rho + \varepsilon)^n$ is a geometric series with ratio $0 < (\rho + \varepsilon) < 1$ and therefore converges. By the Direct Comparison Test, $\sum_{n=M}^{\infty} |a_n|$ converges. Adding finitely many terms to a series does not affect its convergence or divergence, so the series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + \cdots + |a_{M-1}| + \sum_{n=M}^{\infty} |a_n|$$

also converges. Therefore, $\sum a_n$ converges absolutely.

(b) $1 < \rho \leq \infty$. For all indices beyond some integer M , we have $\sqrt[n]{|a_n|} > 1$, and therefore $|a_n| > 1$ for $n > M$. The terms of the series do not converge to zero. The series diverges by the n th-Term Test.

(c) $\rho = 1$. The series $\sum_{n=1}^{\infty} (1/n)$ and $\sum_{n=1}^{\infty} (1/n^2)$ show that the test is not conclusive when $\rho = 1$. The first series diverges and the second converges, but in both cases $\sqrt[n]{|a_n|} \rightarrow 1$. ■

EXAMPLE 3 Consider again the series with terms $a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even.} \end{cases}$
Does $\sum a_n$ converge?

Solution We apply the Root Test, finding that

$$\sqrt[n]{|a_n|} = \begin{cases} \sqrt[n]{n}/2, & n \text{ odd} \\ 1/2, & n \text{ even.} \end{cases}$$

Therefore,

$$\frac{1}{2} \leq \sqrt[n]{|a_n|} \leq \frac{\sqrt[n]{n}}{2}.$$

Since $\sqrt[n]{n} \rightarrow 1$ (Section 9.1, Theorem 5), we have $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1/2$ by the Sandwich Theorem. The limit is less than 1, so the series converges absolutely by the Root Test. ■

EXAMPLE 4 Which of the following series converge, and which diverge?

$$(a) \sum_{n=1}^{\infty} \frac{n^2}{2^n} \qquad (b) \sum_{n=1}^{\infty} \frac{2^n}{n^3} \qquad (c) \sum_{n=1}^{\infty} \left(\frac{1}{1+n} \right)^n$$

Solution We apply the Root Test to each series, noting that each series has positive terms.

$$(a) \sum_{n=1}^{\infty} \frac{n^2}{2^n} \text{ converges because } \sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{(\sqrt[n]{n})^2}{2} \rightarrow \frac{1^2}{2} < 1.$$

$$(b) \sum_{n=1}^{\infty} \frac{2^n}{n^3} \text{ diverges because } \sqrt[n]{\frac{2^n}{n^3}} = \frac{2}{(\sqrt[n]{n})^3} \rightarrow \frac{2}{1^3} > 1.$$

$$(c) \sum_{n=1}^{\infty} \left(\frac{1}{1+n} \right)^n \text{ converges because } \sqrt[n]{\left(\frac{1}{1+n} \right)^n} = \frac{1}{1+n} \rightarrow 0 < 1. \quad \blacksquare$$

EXERCISES 9.5

Using the Ratio Test

In Exercises 1–8, use the Ratio Test to determine whether each series converges absolutely or diverges.

$$1. \sum_{n=1}^{\infty} \frac{2^n}{n!}$$

$$2. \sum_{n=1}^{\infty} (-1)^n \frac{n+2}{3^n}$$

$$3. \sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$$

$$4. \sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}}$$

$$5. \sum_{n=1}^{\infty} \frac{n^4}{(-4)^n}$$

$$6. \sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$$

$$7. \sum_{n=1}^{\infty} (-1)^n \frac{n^2(n+2)!}{n!3^{2n}}$$

$$8. \sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)\ln(n+1)}$$

$$13. \sum_{n=1}^{\infty} \frac{-8}{(3 + (1/n))^{2n}}$$

$$14. \sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$$

$$15. \sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{1}{n}\right)^{n^2}$$

(Hint: $\lim_{n \rightarrow \infty} (1 + x/n)^n = e^x$)

$$16. \sum_{n=2}^{\infty} \frac{(-1)^n}{n^{1+n}}$$

Determining Convergence or Divergence

In Exercises 17–46, use any method to determine whether the series converges or diverges. Give reasons for your answer.

$$17. \sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$$

$$18. \sum_{n=1}^{\infty} (-1)^n n^2 e^{-n}$$

$$19. \sum_{n=1}^{\infty} n!(-e)^{-n}$$

$$20. \sum_{n=1}^{\infty} \frac{n!}{10^n}$$

$$21. \sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$$

$$22. \sum_{n=1}^{\infty} \left(\frac{n-2}{n} \right)^n$$

Using the Root Test

In Exercises 9–16, use the Root Test to determine whether each series converges absolutely or diverges.

$$9. \sum_{n=1}^{\infty} \frac{7}{(2n+5)^n}$$

$$10. \sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$$

$$11. \sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5} \right)^n$$

$$12. \sum_{n=1}^{\infty} \left(-\ln\left(e^2 + \frac{1}{n}\right) \right)^{n+1}$$

23. $\sum_{n=1}^{\infty} \frac{2 + (-1)^n}{1.25^n}$

24. $\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n}$

25. $\sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{3}{n}\right)^n$

26. $\sum_{n=1}^{\infty} \left(1 - \frac{1}{3n}\right)^n$

27. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

28. $\sum_{n=1}^{\infty} \frac{(-\ln n)^n}{n^n}$

29. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)$

30. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

31. $\sum_{n=1}^{\infty} \frac{e^n}{n^e}$

32. $\sum_{n=1}^{\infty} \frac{n \ln n}{(-2)^n}$

33. $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$

34. $\sum_{n=1}^{\infty} e^{-n}(n^3)$

35. $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!n!3^n}$

36. $\sum_{n=1}^{\infty} \frac{n2^n(n+1)!}{3^n n!}$

37. $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$

38. $\sum_{n=1}^{\infty} \frac{n!}{(-n)^n}$

39. $\sum_{n=2}^{\infty} \frac{-n}{(\ln n)^n}$

40. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^{(n/2)}}$

41. $\sum_{n=1}^{\infty} \frac{n! \ln n}{n(n+2)!}$

42. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3 2^n}$

43. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

44. $\sum_{n=1}^{\infty} \frac{(2n+3)(2^n+3)}{3^n+2}$

45. $\sum_{n=3}^{\infty} \frac{2^n}{n^2}$

46. $\sum_{n=3}^{\infty} \frac{2^{n^2}}{n^{2^n}}$

Recursively Defined Terms Which of the series $\sum_{n=1}^{\infty} a_n$ defined by the formulas in Exercises 47–56 converge, and which diverge? Give reasons for your answers.

47. $a_1 = 2, \quad a_{n+1} = \frac{1 + \sin n}{n} a_n$

48. $a_1 = 1, \quad a_{n+1} = \frac{1 + \tan^{-1} n}{n} a_n$

49. $a_1 = \frac{1}{3}, \quad a_{n+1} = \frac{3n-1}{2n+5} a_n$

50. $a_1 = 3, \quad a_{n+1} = \frac{n}{n+1} a_n$

51. $a_1 = 2, \quad a_{n+1} = \frac{2}{n} a_n$

52. $a_1 = 5, \quad a_{n+1} = \frac{\sqrt[n]{n}}{2} a_n$

53. $a_1 = 1, \quad a_{n+1} = \frac{1 + \ln n}{n} a_n$

54. $a_1 = \frac{1}{2}, \quad a_{n+1} = \frac{n + \ln n}{n + 10} a_n$

55. $a_1 = \frac{1}{3}, \quad a_{n+1} = \sqrt[n]{a_n}$

56. $a_1 = \frac{1}{2}, \quad a_{n+1} = (a_n)^{n+1}$

Convergence or Divergence

Which of the series in Exercises 57–64 converge, and which diverge? Give reasons for your answers.

57. $\sum_{n=1}^{\infty} \frac{2^n n! n!}{(2n)!}$

58. $\sum_{n=1}^{\infty} \frac{(-1)^n (3n)!}{n!(n+1)!(n+2)!}$

59. $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$

60. $\sum_{n=1}^{\infty} (-1)^n \frac{(n!)^n}{n^{(n^2)}}$

61. $\sum_{n=1}^{\infty} \frac{n^n}{2^{(n^2)}}$

62. $\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2}$

63. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \dots \cdot (2n-1)}{4^n 2^n n!}$

64. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \dots \cdot (2n-1)}{[2 \cdot 4 \cdot \dots \cdot (2n)](3^n + 1)}$

65. Assume that b_n is a sequence of positive numbers converging to $4/5$. Determine whether the following series converge or diverge.

a. $\sum_{n=1}^{\infty} (b_n)^{1/n}$

b. $\sum_{n=1}^{\infty} \left(\frac{5}{4}\right)^n (b_n)$

c. $\sum_{n=1}^{\infty} (b_n)^n$

d. $\sum_{n=1}^{\infty} \frac{1000^n}{n! + b_n}$

66. Assume that b_n is a sequence of positive numbers converging to $1/3$. Determine whether the following series converge or diverge.

a. $\sum_{n=1}^{\infty} \frac{b_{n+1} b_n}{n 4^n}$

b. $\sum_{n=1}^{\infty} \frac{n^n}{n! b_1^2 b_2^2 \dots b_n^2}$

Theory and Examples

67. Neither the Ratio Test nor the Root Test helps with p -series. Try them on

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

and show that both tests fail to provide information about convergence.

68. Show that neither the Ratio Test nor the Root Test provides information about the convergence of

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p} \quad (p \text{ constant}).$$

69. Let $a_n = \begin{cases} n/2^n, & \text{if } n \text{ is a prime number} \\ 1/2^n, & \text{otherwise.} \end{cases}$

Does $\sum a_n$ converge? Give reasons for your answer.

70. Show that $\sum_{n=1}^{\infty} 2^{(n^2)}/n!$ diverges. Recall from the Laws of Exponents that $2^{(n^2)} = (2^n)^n$.

71. Determine whether the series $\sum_{n=1}^{\infty} c_n$ converges, where

$$c_n = \begin{cases} -1/n, & \text{if } n \text{ is a perfect square,} \\ 1/n^2, & \text{if } n \text{ is not a perfect square.} \end{cases}$$

9.6 Alternating Series and Conditional Convergence

A series in which the terms are alternately positive and negative is an **alternating series**. Here are three examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots \quad (1)$$

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^n 4}{2^n} + \cdots \quad (2)$$

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1} n + \cdots \quad (3)$$

We see from these examples that the n th term of an alternating series is of the form

$$a_n = (-1)^{n+1} u_n \quad \text{or} \quad a_n = (-1)^n u_n,$$

where $u_n = |a_n|$ is a positive number.

Series (1), called the **alternating harmonic series**, converges, as we will see in a moment. Series (2), which is a geometric series with ratio $r = -1/2$, converges to $-2/[1 + (1/2)] = -4/3$. Series (3) diverges because the n th term does not approach zero.

We prove the convergence of the alternating harmonic series by applying the Alternating Series Test. This test is for *convergence* of an alternating series and cannot be used to conclude that such a series diverges. If we multiply $(u_1 - u_2 + u_3 - u_4 + \cdots)$ by -1 , we see that the test is also valid for the alternating series $-u_1 + u_2 - u_3 + u_4 - \cdots$, as with the one in Series (2) given above.

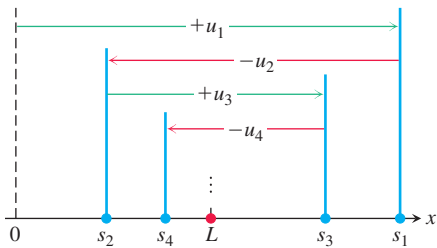


FIGURE 9.15 The partial sums of an alternating series that satisfies the hypotheses of Theorem 15 for $N = 1$ straddle the limit from the beginning.

THEOREM 15—The Alternating Series Test

The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if the following conditions are satisfied:

1. The u_n 's are all positive.
2. The u_n 's are eventually nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N .
3. $u_n \rightarrow 0$.

Proof We look at the case $N = 1$, where we have nonincreasing terms $u_1 \geq u_2 \geq u_3 \geq \cdots$. If n is an even integer, say $n = 2m$, then the sum of the first n terms is

$$s_{2m} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2m-1} - u_{2m}).$$

Written this way, we see that s_{2m} is the sum of m nonnegative terms, since each term in parentheses is positive or zero. In particular, s_{2m+2} equals s_{2m} plus the nonnegative term $(u_{2m+1} - u_{2m+2})$, so $s_{2m+2} \geq s_{2m}$. This shows that the sequence $\{s_{2m}\}$ is nondecreasing. On the other hand, we can write

$$s_{2m} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2m-2} - u_{2m-1}) - u_{2m}.$$

This shows that $s_{2m} \leq u_1$ for every m . Hence the sequence $\{s_{2m}\}$ is both nondecreasing and bounded from above, so it has a limit, which we call L ,

$$\lim_{m \rightarrow \infty} s_{2m} = L. \quad \text{Theorem 6} \quad (4)$$

If n is an odd integer, say $n = 2m + 1$, then the sum of the first n terms is $s_{2m+1} = s_{2m} + u_{2m+1}$. Since $u_n \rightarrow 0$,

$$\lim_{m \rightarrow \infty} u_{2m+1} = 0.$$

Therefore,

$$\lim_{m \rightarrow \infty} s_{2m+1} = \lim_{m \rightarrow \infty} s_{2m} + \lim_{m \rightarrow \infty} u_{2m+1} = L + 0 = L. \quad (5)$$

Combining the results of Equations (4) and (5) gives $\lim_{n \rightarrow \infty} s_n = L$ (Section 9.1, Exercise 143). ■

EXAMPLE 1 The alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

clearly satisfies the three requirements of Theorem 15 with $N = 1$; it therefore converges by the Alternating Series Test. Notice that the test gives no information about what the sum of the series might be. Figure 9.16 shows histograms of the partial sums of the divergent harmonic series and those of the convergent alternating harmonic series. It turns out that the alternating harmonic series converges to $\ln 2$ (Exercise 61 in Section 9.7). ■

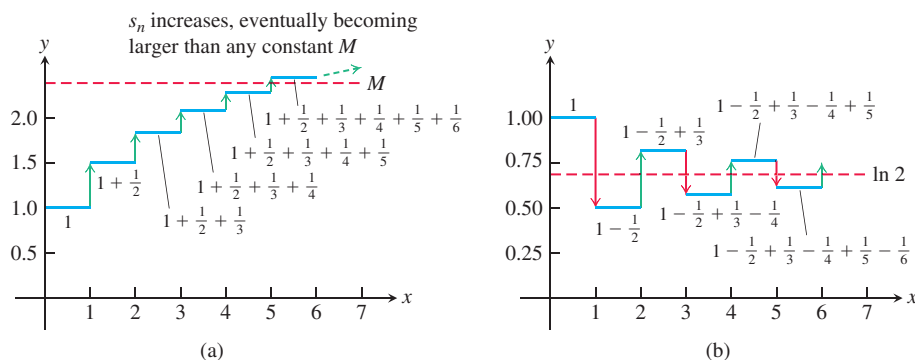


FIGURE 9.16 (a) The harmonic series diverges, with partial sums that eventually exceed any constant. (b) The alternating harmonic series converges to $\ln 2 \approx .693$.

Rather than directly verifying the definition $u_n \geq u_{n+1}$, a second way to show that the sequence $\{u_n\}$ is nonincreasing is to define a differentiable function $f(x)$ satisfying $f(n) = u_n$. That is, the values of f match the values of the sequence at every positive integer n . If $f'(x) \leq 0$ for all x greater than or equal to some positive integer N , then $f(x)$ is nonincreasing for $x \geq N$. It follows that $f(n) \geq f(n+1)$, or $u_n \geq u_{n+1}$, for $n \geq N$.

EXAMPLE 2 We show that the sequence $u_n = 10n/(n^2 + 16)$ is eventually nonincreasing. Define $f(x) = 10x/(x^2 + 16)$. Then, from the Derivative Quotient Rule,

$$f'(x) = \frac{10(16 - x^2)}{(x^2 + 16)^2} \leq 0 \quad \text{whenever } x \geq 4.$$

It follows that $u_n \geq u_{n+1}$ for $n \geq 4$. That is, the sequence $\{u_n\}$ is nonincreasing for $n \geq 4$. ■

A graphical interpretation of the partial sums (Figure 9.15) shows how an alternating series converges to its limit L when the three conditions of Theorem 15 are satisfied with $N = 1$. Starting from the origin of the x -axis, we lay off the positive distance $s_1 = u_1$. To find the point corresponding to $s_2 = u_1 - u_2$, we back up a distance equal to u_2 . Since $u_2 \leq u_1$, we do not back up any farther than the origin. We continue in this seesaw fashion, backing up or going forward as the signs in the series demand. Each forward or backward step is shorter than (or at most the same size as) the preceding step because $u_{n+1} \leq u_n$. And since the n th term approaches zero as n increases, the size of step we take forward or backward gets smaller and smaller. We oscillate back and forth across the limit L , and the amplitude of oscillation approaches zero. The limit L lies between any two successive

sums s_n and s_{n+1} and hence differs from s_n by an amount less than u_{n+1} . If $N > 1$, then the first few forward and backward steps need not get smaller, but from N onward they will decrease. Therefore

$$|L - s_n| < u_{n+1} \quad \text{for } n \geq N,$$

and we can use this fact to make useful estimates of the sums of convergent alternating series.

THEOREM 16—The Alternating Series Estimation Theorem

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ satisfies the three conditions of Theorem 15, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1} u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1} , the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1} , and the remainder, $L - s_n$, has the same sign as the first unused term.

We leave the verification of the sign of the remainder for Exercise 87.

EXAMPLE 3 We try Theorem 16 on a series whose sum we know:

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \frac{1}{256} - \cdots.$$

The theorem says that if we truncate the series after the eighth term, we throw away a total that is positive and less than $1/256$. The sum of the first eight terms is $s_8 = 0.6640625$ and the sum of the first nine terms is $s_9 = 0.66796875$. The sum of the geometric series is

$$\frac{1}{1 - (-1/2)} = \frac{1}{3/2} = \frac{2}{3},$$

and we note that $0.6640625 < (2/3) < 0.66796875$. The difference,

$$(2/3) - 0.6640625 = 0.0026041666 \dots,$$

is positive and is less than $(1/256) = 0.00390625$. ■

Conditional Convergence

If we replace all the negative terms in the alternating series in Example 3, changing them to positive terms instead, we obtain the geometric series $\sum 1/2^n$. The original series and the new series of absolute values both converge (although to different sums). For an absolutely convergent series, changing infinitely many of the negative terms in the series to positive values does not change its property of still being a convergent series. Other convergent series may behave differently. The convergent alternating harmonic series has infinitely many negative terms, but if we change its negative terms to positive values, the resulting series is the divergent harmonic series. So the presence of infinitely many negative terms is essential to the convergence of this alternating harmonic series. The following terminology distinguishes these two types of convergent series.

DEFINITION A series that is convergent but not absolutely convergent is called **conditionally convergent**.

The alternating harmonic series is conditionally convergent, or **converges conditionally**. The next example extends that result to the alternating p -series.

EXAMPLE 4 If p is a positive constant, the sequence $\{1/n^p\}$ is a decreasing sequence with limit zero. Therefore, the alternating p -series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots, \quad p > 0$$

converges.

If $p > 1$, the series converges absolutely as an ordinary p -series. If $0 < p \leq 1$, the series converges conditionally: It converges by the alternating series test, but the corresponding series of absolute values is a divergent p -series. For instance,

$$\text{Absolute convergence } (p = 3/2): \quad 1 - \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} - \frac{1}{4^{3/2}} + \cdots$$

$$\text{Conditional convergence } (p = 1/2): \quad 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots \quad \blacksquare$$

We need to be careful when using a conditionally convergent series. We have seen with the alternating harmonic series that altering the signs of infinitely many terms of a conditionally convergent series can change its convergence status. Even more, simply changing the order of occurrence of infinitely many of its terms can also have a significant effect, as we now discuss.

Rearranging Series

We can always rearrange the terms of a *finite* collection of numbers without changing their sum. The same result is true for an infinite series that is absolutely convergent (see Exercise 96 for an outline of the proof).

THEOREM 17—The Rearrangement Theorem for Absolutely Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \dots, b_n, \dots$ is any arrangement of the sequence $\{a_n\}$, then $\sum_{n=1}^{\infty} b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

On the other hand, if we rearrange the terms of a conditionally convergent series, we can get different results. In fact, for any real number r , a given conditionally convergent series can be rearranged so that its sum is equal to r . (We omit the proof of this.) Here's an example where summing the terms of a conditionally convergent series with different orderings gives different values for the sum.

EXAMPLE 5 We know that the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1}/n$ converges to some number L . Moreover, by Theorem 16, L lies between the successive partial sums $s_2 = 1/2$ and $s_3 = 5/6$, so $L \neq 0$. If we multiply the series by 2, we obtain

$$\begin{aligned} 2L &= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 2 \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \cdots \right) \\ &= 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \cdots. \end{aligned}$$

Now we change the order of this last sum by grouping each pair of terms with the same odd denominator, but leaving the negative terms with the even denominators as they are

placed (so that the denominators are the positive integers in their natural order). This rearrangement gives

$$\begin{aligned} (2-1) - \frac{1}{2} + \left(\frac{2}{3} - \frac{1}{3}\right) - \frac{1}{4} + \left(\frac{2}{5} - \frac{1}{5}\right) - \frac{1}{6} + \left(\frac{2}{7} - \frac{1}{7}\right) - \frac{1}{8} + \cdots \\ = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \cdots\right) \\ = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = L. \end{aligned}$$

So when we rearrange the terms of the conditionally convergent series $\sum_{n=1}^{\infty} 2(-1)^{n+1}/n$, the series becomes $\sum_{n=1}^{\infty} (-1)^{n+1}/n$, which is the alternating harmonic series itself. If the two series are the same, it would imply that $2L = L$, which is clearly false since $L \neq 0$. ■

Example 5 shows that we cannot rearrange the terms of a conditionally convergent series and expect the new series to be the same as the original one. When we use a conditionally convergent series, we must add the terms together in the order in which they are given to obtain a correct result. In contrast, Theorem 17 guarantees that the terms of an absolutely convergent series can be summed in any order without affecting the result.

Summary of Tests to Determine Convergence or Divergence

We have developed a variety of tests to determine convergence or divergence for an infinite series of constants. Other tests that we have not presented are sometimes given in more advanced courses. Here is a summary of the tests we have considered.

- 1. The n th-Term Test for Divergence:** Unless $a_n \rightarrow 0$, the series diverges.
- 2. Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise, it diverges.
- 3. p -series:** $\sum 1/n^p$ converges if $p > 1$; otherwise, it diverges.
- 4. Series with nonnegative terms:** Try the Integral Test or try comparing to a known series with the Direct Comparison Test or the Limit Comparison Test. Try the Ratio or Root Test.
- 5. Series with some negative terms:** Does $\sum |a_n|$ converge by the Ratio or Root Test, or by another of the tests listed above? Remember, absolute convergence implies convergence.
- 6. Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

EXERCISES 9.6

Convergence of Alternating Series

In Exercises 1–14, determine whether the alternating series converges or diverges. Some of the series do not satisfy the conditions of the Alternating Series Test.

1. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

2. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{3/2}}$

3. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n3^n}$

4. $\sum_{n=2}^{\infty} (-1)^n \frac{4}{(\ln n)^2}$

5. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 1}$

6. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 + 5}{n^2 + 4}$

7. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n^2}$

8. $\sum_{n=1}^{\infty} (-1)^n \frac{10^n}{(n+1)!}$

9. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{10}\right)^n$

10. $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\ln n}$

$$11. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n} \quad 12. \sum_{n=1}^{\infty} (-1)^n \ln \left(1 + \frac{1}{n} \right)$$

$$13. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n} + 1}{n + 1} \quad 14. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3\sqrt{n} + 1}{\sqrt{n} + 1}$$

Absolute and Conditional Convergence

Which of the series in Exercises 15–48 converge absolutely, which converge conditionally, and which diverge? Give reasons for your answers.

$$15. \sum_{n=1}^{\infty} (-1)^{n+1} (0.1)^n \quad 16. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.1)^n}{n}$$

$$17. \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}} \quad 18. \sum_{n=1}^{\infty} \frac{(-1)^n}{1 + \sqrt{n}}$$

$$19. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3 + 1} \quad 20. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{2^n}$$

$$21. \sum_{n=1}^{\infty} (-1)^n \frac{1}{n + 3} \quad 22. \sum_{n=1}^{\infty} (-1)^n \frac{\sin n}{n^2}$$

$$23. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3 + n}{5 + n} \quad 24. \sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{n + 5^n}$$

$$25. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1 + n}{n^2} \quad 26. \sum_{n=1}^{\infty} (-1)^{n+1} (\sqrt[4]{10})$$

$$27. \sum_{n=1}^{\infty} (-1)^n n^2 (2/3)^n \quad 28. \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$$

$$29. \sum_{n=1}^{\infty} (-1)^n \frac{\arctan n}{n^2 + 1} \quad 30. \sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n - \ln n}$$

$$31. \sum_{n=1}^{\infty} (-1)^n \frac{n}{n + 1} \quad 32. \sum_{n=1}^{\infty} (-5)^{-n}$$

$$33. \sum_{n=1}^{\infty} \frac{(-100)^n}{n!} \quad 34. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 + 2n + 1}$$

$$35. \sum_{n=1}^{\infty} \frac{\cos n\pi}{n\sqrt{n}} \quad 36. \sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$$

$$37. \sum_{n=1}^{\infty} \frac{(-1)^n (n + 1)^n}{(2n)^n} \quad 38. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (n!)^2}{(2n)!}$$

$$39. \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{2^n n! n} \quad 40. \sum_{n=1}^{\infty} (-1)^n \frac{(n!)^2 3^n}{(2n + 1)!}$$

$$41. \sum_{n=1}^{\infty} (-1)^n (\sqrt{n + 1} - \sqrt{n}) \quad 42. \sum_{n=1}^{\infty} (-1)^n (\sqrt{n^2 + n} - n)$$

$$43. \sum_{n=1}^{\infty} (-1)^n (\sqrt{n + \sqrt{n}} - \sqrt{n})$$

$$44. \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n} + \sqrt{n + 1}}$$

$$45. \sum_{n=1}^{\infty} (-1)^n \operatorname{sech} n \quad 46. \sum_{n=1}^{\infty} (-1)^n \operatorname{csch} n$$

$$47. \frac{1}{4} - \frac{1}{6} + \frac{1}{8} - \frac{1}{10} + \frac{1}{12} - \frac{1}{14} + \cdots$$

$$48. 1 + \frac{1}{4} - \frac{1}{9} - \frac{1}{16} + \frac{1}{25} + \frac{1}{36} - \frac{1}{49} - \frac{1}{64} + \cdots$$

Error Estimation

In Exercises 49–52, estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the entire series.

$$49. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \quad 50. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{10^n}$$

$$51. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.01)^n}{n} \quad \text{As you will see in Section 9.7, the sum is } \ln(1.01).$$

$$52. \frac{1}{1 + t} = \sum_{n=0}^{\infty} (-1)^n t^n, \quad 0 < t < 1$$

In Exercises 53–56, determine how many terms should be used to estimate the sum of the entire series with an error of less than 0.001.

$$53. \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2 + 3} \quad 54. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2 + 1}$$

$$55. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(n + 3\sqrt{n})^3} \quad 56. \sum_{n=1}^{\infty} (-1)^n \frac{1}{\ln(\ln(n + 2))}$$

Determining Convergence or Divergence

In Exercises 57–82, use any method to determine whether the series converges or diverges. Give reasons for your answer.

$$57. \sum_{n=1}^{\infty} \frac{3^n}{n^n} \quad 58. \sum_{n=1}^{\infty} \frac{3^n}{n^3}$$

$$59. \sum_{n=1}^{\infty} \left(\frac{1}{n + 2} - \frac{1}{n + 3} \right) \quad 60. \sum_{n=1}^{\infty} \left(\frac{1}{2n + 1} - \frac{1}{2n + 2} \right)$$

$$61. \sum_{n=0}^{\infty} (-1)^n \frac{(n + 2)!}{(2n)!} \quad 62. \sum_{n=2}^{\infty} \frac{(3n)!}{(n!)^3}$$

$$63. \sum_{n=1}^{\infty} n^{-2/\sqrt{5}} \quad 64. \sum_{n=2}^{\infty} \frac{3}{10 + n^{4/3}}$$

$$65. \sum_{n=1}^{\infty} \left(1 - \frac{2}{n} \right)^{n^2} \quad 66. \sum_{n=0}^{\infty} \left(\frac{n + 1}{n + 2} \right)^n$$

$$67. \sum_{n=1}^{\infty} \frac{n - 2}{n^2 + 3n} \left(-\frac{2}{3} \right)^n \quad 68. \sum_{n=0}^{\infty} \frac{n + 1}{(n + 2)!} \left(\frac{3}{2} \right)^n$$

$$69. \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \cdots$$

$$70. 1 - \frac{1}{8} + \frac{1}{64} - \frac{1}{512} + \frac{1}{4096} - \cdots$$

$$71. \sum_{n=3}^{\infty} \sin \left(\frac{1}{\sqrt{n}} \right) \quad 72. \sum_{n=1}^{\infty} \tan(n^{1/n})$$

$$73. \sum_{n=2}^{\infty} \frac{n}{\ln n} \quad 74. \sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

$$75. \sum_{n=2}^{\infty} \ln \left(\frac{n + 2}{n + 1} \right) \quad 76. \sum_{n=2}^{\infty} \left(\frac{\ln n}{n} \right)^3$$

$$77. \sum_{n=2}^{\infty} \frac{1}{1 + 2 + 2^2 + \cdots + 2^n}$$

$$78. \sum_{n=2}^{\infty} \frac{1 + 3 + 3^2 + \cdots + 3^{n-1}}{1 + 2 + 3 + \cdots + n}$$

$$79. \sum_{n=0}^{\infty} (-1)^n \frac{e^n}{e^n + e^{n^2}}$$

$$80. \sum_{n=0}^{\infty} \frac{(2n+3)(2^n+3)}{3^n+2}$$

$$81. \sum_{n=1}^{\infty} \frac{n^2 3^n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$$

$$82. \sum_{n=1}^{\infty} \frac{4 \cdot 6 \cdot 8 \cdots (2n)}{5^{n+1}(n+2)!}$$

T Approximate the sums in Exercises 83 and 84 with an error of magnitude less than 5×10^{-6} .

$$83. \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \quad \text{As you will see in Section 9.9, the sum is } \cos 1, \text{ the cosine of 1 radian.}$$

$$84. \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \quad \text{As you will see in Section 9.9 the sum is } e^{-1}.$$

Theory and Examples

85. a. The series

$$\frac{1}{3} - \frac{1}{2} + \frac{1}{9} - \frac{1}{4} + \frac{1}{27} - \frac{1}{8} + \cdots + \frac{1}{3^n} - \frac{1}{2^n} + \cdots$$

does not meet one of the conditions of the Alternating Series Test. Which one?

b. Use the Sum Rule for series given in Section 9.2 to find the sum of the series in part (a).

T 86. The limit L of an alternating series that satisfies the conditions of Theorem 15 lies between the values of any two consecutive partial sums. This suggests using the average

$$\frac{s_n + s_{n+1}}{2} = s_n + \frac{1}{2}(-1)^{n+2} a_{n+1}$$

to estimate L . Compute

$$s_{20} + \frac{1}{2} \cdot \frac{1}{21}$$

as an approximation to the sum of the alternating harmonic series. The exact sum is $\ln 2 = 0.69314718 \dots$.

87. **The sign of the remainder of an alternating series that satisfies the conditions of Theorem 15** Prove the assertion in Theorem 16 that whenever an alternating series satisfying the conditions of Theorem 15 is approximated with one of its partial sums, the remainder (the sum of the unused terms) has the same sign as the first unused term. (*Hint:* Group the remainder's terms in consecutive pairs.)

88. Show that the sum of the first $2n$ terms of the series

$$1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \cdots$$

is the same as the sum of the first n terms of the series

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \cdots$$

Do these series converge? What is the sum of the first $2n+1$ terms of the first series? If the series converge, what is their sum?

89. Show that if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges.

90. Show that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|.$$

91. Show that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge absolutely, then so do the following.

$$\text{a. } \sum_{n=1}^{\infty} (a_n + b_n) \quad \text{b. } \sum_{n=1}^{\infty} (a_n - b_n)$$

$$\text{c. } \sum_{n=1}^{\infty} k a_n \quad (k \text{ any number})$$

92. Show by example that $\sum_{n=1}^{\infty} a_n b_n$ may diverge even if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge.

93. Prove that if $\sum a_n$ converges absolutely, then $\sum a_n^2$ converges.

94. Does the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2} \right)$$

converge or diverge? Justify your answer.

T 95. In the alternating harmonic series, suppose the goal is to arrange the terms to get a new series that converges to $-1/2$. Start the new arrangement with the first negative term, which is $-1/2$. Whenever you have a sum that is less than or equal to $-1/2$, start introducing positive terms, taken in order, until the new total is greater than $-1/2$. Then add negative terms until the total is less than or equal to $-1/2$ again. Continue this process until your partial sums have been above the target at least three times and finish at or below it. If s_n is the sum of the first n terms of your new series, plot the points (n, s_n) to illustrate how the sums are behaving.

96. **Outline of the proof of the Rearrangement Theorem (Theorem 17)**

a. Let ε be a positive real number, let $L = \sum_{n=1}^{\infty} a_n$, and let $s_k = \sum_{n=1}^k a_n$. Show that for some index N_1 and for some index $N_2 \geq N_1$,

$$\sum_{n=N_1}^{\infty} |a_n| < \frac{\varepsilon}{2} \quad \text{and} \quad |s_{N_2} - L| < \frac{\varepsilon}{2}.$$

Since all the terms $a_1, a_2, \dots, a_{N_2}$ appear somewhere in the sequence $\{b_n\}$, there is an index $N_3 \geq N_2$ such that if $n \geq N_3$, then $(\sum_{k=1}^n b_k) - s_{N_2}$ is at most a sum of terms a_m with $m \geq N_1$. Therefore, if $n \geq N_3$, then

$$\begin{aligned} \left| \sum_{k=1}^n b_k - L \right| &\leq \left| \sum_{k=1}^n b_k - s_{N_2} \right| + |s_{N_2} - L| \\ &\leq \sum_{k=N_1}^{\infty} |a_k| + |s_{N_2} - L| < \varepsilon. \end{aligned}$$

b. The argument in part (a) shows that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} b_n$ converges and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$. Now show that because $\sum_{n=1}^{\infty} a_n$ converges, $\sum_{n=1}^{\infty} b_n$ converges to $\sum_{n=1}^{\infty} a_n$.

9.7 Power Series

Now that we can test many infinite series of numbers for convergence, we can study sums that look like “infinite polynomials.” We call these sums *power series* because they are defined as infinite series of powers of some variable, in our case x . Like polynomials, power series can be added, subtracted, multiplied, differentiated, and integrated to give new power series. With power series we can extend the methods of calculus to a vast array of functions, making the techniques of calculus applicable in an even wider setting.

Power Series and Convergence

We begin with the formal definition, which specifies the notation and terminology used for power series.

DEFINITIONS A **power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots \quad (1)$$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \cdots + c_n (x - a)^n + \cdots \quad (2)$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \dots, c_n, \dots$ are constants.

Equation (1) is the special case obtained by taking $a = 0$ in Equation (2). We will see that a power series defines a function $f(x)$ on a certain interval where it converges. Moreover, this function will be shown to be continuous and differentiable over the interior of that interval.

EXAMPLE 1 Taking all the coefficients to be 1 in Equation (1) gives the geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

This is the geometric series with first term 1 and ratio x . It converges to $1/(1 - x)$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1 - x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1. \quad (3)$$

Power Series for $\frac{1}{1 - x}$

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

Up to now, we have used Equation (3) as a formula for the sum of the series on the right. We now change the focus: We think of the partial sums of the series on the right as polynomials $P_n(x)$ that approximate the function on the left. For values of x near zero, we need take only a few terms of the series to get a good approximation. As we move toward $x = 1$, or -1 , we must take more terms. Figure 9.17 shows the graphs of $f(x) = 1/(1 - x)$ and the approximating polynomials $y_n = P_n(x)$ for $n = 0, 1, 2$, and 8. The function $f(x) = 1/(1 - x)$ is not continuous on intervals containing $x = 1$, where it has a vertical asymptote. The approximations do not apply when $x \geq 1$.

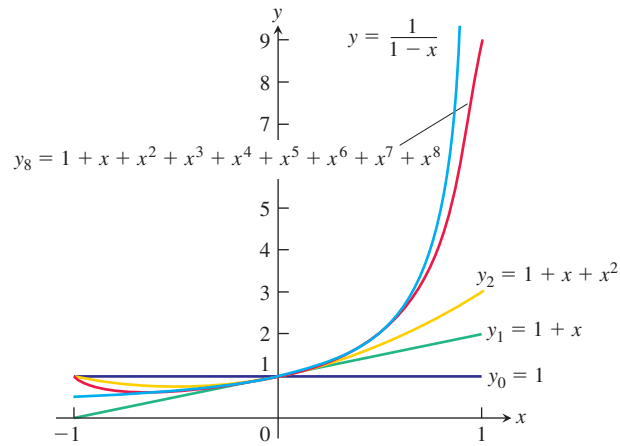


FIGURE 9.17 The graphs of $f(x) = 1/(1-x)$ in Example 1 and four of its polynomial approximations.

EXAMPLE 2 The power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots \quad (4)$$

matches Equation (2) with $a = 2, c_0 = 1, c_1 = -1/2, c_2 = 1/4, \dots, c_n = (-1/2)^n$.

This is a geometric series with first term 1 and ratio $r = -\frac{x-2}{2}$. The series converges for $\left|\frac{x-2}{2}\right| < 1$, which simplifies to $0 < x < 4$. The sum is

$$\frac{1}{1-r} = \frac{1}{1 + \frac{x-2}{2}} = \frac{2}{x},$$

so

$$\frac{2}{x} = 1 - \frac{(x-2)}{2} + \frac{(x-2)^2}{4} - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots, \quad 0 < x < 4.$$

Series (4) generates useful polynomial approximations of $f(x) = 2/x$ for values of x near 2:

$$P_0(x) = 1$$

$$P_1(x) = 1 - \frac{1}{2}(x-2) = 2 - \frac{x}{2}$$

$$P_2(x) = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 = 3 - \frac{3x}{2} + \frac{x^2}{4},$$

and so on (Figure 9.18).

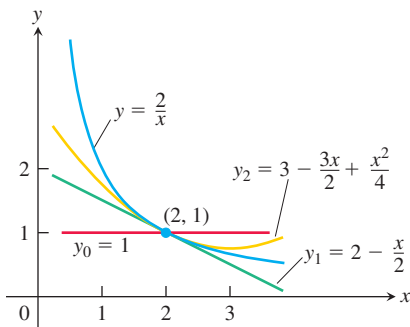


FIGURE 9.18 The graphs of $f(x) = 2/x$ and its first three polynomial approximations (Example 2).

The following example illustrates how we test a power series for convergence by using the Ratio Test to see where it converges and where it diverges.

EXAMPLE 3 For what values of x do the following power series converge?

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

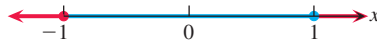
$$(c) \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$(d) \sum_{n=0}^{\infty} n! x^n = 1 + x + 2! x^2 + 3! x^3 + \cdots$$

Solution Apply the Ratio Test to the series $\sum u_n$, where u_n is the n th term of the power series in question.

$$(a) \left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x} \right| = \frac{n}{n+1} |x| \rightarrow |x|.$$

By the Ratio Test, this series converges absolutely for $|x| < 1$, and it diverges for $|x| > 1$. At $x = 1$, we obtain the alternating harmonic series $1 - 1/2 + 1/3 - 1/4 + \cdots$, which converges (though it does not converge absolutely). At $x = -1$, we get $-1 - 1/2 - 1/3 - 1/4 - \cdots$, the negative of the harmonic series, which diverges. Series (a) converges for $-1 < x \leq 1$ and diverges elsewhere. The convergence is absolute for $-1 < x < 1$, but conditional at the point $x = 1$.



We will see in Example 6 that this series converges to the function $\ln(1+x)$ on the interval $(-1, 1]$ (see Figure 9.19).

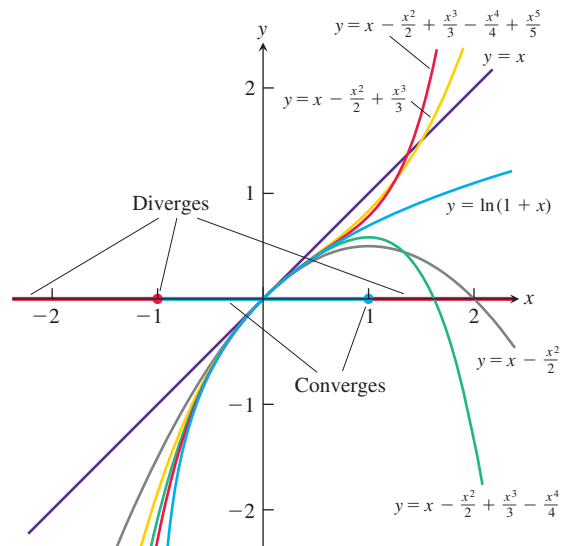
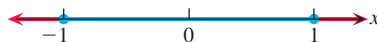


FIGURE 9.19 The power series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ converges on the interval $(-1, 1]$.

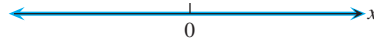
$$(b) \left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{x^{2n+1}}{2n+1} \cdot \frac{2n-1}{x^{2n-1}} \right| = \frac{2n-1}{2n+1} x^2 \rightarrow x^2. \quad 2(n+1)-1 = 2n+1$$

By the Ratio Test, the series converges absolutely for $x^2 < 1$ and diverges for $x^2 > 1$. At $x = 1$ the series becomes $1 - 1/3 + 1/5 - 1/7 + \cdots$, which converges by the Alternating Series Theorem. It also converges at $x = -1$ because it is again an alternating series that satisfies the conditions for convergence. The value at $x = -1$ is the negative of the value at $x = 1$. Series (b) converges for $-1 \leq x \leq 1$ and diverges elsewhere. The convergence is absolute for $-1 < x < 1$, but conditional at the points $x = -1$ and $x = 1$.



$$(c) \left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \rightarrow 0 \text{ for every } x. \quad \frac{n!}{(n+1)!} = \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 2 \cdot 3 \cdots n \cdot (n+1)}$$

The series converges absolutely for all x .



$$(d) \left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = (n+1)|x| \rightarrow \begin{cases} 0 & \text{if } x = 0 \\ \infty & \text{if } x \neq 0. \end{cases}$$

The series diverges for all values of x except $x = 0$.



The previous example illustrated how a power series might converge. The next result shows that if a power series converges at a nonzero value, then it converges over an entire interval of values. The interval might be finite or infinite and might contain one, both, or none of its endpoints. We will see that each endpoint of a finite interval must be tested independently for convergence or divergence.

THEOREM 18—The Convergence Theorem for Power Series

If the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \text{ converges at } x = c \neq 0, \text{ then it converges}$$

absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

Proof The proof uses the Direct Comparison Test, with the given series compared to a converging geometric series.

Suppose the series $\sum_{n=0}^{\infty} a_n c^n$ converges. Then $\lim_{n \rightarrow \infty} a_n c^n = 0$ by the n th-Term Test. Hence, there is an integer N such that $|a_n c^n| < 1$ for all $n > N$, so

$$|a_n| < \frac{1}{|c|^n} \quad \text{for } n > N. \quad (5)$$

Now take any x such that $|x| < |c|$, so that $|x|/|c| < 1$. Multiplying both sides of Equation (5) by $|x|^n$ gives

$$|a_n| |x|^n < \frac{|x|^n}{|c|^n} \quad \text{for } n > N.$$

Since $|x/c| < 1$, it follows that the geometric series $\sum_{n=0}^{\infty} |x/c|^n$ converges. By the Direct Comparison Test (Theorem 10), the series $\sum_{n=0}^{\infty} |a_n| |x|^n$ converges, so the original power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $-|c| < x < |c|$, as claimed by the theorem. (See Figure 9.20.)

Now suppose that the series $\sum_{n=0}^{\infty} a_n x^n$ diverges at $x = d$. If x is a number with $|x| > |d|$ and the series converges at x , then the first half of the theorem shows that the series also converges at d , contrary to our assumption. So the series diverges for all x with $|x| > |d|$. ■

To simplify the notation, Theorem 18 deals with the convergence of series of the form $\sum a_n x^n$. For series of the form $\sum a_n (x - a)^n$, we can replace $x - a$ by t and apply the results to the series $\sum a_n t^n$.

The Radius of Convergence of a Power Series

The theorem we have just proved and the examples we have studied lead to the conclusion that a power series $\sum c_n (x - a)^n$ behaves in one of three possible ways. It might

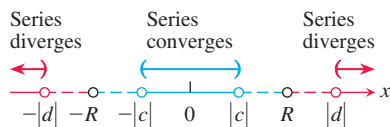


FIGURE 9.20 Convergence of $\sum a_n x^n$ at $x = c$ implies absolute convergence on the interval $-|c| < x < |c|$; divergence at $x = d$ implies divergence for $|x| > |d|$. The corollary to Theorem 18 asserts the existence of a radius of convergence $R \geq 0$. For $|x| < R$ the series converges absolutely, and for $|x| > R$ it diverges.

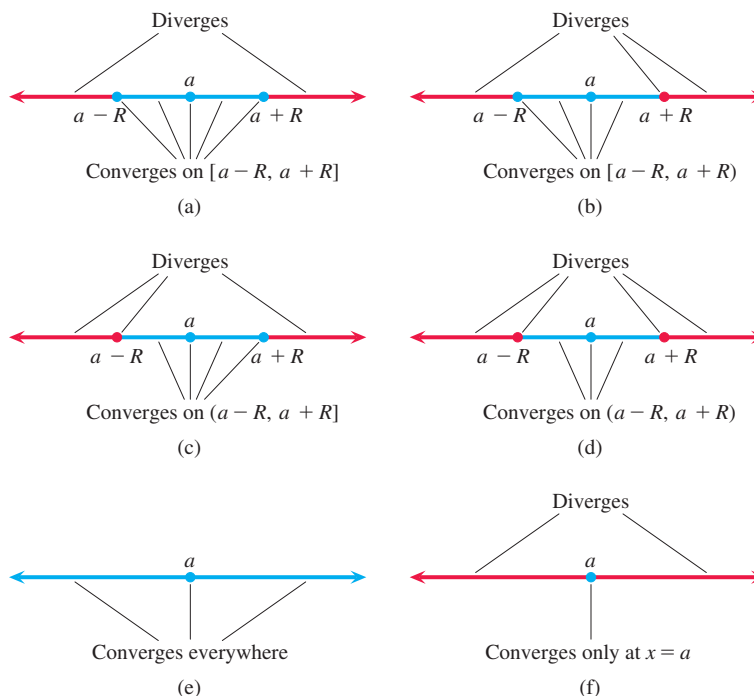


FIGURE 9.21 The six possibilities for an interval of convergence.

converge only at $x = a$, or converge everywhere, or converge on some interval of radius R centered at $x = a$. We prove this as a corollary to Theorem 18. When we also consider the convergence at the endpoints of an interval, we see that there are six different possibilities, shown in Figure 9.21.

Corollary to Theorem 18

The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a-R$ and $x = a+R$.
2. The series converges absolutely for every x ($R = \infty$).
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).

Proof We first consider the case where $a = 0$, so that we have a power series $\sum_{n=0}^{\infty} c_n x^n$ centered at 0. If the series converges everywhere we are in Case 2. If it converges only at $x = 0$ then we are in Case 3. Otherwise there is a nonzero number d such that $\sum_{n=0}^{\infty} c_n d^n$ diverges. Let S be the set of values of x for which $\sum_{n=0}^{\infty} c_n x^n$ converges. The set S does not include any x with $|x| > |d|$, since Theorem 18 implies the series diverges at all such values. So the set S is bounded. By the Completeness Property of the Real Numbers (Appendix A.9) S has a least upper bound R . (This is the smallest number with the property that all elements of S are less than or equal to R .) Since we are not in Case 3, the series converges at some number $b \neq 0$ and, by Theorem 18, also on the open interval $(-|b|, |b|)$. Therefore, $R > 0$.

If $|x| < R$ then there is a number c in S with $|x| < c < R$, since otherwise R would not be the least upper bound for S . The series converges at c since $c \in S$, so by Theorem 18 the series converges absolutely at x .

Now suppose $|x| > R$. If the series converges at x , then Theorem 18 implies it converges absolutely on the open interval $(-|x|, |x|)$, so that S contains this interval. Since R is an upper bound for S , it follows that $|x| \leq R$, which is a contradiction. So if $|x| > R$, then the series diverges. This proves the theorem for power series centered at $a = 0$.

For a power series centered at an arbitrary point $x = a$, set $t = x - a$ and repeat the argument above, replacing x with t . Since $t = 0$ when $x = a$, convergence of the series $\sum_{n=0}^{\infty} |c_n t^n|$ on a radius R open interval centered at $t = 0$ corresponds to convergence of the series $\sum_{n=0}^{\infty} |c_n (x - a)^n|$ on a radius R open interval centered at $x = a$. ■

R is called the **radius of convergence** of the power series, and the interval of radius R centered at $x = a$ is called the **interval of convergence**. The interval of convergence may be open, closed, or half-open, depending on the particular series. At points x with $|x - a| < R$, the series converges absolutely. If the series converges for all values of x , we say its radius of convergence is infinite. If it converges only at $x = a$, we say its radius of convergence is zero.

How to Test a Power Series for Convergence

1. Use the Ratio Test or the Root Test to find the largest open interval where the series converges absolutely,

$$|x - a| < R \quad \text{or} \quad a - R < x < a + R.$$

2. If R is finite, test for convergence or divergence at each endpoint, as in Examples 3a and b.
3. If R is finite, the series diverges for $|x - a| > R$.

Operations on Power Series

On the intersection of their intervals of convergence, two power series can be added and subtracted term by term just like series of constants (Theorem 8). They can be multiplied just as we multiply polynomials, but we often limit the computation of the product to the first few terms, which are the most important. The following result gives a formula for the coefficients in the product, but we omit the proof. (Power series can also be divided in a way similar to division of polynomials, but we do not give a formula for the general coefficient here.)

THEOREM 19—Series Multiplication for Power Series

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^n a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} c_n x^n.$$

Finding the general coefficient c_n in the product of two power series can be tedious, and the term may be unwieldy. The following computation provides an illustration of a

product where we find the first few terms by multiplying the terms of the second series by each term of the first series:

$$\begin{aligned}
 & \left(\sum_{n=0}^{\infty} x^n \right) \cdot \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \right) \\
 &= (1 + x + x^2 + \cdots) \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \right) \quad \text{Multiply second series ...} \\
 &= \underbrace{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \right)}_{\text{by 1}} + \underbrace{\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \cdots \right)}_{\text{by } x} + \underbrace{\left(x^3 - \frac{x^4}{2} + \frac{x^5}{3} - \cdots \right)}_{\text{by } x^2} + \cdots \\
 &= x + \frac{x^2}{2} + \frac{5x^3}{6} + \cdots \quad \text{and gather the first three powers.}
 \end{aligned}$$

We can also substitute a function $f(x)$ for x in a convergent power series.

THEOREM 20 If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, and f is a continuous function, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely on the set of points x that satisfy $|f(x)| < R$.

For example, since $1/(1-x) = \sum_{n=0}^{\infty} x^n$ converges absolutely for $|x| < 1$, it follows from Theorem 20 that $1/(1-4x^2) = \sum_{n=0}^{\infty} (4x^2)^n$ converges absolutely when x satisfies $|4x^2| < 1$, or, equivalently, when $|x| < 1/2$.

Theorem 21 says that a power series can be differentiated term by term at each interior point of its interval of convergence. A proof of a restricted case of the theorem is outlined in Exercise 66.

THEOREM 21 – Term-by-Term Differentiation

If $\sum c_n (x-a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{on the interval} \quad a-R < x < a+R.$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$\begin{aligned}
 f'(x) &= \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}, \\
 f''(x) &= \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2},
 \end{aligned}$$

and so on. Each of these derived series converges at every point of the interval $a-R < x < a+R$.

EXAMPLE 4 Find series for $f'(x)$ and $f''(x)$ if

$$\begin{aligned}
 f(x) &= \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots \\
 &= \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.
 \end{aligned}$$

Solution We differentiate the power series on the right term by term:

$$\begin{aligned} f'(x) &= \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots \\ &= \sum_{n=1}^{\infty} nx^{n-1}, \quad -1 < x < 1; \\ f''(x) &= \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots \\ &= \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \quad -1 < x < 1. \end{aligned}$$

Caution Term-by-term differentiation might not work for series that are not power series. For example, the trigonometric series

$$\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$$

converges for all x . But if we differentiate term by term, we get the series

$$\sum_{n=1}^{\infty} \frac{n! \cos(n!x)}{n^2},$$

which diverges for all x . These are not power series since they are not sums of positive integer powers of x .

It is also true that a power series can be integrated term by term throughout its interval of convergence. The proof is outlined in Exercise 67.

THEOREM 22—Term-by-Term Integration

Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $a - R < x < a + R$ (where $R > 0$). Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ and

$$\int f(x) dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a - R < x < a + R$.

EXAMPLE 5 Identify the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \quad -1 \leq x \leq 1.$$

Solution We differentiate the original series term by term and get

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots, \quad -1 < x < 1. \quad \text{Theorem 21}$$

This is a geometric series with first term 1 and ratio $-x^2$, so

$$f'(x) = \frac{1}{1 - (-x^2)} = \frac{1}{1 + x^2}.$$

We can now integrate $f'(x) = 1/(1 + x^2)$ to get

$$\int f'(x) dx = \int \frac{dx}{1 + x^2} = \arctan x + C.$$

The series for $f(x)$ is zero when $x = 0$, so $C = 0$. Hence

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \arctan x, \quad -1 < x < 1. \quad (6)$$

It can be shown that the series also converges to $\arctan x$ at the endpoints $x = \pm 1$, but we omit the proof. ■

Notice that the original series in Example 5 converges at both endpoints of the original interval of convergence, but Theorem 22 can guarantee only the convergence of the integrated series inside the interval.

EXAMPLE 6 The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval $-1 < t < 1$. Therefore,

$$\begin{aligned} \ln(1+x) &= \int_0^x \frac{1}{1+t} dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big|_0^x && \text{Theorem 22} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \end{aligned}$$

or

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x < 1.$$

It can also be shown that the series converges at $x = 1$ to the number $\ln 2$, but that was not guaranteed by the theorem. A proof of this is outlined in Exercise 63. ■

The Number π as a Series

$$\frac{\pi}{4} = \arctan 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Alternating Harmonic Series Sum

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

EXERCISES 9.7

Intervals of Convergence

In Exercises 1–36, (a) find the series' radius and interval of convergence. For what values of x does the series converge (b) absolutely, (c) conditionally?

1. $\sum_{n=0}^{\infty} x^n$

2. $\sum_{n=0}^{\infty} (x+5)^n$

3. $\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$

4. $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$

5. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$

6. $\sum_{n=0}^{\infty} (2x)^n$

7. $\sum_{n=0}^{\infty} \frac{nx^n}{n+2}$

8. $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$

9. $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n} 3^n}$

10. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$

11. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

12. $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

13. $\sum_{n=1}^{\infty} \frac{4^n x^{2n}}{n}$

14. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3 3^n}$

15. $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$

16. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{\sqrt{n+3}}$

17.
$$\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$$

18.
$$\sum_{n=0}^{\infty} \frac{nx^n}{4^n(n^2+1)}$$

19.
$$\sum_{n=0}^{\infty} \frac{\sqrt{n}x^n}{3^n}$$

20.
$$\sum_{n=1}^{\infty} \sqrt[n]{n}(2x+5)^n$$

21.
$$\sum_{n=1}^{\infty} (2 + (-1)^n) \cdot (x+1)^{n-1}$$

22.
$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n} (x-2)^n}{3n}$$

23.
$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n x^n$$

24.
$$\sum_{n=1}^{\infty} (\ln n) x^n$$

25.
$$\sum_{n=1}^{\infty} n^n x^n$$

26.
$$\sum_{n=0}^{\infty} n!(x-4)^n$$

27.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x+2)^n}{n2^n}$$

28.
$$\sum_{n=0}^{\infty} (-2)^n (n+1)(x-1)^n$$

29.
$$\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$$
 Get the information you need about $\sum 1/(n(\ln n)^2)$ from Section 9.3, Exercise 61.

30.
$$\sum_{n=2}^{\infty} \frac{x^n}{n \ln n}$$
 Get the information you need about $\sum 1/(n \ln n)$ from Section 9.3, Exercise 60.

31.
$$\sum_{n=1}^{\infty} \frac{(4x-5)^{2n+1}}{n^{3/2}}$$

32.
$$\sum_{n=1}^{\infty} \frac{(3x+1)^{n+1}}{2n+2}$$

33.
$$\sum_{n=1}^{\infty} \frac{1}{2 \cdot 4 \cdot 6 \cdots (2n)} x^n$$

34.
$$\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{n^2 \cdot 2^n} x^{n+1}$$

35.
$$\sum_{n=1}^{\infty} \frac{1+2+3+\cdots+n}{1^2+2^2+3^2+\cdots+n^2} x^n$$

36.
$$\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})(x-3)^n$$

In Exercises 37–42, find the series' radius of convergence.

37.
$$\sum_{n=1}^{\infty} \frac{n!}{3 \cdot 6 \cdot 9 \cdots 3n} x^n$$

38.
$$\sum_{n=1}^{\infty} \left(\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{2 \cdot 5 \cdot 8 \cdots (3n-1)} \right)^2 x^n$$

39.
$$\sum_{n=1}^{\infty} \frac{(n!)^2}{2^n (2n)!} x^n$$

40.
$$\sum_{n=1}^{\infty} \frac{n! x^n}{n^n}$$

41.
$$\sum_{n=1}^{\infty} \frac{(n+a)!}{n!(n+b)!} x^n$$
, where a, b are positive integers

42.
$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2} x^n$$

(Hint: Apply the Root Test.)

In Exercises 43–50, use Theorem 20 to find the series' interval of convergence and, within this interval, the sum of the series as a function of x .

43.
$$\sum_{n=0}^{\infty} 3^n x^n$$

44.
$$\sum_{n=0}^{\infty} (e^x - 4)^n$$

45.
$$\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n}$$

46.
$$\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n}$$

47.
$$\sum_{n=0}^{\infty} \left(\frac{\sqrt{x}}{2} - 1 \right)^n$$

48.
$$\sum_{n=0}^{\infty} (\ln x)^n$$

49.
$$\sum_{n=0}^{\infty} \left(\frac{x^2+1}{3} \right)^n$$

50.
$$\sum_{n=0}^{\infty} \left(\frac{x^2-1}{2} \right)^n$$

Using the Geometric Series

51. In Example 2 we represented the function $f(x) = 2/x$ as a power series about $x = 2$. Use a geometric series to represent $f(x)$ as a power series about $x = 1$, and find its interval of convergence.

52. Use a geometric series to represent each of the given functions as a power series about $x = 0$, and find their intervals of convergence.

a.
$$f(x) = \frac{5}{3-x}$$

b.
$$g(x) = \frac{3}{x-2}$$

53. Represent the function $g(x)$ in Exercise 52 as a power series about $x = 5$, and find the interval of convergence.

54. a. Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{8}{4^{n+2}} x^n.$$

b. Represent the power series in part (a) as a power series about $x = 3$ and identify the interval of convergence of the new series. (Later in the chapter you will understand why the new interval of convergence does not necessarily include all of the numbers in the original interval of convergence.)

Theory and Examples

55. For what values of x does the series

$$1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots$$

converge? What is its sum? What series do you get if you differentiate the given series term by term? For what values of x does the new series converge? What is its sum?

56. If you integrate the series in Exercise 55 term by term, what new series do you get? For what values of x does the new series converge, and what is another name for its sum?

57. The series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

converges to $\sin x$ for all x .

a. Find the first six terms of a series for $\cos x$. For what values of x should the series converge?

b. By replacing x by $2x$ in the series for $\sin x$, find a series that converges to $\sin 2x$ for all x .

c. Using the result in part (a) and series multiplication, calculate the first six terms of a series for $2 \sin x \cos x$. Compare your answer with the answer in part (b).

58. The series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

converges to e^x for all x .

a. Find a series for $(d/dx)e^x$. Do you get the series for e^x ? Explain your answer.

- b. Find a series for $\int e^x dx$. Do you get the series for e^x ? Explain your answer.
- c. Replace x by $-x$ in the series for e^x to find a series that converges to e^{-x} for all x . Then multiply the series for e^x and e^{-x} to find the first six terms of a series for $e^{-x} \cdot e^x$.

59. The series

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots$$

converges to $\tan x$ for $-\pi/2 < x < \pi/2$.

- a. Find the first five terms of the series for $\ln |\sec x|$. For what values of x should the series converge?
- b. Find the first five terms of the series for $\sec^2 x$. For what values of x should this series converge?
- c. Check your result in part (b) by squaring the series given for $\sec x$ in Exercise 60.

60. The series

$$\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots$$

converges to $\sec x$ for $-\pi/2 < x < \pi/2$.

- a. Find the first five terms of a power series for the function $\ln |\sec x + \tan x|$. For what values of x should the series converge?
- b. Find the first four terms of a series for $\sec x \tan x$. For what values of x should the series converge?
- c. Check your result in part (b) by multiplying the series for $\sec x$ by the series given for $\tan x$ in Exercise 59.

61. Uniqueness of convergent power series

- a. Show that if two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are convergent and equal for all values of x in an open interval $(-c, c)$, then $a_n = b_n$ for every n . (Hint: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$. Differentiate term by term to show that a_n and b_n both equal $f^{(n)}(0)/(n!)$.)
- b. Show that if $\sum_{n=0}^{\infty} a_n x^n = 0$ for all x in an open interval $(-c, c)$, then $a_n = 0$ for every n .

62. The sum of the series $\sum_{n=0}^{\infty} (n^2/2^n)$ To find the sum of this series, express $1/(1-x)$ as a geometric series, differentiate both sides of the resulting equation with respect to x , multiply both sides of the result by x , differentiate again, multiply by x again, and set x equal to $1/2$. What do you get?

63. The sum of the alternating harmonic series This exercise will show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$$

Let h_n be the n th partial sum of the harmonic series, and let s_n be the n th partial sum of the alternating harmonic series.

- a. Use mathematical induction or algebra to show that

$$s_{2n} = h_{2n} - h_n.$$

- b. Use the results in Exercise 63 in Section 9.3 to conclude that

$$\lim_{n \rightarrow \infty} (h_n - \ln n) = \gamma$$

and

$$\lim_{n \rightarrow \infty} (h_{2n} - \ln 2n) = \gamma,$$

where γ is Euler's constant.

- c. Use these facts to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \lim_{n \rightarrow \infty} s_{2n} = \ln 2.$$

64. Assume that the series $\sum a_n x^n$ converges for $x = 4$ and diverges for $x = 7$. Answer true (T), false (F), or not enough information given (N) for the following statements about the series.

- a. Converges absolutely for $x = -4$
- b. Diverges for $x = 5$
- c. Converges absolutely for $x = -8.5$
- d. Converges for $x = -2$
- e. Diverges for $x = 8$
- f. Diverges for $x = -6$
- g. Converges absolutely for $x = 0$
- h. Converges absolutely for $x = -7.1$

65. Assume that the series $\sum a_n (x-2)^n$ converges for $x = -1$ and diverges for $x = 6$. Answer true (T), false (F), or not enough information given (N) for the following statements about the series.

- a. Converges absolutely for $x = 1$
- b. Diverges for $x = -6$
- c. Diverges for $x = 2$
- d. Converges for $x = 0$
- e. Converges absolutely for $x = 5$
- f. Diverges for $x = 4.9$
- g. Diverges for $x = 5.1$
- h. Converges absolutely for $x = 4$

66. Proof of a special case of Theorem 21 Assume that $a = 0$ in Theorem 21, and assume further that $L = \lim_{n \rightarrow \infty} \frac{|c_{n+1}|}{|c_n|}$ exists.

If $L \neq 0$, then set $R = 1/L$, while if $L = 0$, then set $R = \infty$.

The Ratio Test implies that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for

$-R < x < R$. Let $g(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$. This exercise will

prove that f is differentiable and that $f'(x) = g(x)$, that is

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = g(x) \text{ for any } x \text{ with } -R < x < R.$$

Throughout parts (a)–(h), assume that $-R < x < R$ and that h is small enough that $-R < x+h < R$.

- a. Use the Ratio Test to prove that the series defining $g(x)$ converges.
- b. Apply the Mean Value Theorem to the function $p_n(x) = x^n$ to show that

$$\frac{(x+h)^n - x^n}{h} = n(t_n)^{n-1}$$

for some t_n between x and $x+h$ for $n = 1, 2, 3, \dots$

- c. Show

$$g(x) - \frac{f(x+h) - f(x)}{h} = \sum_{n=2}^{\infty} n c_n (x^{n-1} - (t_n)^{n-1}).$$

- d. Apply the Mean Value Theorem to the function $p_{n-1}(x) = x^{n-1}$ to show that

$$\frac{x^{n-1} - (t_n)^{n-1}}{x - t_n} = (n-1)(d_{n-1})^{n-2}$$

for some d_{n-1} between x and t_n for $n = 2, 3, 4, \dots$

- e. Explain why $|x - t_n| < |h|$ and why

$$|d_{n-1}| \leq \alpha = \max\{|x|, |x + h|\}.$$

- f. Show that

$$\left| g(x) - \frac{f(x+h) - f(x)}{h} \right| \leq |h| \sum_{n=2}^{\infty} |n(n-1)c_n \alpha^{n-2}|.$$

Hint: Multiply each term $nc_n(x^{n-1} - (t_n)^{n-1})$ in part (c)

by $\frac{x - t_n}{x - t_n}$.

- g. Show that $\sum_{n=2}^{\infty} n(n-1)c_n \alpha^{n-2}$ converges.

- h. Let $h \rightarrow 0$ in part (f) to conclude that

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = g(x).$$

67. Proof of Theorem 22 Assume that $a = 0$ in Theorem 22 and assume that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for $-R < x < R$.

Let $g(x) = \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$. This exercise will prove that $g'(x) = f(x)$.

- a. Prove that the series defining $g(x)$ converges for $-R < x < R$.

- b. Use Theorem 21 to show that $g'(x) = f(x)$, that is,

$$\int f(x) dx = g(x) + C.$$

9.8 Taylor and Maclaurin Series

We have seen how geometric series can be used to generate a power series for functions such as $f(x) = 1/(1-x)$ or $g(x) = 3/(x-2)$. Now we expand our capability to represent a function with a power series. This section shows how functions that are infinitely differentiable generate power series called *Taylor series*. In many cases, these series provide useful polynomial approximations of the original functions. Because approximation by polynomials is extremely useful to both mathematicians and scientists, Taylor series are an important application of the theory of infinite series.

Series Representations

We know from Theorem 21 that within its interval of convergence I , the sum of a power series is a continuous function with derivatives of all orders. But what about the other way around? If a function $f(x)$ has derivatives of all orders on an interval, can it be expressed as a power series on at least part of that interval? And if it can, what are its coefficients?

We can answer the last question readily if we assume that $f(x)$ is the sum of a power series about $x = a$,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x-a)^n \\ &= a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots \end{aligned}$$

with a positive radius of convergence. By repeated term-by-term differentiation within the interval of convergence I , we obtain

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1} + \cdots, \\ f''(x) &= 1 \cdot 2a_2 + 2 \cdot 3a_3(x-a) + 3 \cdot 4a_4(x-a)^2 + \cdots, \\ f'''(x) &= 1 \cdot 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4(x-a) + 3 \cdot 4 \cdot 5a_5(x-a)^2 + \cdots, \end{aligned}$$

with the n th derivative being

$$f^{(n)}(x) = n!a_n + \text{a sum of terms with } (x-a) \text{ as a factor.}$$

Since these equations all hold at $x = a$, we have

$$f'(a) = a_1, \quad f''(a) = 1 \cdot 2a_2, \quad f'''(a) = 1 \cdot 2 \cdot 3a_3,$$

and, in general,

$$f^{(n)}(a) = n!a_n.$$

These formulas reveal a pattern in the coefficients of any power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ that converges to the values of f on I (“represents f on I ”). If there is such a series (still an open question), then there is only one such series, and its n th coefficient is

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

If f has a series representation with the center at a , then the series must be

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &+ \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots. \end{aligned} \quad (1)$$

But if we start with an arbitrary function f that is infinitely differentiable on an interval containing $x = a$ and use it to generate the series in Equation (1), does the series converge to $f(x)$ at each x in the interval of convergence? The answer is maybe—for some functions it will, but for other functions it will not (as we will see in Example 4).

HISTORICAL BIOGRAPHIES

Brook Taylor (1685–1731)

Taylor was an ingenious and productive British mathematician. Taylor published his book on calculus *Methodus incrementorum directa et inversa* in 1715 and his book on geometry *Linear Perspective* in the same year. To know more, visit the companion Website.

Colin Maclaurin (1698–1746)

Maclaurin was elected a fellow of the Royal Society of London when he was only 21 years old. His *Treatise of Fluxions* (1742) has been described as the earliest logical and systematic publication of Newton’s methods.

To know more, visit the companion Website.

Taylor and Maclaurin Series

The series on the right-hand side of Equation (1) is the most important and useful series we will study in this chapter.

DEFINITIONS Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &+ \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots. \end{aligned}$$

The **Maclaurin series of f** is the Taylor series generated by f at $x = 0$, or

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

The Maclaurin series generated by f is often just called the Taylor series of f .

EXAMPLE 1 Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

Solution We need to find $f(2)$, $f'(2)$, $f''(2)$, ... Taking derivatives, we get

$$f(x) = x^{-1}, \quad f'(x) = -x^{-2}, \quad f''(x) = 2!x^{-3}, \dots, \quad f^{(n)}(x) = (-1)^n n! x^{-(n+1)},$$

so that

$$f(2) = 2^{-1} = \frac{1}{2}, \quad f'(2) = -\frac{1}{2^2}, \quad \frac{f''(2)}{2!} = 2^{-3} = \frac{1}{2^3}, \dots, \quad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}.$$

The Taylor series is

$$\begin{aligned} f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots \\ = \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots. \end{aligned}$$

This is a geometric series with first term $1/2$ and ratio $r = -(x - 2)/2$. It converges absolutely for $|x - 2| < 2$ and its sum is

$$\frac{1/2}{1 + (x - 2)/2} = \frac{1}{2 + (x - 2)} = \frac{1}{x}.$$

In this example the Taylor series generated by $f(x) = 1/x$ at $a = 2$ converges to $1/x$ for $|x - 2| < 2$, or $0 < x < 4$. ■

Taylor Polynomials

The linearization of a differentiable function f at a point a is the polynomial of degree at most 1 given by

$$P_1(x) = f(a) + f'(a)(x - a).$$

In Section 3.11 we used this linearization to approximate $f(x)$ at values of x near a . If f has derivatives of higher order at a , then it has higher-order polynomial approximations as well, one for each available derivative. These polynomials are called the Taylor polynomials of f .

DEFINITION Let f be a function with derivatives of order k for $k = 1, 2, \dots, N$ in some interval containing a as an interior point. Then, for any integer n from 0 through N , the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x - a)^k + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

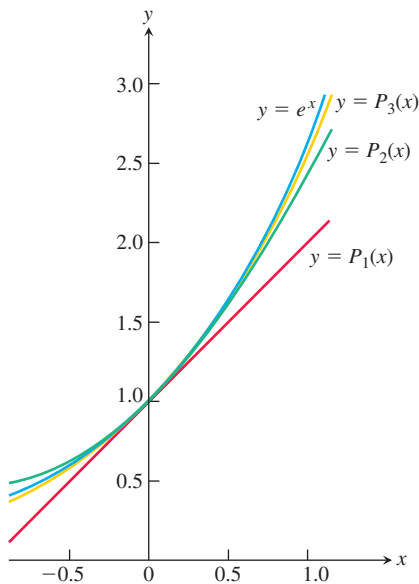


FIGURE 9.22 The graph of $f(x) = e^x$ and its Taylor polynomials

$$P_1(x) = 1 + x$$

$$P_2(x) = 1 + x + (x^2/2!)$$

$$P_3(x) = 1 + x + (x^2/2!) + (x^3/3!).$$

Notice the very close agreement near the center $x = 0$ (Example 2).

We speak of a Taylor polynomial of *order* n rather than *degree* n because $f^{(n)}(a)$ may be zero. The first two Taylor polynomials of $f(x) = \cos x$ at $x = 0$, for example, are $P_0(x) = 1$ and $P_1(x) = 1$. The first-order Taylor polynomial has degree 0, not 1.

Just as the linearization of f at $x = a$ provides the best linear approximation of f in the neighborhood of a , the higher-order Taylor polynomials provide the “best” polynomial approximations of their respective degrees. (See Exercise 51.)

EXAMPLE 2 Find the Taylor series and the Taylor polynomials generated by $f(x) = e^x$ at $x = 0$.

Solution Since $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for every $n = 0, 1, 2, \dots$, the Taylor series generated by f at $x = 0$ (see Figure 9.22) is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots \\ = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + \dots \\ = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \end{aligned}$$

This is also the Maclaurin series for e^x . In the next section we will see that this series converges to e^x at every x .

The Taylor polynomial of order n at $x = 0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!}. \quad \blacksquare$$

EXAMPLE 3 Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$.

Solution The cosine and its derivatives are

$$\begin{aligned} f(x) &= \cos x, & f'(x) &= -\sin x, \\ f''(x) &= -\cos x, & f^{(3)}(x) &= \sin x, \\ &\vdots & &\vdots \\ f^{(2n)}(x) &= (-1)^n \cos x, & f^{(2n+1)}(x) &= (-1)^{n+1} \sin x. \end{aligned}$$

At $x = 0$, the cosines are 1 and the sines are 0, so

$$f^{(2n)}(0) = (-1)^n, \quad f^{(2n+1)}(0) = 0.$$

The Taylor series generated by f at 0 is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ = 1 + 0 \cdot x - \frac{x^2}{2!} + 0 \cdot x^3 + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\ = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}. \end{aligned}$$

This is also the Maclaurin series for $\cos x$. Notice that only even powers of x occur in the Taylor series generated by the cosine function, which is consistent with the fact that it is an even function. In Section 9.9, we will see that the series converges to $\cos x$ at every x .

Because $f^{(2n+1)}(0) = 0$, the Taylor polynomials of orders $2n$ and $2n + 1$ are identical:

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

Figure 9.23 shows how well these polynomials approximate $f(x) = \cos x$ near $x = 0$. Only the right-hand portions of the graphs are given because the graphs are symmetric about the y -axis. ■

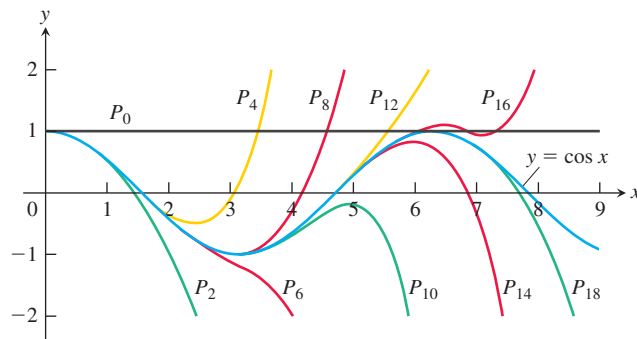


FIGURE 9.23 The polynomials

$$P_{2n}(x) = \sum_{k=0}^n \frac{(-1)^k x^{2k}}{(2k)!}$$

converge to $\cos x$ as $n \rightarrow \infty$. We can deduce the behavior of $\cos x$ arbitrarily far away solely from knowing the values of the cosine and its derivatives at $x = 0$ (Example 3).

EXAMPLE 4 By using mathematical induction (see Exercise 51), it can be shown that

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$$

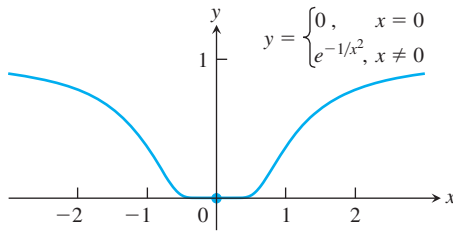


FIGURE 9.24 The graph of the continuous extension of $y = e^{-1/x^2}$ is so flat at the origin that all of its derivatives there are zero (Example 4). Therefore, its Taylor series, which is zero everywhere, is not the function itself.

(Figure 9.24) has derivatives of all orders at $x = 0$ and that $f^{(n)}(0) = 0$ for all n . This means that the Taylor series generated by f at $x = 0$ is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ = 0 + 0 \cdot x + 0 \cdot x^2 + \cdots + 0 \cdot x^n + \cdots \\ = 0 + 0 + \cdots + 0 + \cdots. \end{aligned}$$

The series converges for every x (its sum is 0) but converges to $f(x)$ only at $x = 0$. That is, the Taylor series generated by $f(x)$ in this example is *not* equal to the function $f(x)$ over the entire interval of convergence. ■

Two questions still remain.

1. For what values of x can we normally expect a Taylor series to converge to its generating function?
2. How accurately do a function's Taylor polynomials approximate the function on a given interval?

The answers are provided by a theorem of Taylor in the next section.

EXERCISES 9.8

Finding Taylor Polynomials

In Exercises 1–10, find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a .

1. $f(x) = e^{2x}$, $a = 0$
2. $f(x) = \sin x$, $a = 0$
3. $f(x) = \ln x$, $a = 1$
4. $f(x) = \ln(1 + x)$, $a = 0$
5. $f(x) = 1/x$, $a = 2$
6. $f(x) = 1/(x + 2)$, $a = 0$
7. $f(x) = \sin x$, $a = \pi/4$
8. $f(x) = \tan x$, $a = \pi/4$
9. $f(x) = \sqrt{x}$, $a = 4$
10. $f(x) = \sqrt{1 - x}$, $a = 0$

Finding Taylor Series at $x = 0$ (Maclaurin Series)

Find the Maclaurin series for the functions in Exercises 11–24.

11. e^{-x}
12. xe^x
13. $\frac{1}{1+x}$
14. $\frac{2+x}{1-x}$
15. $\sin 3x$
16. $\sin \frac{x}{2}$
17. $7 \cos(-x)$
18. $5 \cos \pi x$
19. $\cosh x = \frac{e^x + e^{-x}}{2}$
20. $\sinh x = \frac{e^x - e^{-x}}{2}$
21. $x^4 - 2x^3 - 5x + 4$
22. $\frac{x^2}{x+1}$
23. $x \sin x$
24. $(x+1) \ln(x+1)$

Finding Taylor and Maclaurin Series

In Exercises 25–34, find the Taylor series generated by f at $x = a$.

25. $f(x) = x^3 - 2x + 4$, $a = 2$
26. $f(x) = 2x^3 + x^2 + 3x - 8$, $a = 1$
27. $f(x) = x^4 + x^2 + 1$, $a = -2$
28. $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2$, $a = -1$
29. $f(x) = 1/x^2$, $a = 1$
30. $f(x) = 1/(1-x)^3$, $a = 0$
31. $f(x) = e^x$, $a = 2$
32. $f(x) = 2^x$, $a = 1$

33. $f(x) = \cos(2x + (\pi/2))$, $a = \pi/4$

34. $f(x) = \sqrt{x+1}$, $a = 0$

In Exercises 35–40, find the first three nonzero terms of the Maclaurin series for each function.

35. $f(x) = \cos x - (2/(1-x))$

36. $f(x) = (1-x+x^2)e^x$

37. $f(x) = (\sin x) \ln(1+x)$

38. $f(x) = x \sin^2 x$

39. $f(x) = x^4 e^{x^2}$

40. $f(x) = \frac{x^3}{1+2x}$

Quadratic Approximations The Taylor polynomial of order 2 generated by a twice-differentiable function $f(x)$ at $x = a$ is called the *quadratic approximation* of f at $x = a$. In Exercises 41–46, find the (a) linearization (Taylor polynomial of order 1) and (b) quadratic approximation of f at $x = 0$.

41. $f(x) = \ln(\cos x)$

42. $f(x) = e^{\sin x}$

43. $f(x) = 1/\sqrt{1-x^2}$

44. $f(x) = \cosh x$

45. $f(x) = \sin x$

46. $f(x) = \tan x$

47. If m is a small positive number, then the graphs of $y = \sin x$ and $y = mx$ intersect at a point whose x -coordinate is close to π . Find the quadratic approximation to $f(x) = \sin x - mx$ at $x = \pi$, and use this to show that an approximate solution to the equation $\sin x = mx$ is $x \approx \pi/(1+m)$.

Theory and Examples

48. Use the Taylor series generated by e^x at $x = a$ to show that

$$e^x = e^a \left[1 + (x-a) + \frac{(x-a)^2}{2!} + \cdots \right].$$

49. (Continuation of Exercise 48.) Find the Taylor series generated by e^x at $x = 1$. Compare your answer with the formula in Exercise 48.

50. Let $f(x)$ have derivatives through order n at $x = a$. Show that the Taylor polynomial of order n and its first n derivatives have the same values that f and its first n derivatives have at $x = a$.

51. Approximation properties of Taylor polynomials Suppose that $f(x)$ is differentiable on an interval centered at $x = a$ and that $g(x) = b_0 + b_1(x - a) + \cdots + b_n(x - a)^n$ is a polynomial of degree n with constant coefficients $b_0, \dots, b_n$. Let $E(x) = f(x) - g(x)$. Show that if we impose on g the conditions

i) $E(a) = 0$ The approximation error is zero at $x = a$.

ii) $\lim_{x \rightarrow a} \frac{E(x)}{(x - a)^n} = 0$, The error is negligible when compared to $(x - a)^n$.

then

$$g(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

Thus, the Taylor polynomial $P_n(x)$ is the only polynomial of degree less than or equal to n whose error is both zero at $x = a$ and negligible when compared with $(x - a)^n$.

52. Let f be the function from Example 4. Let $p_1(t) = 2t^3$ and $p_2(t) = 4t^6 - 6t^4$.

a. Show that if $x > 0$, then

$$f'(x) = 2x^{-3}e^{-x^{-2}} = p_1(1/x)f(x),$$

while for $x = 0$,

$$f'(0) = \lim_{h \rightarrow 0^+} \frac{f(h) - 0}{h - 0} = \lim_{t \rightarrow \infty} \frac{f(1/t)}{1/t} = 0.$$

b. Show that if $x > 0$, then

$$f''(x) = (4x^{-6} - 6x^{-3})e^{-x^{-2}} = p_2(1/x)f(x),$$

and $f''(0) = 0$.

c. For $n \geq 2$, recursively define polynomials p_n by $p_{n+1}(t) = 2t^3 p_n(t) - t^2 p'_n(t)$. Use mathematical induction to prove that p_n has degree $3n$, $f^{(n)}(x) = p_n(1/x)f(x)$ for $x > 0$, and $f^{(n)}(0) = 0$.

9.9 Convergence of Taylor Series

In the last section we asked when a Taylor series for a function can be expected to converge to the function that generates it. The finite-order Taylor polynomials that approximate the Taylor series provide estimates for the generating function. In order for these estimates to be useful, we need a way to control the possible errors we may encounter when approximating a function with its finite-order Taylor polynomials. How do we bound such possible errors? We answer the question in this section with the following theorem.

THEOREM 23—Taylor's Theorem

If f and its first n derivatives $f', f'', \dots, f^{(n)}$ are continuous on the closed interval between a and b , and $f^{(n)}$ is differentiable on the open interval between a and b , then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b - a)^n + \frac{f^{(n+1)}(c)}{(n + 1)!}(b - a)^{n+1}.$$

Taylor's Theorem is a generalization of the Mean Value Theorem (Exercise 49), and we omit its proof here.

When we apply Taylor's Theorem, we usually want to hold a fixed and treat b as an independent variable. Taylor's formula is easier to use in circumstances like these if we change b to x . Here is a version of the theorem with this change.

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a , then for each positive integer n and for each x in I ,

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x), \quad (1)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \quad (2)$$

When we state Taylor's theorem this way, it says that for each $x \in I$,

$$f(x) = P_n(x) + R_n(x).$$

The function $R_n(x)$ is determined by the value of the $(n + 1)$ st derivative $f^{(n+1)}$ at a point c that depends on both a and x , and that lies somewhere between them. For any value of n we want, the equation gives both a polynomial approximation of f of that order and a formula for the error involved in using that approximation over the interval I .

Equation (1) is called **Taylor's formula**. The function $R_n(x)$ is called the **remainder of order n** or the **error term** for the approximation of f by $P_n(x)$ over I .

If $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ **converges** to f on I , and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k.$$

Often we can estimate R_n without knowing the value of c , as the following example illustrates.

EXAMPLE 1 Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every real value of x .

Solution The function has derivatives of all orders throughout the interval $I = (-\infty, \infty)$. Equations (1) and (2) with $f(x) = e^x$ and $a = 0$ give

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x) \quad \text{Polynomial from Section 9.8, Example 2}$$

and

$$R_n(x) = \frac{e^c}{(n+1)!} x^{n+1} \quad \text{for some } c \text{ between } 0 \text{ and } x.$$

Since e^x is an increasing function of x , e^c lies between $e^0 = 1$ and e^x . When x is negative, so is c , and $e^c < 1$. When x is zero, $e^x = 1$ so that $R_n(x) = 0$. When x is positive, so is c , and $e^c < e^x$. Thus, for $R_n(x)$ given as above,

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \quad \text{when } x \leq 0, \quad e^c < 1 \text{ since } c < 0$$

and

$$|R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!} \quad \text{when } x > 0. \quad e^c < e^x \text{ since } c < x$$

Finally, because

$$\lim_{n \rightarrow \infty} \frac{x^{n+1}}{(n+1)!} = 0 \quad \text{for every } x, \quad \text{Section 9.1, Theorem 5}$$

we see that $\lim_{n \rightarrow \infty} R_n(x) = 0$, and therefore the Taylor series converges to e^x for every x . Thus,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots. \quad (3)$$



The Number e as a Series

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

We can use the result of Example 1 with $x = 1$ to write

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1),$$

where, for some c between 0 and 1,

$$R_n(1) = e^c \frac{1}{(n+1)!} < \frac{3}{(n+1)!}, \quad e^c < e^1 < 3$$

Estimating the Remainder

It is often possible to estimate $R_n(x)$ as we did in Example 1. This method of estimation is so convenient that we state it as a theorem for future reference.

THEOREM 24—The Remainder Estimation Theorem

If there is a positive constant M such that $|f^{(n+1)}(t)| \leq M$ for all t between x and a , inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x - a|^{n+1}}{(n+1)!}.$$

If this inequality holds for every n , and the other conditions of Taylor's Theorem are satisfied by f , then the Taylor series converges to $f(x)$.

The next two examples use Theorem 24 to show that the Taylor series generated by the sine and cosine functions do in fact converge to the functions themselves.

EXAMPLE 2 Show that the Taylor series for $\sin x$ at $x = 0$ converges for all x .

Solution The function and its derivatives are

$$\begin{aligned} f(x) &= \sin x, & f'(x) &= \cos x, \\ f''(x) &= -\sin x, & f'''(x) &= -\cos x, \\ &\vdots & &\vdots \\ f^{(2k)}(x) &= (-1)^k \sin x, & f^{(2k+1)}(x) &= (-1)^k \cos x, \end{aligned}$$

so

$$f^{(2k)}(0) = 0 \quad \text{and} \quad f^{(2k+1)}(0) = (-1)^k.$$

The series has only odd-powered terms, and for $n = 2k + 1$, Taylor's Theorem gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x).$$

All the derivatives of $\sin x$ have absolute values less than or equal to 1, so we can apply the Remainder Estimation Theorem with $M = 1$ to obtain

$$|R_{2k+1}(x)| \leq 1 \cdot \frac{|x|^{2k+2}}{(2k+2)!}.$$

From Theorem 5, Rule 6, we have $(|x|^{2k+2}/(2k+2)!) \rightarrow 0$ as $k \rightarrow \infty$, whatever the value of x , so $R_{2k+1}(x) \rightarrow 0$ and the Maclaurin series for $\sin x$ converges to $\sin x$ for every x . Thus,

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \quad (4)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

EXAMPLE 3 Show that the Taylor series for $\cos x$ at $x = 0$ converges to $\cos x$ for every value of x .

Solution We add the remainder term to the Taylor polynomial for $\cos x$ (Section 9.8, Example 3) to obtain Taylor's formula for $\cos x$ with $n = 2k$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + R_{2k}(x).$$

Because the derivatives of the cosine have absolute value less than or equal to 1, the Remainder Estimation Theorem with $M = 1$ gives

$$|R_{2k}(x)| \leq 1 \cdot \frac{|x|^{2k+1}}{(2k+1)!}.$$

For every value of x , $R_{2k}(x) \rightarrow 0$ as $k \rightarrow \infty$. Therefore, the series converges to $\cos x$ for every value of x . Thus,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad (5)$$

Using Taylor Series

Since every Taylor series is a power series, the operations of adding, subtracting, and multiplying Taylor series are all valid on the intersection of their intervals of convergence.

EXAMPLE 4 Using known series, find the first few terms of the Taylor series for the given function by using power series operations.

(a) $\frac{1}{3}(2x + x \cos x)$ (b) $e^x \cos x$

Solution

$$\begin{aligned} \text{(a)} \quad \frac{1}{3}(2x + x \cos x) &= \frac{2}{3}x + \frac{1}{3}x \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots \right) \quad \text{Taylor series for } \cos x \\ &= \frac{2}{3}x + \frac{1}{3}x - \frac{x^3}{3!} + \frac{x^5}{3 \cdot 4!} - \cdots = x - \frac{x^3}{6} + \frac{x^5}{72} - \cdots \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad e^x \cos x &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right) \quad \text{Multiply the first series by each term of the second series.} \\ &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right) - \left(\frac{x^2}{2!} + \frac{x^3}{2!} + \frac{x^4}{2!2!} + \frac{x^5}{2!3!} + \cdots \right) \\ &\quad + \left(\frac{x^4}{4!} + \frac{x^5}{4!} + \frac{x^6}{2!4!} + \cdots \right) + \cdots \\ &= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots \end{aligned}$$

By Theorem 20, if the Taylor series generated by f at $x = 0$,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots, \quad (6)$$

converges absolutely for $|x| < R$, and if u is a continuous function, then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (u(x))^k \quad (7)$$

obtained by substituting $u(x)$ for x in the Taylor series of Formula (6) converges absolutely on the set of points x where $|u(x)| < R$.

If $u(x) = cx^m$, where $c \neq 0$ and m is a positive integer, then the series of Formula (7) is a power series and can be shown to be the Taylor series generated by $f(u(x))$ at $x = 0$.

For instance, we can find the Taylor series for $\cos 2x$ by substituting $2x$ for x in the Taylor series for $\cos x$:

$$\begin{aligned} \cos 2x &= \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots \quad \text{Eq. (5) with } 2x \text{ for } x \\ &= 1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \cdots \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} x^{2k}}{(2k)!}. \end{aligned}$$

By Theorem 20, this new Taylor series converges for all x .

EXAMPLE 5 For what values of x can we replace $\sin x$ by $x - (x^3/3!)$ and obtain an error whose magnitude is no greater than 3×10^{-4} ?

Solution Here we can take advantage of the fact that the Taylor series for $\sin x$ is an alternating series for every nonzero value of x . According to the Alternating Series Estimation Theorem (Section 9.6), the error in truncating

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

after $(x^3/3!)$ is no greater than

$$\left| \frac{x^5}{5!} \right| = \frac{|x|^5}{120}.$$

Therefore, the error will be less than or equal to 3×10^{-4} if

$$\frac{|x|^5}{120} < 3 \times 10^{-4} \quad \text{or} \quad |x| < \sqrt[5]{360 \times 10^{-4}} \approx 0.514. \quad \text{Rounded down, to be safe}$$

The Alternating Series Estimation Theorem tells us something that the Remainder Estimation Theorem does not: The estimate $x - (x^3/3!)$ for $\sin x$ is an underestimate when x is positive, because then $x^5/120$ is positive.

Figure 9.25 shows the graph of $\sin x$, along with the graphs of a number of its approximating Taylor polynomials. The graph of $P_3(x) = x - (x^3/3!)$ is almost indistinguishable from the sine curve when $0 \leq x \leq 1$. ■

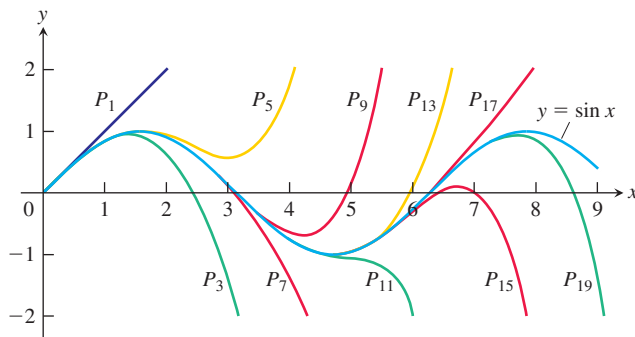


FIGURE 9.25 The polynomials

$$P_{2n+1}(x) = \sum_{k=0}^n \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

converge to $\sin x$ as $n \rightarrow \infty$. Notice how closely $P_3(x)$ approximates the sine curve for $x \leq 1$ (Example 5).

A Proof of Taylor's Theorem

We prove Taylor's theorem assuming $a < b$. The proof for $a > b$ is nearly the same.

The Taylor polynomial

$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

and its first n derivatives match the function f and its first n derivatives at $x = a$. We do not disturb that matching if we add another term of the form $K(x - a)^{n+1}$, where K is any constant, because such a term and its first n derivatives are all equal to zero at $x = a$. The new function

$$\phi_n(x) = P_n(x) + K(x - a)^{n+1}$$

and its first n derivatives still agree with f and its first n derivatives at $x = a$.

We now choose the particular value of K that makes the curve $y = \phi_n(x)$ agree with the original curve $y = f(x)$ at $x = b$. In symbols,

$$f(b) = P_n(b) + K(b - a)^{n+1}, \quad \text{or} \quad K = \frac{f(b) - P_n(b)}{(b - a)^{n+1}}. \quad (8)$$

With K defined by Equation (8), the function

$$F(x) = f(x) - \phi_n(x)$$

measures the difference between the original function f and the approximating function ϕ_n for each x in $[a, b]$.

We now use Rolle's Theorem (Section 4.2). First, because $F(a) = F(b) = 0$ and both F and F' are continuous on $[a, b]$, we know that

$$F'(c_1) = 0 \quad \text{for some } c_1 \text{ in } (a, b).$$

Next, because $F'(a) = F'(c_1) = 0$ and both F' and F'' are continuous on $[a, c_1]$, we know that

$$F''(c_2) = 0 \quad \text{for some } c_2 \text{ in } (a, c_1).$$

Rolle's Theorem, applied successively to $F'', F''', \dots, F^{(n-1)}$, implies the existence of

$$\begin{aligned} c_3 & \text{ in } (a, c_2) & \text{such that } F'''(c_3) &= 0, \\ c_4 & \text{ in } (a, c_3) & \text{such that } F^{(4)}(c_4) &= 0, \\ & \vdots \\ c_n & \text{ in } (a, c_{n-1}) & \text{such that } F^{(n)}(c_n) &= 0. \end{aligned}$$

Finally, because $F^{(n)}$ is continuous on $[a, c_n]$ and differentiable on (a, c_n) , and $F^{(n)}(a) = F^{(n)}(c_n) = 0$, Rolle's Theorem implies that there is a number c_{n+1} in (a, c_n) such that

$$F^{(n+1)}(c_{n+1}) = 0. \quad (9)$$

If we differentiate $F(x) = f(x) - P_n(x) - K(x - a)^{n+1}$ a total of $n + 1$ times, we get

$$F^{(n+1)}(x) = f^{(n+1)}(x) - 0 - (n + 1)!K. \quad (10)$$

Equations (9) and (10) together give

$$K = \frac{f^{(n+1)}(c)}{(n + 1)!} \quad \text{for some number } c = c_{n+1} \text{ in } (a, b). \quad (11)$$

Equations (8) and (11) give

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n + 1)!}(b - a)^{n+1}.$$

This concludes the proof. ■

EXERCISES 9.9

Finding Taylor Series

Use substitution (as in Formula (7)) to find the Taylor series at $x = 0$ of the functions in Exercises 1–12.

1. e^{-5x}
2. $e^{-x/2}$
3. $5 \sin(-x)$
4. $\sin\left(\frac{\pi x}{2}\right)$
5. $\cos 5x^2$
6. $\cos(x/\sqrt{2})$
7. $\ln(1 + x^2)$
8. $\arctan(3x^4)$
9. $\frac{1}{1 + \frac{3}{4}x^3}$
10. $\frac{1}{2 - x}$
11. $\ln(3 + 6x)$
12. $e^{-x^2 + \ln 5}$

Use power series operations to find the Taylor series at $x = 0$ for the functions in Exercises 13–30.

13. xe^x
14. $x^2 \sin x$
15. $\frac{x^2}{2} - 1 + \cos x$
16. $\sin x - x + \frac{x^3}{3!}$
17. $x \cos \pi x$
18. $x^2 \cos(x^2)$
19. $\cos^2 x$ (Hint: $\cos^2 x = (1 + \cos 2x)/2$.)
20. $\sin^2 x$
21. $\frac{x^2}{1 - 2x}$
22. $x \ln(1 + 2x)$
23. $\frac{1}{(1 - x)^2}$
24. $\frac{2}{(1 - x)^3}$
25. $x \arctan x^2$
26. $\sin x \cdot \cos x$
27. $e^x + \frac{1}{1 + x}$
28. $\cos x - \sin x$
29. $\frac{x}{3} \ln(1 + x^2)$
30. $\ln(1 + x) - \ln(1 - x)$

Find the first four nonzero terms in the Maclaurin series for the functions in Exercises 31–38.

31. $e^x \sin x$
32. $\frac{\ln(1 + x)}{1 - x}$
33. $(\arctan x)^2$
34. $\cos^2 x \cdot \sin x$
35. $e^{\sin x}$
36. $\sin(\tan^{-1} x)$
37. $\cos(e^x - 1)$
38. $\cos \sqrt{x} + \ln(\cos x)$

Error Estimates

39. Estimate the error if $P_3(x) = x - (x^3/6)$ is used to estimate the value of $\sin x$ at $x = 0.1$.
40. Estimate the error if $P_4(x) = 1 + x + (x^2/2) + (x^3/6) + (x^4/24)$ is used to estimate the value of e^x at $x = 1/2$.
41. For approximately what values of x can you replace $\sin x$ by $x - (x^3/6)$ with an error of magnitude no greater than 5×10^{-4} ? Give reasons for your answer.
42. If $\cos x$ is replaced by $1 - (x^2/2)$ and $|x| < 0.5$, what estimate can be made of the error? Does $1 - (x^2/2)$ tend to be too large, or too small? Give reasons for your answer.
43. How close is the approximation $\sin x = x$ when $|x| < 10^{-3}$? For which of these values of x is $x < \sin x$?
44. The estimate $\sqrt{1 + x} = 1 + (x/2)$ is used when x is small. Estimate the error when $|x| < 0.01$.
45. The approximation $e^x = 1 + x + (x^2/2)$ is used when x is small. Use the Remainder Estimation Theorem to estimate the error when $|x| < 0.1$.

46. (Continuation of Exercise 45.) When $x < 0$, the series for e^x is an alternating series. Use the Alternating Series Estimation Theorem to estimate the error that results from replacing e^x by $1 + x + (x^2/2)$ when $-0.1 < x < 0$. Compare your estimate with the one you obtained in Exercise 45.

Theory and Examples

47. Use the identity $\sin^2 x = (1 - \cos 2x)/2$ to obtain the Maclaurin series for $\sin^2 x$. Then differentiate this series to obtain the Maclaurin series for $2 \sin x \cos x$. Check that this is the series for $\sin 2x$.
48. (Continuation of Exercise 47.) Use the identity $\cos^2 x = \cos 2x + \sin^2 x$ to obtain a power series for $\cos^2 x$.
49. **Taylor's Theorem and the Mean Value Theorem** Explain how the Mean Value Theorem (Section 4.2, Theorem 4) is a special case of Taylor's Theorem.
50. **Linearizations at inflection points** Show that if the graph of a twice-differentiable function $f(x)$ has an inflection point at $x = a$, then the linearization of f at $x = a$ is also the quadratic approximation of f at $x = a$. This explains why tangent lines fit so well at inflection points.
51. **The (second) second derivative test** Use the equation

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(c_2)}{2}(x - a)^2$$

to establish the following test.

Let f have continuous first and second derivatives and suppose that $f'(a) = 0$. Then

- a. f has a local maximum at a if $f'' \leq 0$ throughout an interval whose interior contains a ;
 - b. f has a local minimum at a if $f'' \geq 0$ throughout an interval whose interior contains a .
52. **A cubic approximation** Use Taylor's formula with $a = 0$ and $n = 3$ to find the standard cubic approximation of $f(x) = 1/(1 - x)$ at $x = 0$. Give an upper bound for the magnitude of the error in the approximation when $|x| \leq 0.1$.
 53. a. Use Taylor's formula with $n = 2$ to find the quadratic approximation of $f(x) = (1 + x)^k$ at $x = 0$ (k a constant).
b. If $k = 3$, for approximately what values of x in the interval $[0, 1]$ will the error in the quadratic approximation be less than $1/100$?

54. Improving approximations of π

- a. Let P be an approximation of π accurate to n decimals. Show that $P + \sin P$ gives an approximation correct to $3n$ decimals. (Hint: Let $P = \pi + x$.)

T b. Try it with a calculator.

55. **The Taylor series generated by** $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is $\sum_{n=0}^{\infty} a_n x^n$. A function defined by a power series $\sum_{n=0}^{\infty} a_n x^n$ with a radius of convergence $R > 0$ has a Taylor series that converges to the function at every point of $(-R, R)$. Show this by showing that the Taylor series generated by $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is the series $\sum_{n=0}^{\infty} a_n x^n$ itself.

An immediate consequence of this is that series like

$$x \sin x = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots$$

and

$$x^2 e^x = x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \cdots,$$

obtained by multiplying Taylor series by powers of x , as well as series obtained by integration and differentiation of convergent power series, are themselves the Taylor series generated by the functions they represent.

56. Taylor series for even functions and odd functions (Continuation of Section 9.7, Exercise 61.) Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for all x in an open interval $(-R, R)$. Show that

- If f is even, then $a_1 = a_3 = a_5 = \cdots = 0$, i.e., the Taylor series for f at $x = 0$ contains only even powers of x .
- If f is odd, then $a_0 = a_2 = a_4 = \cdots = 0$, i.e., the Taylor series for f at $x = 0$ contains only odd powers of x .

COMPUTER EXPLORATIONS

Taylor's formula with $n = 1$ and $a = 0$ gives the linearization of a function at $x = 0$. With $n = 2$ and $n = 3$, we obtain the standard quadratic and cubic approximations. In these exercises we explore the errors associated with these approximations. We seek answers to two questions:

- For what values of x can the function be replaced by each approximation with an error less than 10^{-2} ?
- What is the maximum error we can expect if we replace the function by each approximation over the specified interval?

Using a CAS, perform the following steps to aid in answering questions (a) and (b) for the functions and intervals in Exercises 57–62.

Step 1: Plot the function over the specified interval.

Step 2: Find the Taylor polynomials $P_1(x)$, $P_2(x)$, and $P_3(x)$ at $x = 0$.

Step 3: Calculate the $(n + 1)$ st derivative $f^{(n+1)}(c)$ associated with the remainder term for each Taylor polynomial. Plot the derivative as a function of c over the specified interval and estimate its maximum absolute value, M .

Step 4: Calculate the remainder $R_n(x)$ for each polynomial. Using the estimate M from Step 3 in place of $f^{(n+1)}(c)$, plot $R_n(x)$ over the specified interval. Then estimate the values of x that answer question (a).

Step 5: Compare your estimated error with the actual error $E_n(x) = |f(x) - P_n(x)|$ by plotting $E_n(x)$ over the specified interval. This will help you answer question (b).

Step 6: Graph the function and its three Taylor approximations together. Discuss the graphs in relation to the information discovered in Steps 4 and 5.

$$57. f(x) = \frac{1}{\sqrt{1+x}}, \quad |x| \leq \frac{3}{4}$$

$$58. f(x) = (1+x)^{3/2}, \quad -\frac{1}{2} \leq x \leq 2$$

$$59. f(x) = \frac{x}{x^2 + 1}, \quad |x| \leq 2$$

$$60. f(x) = (\cos x)(\sin 2x), \quad |x| \leq 2$$

$$61. f(x) = e^{-x} \cos 2x, \quad |x| \leq 1$$

$$62. f(x) = e^{x/3} \sin 2x, \quad |x| \leq 2$$

9.10 Applications of Taylor Series

We can use Taylor series to solve problems that would otherwise be intractable. For example, many functions have antiderivatives that cannot be expressed using familiar functions. In this section we show how to evaluate integrals of such functions by giving them as Taylor series. We also show how to use Taylor series to evaluate limits that lead to indeterminate forms and how Taylor series can be used to extend the exponential function from real to complex numbers. We begin with a discussion of the binomial series, which comes from the Taylor series of the function $f(x) = (1+x)^m$, and we conclude the section with Table 9.1, which lists some commonly used Taylor series.

The Binomial Series for Powers and Roots

The Taylor series generated by $f(x) = (1+x)^m$, when m is constant, is

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}x^k + \cdots \quad (1)$$

This series, called the **binomial series**, converges absolutely for $|x| < 1$. To derive the series, we first list the function and its derivatives:

$$\begin{aligned} f(x) &= (1+x)^m \\ f'(x) &= m(1+x)^{m-1} \\ f''(x) &= m(m-1)(1+x)^{m-2} \\ f'''(x) &= m(m-1)(m-2)(1+x)^{m-3} \\ &\vdots \\ f^{(k)}(x) &= m(m-1)(m-2)\cdots(m-k+1)(1+x)^{m-k}. \end{aligned}$$

We then evaluate these at $x = 0$ and substitute into the Taylor series formula to obtain Series (1).

If m is an integer greater than or equal to zero, the series stops after $(m+1)$ terms because the coefficients from $k = m+1$ on are zero.

If m is not a positive integer or zero, the series is infinite and converges for $|x| < 1$. To see why, let u_k be the term involving x^k . Then apply the Ratio Test for absolute convergence to see that

$$\left| \frac{u_{k+1}}{u_k} \right| = \left| \frac{m-k}{k+1} x \right| \rightarrow |x| \quad \text{as } k \rightarrow \infty.$$

Our derivation of the binomial series shows only that it is generated by $(1+x)^m$ and converges for $|x| < 1$. The derivation does not show that the series converges to $(1+x)^m$. It does, but we leave the proof to Exercise 58. The following formulation gives a succinct way to express the series.

The Binomial Series

For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{1} = m, \quad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \quad \text{for } k \geq 3.$$

EXAMPLE 1 If $m = -1$, then

$$\binom{-1}{1} = -1, \quad \binom{-1}{2} = \frac{-1(-2)}{2!} = 1,$$

and

$$\binom{-1}{k} = \frac{-1(-2)(-3)\cdots(-1-k+1)}{k!} = (-1)^k \left(\frac{k!}{k!} \right) = (-1)^k.$$

With these coefficient values and with x replaced by $-x$, the binomial series formula gives the familiar geometric series

$$(1+x)^{-1} = 1 + \sum_{k=1}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots. \quad \blacksquare$$

EXAMPLE 2 We know from Section 3.11, Example 1, that $\sqrt{1+x} \approx 1 + (x/2)$ for $|x|$ small. With $m = 1/2$, the binomial series gives quadratic and higher-order approximations as well, along with error estimates that come from the Alternating Series Estimation Theorem:

$$\begin{aligned}(1+x)^{1/2} &= 1 + \frac{x}{2} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 \\ &\quad + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}x^4 + \cdots \\ &= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots.\end{aligned}$$

Substitution for x gives still other approximations. For example,

$$\begin{aligned}\sqrt{1-x^2} &\approx 1 - \frac{x^2}{2} - \frac{x^4}{8} \quad \text{for } |x^2| \text{ small} \\ \sqrt{1-\frac{1}{x}} &\approx 1 - \frac{1}{2x} - \frac{1}{8x^2} \quad \text{for } \left|\frac{1}{x}\right| \text{ small, that is, } |x| \text{ large.}\end{aligned}$$

Evaluating Nonelementary Integrals

Sometimes we can use a familiar Taylor series to find the sum of a given power series in terms of a known function. For example,

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots = (x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots = \sin x^2.$$

Additional examples are provided in Exercises 59–62.

Taylor series can be used to express nonelementary integrals in terms of series. Integrals like $\int \sin x^2 dx$ arise in the study of the diffraction of light.

EXAMPLE 3 Express $\int \sin x^2 dx$ as a power series.

Solution From the series for $\sin x$, we substitute x^2 for x to obtain

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \cdots.$$

Therefore,

$$\int \sin x^2 dx = C + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \frac{x^{19}}{19 \cdot 9!} - \cdots.$$

EXAMPLE 4 Estimate $\int_0^1 \sin x^2 dx$ with an error of less than 0.001.

Solution From the indefinite integral in Example 3, we easily find that

$$\int_0^1 \sin x^2 dx = \frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \frac{1}{19 \cdot 9!} - \cdots.$$

The series on the right-hand side alternates, and we find by numerical evaluations that

$$\frac{1}{11 \cdot 5!} \approx 0.00076$$

is the first term to be numerically less than 0.001. The sum of the preceding two terms gives

$$\int_0^1 \sin x^2 dx \approx \frac{1}{3} - \frac{1}{42} \approx 0.310.$$

With two more terms, we could estimate

$$\int_0^1 \sin x^2 dx \approx 0.310268$$

with an error of less than 10^{-6} . With only one term beyond that, we have

$$\int_0^1 \sin x^2 dx \approx \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} + \frac{1}{6894720} \approx 0.310268303,$$

with an error of about 1.08×10^{-9} . Guaranteeing this accuracy with the error formula for the Trapezoidal Rule would require using about 8000 subintervals. ■

Arctangents

In Section 9.7, Example 5, we found a series for $\arctan x$ by differentiating to get

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

and then integrating to get

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

However, we did not prove the term-by-term integration theorem on which this conclusion depended. We now derive the series again by integrating both sides of the finite formula

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}, \quad (2)$$

in which the last term comes from adding the remaining terms as a geometric series with first term $a = (-1)^{n+1} t^{2n+2}$ and ratio $r = -t^2$. Integrating both sides of Equation (2) from $t = 0$ to $t = x$ gives

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + R_n(x),$$

where

$$R_n(x) = \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} dt.$$

The denominator of the integrand is greater than or equal to 1; hence

$$|R_n(x)| \leq \int_0^{|x|} t^{2n+2} dt = \frac{|x|^{2n+3}}{2n+3}.$$

If $|x| \leq 1$, the right side of this inequality approaches zero as $n \rightarrow \infty$. Therefore, $\lim_{n \rightarrow \infty} R_n(x) = 0$ if $|x| \leq 1$ and

$$\begin{aligned} \arctan x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \leq 1. \\ \arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \quad |x| \leq 1. \end{aligned} \quad (3)$$

We take this route instead of finding the Taylor series directly because the formulas for the higher-order derivatives of $\arctan x$ are unmanageable. When we put $x = 1$ in Equation (3), we get **Leibniz's formula**:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + \frac{(-1)^n}{2n+1} + \cdots.$$

Because this series converges very slowly, it is not used in approximating π to many decimal places. The series for $\arctan x$ converges most rapidly when x is near zero. For that reason, people who use the series for $\arctan x$ to compute π use various trigonometric identities.

For example, if

$$\alpha = \arctan \frac{1}{2} \quad \text{and} \quad \beta = \arctan \frac{1}{3},$$

then

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = 1 = \tan \frac{\pi}{4},$$

and therefore,

$$\frac{\pi}{4} = \alpha + \beta = \arctan \frac{1}{2} + \arctan \frac{1}{3}.$$

Now Equation (3) may be used with $x = 1/2$ to evaluate $\arctan(1/2)$ and with $x = 1/3$ to give $\arctan(1/3)$. The sum of these results, multiplied by 4, gives π .

Evaluating Indeterminate Forms

We can sometimes evaluate indeterminate forms by expressing the functions involved as Taylor series.

EXAMPLE 5 Evaluate

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1}.$$

Solution We represent $\ln x$ as a Taylor series in powers of $x - 1$. This can be accomplished by calculating the Taylor series generated by $\ln x$ at $x = 1$ directly or by replacing x by $x - 1$ in the series for $\ln(1 + x)$ in Section 9.7, Example 6. Either way, we obtain

$$\ln x = (x - 1) - \frac{1}{2}(x - 1)^2 + \cdots,$$

from which we find that

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1} = \lim_{x \rightarrow 1} \left(1 - \frac{1}{2}(x - 1) + \cdots \right) = 1.$$

Of course, this particular limit can be evaluated just as well using l'Hôpital's Rule. ■

EXAMPLE 6 Evaluate

$$\lim_{x \rightarrow 0} \frac{\sin x - \tan x}{x^3}.$$

Solution The Taylor series for $\sin x$ and $\tan x$, to terms in x^5 , are

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \quad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots.$$

Subtracting the series term by term, it follows that

$$\sin x - \tan x = -\frac{x^3}{2} - \frac{x^5}{8} - \cdots = x^3 \left(-\frac{1}{2} - \frac{x^2}{8} - \cdots \right).$$

Division of both sides by x^3 and taking limits then give

$$\lim_{x \rightarrow 0} \frac{\sin x - \tan x}{x^3} = \lim_{x \rightarrow 0} \left(-\frac{1}{2} - \frac{x^2}{8} - \cdots \right) = -\frac{1}{2}. \quad \blacksquare$$

If we apply series to calculate $\lim_{x \rightarrow 0} ((1/\sin x) - (1/x))$, we not only find the limit successfully but also discover an approximation formula for $\csc x$.

EXAMPLE 7 Find $\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$.

Solution Using algebra and the Taylor series for $\sin x$, we have

$$\begin{aligned} \frac{1}{\sin x} - \frac{1}{x} &= \frac{x - \sin x}{x \sin x} = \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right)}{x \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right)} \\ &= \frac{x^3 \left(\frac{1}{3!} - \frac{x^2}{5!} + \cdots \right)}{x^2 \left(1 - \frac{x^2}{3!} + \cdots \right)} = x \cdot \frac{\frac{1}{3!} - \frac{x^2}{5!} + \cdots}{1 - \frac{x^2}{3!} + \cdots}. \end{aligned}$$

Therefore,

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \rightarrow 0} \left(x \cdot \frac{\frac{1}{3!} - \frac{x^2}{5!} + \cdots}{1 - \frac{x^2}{3!} + \cdots} \right) = 0.$$

From the quotient on the right, we can see that if $|x|$ is small, then

$$\frac{1}{\sin x} - \frac{1}{x} \approx x \cdot \frac{1}{3!} = \frac{x}{6} \quad \text{or} \quad \csc x \approx \frac{1}{x} + \frac{x}{6}. \quad \blacksquare$$

Euler's Identity

A complex number is a number of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$ (see Chapter 18). If we substitute $x = i\theta$ (with θ real) in the Taylor series for e^x and use the relations

$$i^2 = -1, \quad i^3 = i^2 i = -i, \quad i^4 = i^2 i^2 = 1, \quad i^5 = i^4 i = i,$$

and so on to simplify the result, we obtain

$$\begin{aligned} e^{i\theta} &= 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) = \cos \theta + i \sin \theta. \end{aligned}$$

This does not *prove* that $e^{i\theta} = \cos \theta + i \sin \theta$ because we have not yet defined what it means to raise e to an imaginary power. Rather, it tells us how to define $e^{i\theta}$ so that its properties are consistent with the properties of the exponential function for real numbers.

DEFINITION For any real number θ ,

$$e^{i\theta} = \cos \theta + i \sin \theta. \quad (4)$$

Equation (4), called **Euler's identity**, enables us to define e^{a+bi} to be $e^a \cdot e^{bi}$ for any complex number $a + bi$. So

$$e^{a+ib} = e^a(\cos b + i \sin b).$$

One consequence of this identity is the equation

$$e^{i\pi} = -1.$$

When written in the form $e^{i\pi} + 1 = 0$, this equation combines five of the most important constants in mathematics.

TABLE 9.1 Frequently Used Taylor Series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \leq 1$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \leq 1$$

EXERCISES 9.10

Taylor Series

Find the first four nonzero terms of the Taylor series for the functions in Exercises 1–10.

1. $(1+x)^{1/2}$
2. $(1+x)^{1/3}$
3. $(1-x)^{-3}$
4. $(1-2x)^{1/2}$
5. $\left(1+\frac{x}{2}\right)^{-2}$
6. $\left(1-\frac{x}{3}\right)^4$
7. $(1+x^3)^{-1/2}$
8. $(1+x^2)^{-1/3}$
9. $\left(1+\frac{x^2}{2}\right)^{3/2}$
10. $\frac{x}{\sqrt[3]{1+x}}$

Find the binomial series for the functions in Exercises 11–14.

11. $(1+x)^4$
12. $(1+x^2)^3$
13. $(1-2x)^3$
14. $\left(1-\frac{x}{2}\right)^4$

Approximations and Nonelementary Integrals

T In Exercises 15–18, use series to estimate the integrals' values with an error of magnitude less than 10^{-5} . (The answer section gives the integrals' values rounded to seven decimal places.)

15. $\int_0^{0.6} \sin x^2 dx$
16. $\int_0^{0.4} \frac{e^{-x}-1}{x} dx$
17. $\int_0^{0.5} \frac{1}{\sqrt{1+x^4}} dx$
18. $\int_0^{0.35} \sqrt[3]{1+x^2} dx$

T Use series to approximate the values of the integrals in Exercises 19–22 with an error of magnitude less than 10^{-8} .

19. $\int_0^{0.1} \frac{\sin x}{x} dx$
20. $\int_0^{0.1} e^{-x^2} dx$
21. $\int_0^{0.1} \sqrt{1+x^4} dx$
22. $\int_0^1 \frac{1-\cos x}{x^2} dx$

23. Estimate the error if $\cos t^2$ is approximated by $1 - \frac{t^4}{2} + \frac{t^8}{4!}$ in the integral $\int_0^1 \cos t^2 dt$.

24. Estimate the error if $\cos \sqrt{t}$ is approximated by $1 - \frac{t}{2} + \frac{t^2}{4!} - \frac{t^3}{6!}$ in the integral $\int_0^1 \cos \sqrt{t} dt$.

In Exercises 25–28, find a polynomial that will approximate $F(x)$ throughout the given interval with an error of magnitude less than 10^{-3} .

25. $F(x) = \int_0^x \sin t^2 dt, [0, 1]$
26. $F(x) = \int_0^x t^2 e^{-t^2} dt, [0, 1]$
27. $F(x) = \int_0^x \arctan t dt, \quad \text{(a) } [0, 0.5] \quad \text{(b) } [0, 1]$
28. $F(x) = \int_0^x \frac{\ln(1+t)}{t} dt, \quad \text{(a) } [0, 0.5] \quad \text{(b) } [0, 1]$

Indeterminate Forms

Use series to evaluate the limits in Exercises 29–40.

29. $\lim_{x \rightarrow 0} \frac{e^x - (1+x)}{x^2}$
30. $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x}$
31. $\lim_{t \rightarrow 0} \frac{1 - \cos t - (t^2/2)}{t^4}$
32. $\lim_{\theta \rightarrow 0} \frac{\sin \theta - \theta + (\theta^3/6)}{\theta^5}$
33. $\lim_{y \rightarrow 0} \frac{y - \arctan y}{y^3}$
34. $\lim_{y \rightarrow 0} \frac{\tan^{-1} y - \sin y}{y^3 \cos y}$
35. $\lim_{x \rightarrow \infty} x^2(e^{-1/x^2} - 1)$
36. $\lim_{x \rightarrow \infty} (x+1) \sin \frac{1}{x+1}$
37. $\lim_{x \rightarrow 0} \frac{\ln(1+x^2)}{1-\cos x}$
38. $\lim_{x \rightarrow 2} \frac{x^2-4}{\ln(x-1)}$
39. $\lim_{x \rightarrow 0} \frac{\sin 3x^2}{1-\cos 2x}$
40. $\lim_{x \rightarrow 0} \frac{\ln(1+x^3)}{x \cdot \sin x^2}$

Using Table 9.1

In Exercises 41–52, use Table 9.1 to find the sum of each series.

41. $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$
42. $\left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5 + \left(\frac{1}{4}\right)^6 + \dots$
43. $1 - \frac{3^2}{4^2 \cdot 2!} + \frac{3^4}{4^4 \cdot 4!} - \frac{3^6}{4^6 \cdot 6!} + \dots$
44. $\frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \dots$
45. $\frac{\pi}{3} - \frac{\pi^3}{3^3 \cdot 3!} + \frac{\pi^5}{3^5 \cdot 5!} - \frac{\pi^7}{3^7 \cdot 7!} + \dots$
46. $\frac{2}{3} - \frac{2^3}{3^3 \cdot 3} + \frac{2^5}{3^5 \cdot 5} - \frac{2^7}{3^7 \cdot 7} + \dots$
47. $x^3 + x^4 + x^5 + x^6 + \dots$
48. $1 - \frac{3^2 x^2}{2!} + \frac{3^4 x^4}{4!} - \frac{3^6 x^6}{6!} + \dots$
49. $x^3 - x^5 + x^7 - x^9 + x^{11} - \dots$
50. $x^2 - 2x^3 + \frac{2^2 x^4}{2!} - \frac{2^3 x^5}{3!} + \frac{2^4 x^6}{4!} - \dots$
51. $-1 + 2x - 3x^2 + 4x^3 - 5x^4 + \dots$
52. $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \dots$

Theory and Examples

53. Replace x by $-x$ in the Taylor series for $\ln(1+x)$ to obtain a series for $\ln(1-x)$. Then subtract this from the Taylor series for $\ln(1+x)$ to show that for $|x| < 1$,

$$\ln \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right).$$

54. How many terms of the Taylor series for $\ln(1+x)$ should you add to be sure of calculating $\ln(1.1)$ with an error of magnitude less than 10^{-8} ? Give reasons for your answer.

55. According to the Alternating Series Estimation Theorem, how many terms of the Taylor series for $\arctan 1$ would you have to add to be sure of finding $\pi/4$ with an error of magnitude less than 10^{-3} ? Give reasons for your answer.

56. Show that the Taylor series for $f(x) = \arctan x$ diverges for $|x| > 1$.

T 57. **Estimating pi** About how many terms of the Taylor series for $\arctan x$ would you have to use to evaluate each term on the right-hand side of the equation

$$\pi = 48 \arctan \frac{1}{18} + 32 \arctan \frac{1}{57} - 20 \arctan \frac{1}{239}$$

with an error of magnitude less than 10^{-6} ? In contrast, the convergence of $\sum_{n=1}^{\infty} (1/n^2)$ to $\pi^2/6$ is so slow that even 50 terms will not yield two-place accuracy.

58. Use the following steps to prove that the binomial series in Equation (1) converges to $(1+x)^m$.

a. Differentiate the series

$$f(x) = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$$

to show that

$$f'(x) = \frac{mf(x)}{1+x}, \quad -1 < x < 1.$$

b. Define $g(x) = (1+x)^{-m} f(x)$ and show that $g'(x) = 0$.

c. From part (b), show that

$$f(x) = (1+x)^m.$$

59. a. Use the binomial series and the fact that

$$\frac{d}{dx} \arcsin x = (1-x^2)^{-1/2}$$

to generate the first four nonzero terms of the Taylor series for $\arcsin x$. What is the radius of convergence?

b. **Series for arccos x** Use your result in part (a) to find the first five nonzero terms of the Taylor series for $\arccos x$.

60. a. **Series for $\sinh^{-1} x$** Find the first four nonzero terms of the Taylor series for

$$\sinh^{-1} x = \int_0^x \frac{dt}{\sqrt{1+t^2}}.$$

T b. Use the first *three* terms of the series in part (a) to estimate $\sinh^{-1} 0.25$. Give an upper bound for the magnitude of the estimation error.

61. Obtain the Taylor series for $1/(1+x)^2$ from the series for $-1/(1+x)$.

62. Use the Taylor series for $1/(1-x^2)$ to obtain a series for $2x/(1-x^2)^2$.

T 63. **Estimating pi** The English mathematician Wallis discovered the formula

$$\frac{\pi}{4} = \frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdots}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdots}.$$

Find π to two decimal places with this formula.

64. **The complete elliptic integral of the first kind** is the integral

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}},$$

where $0 < k < 1$ is constant.

a. Show that the first four terms of the binomial series for $1/\sqrt{1-x}$ are

$$(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \cdots.$$

b. From part (a) and the reduction integral Formula 67 at the back of the text, show that

$$K = \frac{\pi}{2} \left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 k^6 + \cdots \right].$$

65. **Series for arcsin x** Integrate the binomial series for $(1-x^2)^{-1/2}$ to show that for $|x| < 1$,

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{x^{2n+1}}{2n+1}.$$

66. **Series for arctan x for $|x| > 1$** Derive the series

$$\arctan x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots, \quad x > 1$$

$$\arctan x = -\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots, \quad x < -1,$$

by integrating the series

$$\frac{1}{1+t^2} = \frac{1}{t^2} \cdot \frac{1}{1+(1/t^2)} = \frac{1}{t^2} - \frac{1}{t^4} + \frac{1}{t^6} - \frac{1}{t^8} + \cdots$$

in the first case from x to ∞ and in the second case from $-\infty$ to x .

Euler's Identity

67. Use Equation (4) to write the following powers of e in the form $a + bi$.

a. $e^{-i\pi}$ b. $e^{i\pi/4}$ c. $e^{-i\pi/2}$

68. Use Equation (4) to show that

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

69. Establish the equations in Exercise 68 by combining the formal Taylor series for $e^{i\theta}$ and $e^{-i\theta}$.

70. Show that

a. $\cosh i\theta = \cos \theta$, b. $\sinh i\theta = i \sin \theta$.

71. By multiplying the Taylor series for e^x and $\sin x$, find the terms through x^5 of the Taylor series for $e^x \sin x$. This series is the imaginary part of the series for

$$e^x \cdot e^{ix} = e^{(1+i)x}.$$

Use this fact to check your answer. For what values of x should the series for $e^x \sin x$ converge?

72. When a and b are real, we define $e^{(a+ib)x}$ with the equation

$$e^{(a+ib)x} = e^{ax} \cdot e^{ibx} = e^{ax} (\cos bx + i \sin bx).$$

Differentiate the right-hand side of this equation to show that

$$\frac{d}{dx}e^{(a+ib)x} = (a+ib)e^{(a+ib)x}.$$

Thus the familiar rule $(d/dx)e^{kx} = ke^{kx}$ holds for k complex as well as real.

73. Use the definition of $e^{i\theta}$ to show that for any real numbers θ, θ_1 , and θ_2 ,
- $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$,
 - $e^{-i\theta} = 1/e^{i\theta}$.

74. Two complex numbers $a+ib$ and $c+id$ are equal if and only if $a=c$ and $b=d$. Use this fact to evaluate

$$\int e^{ax} \cos bx \, dx \quad \text{and} \quad \int e^{ax} \sin bx \, dx$$

from

$$\int e^{(a+ib)x} \, dx = \frac{a-ib}{a^2+b^2}e^{(a+ib)x} + C,$$

where $C = C_1 + iC_2$ is a complex constant of integration.

CHAPTER 9 Questions to Guide Your Review

- What is an infinite sequence? What does it mean for such a sequence to converge? To diverge? Give examples.
- What is a monotonic sequence? Under what circumstances does such a sequence have a limit? Give examples.
- What theorems are available for calculating limits of sequences? Give examples.
- What theorem sometimes enables us to use l'Hôpital's Rule to calculate the limit of a sequence? Give an example.
- What are the six commonly occurring limits in Theorem 5 that arise frequently when you work with sequences and series?
- What is an infinite series? What does it mean for such a series to converge? To diverge? Give examples.
- What is a geometric series? When does such a series converge? Diverge? When it does converge, what is its sum? Give examples.
- Besides geometric series, what other convergent and divergent series do you know?
- What is the n th-Term Test for Divergence? What is the idea behind the test?
- What can be said about term-by-term sums and differences of convergent series? About constant multiples of convergent and divergent series?
- What happens if you add a finite number of terms to a convergent series? A divergent series? What happens if you delete a finite number of terms from a convergent series? A divergent series?
- How do you reindex a series? Why might you want to do this?
- Under what circumstances will an infinite series of nonnegative terms converge? Diverge? Why study series of nonnegative terms?
- What is the Integral Test? What is the reasoning behind it? Give an example of its use.
- When do p -series converge? Diverge? How do you know? Give examples of convergent and divergent p -series.
- What are the Direct Comparison Test and the Limit Comparison Test? What is the reasoning behind these tests? Give examples of their use.
- What are the Ratio and Root Tests? Do they always give you the information you need to determine convergence or divergence? Give examples.
- What is absolute convergence? Conditional convergence? How are the two related?
- What is an alternating series? What theorem is available for determining the convergence of such a series?
- How can you estimate the error involved in approximating the sum of an alternating series with one of the series' partial sums? What is the reasoning behind the estimate?
- What do you know about rearranging the terms of an absolutely convergent series? Of a conditionally convergent series?
- What is a power series? How do you test a power series for convergence? What are the possible outcomes?
- What are the basic facts about
 - sums, differences, and products of power series?
 - substitution of a function for x in a power series?
 - term-by-term differentiation of power series?
 - term-by-term integration of power series?
 - Give examples.
- What is the Taylor series generated by a function $f(x)$ at a point $x=a$? What information do you need about f to construct the series? Give an example.
- What is a Maclaurin series?
- Does a Taylor series always converge to its generating function? Explain.
- What are Taylor polynomials? Of what use are they?
- What is Taylor's formula? What does it say about the errors involved in using Taylor polynomials to approximate functions? In particular, what does Taylor's formula say about the error in a linearization? A quadratic approximation?
- What is the binomial series? On what interval does it converge? How is it used?
- How can you sometimes use power series to estimate the values of nonelementary definite integrals? To find limits?
- What are the Taylor series for $1/(1-x)$, $1/(1+x)$, e^x , $\sin x$, $\cos x$, $\ln(1+x)$, and $\arctan x$? How do you estimate the errors involved in replacing these series with their partial sums?

CHAPTER 9 Practice Exercises

Determining Convergence of Sequences

Which of the sequences whose n th terms appear in Exercises 1–18 converge, and which diverge? Find the limit of each convergent sequence.

1. $a_n = 1 + \frac{(-1)^n}{n}$
2. $a_n = \frac{1 - (-1)^n}{\sqrt{n}}$
3. $a_n = \frac{1 - 2^n}{2^n}$
4. $a_n = 1 + (0.9)^n$
5. $a_n = \sin \frac{n\pi}{2}$
6. $a_n = \sin n\pi$
7. $a_n = \frac{\ln(n^2)}{n}$
8. $a_n = \frac{\ln(2n+1)}{n}$
9. $a_n = \frac{n + \ln n}{n}$
10. $a_n = \frac{\ln(2n^3 + 1)}{n}$
11. $a_n = \left(\frac{n-5}{n}\right)^n$
12. $a_n = \left(1 + \frac{1}{n}\right)^{-n}$
13. $a_n = \sqrt[n]{\frac{3^n}{n}}$
14. $a_n = \left(\frac{3}{n}\right)^{1/n}$
15. $a_n = n(2^{1/n} - 1)$
16. $a_n = \sqrt[n]{2n+1}$
17. $a_n = \frac{(n+1)!}{n!}$
18. $a_n = \frac{(-4)^n}{n!}$

Convergent Series

Find the sums of the series in Exercises 19–24.

19. $\sum_{n=3}^{\infty} \frac{1}{(2n-3)(2n-1)}$
20. $\sum_{n=2}^{\infty} \frac{-2}{n(n+1)}$
21. $\sum_{n=1}^{\infty} \frac{9}{(3n-1)(3n+2)}$
22. $\sum_{n=3}^{\infty} \frac{-8}{(4n-3)(4n+1)}$
23. $\sum_{n=0}^{\infty} e^{-n}$
24. $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4^n}$

Determining Convergence of Series

Which of the series in Exercises 25–44 converge absolutely, which converge conditionally, and which diverge? Give reasons for your answers.

25. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
26. $\sum_{n=1}^{\infty} \frac{-5}{n}$
27. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$
28. $\sum_{n=1}^{\infty} \frac{1}{2n^3}$
29. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$
30. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
31. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$
32. $\sum_{n=3}^{\infty} \frac{\ln n}{\ln(\ln n)}$
33. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n^2+1}}$
34. $\sum_{n=1}^{\infty} \frac{(-1)^n 3n^2}{n^3+1}$

35. $\sum_{n=1}^{\infty} \frac{n+1}{n!}$
36. $\sum_{n=1}^{\infty} \frac{(-1)^n (n^2+1)}{2n^2+n-1}$
37. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$
38. $\sum_{n=1}^{\infty} \frac{2^n 3^n}{n^n}$
39. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$
40. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$
41. $1 - \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^4 - \left(\frac{1}{\sqrt{3}}\right)^6 + \left(\frac{1}{\sqrt{3}}\right)^8 - \dots$
42. $\sum_{n=0}^{\infty} \frac{(-1)^n}{e^{-n} + 1}$
43. $\sum_{n=0}^{\infty} \frac{1}{1+r+r^2+\dots+r^n}$, for $-1 < r < 1$
44. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+100} - \sqrt{n}}$
45. Prove that $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln(\ln n))^p}$ converges if and only if $p > 1$.
46. Prove that $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ converges if and only if $p > 1$.

Power Series

In Exercises 47–56, (a) find the series' radius and interval of convergence. Then identify the values of x for which the series converges (b) absolutely and (c) conditionally.

47. $\sum_{n=1}^{\infty} \frac{(x+4)^n}{n3^n}$
48. $\sum_{n=1}^{\infty} \frac{(x-1)^{2n-2}}{(2n-1)!}$
49. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (3x-1)^n}{n^2}$
50. $\sum_{n=0}^{\infty} \frac{(n+1)(2x+1)^n}{(2n+1)2^n}$
51. $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$
52. $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$
53. $\sum_{n=0}^{\infty} \frac{(n+1)x^{2n-1}}{3^n}$
54. $\sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{2n+1}}{2n+1}$
55. $\sum_{n=1}^{\infty} (\operatorname{csch} n) x^n$
56. $\sum_{n=1}^{\infty} (\operatorname{coth} n) x^n$

Maclaurin Series

Each of the series in Exercises 57–62 is the value of the Taylor series at $x = 0$ of a function $f(x)$ at a particular point. What function and what point? What is the sum of the series?

57. $1 - \frac{1}{4} + \frac{1}{16} - \dots + (-1)^n \frac{1}{4^n} + \dots$
58. $\frac{2}{3} - \frac{4}{18} + \frac{8}{81} - \dots + (-1)^{n-1} \frac{2^n}{n3^n} + \dots$
59. $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \dots + (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} + \dots$
60. $1 - \frac{\pi^2}{9 \cdot 2!} + \frac{\pi^4}{81 \cdot 4!} - \dots + (-1)^n \frac{\pi^{2n}}{3^{2n}(2n)!} + \dots$

$$61. 1 + \ln 2 + \frac{(\ln 2)^2}{2!} + \cdots + \frac{(\ln 2)^n}{n!} + \cdots$$

$$62. \frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{1}{45\sqrt{3}} - \cdots \\ + (-1)^{n-1} \frac{1}{(2n-1)(\sqrt{3})^{2n-1}} + \cdots$$

Find Taylor series at $x = 0$ for the functions in Exercises 63–70.

$$63. \frac{1}{1-2x}$$

$$64. \frac{1}{1+x^3}$$

$$65. \sin \pi x$$

$$66. \sin \frac{2x}{3}$$

$$67. \cos(x^{5/3})$$

$$68. \cos \frac{x^3}{\sqrt{5}}$$

$$69. e^{(\pi x/2)}$$

$$70. e^{-x^2}$$

Taylor Series

In Exercises 71–74, find the first four nonzero terms of the Taylor series generated by f at $x = a$.

$$71. f(x) = \sqrt{3+x^2} \quad \text{at } x = -1$$

$$72. f(x) = 1/(1-x) \quad \text{at } x = 2$$

$$73. f(x) = 1/(x+1) \quad \text{at } x = 3$$

$$74. f(x) = 1/x \quad \text{at } x = a > 0$$

Nonelementary Integrals

Use series to approximate the values of the integrals in Exercises 75–78 with an error of magnitude less than 10^{-8} . (The answer section gives the integrals' values rounded to ten decimal places.)

$$75. \int_0^{1/2} e^{-x^3} dx$$

$$76. \int_0^1 x \sin(x^3) dx$$

$$77. \int_0^{1/2} \frac{\tan^{-1} x}{x} dx$$

$$78. \int_0^{1/64} \frac{\arctan x}{\sqrt{x}} dx$$

Using Series to Find Limits

In Exercises 79–84:

a. Use power series to evaluate the limit.

T b. Then use a grapher to support your calculation.

$$79. \lim_{x \rightarrow 0} \frac{7 \sin x}{e^{2x} - 1}$$

$$80. \lim_{\theta \rightarrow 0} \frac{e^\theta - e^{-\theta} - 2\theta}{\theta - \sin \theta}$$

$$81. \lim_{t \rightarrow 0} \left(\frac{1}{2 - 2 \cos t} - \frac{1}{t^2} \right)$$

$$82. \lim_{h \rightarrow 0} \frac{(\sin h)/h - \cos h}{h^2}$$

$$83. \lim_{z \rightarrow 0} \frac{1 - \cos^2 z}{\ln(1-z) + \sin z}$$

$$84. \lim_{y \rightarrow 0} \frac{y^2}{\cos y - \cosh y}$$

Theory and Examples

85. Use a series representation of $\sin 3x$ to find values of r and s for which

$$\lim_{x \rightarrow 0} \left(\frac{\sin 3x}{x^3} + \frac{r}{x^2} + s \right) = 0.$$

T 86. Compare the accuracies of the approximations $\sin x \approx x$ and $\sin x \approx 6x/(6+x^2)$ by comparing the graphs of $f(x) = \sin x - x$ and $g(x) = \sin x - (6x/(6+x^2))$. Describe what you find.

87. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{2 \cdot 5 \cdot 8 \cdots (3n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} x^n.$$

88. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{4 \cdot 9 \cdot 14 \cdots (5n-1)} (x-1)^n.$$

89. Find a closed-form formula for the n th partial sum of the series $\sum_{n=2}^{\infty} \ln(1 - (1/n^2))$ and use it to determine the convergence or divergence of the series.

90. Evaluate $\sum_{k=2}^{\infty} (1/(k^2 - 1))$ by finding the limits as $n \rightarrow \infty$ of the series' n th partial sum.

91. a. Find the interval of convergence of the series

$$y = 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots \\ + \frac{1 \cdot 4 \cdot 7 \cdots (3n-2)}{(3n)!} x^{3n} + \cdots.$$

b. Show that the function defined by the series satisfies a differential equation of the form

$$\frac{d^2 y}{dx^2} = x^a y + b,$$

and find the values of the constants a and b .

92. a. Find the Maclaurin series for the function $x^2/(1+x)$.

b. Does the series converge at $x = 1$? Explain.

93. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of nonnegative numbers, can anything be said about $\sum_{n=1}^{\infty} a_n b_n$? Give reasons for your answer.

94. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent series of nonnegative numbers, can anything be said about $\sum_{n=1}^{\infty} a_n b_n$? Give reasons for your answer.

95. Prove that the sequence $\{x_n\}$ and the series $\sum_{k=1}^{\infty} (x_{k+1} - x_k)$ both converge or both diverge.

96. Prove that $\sum_{n=1}^{\infty} (a_n/(1+a_n))$ converges if $a_n > 0$ for all n and $\sum_{n=1}^{\infty} a_n$ converges.

97. Suppose that $a_1, a_2, a_3, \dots, a_n$ are positive numbers satisfying the following conditions:

i) $a_1 \geq a_2 \geq a_3 \geq \cdots$;

ii) the series $a_2 + a_4 + a_8 + a_{16} + \cdots$ diverges.

Show that the series

$$\frac{a_1}{1} + \frac{a_2}{2} + \frac{a_3}{3} + \cdots$$

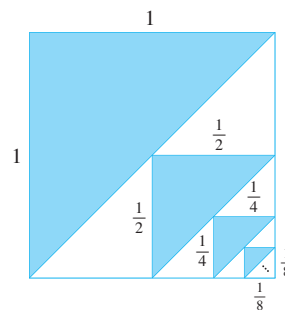
diverges.

98. Use the result in Exercise 97 to show that

$$1 + \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

diverges.

99. Show that if $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.
100. Determine whether $\sum_{n=1}^{\infty} b_n$ converges or diverges.
- $b_1 = 1, \quad b_{n+1} = (-1)^n \frac{n+1}{3n+2} b_n$
 - $b_1 = 3, \quad b_{n+1} = \frac{n}{\ln n} b_n$
101. Assume that $b_n > 0$ and $\sum_{n=1}^{\infty} b_n$ converges. What, if anything, can be said about the following series?
- $\sum_{n=1}^{\infty} \tan(b_n)$
 - $\sum_{n=1}^{\infty} \ln(1 + b_n)$
 - $\sum_{n=1}^{\infty} \ln(2 + b_n)$
102. Consider the convergent series $\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n + e^{cn}}$, where c is a constant. What should c be so that the first 10 terms of the series estimate the sum of the entire series with an error of less than 0.00001?
103. Assume that the following sequence has a limit L . Find the value of L .
- $$4^{1/3}, (4(4^{1/3}))^{1/3}, (4(4(4^{1/3}))^{1/3})^{1/3}, (4(4(4(4^{1/3}))^{1/3})^{1/3})^{1/3}, \dots$$
104. Consider the infinite sequence of shaded right triangles in the accompanying diagram. Compute the total area of the triangles.



CHAPTER 9 Additional and Advanced Exercises

Determining Convergence of Series

Which of the series $\sum_{n=1}^{\infty} a_n$ defined by the formulas in Exercises 1–4 converge, and which diverge? Give reasons for your answers.

- $\sum_{n=1}^{\infty} \frac{1}{(3n-2)^{n+(1/2)}}$
- $\sum_{n=1}^{\infty} \frac{(\tan^{-1} n)^2}{n^2 + 1}$
- $\sum_{n=1}^{\infty} (-1)^n \tanh n$
- $\sum_{n=2}^{\infty} \frac{\log_n(n!)}{n^3}$

Which of the series $\sum_{n=1}^{\infty} a_n$ defined by the formulas in Exercises 5–8 converge, and which diverge? Give reasons for your answers.

5. $a_1 = 1, \quad a_{n+1} = \frac{n(n+1)}{(n+2)(n+3)} a_n$

(Hint: Write out several terms, see which factors cancel, and then generalize.)

6. $a_1 = a_2 = 7, \quad a_{n+1} = \frac{n}{(n-1)(n+1)} a_n \quad \text{if } n \geq 2$

7. $a_1 = a_2 = 1, \quad a_{n+1} = \frac{1}{1+a_n} \quad \text{if } n \geq 2$

8. $a_n = 1/3^n$ if n is odd, $a_n = n/3^n$ if n is even

Choosing Centers for Taylor Series

Taylor's formula

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

expresses the value of f at x in terms of the values of f and its derivatives at $x = a$. In numerical computations, we therefore need a to be a

point where we know the values of f and its derivatives. We also need a to be close enough to the values of f in which we are interested to make $(x-a)^{n+1}$ so small we can neglect the remainder.

In Exercises 9–14, what Taylor series would you choose to represent the function near the given value of x ? (There may be more than one good answer.) Write out the first four nonzero terms of the series you choose.

- $\cos x$ near $x = 1$
- $\sin x$ near $x = 6.3$
- e^x near $x = 0.4$
- $\ln x$ near $x = 1.3$
- $\cos x$ near $x = 69$
- $\arctan x$ near $x = 2$

Theory and Examples

- Let a and b be constants with $0 < a < b$. Does the sequence $\{(a^n + b^n)^{1/n}\}$ converge? If it does converge, what is the limit?
- Find the sum of the infinite series

$$1 + \frac{2}{10} + \frac{3}{10^2} + \frac{7}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \frac{7}{10^6} + \frac{2}{10^7} + \frac{3}{10^8} + \frac{7}{10^9} + \dots$$

- Evaluate

$$\sum_{n=0}^{\infty} \int_n^{n+1} \frac{1}{1+x^2} dx.$$

- Find all values of x for which

$$\sum_{n=1}^{\infty} \frac{nx^n}{(n+1)(2x+1)^n}$$

converges absolutely.

T 19. a. Does the value of

$$\lim_{n \rightarrow \infty} \left(1 - \frac{\cos(a/n)}{n} \right)^n, \quad a \text{ constant},$$

appear to depend on the value of a ? If so, how?

b. Does the value of

$$\lim_{n \rightarrow \infty} \left(1 - \frac{\cos(a/n)}{bn} \right)^n, \quad a \text{ and } b \text{ constant, } b \neq 0,$$

appear to depend on the value of b ? If so, how?

c. Use calculus to confirm your findings in parts (a) and (b).

20. Show that if $\sum_{n=1}^{\infty} a_n$ converges, then

$$\sum_{n=1}^{\infty} \left(\frac{1 + \sin a_n}{2} \right)^n$$

converges.

21. Find a value for the constant b that will make the radius of convergence of the power series

$$\sum_{n=2}^{\infty} \frac{b^n x^n}{\ln n}$$

equal to 5.

22. How do you know that the functions $\sin x$, $\ln x$, and e^x are not polynomials? Give reasons for your answer.

23. Find the value of a for which the limit

$$\lim_{x \rightarrow 0} \frac{\sin(ax) - \sin x - x}{x^3}$$

is finite, and evaluate the limit.

24. Find values of a and b for which

$$\lim_{x \rightarrow 0} \frac{\cos(ax) - b}{2x^2} = -1.$$

25. **Raabe's (or Gauss's) Test** The following test, which we state without proof, is an extension of the Ratio Test.

Raabe's Test: If $\sum_{n=1}^{\infty} u_n$ is a series of positive constants and there exist constants C , K , and N such that

$$\frac{u_n}{u_{n+1}} = 1 + \frac{C}{n} + \frac{f(n)}{n^2},$$

where $|f(n)| < K$ for $n \geq N$, then $\sum_{n=1}^{\infty} u_n$ converges if $C > 1$ and diverges if $C \leq 1$.

Show that the results of Raabe's Test agree with what you know about the series $\sum_{n=1}^{\infty} (1/n^2)$ and $\sum_{n=1}^{\infty} (1/n)$.

26. (Continuation of Exercise 25.) Suppose that the terms of $\sum_{n=1}^{\infty} u_n$ are defined recursively by the formulas

$$u_1 = 1, \quad u_{n+1} = \frac{(2n-1)^2}{(2n)(2n+1)} u_n.$$

Apply Raabe's Test to determine whether the series converges.

27. Suppose that $\sum_{n=1}^{\infty} a_n$ converges, $a_n \neq 1$, and $a_n > 0$ for all n .

a. Show that $\sum_{n=1}^{\infty} a_n^2$ converges.

b. Does $\sum_{n=1}^{\infty} a_n/(1 - a_n)$ converge? Explain.

28. (Continuation of Exercise 27.) If $\sum_{n=1}^{\infty} a_n$ converges, and if $0 < a_n < 1$ for all n , show that $\sum_{n=1}^{\infty} \ln(1 - a_n)$ converges.

(Hint: First show that $|\ln(1 - a_n)| \leq a_n/(1 - a_n)$.)

29. **Nicole Oresme's Theorem** Prove Nicole Oresme's Theorem:

$$1 + \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 3 + \cdots + \frac{n}{2^{n-1}} + \cdots = 4.$$

(Hint: Differentiate both sides of the equation $1/(1 - x) = 1 + \sum_{n=1}^{\infty} x^n$.)

30. a. Find a power series representation of $\frac{e^x - 1}{x}$.

b. By differentiating the series in part (a) term by term, show that $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$.

31. a. Find a power series representation of e^{-x^2} .

b. By differentiating the series in part (a) twice term-by-term, show that $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{2^n n!} = 1$.

32. a. Show that

$$\sum_{n=1}^{\infty} \frac{n(n+1)}{x^n} = \frac{2x^2}{(x-1)^3}$$

for $|x| > 1$ by differentiating the identity

$$\sum_{n=1}^{\infty} x^{n+1} = \frac{x^2}{1-x}$$

twice, multiplying the result by x , and then replacing x by $1/x$.

b. Use part (a) to find the real solution greater than 1 of the equation

$$x = \sum_{n=1}^{\infty} \frac{n(n+1)}{x^n}.$$

33. **Quality control**

a. Differentiate the series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

to obtain a series for $1/(1-x)^2$.

b. In one throw of two dice, the probability of getting a roll of 7 is $p = 1/6$. If you throw the dice repeatedly, the probability that a 7 will appear for the first time at the n th throw is $q^{n-1}p$, where $q = 1 - p = 5/6$. The expected number of throws until a 7 first appears is $\sum_{n=1}^{\infty} nq^{n-1}p$. Find the sum of this series.

c. As an engineer applying statistical control to an industrial operation, you inspect items taken at random from the assembly line. You classify each sampled item as either "good" or "bad." If the probability of an item's being good is p and of an item's being bad is $q = 1 - p$, the probability that the first bad item found is the n th one inspected is $p^{n-1}q$. The average number inspected up to and including the first bad item found is $\sum_{n=1}^{\infty} np^{n-1}q$. Evaluate this sum, assuming $0 < p < 1$.

34. **Expected value** Suppose that a random variable X may assume the values $1, 2, 3, \dots$, with probabilities $p_1, p_2, p_3, \dots$, where p_k is the probability that X equals k ($k = 1, 2, 3, \dots$). Suppose also that $p_k \geq 0$ and that $\sum_{k=1}^{\infty} p_k = 1$. The **expected value** of X ,

denoted by $E(X)$, is the number $\sum_{k=1}^{\infty} kp_k$, provided the series converges. In each of the following cases, show that $\sum_{k=1}^{\infty} p_k = 1$ and find $E(X)$ if it exists. (Hint: See Exercise 33.)

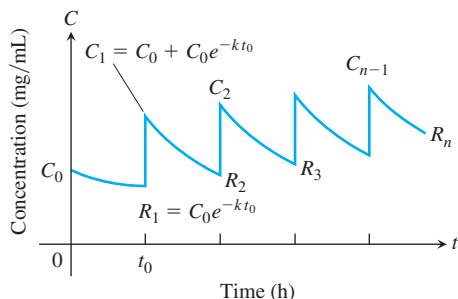
a. $p_k = 2^{-k}$ b. $p_k = \frac{5^{k-1}}{6^k}$

c. $p_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

- T 35. Safe and effective dosage** The concentration in the blood resulting from a single dose of a drug normally decreases with time as the drug is eliminated from the body. Doses may therefore need to be repeated periodically to keep the concentration from dropping below some particular level. One model for the effect of repeated doses gives the residual concentration just before the $(n+1)$ st dose as

$$R_n = C_0 e^{-kt_0} + C_0 e^{-2kt_0} + \cdots + C_0 e^{-nkt_0},$$

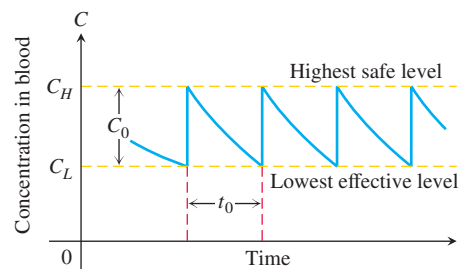
where C_0 = the change in concentration achievable by a single dose (mg/mL), k = the *elimination constant* (h^{-1}), and t_0 = time between doses (h). See the accompanying figure.



- Write R_n in closed form as a single fraction, and find $R = \lim_{n \rightarrow \infty} R_n$.
 - Calculate R_1 and R_{10} for $C_0 = 1 \text{ mg/mL}$, $k = 0.1 \text{ h}^{-1}$, and $t_0 = 10 \text{ h}$. How good an estimate of R is R_{10} ?
 - If $k = 0.01 \text{ h}^{-1}$ and $t_0 = 10 \text{ h}$, find the smallest n such that $R_n > (1/2)R$. Use $C_0 = 1 \text{ mg/mL}$.
- (Source: *Prescribing Safe and Effective Dosage*, B. Horelick and S. Koont, COMAP, Inc., Lexington, MA.)

- 36. Time between drug doses** (Continuation of Exercise 35.) If a drug is known to be ineffective below a concentration C_L and harmful above some higher concentration C_H , we need to find values of C_0 and t_0 that will produce a concentration that is safe (not above C_H) but effective (not below C_L). See the accompanying figure. We therefore want to find values for C_0 and t_0 for which

$$R = C_L \quad \text{and} \quad C_0 + R = C_H.$$



Thus $C_0 = C_H - C_L$. When these values are substituted in the equation for R obtained in part (a) of Exercise 35, the resulting equation simplifies to

$$t_0 = \frac{1}{k} \ln \frac{C_H}{C_L}.$$

To reach an effective level rapidly, one might administer a “loading” dose that would produce a concentration of C_H mg/mL. This could be followed every t_0 hours by a dose that raises the concentration by $C_0 = C_H - C_L$ mg/mL.

- Verify the preceding equation for t_0 .
- If $k = 0.05 \text{ h}^{-1}$ and the highest safe concentration is e times the lowest effective concentration, find the length of time between doses that will ensure safe and effective concentrations.
- Given $C_H = 2 \text{ mg/mL}$, $C_L = 0.5 \text{ mg/mL}$, and $k = 0.02 \text{ h}^{-1}$, determine a scheme for administering the drug.
- Suppose that $k = 0.2 \text{ h}^{-1}$ and that the smallest effective concentration is 0.03 mg/mL . A single dose that produces a concentration of 0.1 mg/mL is administered. About how long will the drug remain effective?

CHAPTER 9 Technology Application Projects

Mathematica/Maple Projects

Projects can be found within MyLab Math.

• Bouncing Ball

The model predicts the height of a bouncing ball, and the time until it stops bouncing.

• Taylor Polynomial Approximations of a Function

A graphical animation shows the convergence of the Taylor polynomials to functions having derivatives of all orders over an interval in their domains.