Mongi Blel

Vector Calculus

2023

University

Mathematematic Department

Vector Calculus

March 26, 2024

PREFACE

This book is intended for students of science, engineering, and mathematics. The basic notions of vector calculus and the integral calculus of functions of several variables are given to help prepare students for their physics courses based on vector calculus. A large number of proofs are not presented in this book, sometimes a sketch of the proof is done.
There are some exercises at the end of each section and at the end of each chapter. These exercises are designed to illustrate the main ideas of the course. a chapter has also been included at the end of this book for solutions to some of these exercises.
As a prerequisite, it is essential that the reader has basic knowledge of differential and integral calculus and linear algebra.

CONTENTS

CHAPTER I

INFINITE SERIES

1 Sequences

1.1 Properties of Convergent Sequences

Definition 1.1

An infinite sequence (or a sequence) of real numbers is a real-valued function $f\colon \mathbb{N} \longrightarrow \mathbb{R}$.
We denote a sequence by listing its terms in order, $(u_n = f(n))_n$. (u_n is called the general term of the sequence.)
Since an infinite subset of $\mathbb{N}$ is in bijection with $\mathbb{N}$, we can suppose that f is defined on an infinite subset of $\mathbb{N}$.

Definition 1.2

A sequence $(u_n)_n$ is called convergent to the real number ℓ if

$$\forall\ \varepsilon > 0,\ \exists N \in \mathbb{N} :\ |u_n - \ell| < \varepsilon \quad \forall n \geq N.$$

ℓ is called the limit of the sequence $(u_n)_n$ and denoted by $\ell = \lim\limits_{n\to+\infty} u_n$.
A sequence which is not convergent is called divergent.

Remark 1 :

1. The limit of a sequence if it exists is unique.
2. If a sequence $(u_n)_n$ converges to a limit ℓ, then the sequence $(v_n)_n$ defined by $v_n = u_{n+p}$, for $n \in \mathbb{N}$ converges also to ℓ.

Definition 1.3

1. A sequence $(u_n)_n$ is called upper bounded, if there exists $M \in \mathbb{R}$ such that, $u_n \leq M,\ \forall n \in \mathbb{N}$.
2. A sequence $(u_n)_n$ is called lower bounded, if there exists $m \in \mathbb{R}$ such that, $u_n \geq m,\ \forall\ n \in \mathbb{N}$.
3. A sequence $(u_n)_n$ is called bounded, if it is upper and lower bounded.

Theorem 1.4

Any convergent sequence is bounded. The converse is not true.

Example 1 :
The sequence defined by: $u_n = (-1)^n$ is bounded but it is not convergent.

1.2 Monotone Sequences

Definition 1.5

1. A sequence $(u_n)_n$ is called increasing if $u_n \leq u_{n+1},\ \forall n \in \mathbb{N}$.
2. A sequence $(u_n)_n$ is called decreasing if $u_n \geq u_{n+1},\ \forall\ n \in \mathbb{N}$.

Theorem 1.6

1. Any increasing upper bounded sequence is convergent.
2. Any decreasing lower bounded sequence is convergent.

Example 2 :

1. The sequence $(u_n = \frac{1}{n})_n$ is convergent and $\lim\limits_{n\to+\infty} \dfrac{1}{n} = 0$.

2. The sequence $(u_n = \dfrac{n^2+3n+1}{2n^2+5n+1})_n$ is convergent and $\lim\limits_{n\to+\infty} u_n = \dfrac{1}{2}$.
   $u_n = \dfrac{n^2+3n+1}{2n^2+5n+1} = \dfrac{n^2(1+\frac{3}{n}+\frac{1}{n^2})}{n^2(2+\frac{5}{n}+\frac{1}{n^2})}$.

3. The sequence $(u_n = a^n)_n$ is convergent if $a \in]-1,1]$ and divergent if $a \in \mathbb{R}\backslash]-1,1]$.
   $\lim\limits_{n\to+\infty} a^n = 0$ if $|a| < 1$. This because $|a|^n = e^{n\ln|a|}$ and $\ln|a| < 0$.

4. The sequence $(u_n = \dfrac{n^2}{2^n})_n$ is convergent and $\lim\limits_{n\to+\infty} u_n = 0$.
   We can take the function $f(x) = \dfrac{x^2}{2^x}$ for $x > 0$. Using l'Hôpital rule, we get $\lim\limits_{x\to+\infty} \dfrac{x^2}{2^x} = \lim\limits_{x\to+\infty} \dfrac{2x}{2^x \ln 2} = \lim\limits_{x\to+\infty} \dfrac{2}{2^x \ln^2 2} = 0$.

Example 3 :
Let $(u_n)_{n\geq 2}$ and $(v_n)_{n\geq 2}$ be the sequences defined by:

$$u_n = 1 + \frac{1}{1!} + \ldots + \frac{1}{n!}, \quad v_n = u_n + \frac{1}{n!}.$$

We have $u_{n+1} - u_n = \dfrac{1}{(n+1)!}$, $v_{n+1} - v_n = \dfrac{2}{(n+1)!} - \dfrac{1}{n!} = \dfrac{1-n}{(n+1)!} \leq 0$, thus the sequence $(u_n)_n$ is increasing and the sequence $(v_n)_n$ is decreasing. Also $u_1 \leq u_n \leq v_n \leq v_2$. Then the sequences $(u_n)_n$ and $(v_n)_n$ are convergent.

1.3 Exercises

Exercise 1 :
Study the convergence of the following sequences

1. $u_n = \dfrac{(-1)^n}{2n+1}$,

2. $u_n = \dfrac{3^n}{4^n+1}$,

3. $u_n = n^2 \sin(\dfrac{1}{n+1})$,

4. $u_n = \sqrt{n+3} - \sqrt{n+1}$,

5. $u_n = (-1)^n \dfrac{n^2+1}{2n^2+3}$,

6. $u_n = \left(1 - \dfrac{2}{n}\right)^n$,

7. $u_n = n\sin(\frac{1}{n+1})$,

Solution to Exercise 1:

1. $u_n = \dfrac{(-1)^n}{2n+1}$, $|u_n| = \dfrac{1}{2n+1}$. Then $\lim\limits_{n\to+\infty} u_n = 0$.

2. $u_n = \dfrac{3^n}{4^n+1} = \dfrac{3^n}{4^n}\dfrac{1}{1+4^{-n}}$. Then $\lim\limits_{n\to+\infty} u_n = 0$.

3. $u_n = n^2\sin(\dfrac{1}{n+1})$, $u_n = \dfrac{n^2}{n+1}\dfrac{\sin(\frac{1}{n+1})}{\frac{1}{n+1}}$. Since $\lim\limits_{n\to+\infty} \dfrac{\sin(\frac{1}{n+1})}{\frac{1}{n+1}} = 1$, then $\lim\limits_{n\to+\infty} u_n = +\infty$.

4. $u_n = \sqrt{n+3} - \sqrt{n+1} = \dfrac{2}{\sqrt{n+3}+\sqrt{n+1}}$. Then $\lim\limits_{n\to+\infty} u_n = 0$.

5. $u_n = (-1)^n \dfrac{n^2+1}{2n^2+3}$. We have $\lim\limits_{n\to+\infty} \dfrac{n^2+1}{2n^2+3} = \dfrac{1}{2}$, then the sequence $(u_n)_n$ is divergent.

6. $u_n = \left(1 - \dfrac{2}{n}\right)^n = e^{n\ln(1-\frac{2}{n})} = e^{2\frac{\ln(1-\frac{2}{n})}{\frac{2}{n}}}$. Since $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = 1$, then $\lim\limits_{n\to+\infty} u_n = \dfrac{1}{e^2}$.

7. $u_n = n\sin(\dfrac{1}{n+1}) = \dfrac{n}{n+1}\dfrac{\sin(\frac{1}{n+1})}{\frac{1}{n+1}}$. Then $\lim\limits_{n\to+\infty} u_n = 1$.

Exercise 2 :

1. Let $(u_n)_n$ be a sequence of real numbers such that $\lim\limits_{n\to+\infty} \dfrac{u_{n+1}}{u_n} = \lambda$, with $|\lambda| < 1$.
   Prove that $\lim_{n\to+\infty} u_n = 0$.

2. Compute the limit of the sequence $(\dfrac{a^n}{n!})_n$, with $a \in \mathbb{R}$.

3. Let $(v_n)_n$ be a sequence of real numbers such that $\lim\limits_{n\to+\infty} \dfrac{v_{n+1}}{v_n} = \lambda$, with $|\lambda| > 1$.
   Prove that the sequence $(v_n)_n$ is divergent.

4. Compute the limit of the sequence $(\dfrac{2^n}{n^2+1})_n$.

Solution to Exercise 2:

1. For $\varepsilon = \frac{1-|\lambda|}{2} > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$, $|u_{n+1}| \leq (|\lambda|+\varepsilon)|u_n|$, then for $n \geq N$, $|u_n| \leq (\dfrac{1+|\lambda|}{2})^{n-N}|u_N|$. Since $\dfrac{1+|\lambda|}{2} < 1$, $\lim\limits_{n\to+\infty} u_n = 0$.

2. If $u_n = \dfrac{a^n}{n!}$, $\lim\limits_{n\to+\infty} \dfrac{u_{n+1}}{u_n} = 0$, then $\lim\limits_{n\to+\infty} \dfrac{a^n}{n!} = 0$.

3. Define $u_n = \dfrac{1}{|v_n|}$ for n large. We have $\lim\limits_{n\to+\infty} \dfrac{u_{n+1}}{u_n} = 0$, then $\lim\limits_{n\to+\infty} u_n = 0$ and $\lim\limits_{n\to+\infty} |v_n| = +\infty$ and the sequence $(v_n)_n$ is divergent.

4. $\lim\limits_{n\to+\infty} \dfrac{2^n}{n^2+1} = +\infty$.

2 Infinite Series

2.1 General Properties of Convergent Series

Definition 2.1

1. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Consider the sequence $(S_n)_n$ defined by: $S_n = \sum\limits_{k=1}^{n} u_k$.

   If the sequence $(S_n)_n$ is convergent, we say that the series $\sum_{n\geq 1} u_n$ is convergent.

   The limit of the sequence $(S_n)_n$ if it exists is denoted by $\sum\limits_{n=1}^{+\infty} u_n$.

2. The series $\sum\limits_{n\geq 1} u_n$ is called divergent if the sequence $(S_n)_n$ is divergent.

Remark 2 :

1. If the series $\sum\limits_{n\geq 1} u_n$ converges, then $\lim\limits_{n\longrightarrow+\infty} u_n = 0$. $(u_n = S_n - S_{n-1}.)$

2. The condition $\lim_{n\longrightarrow+\infty} u_n = 0$ is not, however, sufficient to ensure the convergence of the series $\sum_{n\geq 1} u_n$. For instance, the series $\sum_{n\geq 1} \sqrt{n+1} - \sqrt{n}$ is divergent because $S_n = \sqrt{n+1} - 1$ but $\lim_{n\to+\infty} u_n = 0$.

Example 4 :
Study the convergence of the following series $\sum_{n\geq 1} n\sin(\frac{1}{n})$.
$\lim_{n\to+\infty} n\sin(\frac{1}{n}) = 1$. Then the series $\sum_{n\geq 1} n\sin(\frac{1}{n})$ is divergent.

2.2 Geometric Series

The series $\sum_{n\geq 0} x^n$ is called a geometric series. This series is convergent if and only if $|x| < 1$ and $\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$, for $|x| < 1$.

Example 5 :
Study the convergence of the following series:
$\sum_{n\geq 0} \frac{2^n}{3^n}$, $\sum_{n\geq 1} \frac{n^n}{2^n}$ and $\sum_{n\geq 0} \frac{2^{3n}}{5^n}$.
Solution
$\sum_{n=0}^{+\infty} \frac{2^n}{3^n} = \frac{1}{1-\frac{2}{3}} = 3.$
$\lim_{n\to+\infty} \frac{n^n}{2^n} = +\infty$, then the series $\sum_{n\geq 1} \frac{n^n}{2^n}$ is divergent.
$\frac{2^{3n}}{5^n} = (\frac{8}{5})^n$. Then the series $\sum_{n\geq 0} \frac{2^{3n}}{5^n}$ is divergent.

2.3 Tests of Convergence

There are several standard tests for convergence of a series of non negative terms: The following comparison criterions are based primarily on the fact that an increasing sequence is convergent if, and only, if, it is bounded above. It follows that a series $\sum_{n\geq 1} u_n$ with non negative terms is convergent if, and only, if, the sequence $(S_n)_n$ defined by: $S_n = \sum_{k=1}^{n} u_k$ is bounded.

Theorem 2.2: [Comparison Test]

Let $(u_n)_n$ and $(v_n)_n$ be two sequences with non negative numbers. Assume that there exists an integer $k \in \mathbb{N}$ such that $u_n \leq v_n$, for every $n \geq k$. Then if the series $\sum_{n\geq 1} v_n$ is convergent, the series $\sum_{n\geq 1} u_n$ is also convergent.

Corollary 2.3

Let $(u_n)_n$ and $(v_n)_n$ be two sequences with non negative numbers. Assume that there exists $a > 0$ and $b > 0$ such that $au_n \leq v_n \leq bu_n$ for every $n \geq k$, then the series $\sum_{n\geq 1} u_n$ and $\sum_{n\geq 1} v_n$ converge or diverge simultaneously.

Example 6 :

Study the convergence of the following series:

1. $\sum_{n\geq 0} \frac{1}{1+2^n}$,
2. $\sum_{n\geq 0} \frac{e^n}{1+e^{2n}}$,
3. $\sum_{n\geq 1} \sin(\frac{1}{n^2})$.

Solution

1. $\frac{1}{1+2^n} \leq \frac{1}{2^n}$. $\sum_{n=0}^{+\infty} \frac{1}{2^n} = 2$. Then the series $\sum_{n\geq 0} \frac{1}{1+2^n}$ is convergent.

2. $\frac{e^n}{1+e^{2n}} \leq \frac{1}{e^n}$. $\sum_{n=0}^{+\infty} \frac{1}{e^n} = \frac{e}{e-1}$. Then the series $\sum_{n\geq 0} \frac{e^n}{1+e^{2n}}$ is convergent.

3. For $n \geq 2$, $0 \leq \sin(\frac{1}{n^2}) \leq \frac{1}{n(n-1)}$. $\sum_{n=2}^{+\infty} \frac{1}{n(n-1)} = 1$. Then the series $\sum_{n\geq 1} \sin(\frac{1}{n^2})$ is convergent.

Corollary 2.4

Let $(u_n)_n$ and $(v_n)_n$ be two sequences with non negative numbers. Assume that $\lim_{n\to+\infty} \frac{u_n}{v_n} = \ell$.

1. If $\ell > 0$, the series $\sum_{n\geq 1} u_n$ and $\sum_{n\geq 1} v_n$ converge or diverge simultaneously.
2. If $\ell = 0$, the convergence of the series $\sum_{n\geq 1} v_n$ involves the convergence of the series $\sum_{n\geq 1} u_n$.
3. If $\ell = +\infty$, the convergence of the series $\sum_{n\geq 1} u_n$ involves the convergence of the series $\sum_{n\geq 1} v_n$.

Example 7 :
Study the convergence of the following series:

1. $\sum_{n\geq 1} \frac{3n+\sqrt{n}}{2+n^2+n^{\frac{7}{2}}}$;
2. $\sum_{n\geq 1} \sin(\frac{1}{n^2})$;
3. $\sum_{n\geq 1} \frac{8n^2-7}{e^n(n+1)^2}$;
4. $\sum_{n\geq 1} \frac{\tan^{-1} n}{n}$;
5. $\sum_{n\geq 1} \frac{\ln n}{n^2}$.

Solution

1. $\lim_{n\to+\infty} n^2 \frac{3n+\sqrt{n}}{2+n^2+n^{\frac{7}{2}}} = 0$, then the series $\sum_{n\geq 1} \frac{3n+\sqrt{n}}{2+n^2+n^{\frac{7}{2}}}$ is convergent.
2. $\lim_{n\to+\infty} n^2 \sin(\frac{1}{n^2}) = 1$, then the series $\sum_{n\geq 1} \sin(\frac{1}{n^2})$ is convergent.
3. $\lim_{n\to+\infty} n^2 \frac{8n^2-7}{e^n(n+1)^2} = 0$, then the series $\sum_{n\geq 1} \frac{8n^2-7}{e^n(n+1)^2}$ is convergent.

4. $\lim_{n\to+\infty} n\dfrac{\tan^{-1} n}{n} = \dfrac{\pi}{2}$, then the series $\sum_{n\geq 1} \dfrac{\tan^{-1} n}{n}$ is divergent.

5. $\lim_{n\to+\infty} n^{\frac{3}{2}}\dfrac{\ln n}{n^2} = 0$, then the series $\sum_{n\geq 1} \dfrac{\ln n}{n^2}$ is convergent.

2.4 Integral Test

Theorem 2.5

Let f be a decreasing continuous function on $[1, +\infty[$. For all $n \in \mathbb{N}$, define the $u_n = f(n)$ for all $n \in \mathbb{N}$. Then:

$$\int_1^{+\infty} f(x)dx \text{ converges} \iff \sum_{n\geq 1} u_n \text{ converges.}$$

Example 8 :
Study the convergence of the following series:

1. $\sum_{n\geq 0} \dfrac{\tan^{-1} n}{1+n^2}$; 2. $\sum_{n\geq 1} \dfrac{\ln n}{n}$; 3. $\sum_{n\geq 1} \dfrac{\ln n}{n^2}$.

Solution

1. $\int_1^{+\infty} \dfrac{\tan^{-1} x}{1+x^2} = \dfrac{1}{2}(\tan^{-1} x)^2]_1^{+\infty} = \dfrac{1}{2}(\dfrac{\pi^2}{4} - \dfrac{\pi^2}{16})$. Then the series $\sum_{n\geq 0} \dfrac{\tan^{-1} n}{1+n^2}$ is convergent.

2. $\int_1^{+\infty} \dfrac{\ln x}{x} dx = [\dfrac{1}{2}\ln^2 x]_1^{+\infty} = +\infty$. Then the series $\sum_{n\geq 1} \dfrac{\ln n}{n}$ is divergent.

3. $\int_1^{+\infty} \dfrac{\ln x}{x^2} dx = [-\dfrac{\ln x}{x} - \dfrac{1}{x}]_1^{+\infty} = 1$. Then the series $\sum_{n\geq 1} \dfrac{\ln n}{n^2}$ is convergent.

Corollary 2.6: [Convergence of Riemann Series]

The series $\sum_{n\geq 1} \frac{1}{n^\alpha}$ converges if, and only, if, $\alpha > 1$.

Theorem 2.7

Let $(u_n)_n$ be a sequence with non negative terms. Assume that there exists $0 < a < b$ such that for n large enough, $0 < a \leq n^\alpha u_n \leq b < +\infty$, then the series $\sum_{n\geq 1} u_n$ converges if, and only, if, $\alpha > 1$.

Example 9 :

The series $\sum_{n\geq 1} \frac{\tan^{-1} n}{n}$ is divergent since $\lim_{n\to+\infty} n\frac{\tan^{-1} n}{n} = \frac{\pi}{2}$.

Exercise 1 :

Show that the Bertrand series $\sum_{n\geq 2} \frac{1}{n^\alpha \ln^\beta n}$ converges if, and only, if, $\alpha > 1$ or $\alpha = 1$ and $\beta > 1$.

Solution

If $\alpha \leq 0$, $\lim_{n\to+\infty} \frac{n}{n^\alpha (\ln n)^\beta} = +\infty$, then the series $\sum_{n\geq 2} \frac{1}{n^\alpha \ln^\beta n}$ is divergent.

If $0 < \alpha < 1$, consider $\alpha < \gamma < 1$ and the sequence $(v_n = \frac{1}{n^\gamma})_n$. As $\lim_{n\to+\infty} \frac{n^\gamma}{n^\alpha (\ln n)^\beta} = +\infty$, the series $\sum_{n\geq 2} \frac{1}{n^\alpha (\ln n)^\beta}$ is divergent.

If $\alpha > 1$, consider $1 < \gamma < \alpha$ and the sequence $(v_n = \frac{1}{n^\gamma})_n$. As $\lim_{n\to+\infty} \frac{n^\gamma}{n^\alpha (\ln n)^\beta} = 0$, the series $\sum_{n\geq 2} \frac{1}{n^\alpha (\ln n)^\beta}$ is convergent.

If $\alpha = 1$, consider the sequence $(u_n = \frac{1}{n \ln^\beta n})_n$ and $f(x) = \frac{1}{x \ln^\beta x}$, for $x \geq 2$. The function f is decreasing for x large. Then the series $\sum_{n\geq 2} \frac{1}{n(\ln n)^\beta}$ is convergent if and only if the integral $\int_2^\infty \frac{dx}{x \ln^\beta x}$ is convergent.

The integral $\int_2^\infty \frac{dx}{x \ln^\beta x} \overset{t=\ln x}{=} \int_{\ln 2}^\infty \frac{dt}{t^\beta}$ is convergent if and only if $\beta > 1$. $\square$

Theorem 2.8: (The Ratio Test or D'Alembert's Test)

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of non negative real numbers. Assume that $\lim_{n\longrightarrow+\infty} \frac{u_{n+1}}{u_n} = \ell$ exists. Then

1. If $\ell < 1$, the series $\sum_{n\geq 1} u_n$ is convergent.
2. If $\ell > 1$ the general term of the series does not tend to 0 and the series $\sum_{n\geq 1} u_n$ is divergent.
3. If $\ell = 1$, we can not conclude if the series is convergent or not.

2.5 Root Test or Cauchy Test

Theorem 2.9: (Root Test or the Cauchy Test)

Let $(u_n)_n$ be a sequence of non negative real numbers. Assume that $\lim_{n\to+\infty} \sqrt[n]{u_n} = \ell$ exists.

1. If $\ell < 1$, the series $\sum_{n\geq 1} u_n$ is convergent.
2. If $\ell > 1$, the general term of the series does not tends to 0 and the series $\sum_{n\geq 1} u_n$ is divergent.
3. If $\ell = 1$, we can not conclude if the series is convergent or not.

2.6 Alternating Series

Theorem 2.10: Alternating Series

Let $(u_n)_n$ be a decreasing, non negative sequence and $\lim_{n\to+\infty} u_n = 0$. Then the series $\sum_{n\geq 1}(-1)^n u_n$ is convergent.

Theorem 2.11

Let $(u_n)_n$ be a decreasing, non negative sequence and $\lim_{n\to+\infty} u_n = 0$.

Then for all $m \in \mathbb{N}$, $\left| \sum_{n=m}^{+\infty} (-1)^n u_n \right| \leq a_m$.

Definition 2.12

A series $\sum_{n\geq 1} u_n$ is called absolutely convergent if the series $\sum_{n\geq 1} |u_n|$ converges.

Theorem 2.13

Any absolutely convergent series is convergent but the converse is false.

The series $\sum_{n\geq 1} \frac{(-1)^n}{n}$ is convergent but not absolutely convergent.

Theorem 2.14

Let $(u_n)_n$ be a sequence of real numbers. Assume that $\lim_{n\longrightarrow+\infty} \frac{|u_{n+1}|}{|u_n|} = \ell$ exists. Then

1. If $\ell < 1$, the series $\sum_{n\geq 1} u_n$ is absolutely convergent and then is convergent.
2. If $\ell > 1$ the series $\sum_{n\geq 1} u_n$ is divergent.
3. If $\ell = 1$, we can not conclude if the series is convergent or not.

Theorem 2.15

Let $(u_n)_n$ be a sequence of real numbers. Assume that $\lim_{n\to+\infty} \sqrt[n]{|u_n|} = \ell$ exists.

1. If $\ell < 1$, the series $\sum\limits_{n\geq 1} u_n$ is absolutely convergent and then convergent.
2. If $\ell > 1$, the series $\sum_{n\geq 1} u_n$ is divergent.
3. If $\ell = 1$, we can not conclude if the series is convergent or not.

2.7 Exercises

Exercise 2 :
Study the convergence of the following series:

1. $\sum\limits_{n\geq 0} \dfrac{1}{n!}$
2. $\sum\limits_{n\geq 0} \dfrac{n^n}{n!}$
3. $\sum\limits_{n\geq 0} \dfrac{(n!)^2}{2n!}$
4. $\sum\limits_{n\geq 0} \dfrac{x^n}{n!}$
5. $\sum\limits_{n\geq 0} \dfrac{n^n}{n!}$
6. $\sum\limits_{n\geq 0} \dfrac{n!}{n^n}$

Solution to Exercise 1:

1. If $u_n = \dfrac{1}{n!}$, then $\dfrac{u_{n+1}}{u_n} = \dfrac{1}{n+1}$. The series $\sum\limits_{n\geq 0} \dfrac{1}{n!}$ is convergent.
2. If $u_n = \dfrac{n^n}{n!}$, then $\dfrac{u_{n+1}}{u_n} = (1+\dfrac{1}{n})^n \longrightarrow e > 1$. The series $\sum\limits_{n\geq 0} \dfrac{n^n}{n!}$ is divergent
3. If $u_n = \dfrac{(n!)^2}{2n!}$, then $\dfrac{u_{n+1}}{u_n} = \dfrac{n+1}{2(2n+1)}$. The series $\sum\limits_{n\geq 0} \dfrac{(n!)^2}{2n!}$ is convergent.
4. If $u_n = \dfrac{x^n}{n!}$, then $\dfrac{u_{n+1}}{u_n} = \dfrac{x}{n+1}$. The series $\sum\limits_{n\geq 0} \dfrac{x^n}{n!}$ is convergent.
5. If $u_n = \dfrac{n^n}{n!}$, then $\dfrac{u_{n+1}}{u_n} = \left(1+\dfrac{1}{n}\right)^n \longrightarrow e$. The series $\sum\limits_{n\geq 0} \dfrac{n^n}{n!}$ is divergent.
6. If $u_n = \dfrac{n!}{n^n}$, then $\dfrac{u_{n+1}}{u_n} = \left(1-\dfrac{1}{n+1}\right)^n \longrightarrow \dfrac{1}{e}$. The series $\sum\limits_{n\geq 0} \dfrac{n!}{n^n}$ is convergent.

Exercise 3 :
Study the convergence of the following series:

1. $\sum_{n\geq 1} \frac{e^{2n}}{n^n}$
2. $\sum_{n\geq 1} \left(\frac{4n-5}{2n+1}\right)^n$
3. $\sum_{n\geq 1} \left(\frac{n+1}{n}\right)^n$
4. $\sum_{n\geq 1} \left(\frac{n}{\ln n}\right)^n$

Solution to Exercise 2:

1. If $u_n = \frac{e^{2n}}{n^n}$, then $\sqrt[n]{u_n} = \frac{e}{n}$. The series $\sum_{n\geq 1} \frac{e^{2n}}{n^n}$ is convergent.
2. If $u_n = \left(\frac{4n-5}{2n+1}\right)^n$, then $\sqrt[n]{u_n} = \frac{4n-5}{2n+1}$. The series $\sum_{n\geq 1} \left(\frac{4n-5}{2n+1}\right)^n$ is divergent.
3. If $u_n = \left(\frac{n+1}{n}\right)^n$, then $\lim_{n\to+\infty} u_n = e$. The series $\sum_{n\geq 1} \left(\frac{n+1}{n}\right)^n$ is divergent.
4. If $u_n = \left(\frac{n}{\ln n}\right)^n$, then $\lim_{n\to+\infty} u_n = +\infty$. The series $\sum_{n\geq 1} \left(\frac{n}{\ln n}\right)^n$ is divergent.

Exercise 4 :
Study the convergence of the following series:

1. $\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n}}$,;
2. $\sum_{n\geq 1} \frac{n(-1)^n}{n+1}$;
3. $\sum_{n\geq 2} \frac{(-1)^n \ln n}{n}$.

Solution to Exercise 3:

1. The sequence $\left(\frac{1}{\sqrt{n}}\right)_n$ is decreasing to 0, then the series $\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n}}$ is convergent.

2. As $\lim\limits_{n\to+\infty}\left|\dfrac{n(-1)^n}{n+1}\right| = 1$, the series $\sum\limits_{n\geq 1}\dfrac{n(-1)^n}{n+1}$ is divergent.

3. The sequence $\left(\dfrac{\ln n}{n}\right)_{n\geq 3}$ is decreasing to 0, then the series $\sum\limits_{n\geq 2}\dfrac{(-1)^n\ln n}{n}$ is convergent.

Exercise 5 :
Study the convergence of the following series:

1. $\sum\limits_{n\geq 1}\dfrac{\sin n^2}{1+n^2}$;
2. $\sum\limits_{n\geq 2}\dfrac{(-1)^n}{n\ln^2 n}$.

Solution to Exercise 4:

1. As $\left|\dfrac{\sin n^2}{1+n^2}\right| \leq \dfrac{1}{1+n^2}$ and the series $\sum\limits_{n\geq 1}\dfrac{1}{1+n^2}$ is convergent, then the series $\sum\limits_{n\geq 1}\dfrac{\sin n^2}{1+n^2}$ is absolutely convergent.

2. As $\left|\dfrac{(-1)^n}{n\ln^2 n}\right| \leq \dfrac{1}{n\ln^2 n}$ and the series $\sum\limits_{n\geq 2}\dfrac{1}{n\ln^2 n}$ is convergent, then the series $\sum\limits_{n\geq 2}\dfrac{(-1)^n}{n\ln^2 n}$ is absolutely convergent.

Exercise 6 :
Find the sum of the following series
$\sum\limits_{n=3}^{+\infty}\left(\dfrac{2^{3n}}{3^{2n}}+\dfrac{1}{n^2-3n+2}\right)$

Solution to Exercise 5:

$$\begin{aligned}\sum_{n=3}^{+\infty}\left(\frac{2^{3n}}{3^{2n}}+\frac{1}{n^2-3n+2}\right) &= \sum_{n=3}^{+\infty}\frac{8^n}{9^n}+\sum_{n=3}^{+\infty}\left(\frac{1}{n-2}-\frac{1}{n-1}\right)\\ &= \frac{8^3}{9^2}+1.\end{aligned}$$

3 Power Series

Definition 3.1

Let $f_n(x) = a_n(x - x_0)^n$; with $(a_n)_n$ a sequence of real numbers. The series $\sum_{n\geq 1} a_n(x - x_0)^n$ is called a power series centered at x_0.

We denote $S(x) = \sum_{n=1}^{+\infty} a_n(x - x_0)^n$, whenever x where the series converges.

Let $\sum_{n\geq 1} a_n(x-x_0)^n$ be a power series, we look for its domain of convergence. The series converges at least for $x = x_0$. In which follows, we consider the series centered at 0.

Theorem 3.2: Abel's lemma

If the power series $\sum_{n\geq 1} a_n x^n$ is convergent for $x = x_0$, $x_0 \neq 0$, then the series $\sum_{n\geq 1} a_n x^n$ is absolutely convergent on the interval $] - |x_0|, |x_0|[$,

Corollary 3.3

If the power series $\sum_{n\geq 1} a_n x^n$ diverges for $x = x_0$, then it diverges for every x such that $|x| > |x_0|$.

3.1 Radius of Convergence of Power Series

Theorem 3.4

For every power series $\sum_{n\geq 1} a_n x^n$, there exists a unique $R \in [0, +\infty]$ which fulfills:

1. For every $x \in \mathbb{R}$, such that $|x| < R$, the series $\sum_{n \geq 1} a_n x^n$ is absolutely convergent.
2. For every $x \in \mathbb{R}$, such that $|x| > R$, the sequence $(a_n x^n)_n$ is not bounded and then the series $\sum_{n \geq 1} a_n x^n$ diverges.

   The number R is called the radius of convergence of the power series and $]-R, R[= \{x \in \mathbb{R};\ |x| < R\}$ is called the open interval of convergence of the power series.

Theorem 3.5

Let $\sum_{n \geq 1} a_n x^n$ be a power series with R its radius of convergence. Then

1. If $R = \lim_{n \to +\infty} |\frac{a_n}{a_{n+1}}|$, if the limit exists.
2. $R = \frac{1}{\lim_{n \to +\infty} \sqrt[n]{|a_n|}}$, if the limit exists.

Theorem 3.6

Let $\sum_{n \geq 0} a_n x^n$ be a power series which $R > 0$ as radius of convergence, then the function $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ is differentiable on $]-R, R[$ and $f'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}$ and this series has R as radius of convergence.

Corollary 3.7

If $f(x) = \sum_{n=0}^{+\infty} a_n x^n$, then f is infinitely continuously differentiable on

$]-R, R[$; $a_n = \dfrac{f^{(n)}(0)}{n!}$ and $f(x) = \displaystyle\sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n$. (This series is called the Taylor's series of f at 0.)

Example 10 :

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \qquad \forall x \in \mathbb{R}.$$

$$e^{-x} = \sum_{n=0}^{+\infty} \frac{(-1)^n x^n}{n!} \qquad \forall x \in \mathbb{R}.$$

$$\cosh x = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \qquad \forall x \in \mathbb{R}.$$

$$\sinh x = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad \forall x \in \mathbb{R}.$$

For $|x| < 1$,

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n \quad \text{and} \quad \frac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^n x^n.$$

By integration, we have

$$\ln(1+x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{(n+1)} \quad \text{and} \quad \ln(1-x) = -\sum_{n=0}^{+\infty} \frac{x^{n+1}}{(n+1)}.$$

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x} = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)}.$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n} \text{ and } \tan^{-1} x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)}, \quad |x| < 1.$$

$$\cos x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!} \text{ and } \sin x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

The power series of $f(x) = \dfrac{4}{3-2x}$ centered at $c = 1$.

$$f(x) = \frac{4}{3-2x} = \frac{4}{1-2(x-1)} = \sum_{n=0}^{+\infty} 42^n (x-1)^n, \text{ for } |x-1| < \tfrac{1}{2}.$$

The power series of $f(x) = \dfrac{(x-3)^2}{4-2x}$ centered at $c=3$.

$f(x) = \dfrac{(x-3)^2}{4-2x} = -\dfrac{(x-3)^2}{2+2(x-3)} = \displaystyle\sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{2}(x-3)^{n+2}$, for $|x-3|<1$.

Taylor series of $\sin x$ at $\frac{\pi}{4}$. We have $\sin^{(n)}(x) = \sin(x + \dfrac{n\pi}{2})$, then $\sin^{(n)}(\dfrac{\pi}{4}) = \sin(\dfrac{\pi}{4} + \dfrac{n\pi}{2}) = \dfrac{\sqrt{2}}{2}$ if $n = 4p$ or $n = 4p+1$, $p \geq 0$ and $\sin^{(n)}(\dfrac{\pi}{4}) = -\dfrac{\sqrt{2}}{2}$ if $n = 4p+2$ or $n = 4p+3$, $p \geq 0$

$$\begin{aligned}
\sin x &= \sum_{n=0}^{+\infty} \frac{\sqrt{2}}{2(4n)!}(x-\frac{\pi}{4})^{4n} + \sum_{n=0}^{+\infty} \frac{\sqrt{2}}{2(4n+1)!}(x-\frac{\pi}{4})^{4n+1} \\
&\quad - \sum_{n=0}^{+\infty} \frac{\sqrt{2}}{2(4n+2)!}(x-\frac{\pi}{4})^{4n+2} - \sum_{n=0}^{+\infty} \frac{\sqrt{2}}{2(4n+3)!}(x-\frac{\pi}{4})^{4n+3}
\end{aligned}$$

e^x at $c=2$

$e^x = e^{2+(x-2)} = e^2 \displaystyle\sum_{n=0}^{+\infty} \frac{1}{n!}(x-2)^n$.

3.2 Approximation of Alternate Series

Recall that if $(a_n)_n$ is a non negative decreasing sequence such that $\displaystyle\lim_{n\to+\infty} a_n = 0$, then $\left|\displaystyle\sum_{k=n}^{+\infty} (-1)^k a_k\right| \leq a_n$.

Approximate $\displaystyle\int_0^1 \frac{1-\cos x}{x^2}dx$.

$\dfrac{1-\cos x}{x^2} = \displaystyle\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+2)!}x^{2n} = \frac{1}{2} - \frac{x^2}{4!} + \frac{x^4}{6!} + R(x)$, with $|R(x)| \leq \dfrac{x^6}{8!}$.

Then

$$\begin{aligned}
\int_0^1 \frac{1-\cos x}{x^2}dx &= \int_0^1 \left(\frac{1}{2} - \frac{x^2}{4!} + \frac{x^4}{6!}\right) dx + R \\
&= \frac{1}{2} - \frac{3}{3.4!} + \frac{1}{5.6!} + R
\end{aligned}$$

where $|R| \leq \displaystyle\int_0^1 \frac{x^6}{8!}dx = \frac{1}{6.8!}$.

Hence $\displaystyle\int_0^1 \frac{1-\cos x}{x^2}dx \approx \frac{1}{2} - \frac{3}{3.4!} + \frac{1}{5.6!}$.

3.3 Exercises

Exercise 1 :

Find the sum of the following series and precise its radius of convergence:

1. $\sum_{n=0}^{+\infty} \frac{x^n}{2n-1}$,
2. $\sum_{n=1}^{+\infty} n^2 x^n$,
3. $\sum_{n=0}^{+\infty} \frac{n^2+1}{n!} x^n$,
4. $\sum_{n=0}^{+\infty} \frac{x^n}{(n+1)(n+3)}$,
5. $\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{4n^2-1}$,
6. $\sum_{n=1}^{+\infty} \frac{x^n}{n} \cosh(na), a > 0$
7. $\sum_{n=1}^{+\infty} \frac{x^n \sin n\theta}{2^n}$,
8. $\sum_{n=1}^{+\infty} \frac{x^n \cos n\theta}{n2^n}$,
9. $\sum_{n=1}^{+\infty} \frac{nx^n \sin^2(n\theta)}{2^n}$,
10. $\sum_{n=0}^{+\infty} \frac{n^2+1}{n+1} x^n$,
11. $\sum_{n=0}^{+\infty} \frac{x^n}{(2n)!}$,
12. $\sum_{n=0}^{+\infty} \frac{\sin^2(n\theta)}{n!} x^{2n}$,
13. $\sum_{n\geq 0} (2n+1)x^n$,
14. $\sum_{n=0}^{+\infty} \frac{x^{3n}}{(3n)!}$,
15. $\sum_{n=0}^{+\infty} (n^2+1)\frac{x^n}{n!}$,
16. $\sum_{n=0}^{+\infty} \frac{nx^n}{3^n(n+1)}$,
17. $\sum_{n=0}^{+\infty} (-1)^n \frac{(n^2+1)x^n}{n!}$,
18. $\sum_{n=0}^{+\infty} \frac{nx^n}{3^n(n+1)}$,
19. $\sum_{n=1}^{+\infty} \frac{(-1)^n x^n}{3n+1}$.

Solution to Exercise 1:

1. Recall that if $h(x) = \sum_{n=1}^{+\infty} \frac{x^{2n-1}}{2n-1}$, with $|x| < 1$, then $h'(x) = \sum_{n=1}^{+\infty} x^{2n-2} = \frac{1}{1-x^2}$ and $h(x) = \frac{1}{2} \ln \frac{1+x}{1-x}$.

   Let $f(x) = \sum_{n=0}^{+\infty} \frac{x^n}{2n-1}$, $R = \lim_{n\to+\infty} \frac{2n+1}{2n-1} = 1$ and for $x \geq 0$ we set $x = t^2$ $f(x) = -1 + t\sum_{n=1}^{+\infty} \frac{t^{2n-1}}{2n-1} = -1 + \frac{\sqrt{x}}{2} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}}$.

   For $x \leq 0$, we set $x = -t^2$.

---

2. Recall that for $|x| < 1$

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n \tag{3.1}$$

If $f(x) = \sum_{n=1}^{+\infty} x^n = \frac{1}{1-x}$, $f'(x) = \sum_{n=1}^{+\infty} nx^{n-1} = \sum_{n=1}^{+\infty} nx^n + \sum_{n=0}^{+\infty} x^n$. Then

$$\sum_{n=1}^{+\infty} nx^n = \frac{1}{(1-x)^2} - \frac{1}{1-x} \tag{3.2}$$

$f^{''}(x) = \sum_{n=1}^{+\infty} n^2 x^{n-1} + \sum_{n=1}^{+\infty} nx^{n-1} = \sum_{n=0}^{+\infty} (n^2+2n+1)x^n + \sum_{n=0}^{+\infty} (n+1)x^n$.
Then

$$\sum_{n=1}^{+\infty} n^2 x^n = \frac{2}{(1-x)^3} - 3\sum_{n=0}^{+\infty} nx^n - 2\sum_{n=0}^{+\infty} x^n = \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{1-x}. \tag{3.3}$$

---

3. $R = \lim_{n\to+\infty} \frac{(n^2+1)(n+1)}{(n+1)^2+1} = +\infty.$

$$\begin{aligned}
\sum_{n=0}^{+\infty} \frac{n^2+1}{n!} x^n &= \sum_{n=1}^{+\infty} \frac{n}{(n-1)!} x^n + e^x \\
&= e^x + x\sum_{n=0}^{+\infty} \frac{n+1}{n!} x^n \\
&= e^x + xe^x + x\sum_{n=1}^{+\infty} \frac{x^n}{(n-1)!} \\
&= e^x + xe^x + x^2 \sum_{n=0}^{+\infty} \frac{1}{n!} x^n = e^x(1+x+x^2).
\end{aligned}$$

---

4. $R = \lim_{n\to+\infty} \frac{(n+2)(n+4)}{(n+1)(n+3)} = 1.$

For $x \neq 0$

$$\begin{aligned}\sum_{n=0}^{+\infty}\frac{x^n}{(n+1)(n+3)} &= \frac{1}{2}\sum_{n=0}^{+\infty}\frac{x^n}{n+1}+\frac{1}{2}\sum_{n=0}^{+\infty}\frac{x^n}{n+3}\\ &= \frac{1}{2x}\sum_{n=0}^{+\infty}\frac{x^{n+1}}{n+1}+\frac{1}{2x^3}\sum_{n=0}^{+\infty}\frac{x^{n+3}}{n+3}\\ &= -\frac{1}{2x}\ln(1-x)+\frac{1}{2x^3}(x-\frac{x^2}{2}-\ln(1-x)).\end{aligned}$$

---

5. If $x^2 = x$, the radius of convergence of the power series $\sum_{n\geq 0}\frac{(-1)^n x^n}{4n^2-1}$ is $R = \lim_{n\to+\infty}\frac{4(n+1)^2-1}{4n^2-1} = 1$. Then the radius of convergence of the power series $\sum_{n\geq 0}\frac{(-1)^n x^{2n+1}}{4n^2-1}$ is 1. For $x \neq 0$

$$\frac{d}{dx}\sum_{n=1}^{+\infty}\frac{(-1)^n x^{2n+1}}{4n^2-1} = \sum_{n=1}^{+\infty}\frac{(-1)^n x^{2n}}{2n-1} = x\sum_{n=1}^{+\infty}\frac{(-1)^n x^{2n-1}}{2n-1} = x\tan^{-1}x.$$

$$\begin{aligned}\sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n+1}}{4n^2-1} &= -1+\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int_0^x\frac{t^2}{1+t^2}dt\\ &= -1+\frac{x^2}{2}\tan^{-1}x-\frac{x}{2}+\frac{1}{2}\tan^{-1}x.\end{aligned}$$

---

6. $R = \lim_{n\to+\infty}\frac{(n+1)\cosh(na)}{n\cosh((n+1)a)} = e^a.$

$$\begin{aligned}\sum_{n=1}^{+\infty}\frac{x^n}{n}\cosh(na) &= \frac{1}{2}\sum_{n=1}^{+\infty}\frac{x^n e^{na}}{n}+\frac{1}{2}\sum_{n=1}^{+\infty}\frac{x^n e^{-na}}{n}\\ &= -\frac{1}{2}\ln(e^a-x)-\frac{1}{2}\ln|e^{-a}-x| = -\frac{1}{2}\ln(e^a-x)|e^{-a}-x|.\end{aligned}$$

---

7. $\sum_{n=1}^{+\infty}\frac{x^n\sin n\theta}{2^n} = \operatorname{Im}\sum_{n=1}^{+\infty}\frac{x^n e^{in\theta}}{2^n}$, $R = 2$.

$$\begin{aligned}\sum_{n=1}^{+\infty} \frac{x^n \sin n\theta}{2^n} &= \operatorname{Im} \sum_{n=1}^{+\infty} \frac{x^n e^{in\theta}}{2^n} \\ &= \frac{2x\cos\theta}{x^2+4-4x\cos\theta}\end{aligned}$$

---

8. $\displaystyle\sum_{n=1}^{+\infty} \frac{x^n \cos n\theta}{n2^n} = \operatorname{Re} \sum_{n=1}^{+\infty} \frac{x^n e^{in\theta}}{n2^n}$, $R = 2$.

$$\begin{aligned}\sum_{n=1}^{+\infty} \frac{x^n \cos n\theta}{n2^n} &= \operatorname{Re} \sum_{n=1}^{+\infty} \frac{x^n e^{in\theta}}{n2^n} \\ &= \ln 2 - \ln(x^2+4-4x\cos\theta)\end{aligned}$$

---

9. $R = 2$.

$$\begin{aligned}\sum_{n=1}^{+\infty} \frac{nx^n \sin^2(n\theta)}{2^n} &= \sum_{n=1}^{+\infty} \frac{nx^n(1+\cos(2n\theta))}{2^{n+1}} \\ &= \frac{x}{4(1-x)^2} + \operatorname{Re} \frac{xe^{2i\theta}}{2(2-xe^{2i\theta})} \\ &= \frac{x}{4(1-x)^2} + \frac{2x\cos(2\theta)-x^2}{2(x^2+4-4x\cos(2\theta))}.\end{aligned}$$

---

10. $R = 1$.

$$\begin{aligned}\sum_{n=0}^{+\infty} \frac{n^2+1}{n+1} x^n &= \sum_{n=0}^{+\infty} (n+1)^2 x^n - 2\sum_{n=0}^{+\infty} x^n + 2\sum_{n=0}^{+\infty} \frac{x^n}{n+1} \\ &= -\frac{2}{1-x} - 2\frac{\ln(1-x)}{x}.\end{aligned}$$

---

11. $R = +\infty$.

$$\sum_{n=0}^{+\infty} \frac{x^n}{(2n)!} = \begin{cases} \cosh(\sqrt{x}) & \text{if } x \geq 0 \\ \cos(\sqrt{-x}) & \text{if } x \leq 0 \end{cases}$$

---

12. $R = +\infty$.

$$\sum_{n=0}^{+\infty} \frac{\sin^2(n\theta)}{n!} x^{2n} = \frac{1}{2} \sum_{n=0}^{+\infty} \frac{1-\cos(2n\theta)}{n!} x^{2n} = \frac{1}{2} e^{x^2} - \frac{1}{2} \operatorname{Re} e^{x^2 e^{2\mathrm{i}\theta}}$$
$$= \frac{1}{2} e^{x^2} - \frac{1}{2} e^{x^2 \cos(2\theta)} \cos(x^2 \sin(2\theta)).$$

---

13. $R = 1$
$$\sum_{n=0}^{+\infty} (2n+1) x^n = \frac{2x}{(1-x)^2} + \frac{1}{1-x}.$$

---

14. $R = +\infty$.
If $j = e^{\frac{2\mathrm{i}\pi}{3}}$,

$$\sum_{n=0}^{+\infty} \frac{(jx)^n}{n!} = e^{jx} = \sum_{n=0}^{+\infty} \frac{x^{3n}}{(3n)!} + j \sum_{n=0}^{+\infty} \frac{x^{3n+1}}{(3n+1)!} + j^2 \sum_{n=0}^{+\infty} \frac{x^{3n+2}}{(3n+2)!}.$$

---

15. $R = +\infty$
$$\sum_{n=0}^{+\infty} (n^2+1) \frac{x^n}{n!} = e^x (1 + x + x^2).$$

---

16. $R = +\infty$
$$\sum_{n=0}^{+\infty} \frac{x^n \cos n\theta}{n!} = \operatorname{Re} e^{x e^{\mathrm{i}\theta}} = e^{x \cos\theta} \cos(x \sin\theta).$$

---

17. $R = +\infty$
$$\sum_{n=0}^{+\infty} \frac{x^n \sin n\theta}{n!} = \operatorname{Im} e^{x e^{\mathrm{i}\theta}} = e^{x \cos\theta} \sin(x \sin\theta).$$

---

18. $R = 3$. For $x \neq 0$
$$\sum_{n=0}^{+\infty} \frac{n x^n}{3^n (n+1)} = \frac{3}{3-x} + \frac{3\ln(3-x) - 3\ln 3}{x}.$$

---

19. $R=1$, and set $x=t^3$. For $x\neq 0$

$$\begin{aligned}\sum_{n=0}^{+\infty}\frac{(-1)^n x^n}{3n+1} &= \frac{1}{t}\sum_{n=0}^{+\infty}\frac{(-1)^n t^{3n+1}}{3n+1}\\ &= \frac{1}{t}\int_0^t\frac{ds}{1+s^3}=\frac{1}{3t}\ln(1+t)+\frac{1}{3t}\int_0^t\frac{-bs+2}{1-s+s^2}\\ &= \frac{1}{3t}\ln(1+t)-\frac{1}{6t}\ln(1-t+t^2)+\frac{1}{t\sqrt{3}}\tan^{-1}(\frac{2t-1}{\sqrt{3}})+\frac{\pi}{6t\sqrt{3}}.\end{aligned}$$

Exercise 2 :

1. Define the sequences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ by: $\begin{cases}u_0=1\\ v_0=0\end{cases}$ and $\begin{cases}u_{n+1}=u_n+2v_n\\ v_{n+1}=u_n+v_n.\end{cases}$

   Determine the radius of convergence and the sum of the power series $\sum_{n\geq 0}u_n x^n$.

2. Determine the radius of convergence of the power series:

   $\sum_{n\geq 0}a_n x^n$; with $a_{2n}=0$and $a_{2n+1}=\dfrac{(-1)^n}{(2n-1)(2n+1)}$.

   Let $f(x)=\sum_{n=1}^{+\infty}a_n x^n$, give a simple expression of the derivative $f'(x)$ in term of x and $\tan^{-1}x$.

   Deduce $f(x)$.

Solution to Exercise 2:

1. Remarque that $u_{n+1}-\sqrt{2}v_{n+1}=(1-\sqrt{2})(u_n-\sqrt{2}v_n)$ and $u_{n+1}+\sqrt{2}v_{n+1}=(1+\sqrt{2})(u_n+\sqrt{2}v_n)$. Then

$$u_n=\sqrt{2}v_n+(1-\sqrt{2})^n,\quad u_n=-\sqrt{2}v_n+(1+\sqrt{2})^n.$$

   We deduce that $u_n=\frac{1}{2}(1-\sqrt{2})^n+\frac{1}{2}(1+\sqrt{2})^n$. The radius of convergence of the series $\sum_{n\geq 0}u_n x^n$ is $R=1+\sqrt{2}$ and

$$\sum_{n=0}^{+\infty}u_n x^n=\frac{1}{2}\frac{1}{1-(1-\sqrt{2})x}+\frac{1}{2}\frac{1}{1-(1+\sqrt{2})x}.$$

2. 1 is the radius of convergence of the power series $\sum_{n\geq 0} a_n x^n$.

   $f'(x) = -1 - x\tan^{-1} x$, then $f(x) = -\frac{x}{2} - \frac{(1+x^2)}{2}\tan^{-1}(x)$. We can also compute $f(x)$ as follows:

$$\begin{aligned}
\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{(2n-1)(2n+1)} &= \frac{1}{2}\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{2n-1} - \frac{1}{2}\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \\
&= -\frac{x}{2} - \frac{(1+x^2)}{2}\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \\
&= -\frac{x}{2} - \frac{(1+x^2)}{2}\tan^{-1}(x).
\end{aligned}$$

Exercise 3 :
Give the expansion in power series in a neighborhood of 0 of the following functions

1. $x \longmapsto \dfrac{\ln(1+x)}{1+x}$.
2. $f(x) = (\sin^{-1} x)^2$. (We will be able to show that f fulfills a differential equation of order 2.)
3. $\dfrac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}}$.
4. $\ln(1 - 2x\cos\alpha + x^2)$.
5. $e^{2x}\cos x$.

Solution to Exercise 3:

1. $\dfrac{1}{1+x} = \sum_{n=0}^{+\infty}(-1)^n x^n$, $\ln(1+x) = \sum_{n=0}^{+\infty}(-1)^n \dfrac{x^{n+1}}{n+1}$, then the product of the series yields

   $\dfrac{\ln(1+x)}{1+x} = x\sum_{n=0}^{+\infty} c_n x^n$, with $c_n = (-1)^n \sum_{k=0}^{n} \dfrac{1}{k+1}$.

2. The function f fulfills the following differential equation

$$(1-x^2)y'' - xy' = 2.$$

If $f(x)=\sum_{n=0}^{+\infty} a_n x^n$, then $a_0=a_1=0$ and $a_2=2$. Moreover $(n+1)(n+2)a_{n+2}=n^2a_n$ for $n\geq 1$. then Thus $a_{2n}=0$, $a_{2n+1}=\frac{(2n-1)^2a_{2n-1}}{2n(2n+1)}$, and $a_{2n+1}=2\frac{(2^n n!)^2}{(2n+1)!}$. Then

$$(\sin^{-1}x)^2=\sum_{n=0}^{+\infty} 2\frac{(2^n n!)^2}{(2n+1)!}x^{2n+2}.$$

3. $\dfrac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}}=-\dfrac{d}{dx}(\sin^{-1}\sqrt{x})^2=2\displaystyle\sum_{n=0}^{+\infty}\frac{(2^n(n+1)!)^2}{(2n+1)!}x^{n+1}$

4. If $g(x)=\ln(1-2x\cos\alpha+x^2)$, then $g'(x)=\dfrac{1}{x-e^{\mathrm{i}\alpha}}+\dfrac{1}{x-e^{-\mathrm{i}\alpha}}$. As

$$\frac{1}{x-e^{\mathrm{i}\alpha}}=-e^{-\mathrm{i}\alpha}\sum_{n=0}^{+\infty}x^ne^{-\mathrm{i}n\alpha}.$$

Then

$$g(x)=\ln(1-2x\cos\alpha+x^2)=\sum_{n=0}^{+\infty}-2\frac{x^{n+1}}{n+1}\cos(n+1)\alpha.$$

5. $e^{2x}\cos x=\operatorname{Re}e^{x(2+\mathrm{i})}=\operatorname{Re}\displaystyle\sum_{n=0}^{+\infty}\frac{(2+\mathrm{i})^n}{n!}x^n$. Set $(2+\mathrm{i})=\sqrt{5}e^{\mathrm{i}\theta}$, thus

$$e^{2x}\cos x=\sum_{n=0}^{+\infty}\frac{(\sqrt{5})^n}{n!}x^n\cos n\theta.$$

Exercise 4 :
Give the expansion in power series of the function $f(x)=\dfrac{x}{1-x-x^2}$.

Solution to Exercise 4:
$f(x)=\dfrac{x}{1-x-x^2}=\dfrac{-x}{(x-\alpha)(x-\beta)}=\dfrac{a}{x-\alpha}+\dfrac{b}{x-\beta}$,
where $a=-\dfrac{1+\sqrt{5}}{2\sqrt{5}}$, $b=-\dfrac{1-\sqrt{5}}{2\sqrt{5}}$, $\alpha=-\dfrac{1+\sqrt{5}}{2}$, and $\beta=-\dfrac{1-\sqrt{5}}{2}$.

$$\begin{aligned} f(x)=\frac{x}{1-x-x^2} &= -\frac{a}{\alpha}\sum_{n=0}^{+\infty}\frac{x^n}{\alpha^n}-\frac{a}{\beta}\sum_{n=0}^{+\infty}\frac{x^n}{\beta^n} \\ &= -\frac{1}{2\sqrt{5}}\sum_{n=0}^{+\infty}\frac{(-2)^n x^n}{(1+\sqrt{5})^n}-\frac{1}{2\sqrt{5}}\sum_{n=0}^{+\infty}\frac{(-2)^n x^n}{(1-\sqrt{5})^n}. \end{aligned}$$

Exercise 5 :
Give the expansion in power series of the following functions in a neighborhood of 0 and determine the corresponding radius of convergence:

1. $\dfrac{1}{1-x}$,
2. $\dfrac{1}{(1-x)^2}$,
3. $\ln(1-x)$,
4. $\dfrac{1}{1-x^2}$,
5. $\dfrac{1}{(x-2)(x-3)}$,
6. $\dfrac{1}{1+x^2}$
7. $\ln(1+x+x^2)$
8. $\tan^{-1} x$,
9. $\sin^3 x$,
10. $\cos^2 x$,
11. $\cos^4 x$,
12. $\cosh^2 x$,
13. $\sinh^3 x$,
14. $(x-1)\ln(x^2-5x+6)$,
15. $x\ln(x+\sqrt{x^2+1})$,
16. $\dfrac{x-2}{x^3-x^2-x+1}$,
17. $\dfrac{1}{1+x-2x^3}$,
18. $\dfrac{1-x}{(1+2x-x^2)^2}$,
19. $\sqrt{\dfrac{1-x}{1+x}}$,
20. $\tan^{-1}(x+1)$,
21. $\tan^{-1}(x+\sqrt{3})$,
22. $\displaystyle\int_0^x \frac{\ln(t^2-\frac{5}{2}t+1)}{t}\,dt$,
23. $\left(\dfrac{(1+x)\sin x}{x}\right)^2$,
24. $\displaystyle\int_x^{2x} e^{-t^2}\,dt$,
25. $\displaystyle e^{-2x^2}\int_0^x e^{2t^2}\,dt$,
26. $\dfrac{e^x}{1-x}$,
27. $\dfrac{e^{x^2}}{1-x}$,
28. $\displaystyle\int_0^x \frac{\cos t-1}{t^2}dt$,
29. $\ln\left(\dfrac{1+x}{2-x}\right)$
30. $\ln\sqrt{1-2x\cosh a+x^2}$.

Solution to Exercise 5:

1. $\displaystyle\frac{1}{1-x}=\sum_{n=0}^{+\infty}x^n,\ R=1.$

2. $\displaystyle\frac{1}{(1-x)^2}=\frac{d}{dx}\left(\frac{1}{1-x}\right)=\sum_{n=0}^{+\infty}(n+1)x^n,\ R=1.$

3. $\dfrac{d}{dx}\ln(1-x) = -\dfrac{1}{1-x}$, then

$$\ln(1-x) = -\sum_{n=1}^{+\infty}\frac{x^n}{n}, \quad R = 1.$$

4. $\dfrac{1}{1-x^2} = \sum_{n=0}^{+\infty} x^{2n}$, $R = 1$.

5. $\dfrac{1}{(x-2)(x-3)} = \dfrac{1}{x-3} - \dfrac{1}{x-2} = \sum_{n=0}^{+\infty}\left(\dfrac{1}{2^{n+1}} - \dfrac{1}{3^{n+1}}\right)x^n$, $R = 2$.

6. $\dfrac{1}{1+x^2} = \sum_{n=0}^{+\infty}(-1)^n x^{2n}$, $R = 1$.

7. For $|x| < 1$, $\ln(1+x+x^2) + \ln(1-x) = \ln(1-x^3) = -\sum_{n=1}^{+\infty}\dfrac{x^{3n}}{n}$. Then

$$\ln(1+x+x^2) = -\sum_{n=1}^{+\infty}\frac{x^{3n}}{n} + \sum_{n=1}^{+\infty}\frac{x^n}{n} = \sum_{n=1}^{+\infty} a_n\frac{x^n}{n},$$

where $a_{3n} = 0$, $a_{3n+1} = a_{3n+2} = -1$.

8. $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}$, then

$$\tan^{-1}x = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{2n+1}, \quad R = 1.$$

9. $R = +\infty$, we linearize $\sin^3 x$,

$$\begin{aligned}
\sin^3 x &= \sin x(\frac{1-\cos(2x)}{2} = \frac{1}{2}\sin x - \frac{1}{2}\sin x\cos(2x) \\
&= \frac{1}{2}\sin x - \frac{1}{4}\sin(3x) + \frac{1}{4}\sin(x) = \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)
\end{aligned}$$

Or

$$\begin{aligned}
\sin^3 x &= \frac{-1}{8\mathrm{i}}(e^{\mathrm{i}x} - e^{-\mathrm{i}x})^3 = \frac{-1}{8\mathrm{i}}(e^{3\mathrm{i}x} - 3e^{\mathrm{i}x} + 3e^{-\mathrm{i}x} - e^{-3\mathrm{i}x}) \\
&= \frac{1}{4}\left(3\sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{+\infty}\frac{(-1)^n 3^{2n+1}x^{2n+1}}{(2n+1)!}\right).
\end{aligned}$$

10. $R=+\infty$, $\cos^2 x = \dfrac{1+\cos(2x)}{2} = 1+\dfrac{1}{2}\sum_{n=1}^{+\infty}\dfrac{(-1)^n x^{2n}}{(2n)!}$.

11. $R=+\infty$,

$$\begin{aligned}\cos^4 x &= \frac{1}{4}(1+\cos(2x))^2 = \frac{1}{4}(1+\cos^2(2x)+2\cos(2x))\\ &= 1+\frac{1}{4}\left(\frac{1}{2}\sum_{n=1}^{+\infty}\frac{(-1)^n 2^{2n}x^{2n}}{(2n)!}+\sum_{n=1}^{+\infty}\frac{(-1)^n x^{2n}}{(2n)!}\right)\end{aligned}$$

12. $R=+\infty$, $\cosh^2 x = \dfrac{1}{2}(1+\cosh(2x)) = \dfrac{1}{2}(1+\sum_{n=0}^{+\infty}\dfrac{x^{2n}}{(2n)!} = 1+\dfrac{1}{2}\sum_{n=1}^{+\infty}\dfrac{x^{2n}}{(2n)!}$.

13. $R=+\infty$,

$$\begin{aligned}\sinh^3 x &= \frac{1}{8}(e^x-e^{-x})^3 = \frac{1}{8}(e^{3x}-3e^x+3e^{-x}-e^{-3x})\\ &= \frac{1}{4}\left(\sum_{n=0}^{+\infty}\frac{3^{2n+1}x^{2n+1}}{(2n+1)!}-3\sum_{n=0}^{+\infty}\frac{x^{2n+1}}{(2n+1)!}\right).\end{aligned}$$

14. $R=2$,

$$\begin{aligned}(x-1)\ln(x^2-5x+6) &= (x-1)\ln(2-x)(3-x)\\ &= (x-1)(\ln 2-\sum_{n=1}^{+\infty}\frac{x^n}{n2^n})+(x-1)(\ln 3-\sum_{n=1}^{+\infty}\frac{x^n}{n3^n})\\ &= (x-1)\left(\ln 6-\sum_{n=1}^{+\infty}\left(\frac{1}{2^n}+\frac{1}{3^n}\right)\frac{x^n}{n}\right).\end{aligned}$$

15. Let $f(x)=(1+x)^\alpha$, with $\alpha\notin\mathbb{N}$ and $x\in]-1,1[$. f is C^∞ and $f'(x)=\alpha(1+x)^{\alpha-1}$, then f is a solution of the following differential equation:

$$(1+x)y'-\alpha y=0. \tag{3.4}$$

We look for a power series $\sum_{n\geq 0} a_n x^n$ such that the function $S=\sum_{n=0}^{+\infty} a_n x^n$ is a solution of the differential equation (3.4).

$(1+x)\sum_{n=0}^{+\infty} n a_n x^{n-1} - \alpha\sum_{n=0}^{+\infty} a_n x^n = 0$, which yields that

$$(n+1)a_{n+1}+na_n-\alpha a_n=0 \iff a_{n+1}=\frac{\alpha-n}{n+1}a_n \ \forall n\geq 0.$$

Then

$$a_n = \alpha\frac{\alpha-1}{2}\dots\frac{\alpha-n}{n+1}a_0.$$

$$S(x) = a_0(1+\sum_{n=1}^{+\infty}\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}x^n),$$

and

$$(1+x)^\alpha = 1+\sum_{n=1}^{+\infty}\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}x^n.$$

The radius of convergence of this series is $R = 1$.

For $\alpha = \frac{-1}{2}$, we have:

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{+\infty}\frac{C_{2n}^n}{4^n}x^n.$$

$$\frac{1}{\sqrt{1+x}} = \sum_{n=0}^{+\infty}\frac{(-1)^n C_{2n}^n}{4^n}x^n.$$

$$\sqrt{1+x} = 1+\frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n C_{2n}^n}{4^n}\frac{x^{n+1}}{n+1}.$$

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{+\infty}\frac{C_{2n}^n}{4^n}x^{2n}.$$

$$\frac{1}{\sqrt{1+x^2}} = \sum_{n=0}^{+\infty}\frac{(-1)^n C_{2n}^n}{4^n}x^{2n}.$$

$$\sin^{-1}x = \sum_{n=0}^{+\infty}\frac{C_{2n}^n}{4^n}\frac{x^{2n+1}}{2n+1}.$$

$$\cos^{-1}x = \frac{\pi}{2}-\sum_{n=0}^{+\infty}\frac{C_{2n}^n}{4^n}\frac{x^{2n+1}}{2n+1}.$$

$$\sinh^{-1}x = \ln(x+\sqrt{1+x^2}) = \sum_{n=0}^{+\infty}\frac{(-1)^n C_{2n}^n}{4^n}\frac{x^{2n+1}}{2n+1}.$$

$$x\ln(x+\sqrt{x^2+1}) = \sum_{n=0}^{+\infty}\frac{(-1)^n C_{2n}^n}{4^n}\frac{x^{2n+2}}{2n+1}.$$

16.

$$\begin{aligned}\frac{x-2}{x^3-x^2-x+1} &= \frac{x-2}{(1-x)^2(1+x)} = -\frac{3}{4(1-x)} - \frac{1}{2(1-x)^2} - \frac{3}{4(1+x)}\\ &= -\frac{3}{4}\sum_{n=0}^{+\infty}x^n - \frac{1}{2}\sum_{n=0}^{+\infty}(n+1)x^n - \frac{3}{4}\sum_{n=0}^{+\infty}(-1)^n x^n.\end{aligned}$$

17.

$$\begin{aligned}\frac{1}{1+x-2x^3} &= \frac{1}{(1-x)(1+2x+2x^2)} = \frac{1}{5(1-x)} + \frac{x+4}{5(1+2x+2x^2)}\\ &= \frac{1}{5}\sum_{n=0}^{+\infty}x^n + \frac{1}{10}\sum_{n=0}^{+\infty}(-1)^n 2^{\frac{n+1}{2}}(\cos(n+1)\frac{\pi}{4} + 7\sin(n+1)\frac{\pi}{4})x^n\end{aligned}$$

18.

$$\begin{aligned}\frac{1-x}{(1+2x-x^2)^2} &= -\frac{1}{2}\frac{d}{dx}\frac{1}{1+2x-x^2} = -\frac{d}{dx}\sum_{n=0}^{\infty}(\frac{(2-\sqrt{2})^n}{2^{n+2}} + \frac{(2+\sqrt{2})^n}{2^{n+2}})x^n\\ &= -\sum_{n=0}^{\infty}n(\frac{(2-\sqrt{2})^n}{2^{n+2}} + \frac{(2+\sqrt{2})^n}{2^{n+2}})x^{n-1}.\end{aligned}$$

19. $\sqrt{\frac{1-x}{1+x}} = \frac{(1-x)}{\sqrt{1-x^2}} = (1-x)\sum_{n=0}^{+\infty}\frac{C_{2n}^n}{4^n}x^{2n}$

20.

$$\begin{aligned}\tan^{-1}(x+1) &= \frac{\pi}{4} + \int_0^x \frac{dt}{2+2t+t^2}\\ &= \frac{\pi}{4} + \frac{\mathrm{i}}{2}\int_0^x \frac{dt}{1+\mathrm{i}+t} - \frac{\mathrm{i}}{2}\int_0^x \frac{dt}{1-\mathrm{i}+t}\\ &= \frac{\pi}{4} + \frac{\mathrm{i}}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n x^{n+1}}{(n+1)(1+\mathrm{i})^{n+1}} - \frac{\mathrm{i}}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n x^{n+1}}{(n+1)(1-\mathrm{i})^{n+1}}\\ &= \frac{\pi}{4} + \frac{\mathrm{i}}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n e^{-\mathrm{i}\frac{(n+1)\pi}{4}}x^{n+1}}{(n+1)2^{\frac{n+1}{2}}} - \frac{\mathrm{i}}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n e^{\mathrm{i}\frac{(n+1)\pi}{4}}x^{n+1}}{(n+1)2^{\frac{n+1}{2}}}\\ &= \frac{\pi}{4} + \sum_{n=0}^{+\infty}\frac{(-1)^n\sin(\frac{(n+1)\pi}{4})x^{n+1}}{(n+1)2^{\frac{n+1}{2}}}\end{aligned}$$

21.

$$\begin{aligned}
\tan^{-1}(x+\sqrt{3}) &= \frac{\pi}{3} + \int_0^x \frac{dt}{4+2\sqrt{3}t+t^2} \\
&= \frac{\pi}{3} + \frac{\mathrm{i}}{2}\int_0^x \frac{dt}{\sqrt{3}+\mathrm{i}+t} - \frac{\mathrm{i}}{2}\int_0^x \frac{dt}{\sqrt{3}-\mathrm{i}+t} \\
&= \frac{\pi}{3} + \frac{\mathrm{i}}{2}\sum_{n=0}^{+\infty} \frac{(-1)^n x^{n+1}}{(n+1)(\sqrt{3}+\mathrm{i})^{n+1}} - \frac{\mathrm{i}}{2}\sum_{n=0}^{+\infty} \frac{(-1)^n x^{n+1}}{(n+1)(\sqrt{3}-\mathrm{i})^{n+1}} \\
&= \frac{\pi}{3} + \frac{\mathrm{i}}{2}\sum_{n=0}^{+\infty} \frac{(-1)^n e^{-\mathrm{i}\frac{(n+1)\pi}{6}} x^{n+1}}{(n+1)2^{\frac{n+1}{2}}} - \frac{\mathrm{i}}{2}\sum_{n=0}^{+\infty} \frac{(-1)^n e^{\mathrm{i}\frac{(n+1)\pi}{6}} x^{n+1}}{(n+1)2^{\frac{n+1}{2}}} \\
&= \frac{\pi}{3} + \sum_{n=0}^{+\infty} \frac{(-1)^n \sin(\frac{(n+1)\pi}{6}) x^{n+1}}{(n+1)2^{\frac{n+1}{2}}}
\end{aligned}$$

22. $R = \frac{1}{2}$.

$$\begin{aligned}
\int_0^x \frac{\ln(t^2 - \frac{5}{2}t + 1)}{t}\, dt &= \int_0^x \frac{\ln(2-t)(\frac{1}{2}-t)}{t}\, dt \\
&= -\sum_{n=1}^{+\infty} \int_0^x \left(\frac{t^{n-1}}{n2^n} + \frac{2^n t^{n-1}}{n} \right) dt \\
&= -\sum_{n=1}^{+\infty} \left(\frac{x^n}{n^2 2^n} + \frac{2^n x^n}{n^2} \right) dt.
\end{aligned}$$

23. $\left(\frac{(1+x)\sin x}{x}\right)^2 = (1+x)^2 \frac{1-\cos(2x)}{2x^2} = \frac{1}{2}(1+x)^2 \sum_{n=1}^{+\infty} \frac{(-1)^{n+1} x^{2(n-1)}}{2n!}$.

$R = +\infty$.

24. $R = +\infty$.

$$\int_x^{2x} e^{-t^2}\, dt = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!(2n+1)} (2^{2n+1} - 1) x^{2n+1}$$

25. $R = +\infty$.

$$e^{-2x^2} \int_0^x e^{2t^2}\, dt = \sum_{n=0}^{+\infty} c_n x^{2n+1}, \text{ where}$$

$$c_n = \frac{2^n}{n!} \sum_{k=0}^{n} \frac{n!}{k!(2k+1)(n-k)} = \frac{2^n}{n!(n+1)}.$$

26. $R = 1$.
$$\frac{e^x}{1-x} = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \sum_{n=0}^{+\infty} x^n = \sum_{n=0}^{+\infty} c_n x^n, \text{ where } c_n = \sum_{k=0}^{n} \frac{1}{k!}.$$

27. $R = 1$.
$$\frac{e^{x^2}}{1-x} = \sum_{n=0}^{+\infty} \frac{x^{2n}}{n!} \sum_{n=0}^{+\infty} x^n = \sum_{n=0}^{+\infty} c_n x^n, \text{ where } c_{2n} = \sum_{k=0}^{n} \frac{1}{k!}, \text{ and } c_{2n+1} = \sum_{k=0}^{n} \frac{1}{k!}.$$

28. $R = +\infty$.
$$\cos t = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{2n!} \text{ and}$$
$$\int_0^x \frac{\cos t - 1}{t^2} dt = \sum_{n=1}^{+\infty} \frac{(-1)^n x^{2n-1}}{2n!(2n-1)}.$$

29. $R = 1$.
$$\ln\left(\frac{1+x}{2-x}\right) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1} x^n}{n} - \ln 2 + \sum_{n=1}^{+\infty} \frac{x^n}{n 2^n}.$$

30. If $a > 0$, $R = e^{-a}$.
$$\begin{aligned} \ln\sqrt{1 - 2x\cosh a + x^2} &= \frac{1}{2}\ln(e^{-a} - x)(e^a - x) \\ &= -\frac{1}{2}\sum_{n=1}^{+\infty} \frac{e^{na} x^n}{n} - \frac{1}{2}\sum_{n=1}^{+\infty} \frac{e^{-na} x^n}{n} \\ &= -\sum_{n=1}^{+\infty} \frac{\cosh(na) x^n}{n}. \end{aligned}$$

Exercise 6 :
Give the expansion in power series the following functions at the corresponding point x_0.

1. $f(x) = \cos x$, $(x_0 = \frac{\pi}{4})$,
2. $f(x) = (1 - x^3)^{-\frac{1}{2}}$, $(x_0 = 0)$,

Solution to Exercise 6:

1. $f(x) = \cos x$, $(x_0 = \dfrac{\pi}{4})$,

$$\begin{aligned}\cos x &= \cos(x - \frac{\pi}{4} + \frac{\pi}{4}) = \frac{\sqrt{2}}{2}(\cos(x - \frac{\pi}{4}) - \sin(x - \frac{\pi}{4})) \\ &= \frac{\sqrt{2}}{2}(\sum_{n=0}^{+\infty} \frac{(-1)^n (x - \frac{\pi}{4})^{2n}}{(2n)!} - \sum_{n=0}^{+\infty} \frac{(-1)^n (x - \frac{\pi}{4})^{2n+1}}{(2n+1)!}).\end{aligned}$$

2. $f(x) = (1 - x^3)^{-\frac{1}{2}}$, $(x_0 = 0)$,

$$(1 - x^3)^{-\frac{1}{2}} = \sum_{n=0}^{+\infty} \frac{C_{2n}^n}{4^n} x^{3n}.$$

Exercise 7 :

1. Consider the sequence (a_n) defined by: $a_0 = 1, a_1 = 2, a_{n+2} - 7a_{n+1} + 12a_n = 0$.

   a) Compute $F(x) = \displaystyle\sum_{n=0}^{+\infty} a_n x^n$.

   b) Deduce the expression of a_n.

2. Consider the sequence (a_n) defined by: $a_0 = 1, a_1 = 2, a_{n+2} - 7a_{n+1} + 12a_n = n$.

   Compute the expression of a_n.

3. Consider the sequence $(a_n)_n$ defined by: $a_0 = 1, a_1 = 2, a_{n+2} - 8a_{n+1} + 16a_n = 0$.

   Find the expression of a_n.

Solution to Exercise 7:

1. a)

$$\begin{aligned}
F(x) &= \sum_{n=0}^{+\infty} a_n x^n \\
&= 1 + 2x + \sum_{n=0}^{+\infty} a_{n+2} x^{n+2} \\
&= 1 + 2x + 7x \sum_{n=0}^{+\infty} a_{n+1} x^{n+1} - 12x^2 \sum_{n=0}^{+\infty} a_n x^n \\
&= 1 + 2x - 12x^2 F(x) + 7x \sum_{n=1}^{+\infty} a_n x^n \\
&= 1 + 2x - 12x^2 F(x) + 7x(F(x) - 1).
\end{aligned}$$

Then $F(x) = \dfrac{1-5x}{1-7x+12x^2} = \dfrac{2}{1-3x} - \dfrac{1}{1-4x}$.

b) $F(x) = \dfrac{2}{1-3x} - \dfrac{1}{1-4x} = 2\sum_{n=0}^{+\infty} 3^n x^n - \sum_{n=0}^{+\infty} 4^n x^n = \sum_{n=0}^{+\infty} (2.3^n - 4^n) x^n$.
Then $a_n = (2.3^n - 4^n)$ for all $n \in \mathbb{N}$.

2. Consider the function $G(x) = \sum_{n=0}^{+\infty} a_n x^n$.

$$\begin{aligned}
G(x) &= \sum_{n=0}^{+\infty} a_n x^n \\
&= 1 + 2x + \sum_{n=0}^{+\infty} a_{n+2} x^{n+2} \\
&= 1 + 2x + 7x \sum_{n=0}^{+\infty} a_{n+1} x^{n+1} - 12x^2 \sum_{n=0}^{+\infty} a_n x^n + \sum_{n=0}^{+\infty} n x^n \\
&= 1 + 2x - 12x^2 G(x) + 7x \sum_{n=1}^{+\infty} a_n x^n + \frac{x}{(1-x)^2} \\
&= 1 + 2x - 12x^2 G(x) + 7x(G(x) - 1) + \frac{x}{(1-x)^2}.
\end{aligned}$$

Then $G(x) = -\dfrac{1}{4(1-3x)} + \dfrac{7}{9(1-4x)} + \dfrac{11}{36(1-x)} + \dfrac{1}{6(1-x)^2}$.

$$\begin{aligned} G(x) &= -\frac{1}{4}\sum_{n=0}^{+\infty} 3^n x^n + \frac{7}{9}\sum_{n=0}^{+\infty} 4^n x^n + \frac{11}{36}\sum_{n=0}^{+\infty} x^n + \frac{1}{6}\sum_{n=0}^{+\infty}(n+1)x^n \\ &= \sum_{n=0}^{+\infty}\left(\frac{n+1}{6} + \frac{11}{36} + \frac{7.4^n}{9} - \frac{3^n}{4}\right)x^n. \end{aligned}$$

Then $a_n = (\frac{n+1}{6} + \frac{11}{36} + \frac{7.4^n}{9} - \frac{3^n}{4})$ for all $n \in \mathbb{N}$.

3. Consider the function $H(x) = \sum_{n=0}^{+\infty} a_n x^n$.

$$\begin{aligned} H(x) &= \sum_{n=0}^{+\infty} a_n x^n \\ &= 1 + 2x + \sum_{n=0}^{+\infty} a_{n+2} x^{n+2} \\ &= 1 + 2x + 8x\sum_{n=0}^{+\infty} a_{n+1} x^{n+1} - 16x^2 \sum_{n=0}^{+\infty} a_n x^n \\ &= 1 + 2x - 12x^2 H(x) + 8x\sum_{n=1}^{+\infty} a_n x^n \\ &= 1 + 2x - 12x^2 H(x) + 8x(H(x) - 1). \end{aligned}$$

Then $H(x) = \frac{1-6x}{12x^2 - 8x + 1} = \frac{1}{(1-2x)} = \sum_{n=0}^{+\infty} 2^n x^n$ and $a_n = 2^n$ for all $n \in \mathbb{N}$.

CHAPTER II

MULTIPLE INTEGRALS

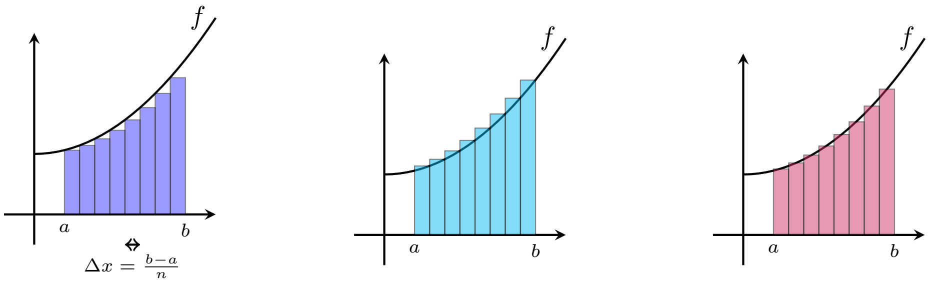
1 Double Integrals

1.1 Introduction

In calculus of one variable, we use the theory of Riemann integration to compute the integral

$$\int_a^b f(x)dx$$

for $f\colon [a,b] \longrightarrow \mathbb{R}^+$ as the area under the curve f from $x = a$ to $x = b$. This integral is approximated by areas of a collection of rectangles.
If the function is not necessary non negative, the definite integral is equal to the area above the $x-$axis minus the area below the $x-$axis.



This concept can be extended to functions of several variables. Consider for example a function of 2 variables $z = f(x,y)$. The definite integral is denoted by

$$\iint_{\mathscr{R}} f(x,y)dA$$

where $\mathscr{R}$ is the region of integration in the $xy-$plane. The definite integral is equal to the volume under the surface $z = f(x, y)$ and above $xy-$plane for x and y in the region $\mathscr{R}$.

For general $f(x, y)$, the definite integral is equal to the volume above the $xy-$plane minus the volume below the $xy-$plane.

Definition 1.1

Let $f : \mathscr{R} \longrightarrow \mathbb{R}$ be a function of two variables defined on a region $\mathscr{R}$ and let $P = (R_k)_{1 \leq k \leq n}$ be an inner partition of $\mathscr{R}$. For any mark $(x_k, y_k) \in P_k$, consider the Riemann sum

$$R(f, \mathscr{R}, P) = \sum_{k=1}^{n} f(x_k, y_k) A_k,$$

where A_k is the area of the rectangle P_k.

Theorem 1.2

Let $f : \mathscr{R} \longrightarrow \mathbb{R}$ be a function of two variables defined on a region $\mathscr{R}$. The double integral of f over $\mathscr{R}$, denoted by $\displaystyle\iint_{\mathscr{R}} f(x, y)dxdy$, is $\displaystyle\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k) A_k$, provided the limit exists.
If the double integral of f over $\mathscr{R}$ exists, then f is said to be integrable over $\mathscr{R}$. It can be proved that if f is continuous on $\mathscr{R}$, then f is integrable over $\mathscr{R}$.

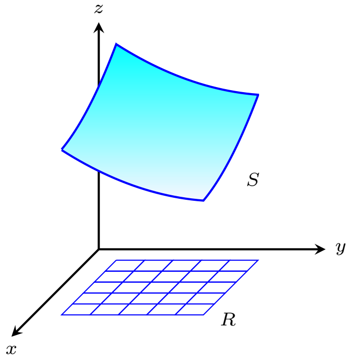
Remark 3 : (**Geometric Interpretation of the Double Integral**)
A useful geometric interpretation of the double integral for a non negative continuous function f throughout a region $\mathscr{R}$.
Let $S = \{(x, y, z) \in \mathbb{R}^3 : \; z = f(x, y)\}$ the surface defined by f and let V be the solid that lies under S and over $\mathscr{R}$. The volume of V is $\displaystyle\iint_{\mathscr{R}} f(x, y)dxdy$.

To find the volume between the rectangle $\mathscr{R} = [a, b] \times [c, d]$ in the xy plane and the surface

$$S = \{(x, y, z) : (x, y) \in \mathscr{R}, \; z = f(x, y)\}$$

where f is a continuous function, we proceed as follows: We divide the rectangle R into an $n \times n$ subsquares $R_{i,j}$. For each subsquare $R_{i,j}$, we make a box of height $f(x_i, y_j)$, where $(x_i, y_j) \in R_{i,j}$.

We approximate this volume by adding up the volumes of the n^2 boxes. The exact volume is obtained by taking the limit when $n \to +\infty$.
If Δ_n is the area of rectangle $R_{i,j}$, the approximate volume under the surface is $\sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i, y_j)\Delta_n$.



If $\mathscr{R} = [a, b] \times [c, d]$ and $S = \{(x, y, z) : z = f(x, y),\ (x, y) \in \mathscr{R}\}$, then the volume between $\mathscr{R}$ and S is

$$V = \lim_{n\to\infty} \sum_{i,j=1}^{n} f(x_i, y_j)\, \Delta_n.$$

We denote this integral by: $= \iint_{\mathscr{R}} f(x, y)dxdy.$

We compute this volume by slicing the three-dimensional region. Suppose the slices are parallel to the $y-$axis. An example of slice between x and $x + dx$. The cross sectional area is the area under the curve $f(x, y)$ for fixed x and y varying between c and d. This area is given by the integral $A(x) = \int_c^d f(x, y)dy$. The volume of the slice between x and $x + dx$ is $A(x)dx$. The total volume is the sum of the volumes of all the slices between $x = a$ and $x = b$:

$$V = \int_a^b A(x)dx = \int_a^b \left(\int_c^d f(x, y)dy\right) dx.$$

Also, we can make slices parallel to the $x-$axis. In this case the volume is given by

$$V = \int_c^d \left(\int_a^b f(x, y)dx\right) dy.$$

We denote this volume as

$$V = \int_c^d \int_a^b f(x, y)dydx.$$

This interpretation is justified by the following theorem

Theorem 1.3: [Fubini's Theorem]

Let $\mathscr{R} = [a, b] \times [c, d]$ be a rectangle in $\mathbb{R}^2$ and f a continuous function on $\mathscr{R}$.

$$\int_{\mathscr{R}} f(x, y)dxdy = \int_a^b \left(\int_c^d f(x, y)dy \right) dx = \int_c^d \left(\int_a^b f(x, y)dx \right) dy.$$

Example 11 :

Consider the double integral: $A = \iint_{\mathscr{R}} (x^2y + xy^2)dxdy$, where $\mathscr{R} = [0, 1] \times [0, 2]$.

$$\begin{aligned} A &= \int_0^2 \left(\int_0^1 (x^2y + xy^2)dx \right) dy \\ &= \int_0^2 \frac{1}{3}y + \frac{1}{2}y^2 dy \\ &= \frac{2}{3} + \frac{4}{3} = 2. \end{aligned}$$

Also

$$\begin{aligned} A &= \int_0^1 \left(\int_0^2 (x^2y + xy^2)dy \right) dx \\ &= \int_0^1 (2x^2 + \frac{8}{3}x)dx = \frac{2}{3} + \frac{4}{3} = 2. \end{aligned}$$

1.2 Double Integrals over General Regions

Definition 1.4

A subset Ω of $\mathbb{R}^2$ is called elementary if there exist $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, and functions φ_1, φ_2 continuous on $[a, b]$ and ψ_1, ψ_2 continuous on $[c, d]$ such that $\varphi_1(x) \leq \varphi_2(x)$ for all $x \in [a, b]$, $\psi_1(y) \leq \psi_2(y)$ for all $y \in [c, d]$ and

$$\begin{aligned} \Omega &= \left\{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\right\} \\ &= \left\{(x, y) \in \mathbb{R}^2 | c \leq y \leq d, \psi_1(y) \leq x \leq \psi_2(y)\right\} \end{aligned}$$

Example 12 :

1. The rectangle $[a,b]\times[c,d]$, with $a<b$ and $c<d$ is an elementary subset of $\mathbb{R}^2$.

2. The unit disc $D=\left\{(x,y)\in\mathbb{R}^2|x^2+y^2\leq 1\right\}$ can be written as $D=\left\{(x,y)\in\mathbb{R}^2|-1\leq x\leq 1,-\sqrt{1-x^2}\leq y\leq\sqrt{1-x^2}\right\}$ and
$D=\left\{(x,y)\in\mathbb{R}^2|-1\leq y\leq 1,-\sqrt{1-y^2}\leq x\leq\sqrt{1-y^2}\right\}$.

3. The ring $\left\{(x,y)\in\mathbb{R}^2|1\leq\sqrt{x^2+y^2}\leq 2\right\}$ is a simple domain of $\mathbb{R}^2$.

Theorem 1.5: [Fubini's Theorem]

Let Ω be an elementary subset of $\mathbb{R}^2$ and f a continuous function on A. With the same notations as in definition (**??**), we have

$$\iint_\Omega f(x,y)dxdy=\int_a^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)}f(x,y)dy\right)dx=\int_c^d\left(\int_{\psi_1(y)}^{\psi_2(y)}f(x,y)dx\right)dy$$

Example 13 :
Consider the function $f(x,y)=xy^2$ on the rectangle $\mathscr{R}=[0,1]\times[1,2]$. We have

$$\begin{aligned}\iint_{\mathscr{R}}xy^2dxdy &= \int_0^1\left(\int_1^2 xy^2dy\right)dx\\ &= \int_0^1\left(\frac{8x}{3}-\frac{x}{3}\right)dx=\int_0^1\frac{7x}{3}dx=\frac{7}{6}.\end{aligned}$$

Also we have

$$\begin{aligned}\iint_{\mathscr{R}}xy^2dxdy &= \int_1^2\left(\int_0^1 xy^2dx\right)dy\\ &= \int_1^2\frac{y^2}{2}dy=\frac{8}{6}-\frac{1}{6}=\frac{7}{6}.\end{aligned}$$

Definition 1.6

A subset Ω of $\mathbb{R}^2$ is called simple if it is a finite union of elementary subsets $\Omega_1, \ldots, \Omega_n$ with disjoint interiors.

$$\forall 1 \leq i, j \leq n, \quad i \neq j, \quad \Omega_i \cap \Omega_j = \emptyset.$$

If f is a continuous function on Ω, we define

$$\iint_{\Omega} f(x,y)dxdy = \sum_{k=1}^{n} \iint_{\Omega_k} f(x,y)dxdy.$$

1.3 Exercises

Exercise 8 :
Compute the following integrals

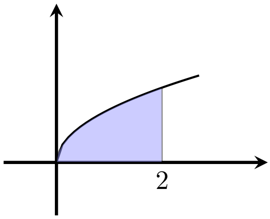
1. $\displaystyle\iint_R x\sec^2 y dxdy,\ R = [0,2] \times [0, \pi/4]$
2. $\displaystyle\iint_R \frac{xy^2}{x^2+1} dxdy,\ R = [0,1] \times [-3,3]$
3. $\displaystyle\iint_R \frac{1}{1+x+y} dxdy,\ R = [1,3] \times [1,2]$

Exercise 9 :
For the following integrals sketch the region of integration and so write equivalent integrals with the order of integration reversed. Evaluate the integrals both ways.

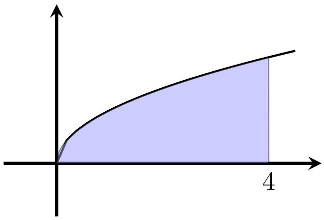
1. $\displaystyle\int_0^{\sqrt{2}} \int_{y^2}^{2} y\, dx\, dy;$
2. $\displaystyle\int_0^{4} \int_0^{\sqrt{x}} y\sqrt{x}\, dydx;$
3. $\displaystyle\int_0^{1} \int_{-y}^{y^2} x\, dx\, dy.$

Solution

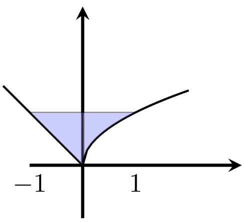
$$\int_0^{\sqrt{2}} \int_{y^2}^{2} y\, dx\, dy = \int_0^{2} \int_0^{\sqrt{x}} y\, dy\, dx = \frac{1}{2} \int_0^{2} x dx = 1.$$



$$\int_0^4 \int_0^{\sqrt{x}} y\sqrt{x}\, dydx = \int_0^2 \int_{y^2}^4 y\sqrt{x}\, dxdy = \frac{2}{3}\int_0^2 (8-y^3)ydy = \frac{42}{5}.$$



$$\int_0^1 \int_{-y}^{y^2} x\, dx\, dy = \int_{-1}^0 \int_{-x}^1 dydx + \int_0^1 \int_{\sqrt{x}}^1 dydx = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$



Exercise 10 :

Reverse the order of integration and hence evaluate:

$$\int_0^{\pi} \int_y^{\pi} x^{-1} \sin x\, dx\, dy\ .$$

Solution

$$\int_0^{\pi} \int_y^{\pi} x^{-1} \sin x\, dx\, dy = \int_0^{\pi} \int_0^{x} x^{-1} \sin x\, dy\, dx = \int_0^{\pi} \sin x dx = 2.$$

Exercise 11 :

For the following integrals sketch the region of integration and so write equivalent integrals with the order of integration reversed. Evaluate the integrals both ways.

1. $\displaystyle\int_0^{\sqrt{2}}\int_{y^2}^{2} y\,dx\,dy,$

2. $\displaystyle\int_0^{4}\int_0^{\sqrt{x}} y\sqrt{x}\,dy\,dx,$

3. $\displaystyle\int_0^{1}\int_{-y}^{y^2} x\,dx\,dy\ .$

Solution

1. $\displaystyle\int_0^{\sqrt{2}}\int_{y^2}^{2} y\,dx\,dy = \int_0^{\sqrt{2}} y(2-y^2)dy = 1.$ Moreover, $\displaystyle\int_0^{\sqrt{2}}\int_{y^2}^{2} y\,dx\,dy = \int_0^{2}\int_0^{\sqrt{x}} ydy\,dx = \int_0^{2}\frac{1}{2}xdx = 1.$

2. $\displaystyle\int_0^{4}\int_0^{\sqrt{x}} y\sqrt{x}\,dy\,dx = \frac{1}{2}\int_0^{4} x^{\frac{3}{2}}\,dx = \frac{32}{5}.$ Moreover, $\displaystyle\int_0^{4}\int_0^{\sqrt{x}} y\sqrt{x}\,dy\,dx = \int_0^{2} y\int_{y^2}^{4}\sqrt{x}\,dx\,dy =$

3. $\displaystyle\int_0^{1}\int_{-y}^{y^2} x\,dx\,dy = \int_0^{1}(y^4-y^2)dy = -\frac{1}{15}.$ Moreover, $\displaystyle\int_0^{1}\int_{-y}^{y^2} x\,dx\,dy = \int_{-1}^{0} x\int_{-x}^{1} dydx + \int_0^{1} x\int_{\sqrt{x}}^{1} dydx = -\frac{1}{6}+\frac{1}{10} = -\frac{1}{15}.$

Exercise 12 :

Reverse the order of integration and hence evaluate:

$$\int_0^{\pi}\int_y^{\pi}\frac{\sin x}{x}\,dx\,dy\ .$$

$$\begin{aligned}\int_0^{\pi}\int_y^{\pi}\frac{\sin x}{x}\,dx\,dy &= \int_0^{\pi}\int_0^{x}\frac{\sin x}{x}\,dy\,dx\\ &= \int_0^{\pi}\sin x dx = 2.\end{aligned}$$

1.4 Area and Volume

Definition 1.7

Let Ω be an elementary subset of $\mathbb{R}^2$. The area of A is defined by:

$$\iint_{\Omega} 1 dx dy.$$

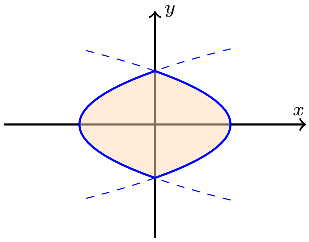
Example 14 :
Consider the triangle $T = \left\{(x, y) \in [0,1]^2 | x + y \leq 1\right\}$. We have:

$$\text{Area}(T) = \int_0^1 \left(\int_0^{1-x} 1 dy\right) dx = \int_0^1 (1-x) dx = \frac{1}{2}.$$

Example 15 :
Consider the disc D of center 0 and radius 1. Then

$$\begin{aligned}
\text{Area}(D) &= \int_{x=-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 dy\right) dx \\
&= 2\int_{x=-1}^{1} \sqrt{1-x^2} dx = 2\int_{-\frac{\pi}{2}}^{\frac{x}{2}} \sqrt{1-\sin^2(\theta)} \cos(\theta) d\theta \\
&= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta) d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos(2\theta)) d\theta = \pi
\end{aligned}$$

Example 16 :
Consider the domain $\Omega = \{-\sqrt{2} \leq y \leq \sqrt{2},\ -2+y^2 \leq x \leq 2+y^2\}$.



$$\begin{aligned}
\iint_{\Omega} dx dy &= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\int_{-2+y^2}^{2-y^2} 1\, dx\right) dy \\
&= \frac{16}{3}\sqrt{2}.
\end{aligned}$$

1.5 Volumes Under Surfaces

If $R = [-1,1] \times [0,2]$ and $f(x,y) = \sqrt{1-x^2}$. The volume between R and the surface $S = \{(x,y,z) : z = f(x,y),\ (x,y) \in R\}$, is

$$V = \iint_R f(x,y)dxdy = \int_{-1}^{1}\left(\int_0^2 \sqrt{1-x^2}dy\right)dx = \pi.$$

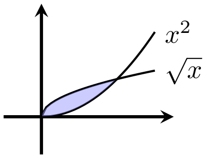
Example 17 :
The volume of the solid that lies under the graph of the function $f(x,y) = 4x^2 + y^2$ and over the region in the $xy-$plane bounded by the polygon with vertices $(0,0), (0,1)$ and $(2,1)$.

$$V = \int_0^2 \int_0^1 (4x^2 + y^2)dydx = \frac{34}{3}.$$

1.6 Exercises

Exercise 13 :
Consider the region R bounded by the graphs of $y = \sqrt{x}$ and $y = x^3$. Compute $\iint_R x^2 ydxdy$.

Solution to Exercise 8:



If f is a continuous function on the region R,

$$\iint_R f(x,y)dxdy = \int_0^1 \left(\int_{x^3}^{\sqrt{x}} f(x,y)dy\right)dx$$

and

$$\iint_R f(x,y)dxdy = \int_0^1 \left(\int_{y^2}^{y^{\frac{1}{3}}} f(x,y)dx\right)dy.$$

Then $\iint_R xydxdy = \int_0^1 y\left(\int_{y^2}^{y^{\frac{1}{3}}} x^2dx\right)dy = \frac{5}{72}$.

Exercise 14 :
Compute the following integrals

1. $\displaystyle\int_1^e \left(\int_0^{\ln x} ydy\right)dx,$

2. $\displaystyle\int_0^8 \int_{\sqrt[3]{y}}^2 \frac{y}{\sqrt{16+x^7}} dxdy,$

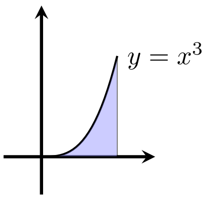
3. $\displaystyle\int_0^1 \int_{2x}^2 e^{y^2} dxdy.$

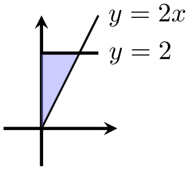
Solution to Exercise 9:

1.
$$\begin{aligned}\int_1^e \left(\int_0^{\ln x} ydy\right) dx &= \int_1^e \int_0^{\ln x} ydxdy \\ &= \int_1^e \frac{1}{2}\ln^2 xdx \\ &= \frac{1}{2}x\ln^2 x - x\ln x + x\Big]_1^e = \frac{1}{2}(e-2).\end{aligned}$$

2.

3. Changing to vertical strip, $0 \le y \le x^3$, $0 \le x \le 2$,
$$\begin{aligned}\int_0^8 \int_{\sqrt[3]{y}}^2 \frac{y}{\sqrt{16+x^7}} dxdy &= \int_0^2 \left(\int_0^{x^3} \frac{y}{\sqrt{16+x^7}} dy\right) dx \\ &= \frac{1}{2}\int_0^2 \frac{x^3}{16+x^7} dx = \frac{8}{7}.\end{aligned}$$





4.

Changing to horizontal strip, $0 \le y \le 1$, $0 \le x \le \frac{y}{2}$,
$$\begin{aligned}\int_0^1 \int_{2x}^2 e^{y^2} dxdy &= \int_0^2 \int_0^{\frac{y}{2}} e^{y^2} dxdy \\ &= \frac{1}{2}\int_0^2 ye^{y^2} dy = \frac{1}{4}(e^4 - 1).\end{aligned}$$

Exercise 15 :
Compute the following integrals

1. $\iint_R f(x,y)dxdy$, where $f(x,y) = x+y$ and $R = [0,1]\times[1,2]$,
2. $\iint_S f(x,y)\,dxdy$, where $f(x,y) = xy$ and S the region bounded by the curves $x = y^2$ and $x = y$ $S = \{(x,y)\in\mathbb{R}^2 : y^2 \le x \le y,\ 0\le y\le 1\}$.
3. $\iint_S f(x,y)dxdy$, where $f(x,y) = x+y$ and $S = \{(x,y)\in\mathbb{R}^2 :\ -1\le x\le 1,\ 0\le y\le 1+|x|\}$.

Solution to Exercise 10:

1.
$$\begin{aligned}\iint_R f(x,y)dxdy &= \int_1^2\left(\int_0^1 f(x,y)\,dx\right)dy\\ &= \int_1^2 (\frac{1}{2}+y)dy = 2.\end{aligned}$$

   Also,
$$\begin{aligned}\iint_R f(x,y)dxdy &= \int_0^1\left(\int_1^2 f(x,y)\,dy\right)dx\\ &= \int_0^1 (x+\frac{3}{2})dx = 2.\end{aligned}$$

2.
$$\begin{aligned}\iint_S xy\,dxdy &= \int_0^1 dy\int_{y^2}^{y} xy\,dx\\ &= \int_0^1 y\left(\int_{y^2}^{y} x\,dx\right)dy\\ &= \frac{1}{2}\int_0^1 (y^3-y^5)\,dy = \frac{1}{24}.\end{aligned}$$

   In this case we can also represent S in the form $S = \{(x,y)\in\mathbb{R}^2 :\ x\le y\le\sqrt{x},\ 0\le x\le 1\}$. Hence,
$$\begin{aligned}\iint_S xy\,dxdy &= \int_0^1 x\,dx\int_x^{\sqrt{x}} y\,dy = \int_0^1\left(\frac{y^2}{2}\Big|_{y=x}^{\sqrt{x}}\right)x\,dx\\ &= \frac{1}{2}\int_0^1 (x^2-x^3)\,dx = \frac{1}{24}.\end{aligned}$$

3. $\iint_S f(x,y)dxdy = \iint_{S_1} f(x+y)dxdy + \iint_{S_2} f(x+y)dxdy$, where $S_1 = \{(x,y) \in \mathbb{R}^2 : \; 0 \le x \le 1, \; 0 \le y \le 1+x\}$ and $S_2 = \{(x,y) \in \mathbb{R}^2 : \; -1 \le x \le 0, \; 0 \le y \le 1-x\}$.

$$\begin{aligned} \iint_{S_1} (x+y)dxdy &= \int_0^1 dx \int_0^{1+x} (x+y)\, dy \\ &= \frac{1}{2}\int_0^1 \left[(2x+1)^2 - x^2\right] dx \\ &= \frac{1}{2}\left[\frac{(2x+1)^3}{6} - \frac{x^3}{3}\right]\Bigg|_0^1 = 2 \end{aligned}$$

and

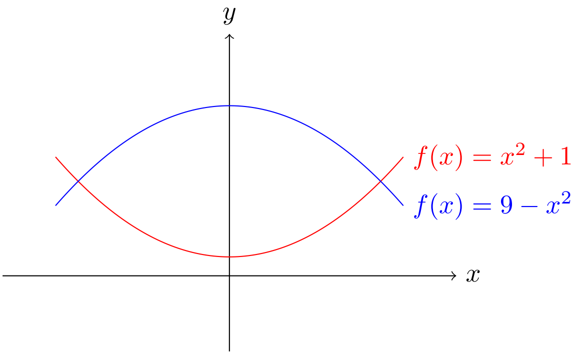
$$\begin{aligned} \iint_{S_2} (x+y)dxdy &= \int_{-1}^0 dx \int_0^{1-x} (x+y)\, dy \\ &= \frac{1}{2}\int_{-1}^0 (1-x^2)\, dx = \frac{1}{2}\left(x - \frac{x^3}{3}\right)\Bigg|_{-1}^0 = \frac{1}{3}. \end{aligned}$$

Therefore, $\iint_S (x+y)dxdy = 2 + \frac{1}{3} = \frac{7}{3}$.

Exercise 16 :

Compute the area of the region bounded by the curves $y = x^2+1$ and $y = 9-x^2$. $S = \{(x,y) \in \mathbb{R}^2 : \; x^2+1 \le y \le 9-x^2, -2 \le x \le 2\}$.

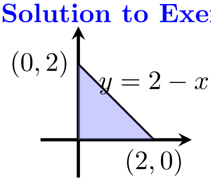
Solution to Exercise 11:



$$\begin{aligned} A = \iint_S dxdy &= \int_{-2}^{2} dx \int_{x^2+1}^{9-x^2} dy \\ &= \int_{-2}^{2} \left[(9-x^2)-(x^2+1)\right] dx \\ &= \int_{-2}^{2} (8-2x^2)\, dx = \frac{64}{3}. \end{aligned}$$

Exercise 17 :
Compute the volume of the solid in the first octant bounded by the graphs of equations $z = 4 - x^2$, $x + y = 2$, $x = 0$, $y = 0$, $z = 0$.

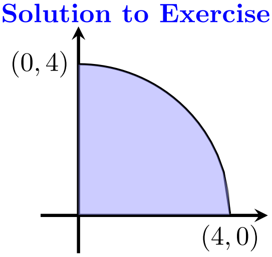
Solution to Exercise 12:



$$V = \int_0^2 \int_0^{2-x} (4 - x^2) dy dx = \frac{20}{3}.$$

Exercise 18 :
Compute the volume of the solid in the first octant bounded by the graphs of equations $z = x$, $x^2 + y^2 = 16$, $x = 0$, $y = 0$.

Solution to Exercise 13:



$$V = \int_0^4 \int_0^{\sqrt{16-x^2}} x dy dx = \frac{64}{3}.$$

Exercise 19 :
Evaluate the following integrals:

1. $\iint_T x^2 y^3 dx dy$, where $T = \left\{(x, y) \in [0, 1]^2 :\ y \leq x\right\}$.

2. $\iint_\Omega (x+3y)dxdy$, where $\Omega = \{1 \le x \le 3,\ \frac{x}{3} \le y \le \frac{4x}{3}\}$.

3. $\int_1^2 \int_{1-x}^{\sqrt{x}} x^2 y dy dx$,

4. $\int_0^1 \int_{x^3}^{\sqrt{x}} xy dy dx$.

Solution to Exercise 14:

1.

$$\begin{aligned}\iint_T x^2y^3dxdy &= \int_0^1 \left(\int_0^x x^2y^3dy\right)dx \\ &= \int_0^1 \frac{x^6}{4}dx = \frac{1}{28}.\end{aligned}$$

Now using horizontal strips, we get:

$$\begin{aligned}\iint_T x^2y^3dxdy &= \int_0^1 \left(\int_y^1 x^2y^3dx\right)dy \\ &= \frac{1}{3}\int_0^1 y^3(1-y^3)dy = \frac{1}{28}.\end{aligned}$$

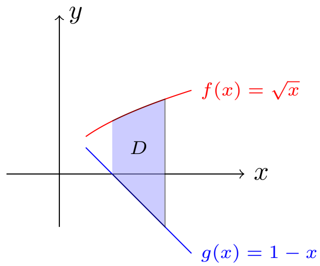
2.

$$\begin{aligned}\iint_\Omega (x+3y)dxdy &= \int_1^3 \left(\int_{\frac{x}{3}}^{\frac{4x}{3}} (x+3y)dy\right) dx \\ &= \frac{7}{2}\int_1^3 x^2dx = \frac{91}{3}.\end{aligned}$$

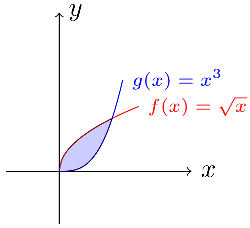
3.

$$\begin{aligned}\int_1^2 \int_{1-x}^{\sqrt{x}} x^2ydydx &= \int_1^2 \frac{1}{2}x^2(x-(1-x)^2)dx \\ &= \frac{1}{2}\int_1^2 (3x^3 - x^2 - x^4)dx = 6 - \frac{7}{6} - \frac{3}{2} - \frac{31}{10}.\end{aligned}$$

Consider the domain $D = \{(x,y) \in \mathbb{R}^2 : 1-x \le y \le \sqrt{x}, 1 \le x \le 2\}$. We have $\int_1^2 \int_{1-x}^{\sqrt{x}} x^2ydydx = \iint_D x^2ydxdy$.



4. $\displaystyle\iint_D xydxdy = \int_0^1 \int_{x^3}^{\sqrt{x}} xydydx = \frac{1}{2}\int_0^1 x^2(1-x^5)dx = \frac{5}{48}$, where $D = \{(x,y) \in \mathbb{R}^2 : x^3 \leq y \leq \sqrt{x},\ 0 \leq x \leq 1\}$.



Now using horizontal strips, we get:

$$\int_0^1 \int_{x^3}^{\sqrt{x}} xydydx = \int_0^1 \int_{y^2}^{y^{\frac{1}{3}}} xydxdy = \frac{1}{2}\int_0^1 y(y^{\frac{2}{3}} - y^4)dy = \frac{5}{48}.$$

Exercise 20 :

Evaluation the area of the domain $D = \{(x,y) \in \mathbb{R}^2 : 1-x \leq y \leq \sqrt{x}, 1 \leq x \leq 2\}$.

Solution to Exercise 15:

The area of D is $\displaystyle A = \iint_D dxdy = \int_1^2 \int_{1-x}^{\sqrt{x}} dydx = \frac{4\sqrt{2}}{3} - \frac{1}{6}$.

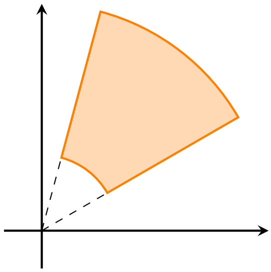
Now using horizontal strips, we get:

$$\begin{aligned} A &= \iint_D dxdy \\ &= \int_{-1}^0 \int_{1-y}^2 dxdy + \int_0^1 \int_1^2 dxdy + \int_1^{\sqrt{2}} \int_{y^2}^2 dxdy \\ &= \frac{1}{2} + 1 + \frac{4\sqrt{2}}{3} - \frac{5}{3} = \frac{4\sqrt{2}}{3} - \frac{1}{6}. \end{aligned}$$

2 Double Integrals in Polar Coordinates

Polar coordinates are defined by $x = r\cos\theta, y = r\sin\theta$. The area of the shaded region

$$R = \{(r,\theta) : a \le r \le b, \quad \alpha \le \theta \le \beta\}.$$



The integral of a continuous function $f(x,y)$ over a polar rectangle R given by $a \le r \le b$, $\alpha \le r \le \beta$, is

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Example 18 :

1. Find $\displaystyle\iint_R (2x-y)\,dA$ if R is the region in the first quadrant bounded by the circle $x^2+y^2=4$ and the lines $x=0$ and $y=x$.

2. Find $\displaystyle\iint_R e^{-x^2-y^2}\,dA$ if D is the region bounded by the semicircle $x = \sqrt{4-y^2}$ and the y-axis.

Theorem 2.1

If f is continuous over a polar region of the form

$$D = \{(r,\theta) : \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

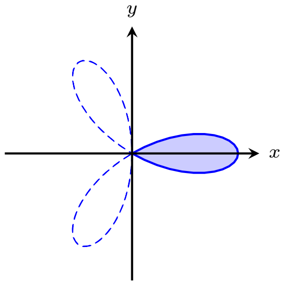
then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Example 19 :

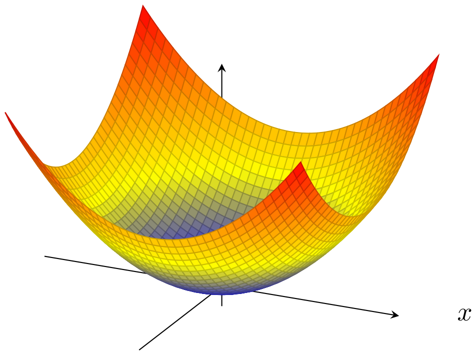
The area of one loop of the rose $r = \cos(3\theta)$

$$A = \int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r dr d\theta = \frac{\pi}{2}.$$



Example 20 :
The volume under the paraboloid $z = f(x, y) = x^2 + y^2$ and above the disc $x^2 + y^2 < 3$.
In polar coordinates, the disc is parameterized by: $x = r\cos\theta$, $y = r\sin\theta$, with $r \in [0, 3]$ and $\theta \in [0, 2\pi]$. Si in polar coordinates $f(x, y)$ is transformed to $g(r, \theta) = r^2$. Hence the volume under the paraboloid $z = x^2 + y^2$ and above the disc $x^2 + y^2 < 3$ is

$$V = \int_0^3 \int_0^{2\pi} r^2 dr d\theta = 18\pi.$$



Example 21 :
Evaluation of the following integral by converting to polar coordinates
$\displaystyle\int_0^a \int_0^{\sqrt{a^2-y^2}} y dx dy$

$$\begin{aligned} \int_0^a \int_0^{\sqrt{a^2-y^2}} y dx dy &= \int_0^a \int_0^{\frac{\pi}{2}} r^2 \sin\theta dr d\theta \\ &= \frac{a^3}{3}. \end{aligned}$$

Example 22 :
Evaluation of the following integral by converting to polar coordinates $\displaystyle\int_1^2 \int_0^x \frac{1}{x^2+y^2} dy dx$

$$\int_1^2 \int_0^x \frac{1}{x^2+y^2} dy dx =$$

2.1 Exercises

Exercise 21 :
Evaluation of the following integrals by converting to polar coordinates

1. $\displaystyle\int_0^2\int_0^{\sqrt{4-y^2}} \cos(x^2+y^2)dxdy,$

2. $\displaystyle\int_0^2\int_0^{\sqrt{2x-x^2}} xydydx,$

3. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}} (x^2+y^2)^{\frac{3}{2}}dydx,$

4. $\displaystyle\int_{-a}^a\int_0^{\sqrt{a^2-x^2}} (x^2+y^2)^{\frac{3}{2}}dydx,$

5. $\displaystyle\int_0^a\int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,\tan^{-1}(y/x)\,dx\,dy.$

Solution to Exercise 16:

1.

$$\begin{aligned}\int_0^2\int_0^{\sqrt{4-y^2}} \cos(x^2+y^2)dxdy &= \int_0^2\int_0^{\frac{\pi}{2}} r\cos(r^2)drd\theta\\ &= \frac{\pi}{4}\sin 4.\end{aligned}$$

2.

$$\begin{aligned}\int_0^2\int_0^{\sqrt{2x-x^2}} xydydx &= \int_0^2\int_0^{\sqrt{4-(x-1)^2}} rdrd\theta\\ &= .\end{aligned}$$

3. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}} (x^2+y^2)^{\frac{3}{2}}dydx = \int_0^3\int_0^{\frac{\pi}{2}} r^4drd\theta = \frac{3^5\pi}{10};$

4. $\displaystyle\int_{-a}^a\int_0^{\sqrt{a^2-x^2}} (x^2+y^2)^{\frac{3}{2}}dydx = \int_0^a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^4drd\theta = \frac{a^5\pi}{5},$

5. $\displaystyle\int_0^a\int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,\tan^{-1}(y/x)\,dx\,dy = \int_0^a\int_0^{\frac{\pi}{2}} r^2\theta rdrd\theta = \frac{\pi^2a^4}{32}.$

3 Surface Area

Consider a surface S defined by $z = f(x,y)$, for (x,y) in a closed region $R \in \mathbb{R}^2$. We assume that $f(x,y) \geq 0$ and f is continuously differentiable. We assume also that no normal vector to S is parallel to the $xy-$plane. $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \neq (0,0)$ for all $(x,y) \in R$. The surface area of S is

$$A = \iint_R \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2} dxdy.$$

Example 23 :
Consider the surface $z = 1 - x^2 - y^2$, for $z \geq 0$.
We have $\frac{\partial f}{\partial x} = -2x$ and $\frac{\partial f}{\partial y} = -2y$. Then

$$\begin{aligned} A &= \iint_{x^2+y^2<1} \sqrt{1+4x^2+4y^2}dxdy \\ &= \int_0^{2\pi}\int_0^1 r\sqrt{1+4r^2}drd\theta \\ &= 2\pi\frac{1}{12}\left[(1+4r^2)^{\frac{3}{2}}\right]_0^1 = \frac{\pi.(5^{\frac{3}{2}}-1)}{6} \end{aligned}$$

3.1 Exercises

Exercise 22 :
Compute the surface area of the following sets

1. The sphere $S = \{(x,y,z) \in \mathbb{R}^3 : \; x^2 + y^2 + z^2 = R^2\}$,

2. The surface $S = \{(x,y,z) \in \mathbb{R}^3 : \; z = 13 - 4x^2 - 4y^2\}$ on the domain $z = 1$, $x > 0$ and $y < 0$.

Solution to Exercise 17:

1. The surface area of S is the double of the area of the surface of the upper half sphere $S' = \{(x,y,z) \in \mathbb{R}^3 : \; z = \sqrt{1 - x^2 - y^2}\}$.
   We have $\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{R^2 - x^2 - y^2}}$ and $\frac{\partial z}{\partial y} = -\frac{y}{\sqrt{R^2 - x^2 - y^2}}$.
   The area is

$$
\begin{aligned}
A &= 2\iint_{x^2+y^2<R^2}\sqrt{1+\frac{x^2+y^2}{R^2-x^2-y^2}}dxdy\\
&= 2R\int_0^{2\pi}\int_0^R\frac{r}{\sqrt{R^2-r^2}}drd\theta = 4\pi R^2.
\end{aligned}
$$

2. We have $\dfrac{\partial z}{\partial x} = -8x$ and $\dfrac{\partial z}{\partial y} = -8y$. Then the area of S is

$$
\begin{aligned}
A &= \iint_{x^2+y^2<3, x>0, y<0}\sqrt{1+16(x^2+y^2)}dxdy\\
&= \int_0^{\frac{\pi}{2}}\int_0^{\sqrt{3}}\sqrt{1+16r^2}rdrd\theta\\
&= \frac{\pi}{2}\frac{1}{48}\left[(1+16r^2)^{\frac{3}{2}}\right]_0^{\sqrt{3}} = \frac{\pi}{2}.
\end{aligned}
$$

4 Triple Integrals

In which follows, we consider continuous functions $f\colon D \longrightarrow \mathbb{R}$, where D is a domain of $\mathbb{R}^3$.

4.1 Triple Integral Over Rectangular Domain

If the domain D is rectangular, $D = [a, b] \times [c, d] \times [r, s]$, then

$$
\begin{aligned}
\int_D f(x,y,z)dxdydz &= \int_a^b\int_c^d\int_r^s f(x,y,z)dzdydx\\
&= \int_a^b\int_r^s\int_c^d f(x,y,z)dydzdx\\
&= \int_c^d\int_a^b\int_r^s f(x,y,z)dzdxdy\\
&= \int_c^d\int_r^s\int_a^b f(x,y,z)dxdzdy\\
&= \int_r^s\int_a^b\int_c^d f(x,y,z)dydxdz\\
&= \int_r^s\int_c^d\int_a^b f(x,y,z)dxdydz.
\end{aligned}
$$

This is the Fubini's Theorem.

4.2 Triple Integral Over General Bounded Domain

The general bounded domains considered in this course are of three types. First, consider the bounded region $D \subset \mathbb{R}^3$ of the form

$$D = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in E : f_1(x, y) \leq z \leq f_2(x, y)\},$$

where E is a bounded domain in the $xy-$plane. Hence

$$\iiint_D f(x, y, z)dxdydz := \iint_E \left(\int_{f_1(x,y)}^{f_2(x,y)} f(x, y, z)dz \right) dxdy.$$

Second, consider the bounded region $D \subset \mathbb{R}^3$ of the form

$$D = \{(x, y, z) \in \mathbb{R}^3 : (x, z) \in E : f_1(x, z) \leq y \leq f_2(x, z)\},$$

where E is a bounded domain in the $xz-$plane. Hence

$$\iiint_D f(x, y, z)dxdydz := \iint_E \left(\int_{f_1(x,z)}^{f_2(x,z)} f(x, y, z)dy \right) dxdz.$$

Third, consider the bounded region $D \subset \mathbb{R}^3$ of the form

$$D = \{(x, y, z) \in \mathbb{R}^3 : (y, z) \in E : f_1(y, z) \leq x \leq f_2(y, z)\},$$

where E is a bounded domain in the $yz-$plane. Hence

$$\iiint_D f(x, y, z)dxdydz := \iint_E \left(\int_{f_1(y,z)}^{f_2(y,z)} f(x, y, z)dx \right) dydz.$$

Example 24 :
Consider the integral $I = \iiint_D f(x, y, z)dxdydz$, where D is the domain delimited by the planes of equations $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$ and $f(x, y, z) = (x + y + z)^2$.
The projection of the domain D on the $xy-$plane is the domain E delimited by the axes and the line of equation $x + y = 1$. Then if $(x, y) \in E$, we have

$$I_z = \int_0^{1-x-y} (x + y + z)^2 dz = \frac{1}{3}(1 - (x + y)^3).$$

Hence

$$I = \frac{1}{3} \int_0^1 \int_0^{1-x} (1 - (x + y)^3)dydx = \frac{1}{10}.$$

Theorem 4.1: (Properties of Triple Integrals)

Let D be a closed, bounded region in space, and let D_1 and D_2 be disjoint regions such that $D = D_1 \cup D_2$. Then

$$\iiint_D dV = \iiint_{D_1} dV + \iiint_{D_2} dV.$$

Example 25 :

Consider the integral $I = \displaystyle\iiint_D f(x, y, z)dxdydz$, where D is the domain delimited by the planes of equations $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$ and $f(x, y, z) = e^{x+y+z}$.

The projection of the domain D on the $xy-$plane is the domain E delimited by the axes and the line of equation $x + y = 1$. Then if $(x, y) \in E$, we have

$$I_z = \int_0^{1-x-y} e^{x+y+z} dz = e - e^{x+y}.$$

Hence

$$I = \int_0^1 \int_0^{1-x} e - e^{x+y} dydx = \frac{e}{2} - 1.$$

4.3 Exercises

Exercise 23 :

Compute the integral $I = \displaystyle\iiint_D f(x, y, z)dxdydz$, where $D = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq z \leq x^2 + y^2,\ 0 \leq y \leq x \leq 1\}$ and $f(x, y, z) = x + y + z$.

Solution to Exercise 18:

The projection of the domain D on the $xy-$plane is the domain E delimited by the $x-$axes and the lines of equations $x = y$ and $x = 1$. Then if $(x, y) \in E$, we have

$$I_z = \int_0^{x^2+y^2} (x + y + z)dz = (x + y)(x^2 + y^2) + \frac{1}{2}(x^2 + y^2)^2.$$

Hence

$$I = \int_0^1 \int_0^x (x + y)(x^2 + y^2) + \frac{1}{2}(x^2 + y^2)^2 dydx = \frac{103}{180}.$$

Exercise 24 :

Compute the integral $I = \iiint_D f(x,y,z)dxdydz$, where D is the domain delimited by the planes of equations $x=0$, $y=0$, $z=0$, $x+z=1$ and $y+z=1$ and $f(x,y,z)=(x-y+z)^2$.

Solution to Exercise 19:

We separate the domain D by the plane $x=y$. On the domain D_1, $x \geq y$ and on the domain D_2, $x \leq y$.

The projection of the domain D_1 on the $xy-$plane is the domain E_1 delimited by the $x-$axes and the lines of equations $x=y$ and $x=1$. Then if $(x,y) \in E_1$, we have

$$I_{1,z} = \int_0^{1-x} (x-y+z)^2 dz = \frac{1}{3}\left((1-y)^3 - (x-y)^3\right).$$

Hence

$$I_1 = \frac{1}{3}\int_0^1 \int_0^x \left((1-y)^3 - (x-y)^3\right) dydx = \frac{1}{20}.$$

The projection of the domain D_2 on the $xy-$plane is the domain E_2 delimited by the $y-$axes and the lines of equations $x=y$ and $y=1$. Then if $(x,y) \in E_y$, we have

$$I_{2,z} = \int_0^{1-y} (x-y+z)^2 dz = \frac{1}{3}\left((x+1-2y)^3 - (x-y)^3\right).$$

Hence

$$I_2 = \frac{1}{3}\int_0^1 \int_0^y \left((x+1-2y)^3 - (x-y)^3\right) dxdy = \frac{1}{60}.$$

Then $I = \iiint_D f(x,y,z)dxdydz = \frac{1}{15}$.

Exercise 25 :

Find the volume of the tetrahedron with corners at $(0,0,0)$, $(0,3,0)$, $(2,3,0)$, and $(2,3,5)$.

Solution to Exercise 20:

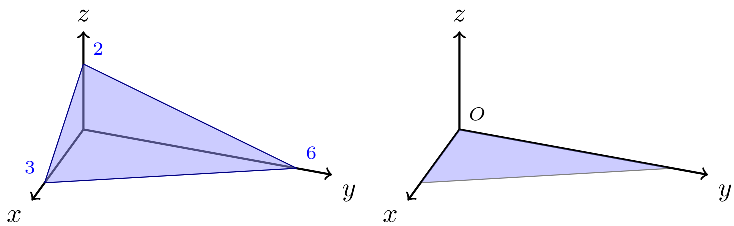
The solid is defined as follows: $0 \leq x \leq 2$, $\frac{3}{2}x \leq y \leq 3$, $0 \leq z \leq \frac{5}{2}x$. The lower y limit comes from the equation of the line $y = \frac{3}{2}x$ that forms one edge of the tetrahedron in the $x-y$ plane; the upper z limit comes from the equation of the plane $z = \frac{5}{2}x$ that forms the "upper" side of the tetrahedron. Now the volume is

$$\begin{aligned}\int_0^2 \int_{\frac{3}{2}x}^3 \int_0^{\frac{5}{2}x} dz\,dy\,dx &= \int_0^2 \int_{\frac{3}{2}x}^3 \frac{5x}{2}\,dy\,dx \\ &= \int_0^2 \frac{15x}{2} - \frac{15x^2}{4}\,dx = 5.\end{aligned}$$

Exercise 26 :
Find the volume of the space region in the first octant bounded by the plane $z = 2 - \frac{y}{3} - 2\frac{x}{3}$, using the order of integration $dzdydx$.

Solution to Exercise 21:

1. Starting with the order of integration $dzdydx$, we need to first find bounds on z. The region D is bounded below by the plane $z = 0$ (because we are restricted to the first octant) and above by $z = 2 - \frac{y}{3} - 2\frac{x}{3}$, $0 \le z \le 2 - \frac{y}{3} - 2\frac{x}{3}$.
   To find the bounds on y and x, we project the region onto the xy plane. By setting $z = 0$, we have $0 = 2 - \frac{y}{3} - 2\frac{x}{3} \Rightarrow y = 6 - 2x$. Secondly, we know this is going to be a straight line between the points $(3, 0)$ and $(0, 6)$ in the xy plane.)



   The region R, in the integration order of $dydx$, with bounds $0 \le y \le 6-2x$ and $0 \le x \le 3$. Thus the volume V of the region D is:

$$\begin{aligned} V &= \iiint_D dV = \int_0^3 \int_0^{6-2x} \int_0^{2-\frac{y}{3}-\frac{2x}{3}} dzdydz \\ &= \int_0^3 \int_0^{6-2x} \left(\int_0^{2-\frac{y}{3}-\frac{2x}{3}} dz \right) dydz \\ &= \int_0^3 \int_0^{6-2x} \left(2 - \frac{y}{3} - \frac{2x}{3} \right) dydz \\ &= \int_0^3 6 - 4x + \frac{2}{3}x^2 dx = 6. \end{aligned}$$

2. Now consider the volume using the order of integration $dzdxdy$. The bounds on z are the same as before, $0 \le z \le 2 - \frac{y}{3} - 2\frac{x}{3}$. Taking the projection on the xy plane, this gives the bounds $0 \le x \le 3 - \frac{y}{2}$ and $0 \le y \le 6$. Thus the volume is given by the triple integral

$$V = \int_0^6 \int_0^{3-\frac{y}{2}} \int_0^{2-\frac{y}{3}-\frac{2x}{3}} dzdxdy.$$

3. Now consider the volume using the order of integration $dxdydz$. The bounds on x are: $0 \leq x \leq 3 - \frac{y}{2} - \frac{3z}{2}$. The projection of the region D on the yz plane, we find the equation of the line $z = 2 - \frac{y}{3}$ by setting $x = 0$ in the equation $x = 3 - \frac{y}{2} - \frac{3z}{2}$. The bound of y are $y = 0$ and $y = 6 - 3z$; the points that bound z are 0 and 2. Thus the triple integral giving volume is: $0 \leq x \leq 3 - \frac{y}{2} - \frac{3z}{2}$, $0 \leq y \leq 6 - 3z$, $0 \leq z \leq 2$. Then

$$V = \int_0^2 \int_0^{6-3z} \int_0^{3-\frac{y}{2}-\frac{3z}{2}} dx\, dy\, dz.$$

Exercise 27 :

Set up the triple integrals that find the volume of the space region D bounded by the surfaces $x^2 + y^2 = 1$, $z = 0$ and $z = -y$, $z \geq 0$, with the orders of integration $dzdydx$, $dydxdz$ and $dxdzdy$.

Solution to Exercise 22:

The order $dzdydx$. The region D is bounded below by the plane $z = 0$ and above by the plane $z = -y$. The cylinder $x^2 + y^2 = 1$ does not offer any bounds in the $z-$direction, as that surface is parallel to the $z-$axis. Thus $0 \leq z \leq -y$. The projection of the region on the xy plane, is a part of the circle with equation $x^2+y^2 = 1$. As a function of x, this half circle has equation $y = -\sqrt{1-x^2}$. Thus y is bounded below by $-\sqrt{1-x^2}$ and above by $y = 0$: $-\sqrt{1-x^2} \leq y \leq 0$.

The x bounds of the half circle are $-1 \leq x \leq 1$. The volume is:

$$V = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^0 \int_0^{-y} dz\, dy\, dx = .$$

In the order $dydxdz$. The region is bounded below in the $y-$direction by the surface $x^2 + y^2 = 1$, then $y = -\sqrt{1-x^2}$ and above by the surface $y = -z$. Thus $-\sqrt{1-x^2} \leq y \leq -z$.

The projection on the xz plane gives the half circle has equation $x^2+z^2 = 1$. (We find this curve by solving each surface for y^2, then $y^2 = 1-x^2$ and $y = -z$, hence $y^2 = z^2$. Thus $x^2 + z^2 = 1$.) It is bounded below by $x = -\sqrt{1-z^2}$ and above by $x = \sqrt{1-z^2}$, where z is bounded by $0 \leq z \leq 1$. We have: $-\sqrt{1-x^2} \leq y \leq -z$, $-\sqrt{1-z^2} \leq x \leq \sqrt{1-z^2}$, $0 \leq z \leq 1$ and

$$V = \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-x^2}}^{-z} dy\, dx\, dz.$$

Exercise 28 :

Evaluate the integral $\displaystyle\int_0^1 \int_{x^2}^{x} \int_{x^2-y}^{2x+3y} \big(xy + 2xz\big) dz\, dy\, dx.$

Solution to Exercise 23:

$$\begin{aligned}
\int_0^1\int_{x^2}^{x}\int_{x^2-y}^{2x+3y}(xy+2xz)\,dz\,dy\,dx &= =\int_0^1\int_{x^2}^{x}\Big(\int_{x^2-y}^{2x+3y}(xy+2xz)\,dz\Big)\,dy\,dx\\
&= \int_0^1\int_{x^2}^{x}\Big((xyz+xz^2)\Big|_{x^2-y}^{2x+3y}\Big)\,dy\,dx\\
&= \int_0^1\int_{x^2}^{x}\Big(xy(2x+3y)+x(2x+3y)^2-\Big(xy(x^2-y)+x(x^2-y)^2\Big)\Big)\,dy\,dx\\
&= \int_0^1\int_{x^2}^{x}\Big(-x^5+x^3y+4x^3+14x^2y+12xy^2\Big)\,dy\,dx\\
&= \int_0^1\Big(-\frac{7}{2}x^7-8x^6-\frac{7}{2}x^5+15x^4\Big)\,dx\\
&= \frac{281}{336}.
\end{aligned}$$

Exercise 29 :
Evaluate the following integrals

1. $\displaystyle\int_0^1\int_0^x\int_0^{x+y} 2x+y-1\,dz\,dy\,dx,$
2. $\displaystyle\int_0^2\int_{-1}^{x^2}\int_1^{y} xyz\,dz\,dy\,dx,$
3. $\displaystyle\int_0^1\int_0^x\int_0^{\ln y} e^{x+y+z}\,dz\,dy\,dx,$
4. $\displaystyle\int_0^1\int_0^{y^2}\int_0^{x+y} x\,dz\,dx\,dy.$
5. $\displaystyle\int_1^2\int_y^{y^2}\int_0^{\ln(y+z)} e^{x}\,dx\,dz\,dy.$
6. $\displaystyle\int_0^{\pi}\int_0^{\pi/2}\int_0^1 z\sin x+z\cos y\,dz\,dy\,dx.$
7. $\displaystyle\int_{-3}^3\int_0^{\sqrt{9-y^2}}\int_0^{9-x^2-y^2} dzdxdy.$

Exercise 30 :
Compute $\displaystyle\int\int\int x+y+z\,dV$ over the region $x^2+y^2+z^2\le 1$ in the first octant.

Exercise 31 :
Consider the iterated integral

$$\int_{-1}^1\int_0^{1-x^2}\int_0^{y} dzdydx.$$

The bounds for this integral describe a region in space which satisfies the 3 inequalities $-1 \leq x \leq 1$, $0 \leq y \leq 1 - x^2$, and $0 \leq z \leq y$.

1. Draw the solid domain D in space described by the bounds of the iterated integral.

2. There are 5 other iterated integrals equivalent to this one. Set up the integrals that use the bounds $dydxdz$ and $dxdzdy$. We'll create the other 3 in class (though you are welcome to include them as part of your presentation).

Exercise 32 :
In each problem below, you'll be given enough information to determine a solid domain D in space. Draw the solid D and then set up an iterated integral (pick any order you want) that would give the volume of D. You don't need to evaluate the integral, rather you just need to set them up.

1. The region D under the surface $z = y^2$, above the xy-plane, and bounded by the planes $y = -1$, $y = 1$, $x = 0$, and $x = 4$.

2. The region D in the first octant that is bounded by the coordinate planes, the plane $y + z = 2$, and the surface $x = 4 - y^2$.

3. The pyramid D in the first octant that is below the planes $\frac{x}{3} + \frac{z}{2} = 1$ and $\frac{y}{5} + \frac{z}{2} = 1$. [Hint, don't let z be the inside bound.]

4. The region D that is inside both right circular cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$.

Example 26 :
Compute $\iiint_D f(x, y, z)dxdydz$, for $D = \{0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$, $f(x, y, z) = \dfrac{1}{(x + y + z + 1)^3}$,

Example 27 :
Compute $\iiint_D f(x, y, z)dxdydz$, for

1. $D = \{x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1\}$, $f(x, y, z) = \dfrac{1}{(x + y + z + 1)^2}$.

2. $D = \{x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1$, $f(x, y, z) = xyz$,

Example 28 :

Compute $\iiint_D f(x,y,z)dxdydz$, for $D = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \right\}$, $f(x,y,z) = x^2 + y^2$,

Example 29 :
Compute $\iiint_D \frac{\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z}{(1+x^2z^2)(1+y^2z^2)}$, with $D = \{(x,y,z) :\ 0 \le x \le 1, 0 \le y \le 1, 0 \le z\}$.
Deduce $\int_0^{+\infty} \left(\frac{\tan^{-1} t}{t} \right)^2 \mathrm{d}t$.

Example 30 :
Let $I = \int_0^1 \frac{\ln(1+x)}{1+x^2} dx$. Compute $J = \iint_D \frac{x\ \mathrm{d}x\ \mathrm{d}y}{(1+x^2)(1+xy)}$, with $D = \{(x,y) :\ 0 \le x \le 1, 0 \le y \le 1\}$ in two manner and deduce the value of I.

5 Centre of Mass and Moment of Inertia

Definition 5.1

Consider a thin plate T of density $\rho(x,y)$ which takes the form of a simple domain D of $\mathbb{R}^2$.
The mass of T is

$$M = \iint_D \rho(x,y)dxdy.$$

The moment of T with respect to the $x-$axis, respectively to the $y-$axis are defined by:

$$M_x = \iint_D y\rho(x,y)dxdy, \qquad M_y = \iint_D x\rho(x,y)dxdy.$$

Definition 5.2

The center of mass or the centroid of T is

$$(\bar{x}, \bar{y}) = \frac{1}{M}(M_y, M_x).$$

In particular if $\rho = 1$, the mass M is the area of D and the center of mass of D is the point G of coordinates

$$(x_G, y_G) = \frac{1}{\mathrm{Area}(D)} \left(\iint_D xdxdy, \iint_D ydxdy \right).$$

Example 31 :
The center of mass of the disc D of center (a,b) and radius R is the point (a,b). Indeed, we have

$$\begin{aligned}
\iint_D x dx dy &= \int_{a-R}^{a+R}\int_{b-\sqrt{R^2-(x-a)^2}}^{b+\sqrt{R^2-(x-a)^2}} x dy dx \\
&= \int_{a-R}^{a+R} 2x\sqrt{R^2-(x-a)^2}dx \\
&= \int_{-R}^{R} 2x\sqrt{R^2-x^2}dx + \int_{-R}^{R} 2a\sqrt{R^2-x^2}dx \\
&= a\,\text{Aire}(D).
\end{aligned}$$

Example 32 :
Consider the triangle with vertices $(0,0)$, $(0,2)$, $(3,0)$ and with density $\rho(x,y) = xy$. Find the total mass and center of mass.

$$\begin{aligned}
M &= \iint_T \rho(x,y)dxdy = \int_0^3\int_0^{2-x} xy\,dy\,dx \\
&= \frac{9}{8}.
\end{aligned}$$

$$\begin{aligned}
\overline{x} &= \frac{8}{9}\iint_T x\rho(x,y)dxdy = \frac{8}{9}\int_0^3\int_0^{2-x} x^2y\,dy\,dx \\
&= \frac{8}{5}.
\end{aligned}$$

$$\begin{aligned}
\overline{y} &= \frac{8}{9}\iint_T y\rho(x,y)dxdy = \frac{8}{9}\int_0^3\int_0^{2-x} xy^2\,dy\,dx \\
&= \frac{3}{10}.
\end{aligned}$$

5.1 Moment of Inertia of a Lamina

Definition 5.3

Consider a thin plate T of density $\rho(x,y)$ which takes the form of a simple domain D of $\mathbb{R}^2$.

The moment about the $x-$axis is $I_x = \iint_D y^2\rho(x,y)dxdy$.
The moment about the $y-$axis is $I_y = \iint_D x^2\rho(x,y)dxdy$.
The moment about the origin is $I_0 = \iint_D (x^2+y^2)\rho(x,y)dxdy = I_x+I_y$.

5.2 Centres of Mass of Solid

Definition 5.4

Consider a solid D of density $\rho(x,y,z)$.
The mass of D is

$$M = \iint_D \rho(x,y,z)dxdydz.$$

The moment of D with respect to the $xy-$plane, respectively to the yz and the $xz-$planes are defined by:

$$M_{xy} = \iiint_D z\rho(x,y,z)dxdyz, \qquad M_{yz} = \iiint_D x\rho(x,y,z)dxdyz,$$

$$M_{xz} = \iiint_D y\rho(x,y,z)dxdyz.$$

Definition 5.5

The center of mass or the centroid of D is

$$(\bar{x},\bar{y},\bar{z}) = \frac{1}{M}(M_{yz}, M_{xz}, M_{xy}).$$

In particular if $\rho = 1$, the mass M is the volume of D and the center of mass of D is the point G of coordinates

$$(x_G, y_G, z_G) = \frac{1}{\text{Volume}(D)}\left(\iiint_D xdxdydz, \iiint_D ydxdydz, \iiint_D zdxdydz\right).$$

Definition 5.6

Consider a solid D of density $\rho(x, y, z)$.
The moment about the $x-$axis is $I_x = \iint_D (y^2 + z^2)\rho(x, y, z)dxdydz$.
The moment about the $y-$axis is $I_y = \iint_D (x^2 + z^2)\rho(x, y, z)dxdydz$.
The moment about the $z-$axis is $I_z = \iint_D (x^2 + y^2)\rho(x, y, z)dxdydz$.
The moment about the origin is $I_0 = I_x + I_y + I_z$.

Example 33 :
Suppose the density of an object is given by $\rho(x, y, z) = xz$, and the object occupies the tetrahedron with corners $(0, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, and $(0, 1, 1)$. Find the mass and center of mass of the object.

As usual, the mass is the integral of density over the region:

$$\begin{aligned} M &= \int_0^1 \int_x^1 \int_0^{y-x} xz\,dz\,dy\,dx \\ &= \int_0^1 \int_x^1 \frac{x(y-x)^2}{2}\,dy\,dx \\ &= \frac{1}{2}\int_0^1 \frac{x(1-x)^3}{3}\,dx \\ &= \frac{1}{6}\int_0^1 x - 3x^2 + 3x^3 - x^4\,dx = \frac{1}{120}. \end{aligned}$$

We compute moments as before, except now there is a third moment:

$$\begin{aligned} M_{xy} &= \int_0^1 \int_x^1 \int_0^{y-x} xz^2\,dz\,dy\,dx = \frac{1}{360}, \\ M_{xz} &= \int_0^1 \int_x^1 \int_0^{y-x} xyz\,dz\,dy\,dx = \frac{1}{144}, \\ M_{yz} &= \int_0^1 \int_x^1 \int_0^{y-x} x^2z\,dz\,dy\,dx = \frac{1}{360}. \end{aligned}$$

Finally, the coordinates of the center of mass are $\bar{x} = \frac{M_{yz}}{M} = \frac{1}{3}$, $\bar{y} = \frac{M_{xz}}{M} = \frac{5}{6}$, and $\bar{z} = \frac{M_{xy}}{M} = \frac{1}{3}$.

Example 34 :
Find the mass and center of mass of the solid with density $\rho(x, y, z)$ and the given shape.

1. $\rho(x,y,z)=4$, solid bounded by $z=x^2+y^2$ and $z=4$
2. $\rho(x,y,z)=2+x$, solid bounded by $z=x^2+y^2$ and $z=4$
3. $\rho(x,y,z)=10+x$, tetrahedron bounded by $x+3y+z=6$ and the coordinate planes
4. $\rho(x,y,z)=1+x$, tetrahedron bound by $2x+y+4z=4$ and the coordinate planes.

5.3 Exercises

Example 35 :
The center of mass of the disc D of center (a,b) and radius R is the point (a,b). Indeed, we have

$$\begin{aligned}
\iint_D x dx dy &= \int_{a-R}^{a+R}\int_{b-\sqrt{R^2-(x-a)^2}}^{b+\sqrt{R^2-(x-a)^2}} x dy dx \\
&= \int_{a-R}^{a+R} 2x\sqrt{R^2-(x-a)^2} dx \\
&= \int_{-R}^{R} 2x\sqrt{R^2-x^2} dx + \int_{-R}^{R} 2a\sqrt{R^2-x^2} dx \\
&= a\,\mathrm{Aire}(D)
\end{aligned}$$

Exercise 33 :
Find the center of mass of a thin, uniform plate whose shape is the region between $y=\cos x$ and the $x-$axis between $x=-\frac{\pi}{2}$ and $x=\frac{\pi}{2}$.

Solution to Exercise 24:
The density is constant, we can take $\sigma(x,y)=1$. Then $\bar{x}=0$.
$M=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\cos x} dy\,dx=2$, $M_x=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\cos x} y\,dy\,dx=\frac{\pi}{4}$ and
$M_y=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\cos x} x\,dy\,dx=0$. So $\bar{x}=0$ and $\bar{y}=\frac{\pi}{8}$.

Exercise 34 :
Find the center of mass of a two-dimensional plate that occupies the quarter circle $x^2+y^2\le 1$ in the first quadrant and has density $k(x^2+y^2)$.

Solution to Exercise 25:

$$M=\int_0^1\int_0^{\sqrt{1-x^2}} k(x^2+y^2)\,dy\,dx = k\int_0^1 x^2\sqrt{1-x^2}+\frac{(1-x^2)^{\frac{3}{2}}}{3}\,dx=\frac{k\pi}{8}.$$

$$M_x=k\int_0^{\frac{\pi}{2}}\int_0^1 r^4\sin\theta\,dr\,d\theta=\frac{k}{5},\quad M_y=k\int_0^{\frac{\pi}{2}}\int_0^1 r^4\cos\theta\,dr\,d\theta=\frac{k}{5}.$$

Then , $\bar{x} = \bar{y} = \dfrac{8}{5\pi}$.

Exercise 35 :
Find the center of mass of a two-dimensional plate that occupies the square $[0,1] \times [0,1]$ and has density function xy.

Solution to Exercise 26:

Exercise 36 :
Find the center of mass of a two-dimensional plate that occupies the triangle $0 \leq x \leq 1$, $0 \leq y \leq x$, and has density function xy.

Solution to Exercise 27:

Exercise 37 :
Find the center of mass of a two-dimensional plate that occupies the upper unit semicircle centered at (0,0) and has density function y.

Solution to Exercise 28:

Exercise 38 :
Find the center of mass of a two-dimensional plate that occupies the upper unit semicircle centered at $(0,0)$ and has density function x^2.

Solution to Exercise 29:

Exercise 39 :
Find the center of mass of a two-dimensional plate that occupies the triangle formed by $x = 2, y = x$, and $y = 2x$ and has density function $2x$.

Solution to Exercise 30:

Exercise 40 :
Find the center of mass of a two-dimensional plate that occupies the triangle formed by $x = 0$, $y = x$, and $2x + y = 6$ and has density function x^2.

Solution to Exercise 31:

Exercise 41 :
Find the center of mass of a two-dimensional plate that occupies the region enclosed by the parabolas $x = y^2$, $y = x^2$ and has density function $\sqrt{x}$.

Solution to Exercise 32:

Exercise 42 :
Find the centroid of the area in the first quadrant bounded by $x^2 - 8y + 4 = 0, x^2 = 4y$, and $x = 0$. (Recall that the centroid is the center of mass when the density is 1 everywhere.)

Solution to Exercise 33:

Exercise 43 :

Find the centroid of one loop of the three-leaf rose $r = \cos(3\theta)$. (Recall that the centroid is the center of mass when the density is 1 everywhere, and that the mass in this case is the same as the area, which was the subject of exercise 11 in section 15.2.) The computations of the integrals for the moments M_x and M_y are elementary but quite long; Sage can help.

Solution to Exercise 34:

Exercise 44 :
Find the center of mass of a two dimensional object that occupies the region $0 \leq x \leq \pi, 0 \leq y \leq \sin x$, with density $\sigma = 1$.

Solution to Exercise 35:

Exercise 45 :
A two-dimensional object has shape given by $r = 1+\cos\theta$ and density $\sigma(r, \theta) = 2 + \cos\theta$. Set up the three integrals required to compute the center of mass.

Solution to Exercise 36:

Exercise 46 :
A two-dimensional object has shape given by $r = \cos\theta$ and density $\sigma(r, \theta) = r + 1$. Set up the three integrals required to compute the center of mass.

Solution to Exercise 37:

Exercise 47 :
Find the mass of a cube with edge length 2 and density equal to the square of the distance from one corner.

Solution to Exercise 38:

Exercise 48 :
Find the mass of a cube with edge length 2 and density equal to the square of the distance from one edge.

Solution to Exercise 39:

Exercise 49 :
An object occupies the volume of the upper hemisphere of $x^2 + y^2 + z^2 = 4$ and has density z at (x, y, z). Find the center of mass.

Solution to Exercise 40:

Exercise 50 :
Find the mass of a cube with edge length 2 and density equal to the square of the distance from one corner.

Solution to Exercise 41:

Exercise 51 :
Find the mass of a cube with edge length 2 and density equal to the square of the distance from one edge.

Solution to Exercise 42:

Exercise 52 :
An object occupies the volume of the upper hemisphere of $x^2 + y^2 + z^2 = 4$ and has density z at (x, y, z). Find the center of mass.

Solution to Exercise 43:

Exercise 53 :
An object occupies the volume of the pyramid with corners at $(1, 1, 0)$, $(1, -1, 0)$, $(-1, -1, 0)$, $(-1, 1, 0)$, and $(0, 0, 2)$ and has density $x^2 + y^2$ at (x, y, z). Find the center of mass.

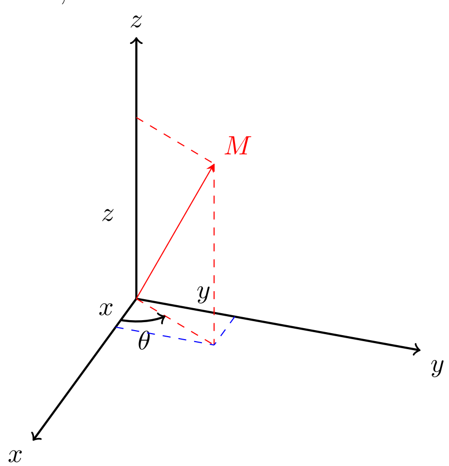
Solution to Exercise 44:

Exercise 54 :
Consider the triangular wedge D that is in the first octant, bounded by the planes $\frac{y}{7} + \frac{z}{5} = 1$ and $x = 12$. In the yz plane, the wedge forms a triangle that passes through the points $(0, 0, 0)$, $(0, 7, 0)$, and $(0, 0, 5)$. Set up integral formulas that would give the centroid $(\bar{x}, \bar{y}, \bar{z})$ of D. Actually compute the integrals for $\bar{y}$. Then state $\bar{x}$ and $\bar{z}$ by using symmetry arguments.

Solution to Exercise 45:

6 Cylindrical Coordinates

The cylindrical coordinate system is just the polar coordinate system plus the z coordinate. The volume of a typical small unit of volume is $r\Delta r\Delta\theta\Delta z$, or in the limit, $r\,dr\,d\theta\,dz$.



Example 36 :

The volume under $z = \sqrt{1-r^2}$ above the quarter circle inside $x^2 + y^2 = 1$ in the first quadrant.

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\sqrt{1-r^2}} r\, dz\, dr\, d\theta &= \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{1-r^2}\, r\, dr\, d\theta \\
&= \frac{\pi}{6}.
\end{aligned}$$

Example 37 :
Consider the solid D delimited by both the cylinder $x^2+y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$, and with density $\rho(x, y, z) = x^2y^2$ at (x, y, z). Compute its total mass.

$$\begin{aligned}
\int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^5 \cos^2\theta \sin^2\theta\, dz\, dr\, d\theta &= \frac{1}{2}\int_0^{2\pi} \int_0^1 \sqrt{4-r^2}\, r^5 \sin^2(2\theta)\, dr\, d\theta \\
&= \frac{\pi}{12}.
\end{aligned}$$

6.1 Double Integrals and Cylindrical Coordinates

Example 38 :
Compute the volume under the surface $z = \sqrt{4 - x^2 - y^2}$ and above the quarter of the disc $x^2 + y^2 < 4$ in the first quadrant.

$$V = \int_0^{\frac{\pi}{2}} \int_0^2 \sqrt{4-r^2}\, r\, dr\, d\theta = \frac{4\pi}{3}.$$

Example 39 :
Compute the volume under $z = \sqrt{4-r^2}$ above the disc defined by $0 \leq r < 2\cos\theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

$$\begin{aligned}
V &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \sqrt{4-r^2}\, r\, dr\, d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \left(\frac{8}{3} - \frac{8}{3}\sin^3\theta\right) d\theta = \frac{8\pi}{3} - \frac{32}{9}.
\end{aligned}$$

Example 40 :
Compute the area outside the circle r=2 and inside $r = 4\sin\theta$.

The region is described by $\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$ and $2 \leq r \leq 4\sin\theta$, so the integral is:

$$A = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_2^{4\sin\theta} r dr\, d\theta = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 8\sin^2\theta - 2\, d\theta = \frac{4\pi}{3} + 2\sqrt{3}.$$

6.2 Exercises

Exercise 55 :
Evaluate $\displaystyle\int_0^1\int_0^x\int_0^{\sqrt{x^2+y^2}}\frac{(x^2+y^2)^{\frac{3}{2}}}{x^2+y^2+z^2}\,dz\,dy\,dx.$

Solution to Exercise 46:
In cylindrical coordinates, we have:

$$\begin{aligned}\int_0^1\int_0^x\int_0^{\sqrt{x^2+y^2}}\frac{(x^2+y^2)^{\frac{3}{2}}}{x^2+y^2+z^2}\,dz\,dy\,dx &= \int_0^1\int_0^{\frac{\pi}{4}}\int_0^r\frac{r^3}{r^2+z^2}\,dz\,d\theta\,dr\\ &= \frac{\pi}{4}\int_0^1 r^2[\tan^{-1}(\frac{z}{r})]_0^r dr=\frac{\pi^2}{48}.\end{aligned}$$

Exercise 56 :
Evaluate $\displaystyle\int_{-1}^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx.$

Solution to Exercise 47:
In cylindrical coordinates, we have:

$$\begin{aligned}&\int_{-1}^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx\\ =&\int_0^\pi\int_0^1\int_r^{\sqrt{2-r^2}} r\sqrt{r^2+z^2}dzdrd\theta\\ =&\frac{\pi}{3}\int_0^1 2^{\frac{3}{2}}r(1-r^3)dr=\frac{2^{\frac{3}{2}}\pi}{10}.\end{aligned}$$

Exercise 57 :
Evaluate $\displaystyle\int\int\int x^2\,dxdydz$ over the interior of the cylinder $x^2+y^2=1$ between $z=0$ and $z=5$.

Solution to Exercise 48:
In cylindrical coordinates, we have:

$$\int\int\int x^2\,dxdydz=\int_0^1\int_0^{2\pi}\int_0^5 r^3\cos^2\theta dzd\theta dr=\frac{5\pi}{4}.$$

Exercise 58 :
Evaluate $\displaystyle\int\int\int xy\,dxdydz$ over the interior of the cylinder $x^2+y^2=1$ between $z=0$ and $z=5$.

Solution to Exercise 49:
In cylindrical coordinates, we have:

$$\int\int\int xy\,dxdydz=\int_0^1\int_0^{2\pi}\int_0^5 r^3\cos\theta\sin\theta dzd\theta dr=0.$$

Exercise 59 :
Evaluate $\displaystyle\int\int\int z\,dxdydz$ over the region above the $xy-$ plane, inside $x^2 + y^2 - 2x = 0$ and under $x^2 + y^2 + z^2 = 4$.

Solution to Exercise 50:
In cylindrical coordinates, we have:

$$\begin{aligned}\int\int\int z\,dxdydz &= \int_0^{2\pi}\int_0^{2\cos\theta}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} zdzrdrd\theta \\ &= \int_0^{2\pi}\int_0^{2\cos\theta} r(4-r^2)drd\theta \\ &= \int_0^{2\pi} \cos^2\theta d\theta = \pi.\end{aligned}$$

Exercise 60 :
Evaluate $\displaystyle\int\int\int yz\,dxdydz$ over the region in the first octant, inside $x^2+y^2-2x = 0$ and under $x^2 + y^2 + z^2 = 4$.

Solution to Exercise 51:
In cylindrical coordinates, we have:

$$\begin{aligned}\int\int\int yz\,dxdydz &= \int_0^{2\pi}\int_0^{2\cos\theta}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} zdzr^2\sin\theta drd\theta \\ &= \int_0^{2\pi}\int_0^{2\cos\theta} (4-r^2)r^2\sin\theta drd\theta \\ &= 0.\end{aligned}$$

Exercise 61 :
Evaluate $\displaystyle\int\int\int (x^2+y^2)\,dxdydz$ over the interior of $x^2 + y^2 + z^2 = 4$.

Solution to Exercise 52:
In cylindrical coordinates, we have:

$$\begin{aligned}\int\int\int (x^2+y^2)\,dxdydz &= \int_0^{2\pi}\int_0^{2}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^3dzdrd\theta \\ &= 4\pi\int_0^{2} r^3\sqrt{4-r^2}dr = \frac{256\pi}{15}.\end{aligned}$$

Exercise 62 :
Evaluate $\displaystyle\int\int\int \sqrt{x^2+y^2}\,dxdydz$ over the interior of $x^2 + y^2 + z^2 = 4$.

Solution to Exercise 53:
In cylindrical coordinates, we have:

$$\begin{aligned}\int\int\int \sqrt{x^2+y^2}\,dxdydz &= \int_0^{2\pi}\int_0^2\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^2 dzdrd\theta \\ &= 4\pi\int_0^2 r^2\sqrt{4-r^2}dr \\ &\overset{r=2\sin t}{=} 16\pi\int_0^{\frac{\pi}{2}} \sin^t dt = 4\pi^2.\end{aligned}$$

Exercise 63 :
Compute $\int\int\int (x+y+z)\,dxdydz$ over the region inside $x^2+y^2+z^2=1$ in the first octant.

Solution to Exercise 54:
In cylindrical coordinates, we have:

$$\begin{aligned}\int\int\int (x+y+z)\,dxdydz &= \int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} (r\cos\theta + r\sin\theta + z) rdzdrd\theta \\ &= \int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} zrdzdrd\theta \\ &= 4\pi\int_0^1 r\sqrt{1-r^2}dr = \frac{4\pi}{3}.\end{aligned}$$

Exercise 64 :
Compute the mass of a right circular cone of height h and base radius R if the density is proportional to the distance from the base.

Solution to Exercise 55:
In cylindrical coordinates, we have:

$$M = \int_0^R\int_0^{2\pi}\int_0^h zdzrdrd\theta = \frac{1}{2}\pi h^2 R^2.$$

Exercise 65 :
Compute the mass of a right circular cone of height h and base radius R if the density is proportional to the distance from its axis of symmetry.

Solution to Exercise 56:
In cylindrical coordinates, we have:

$$M = \int_0^R\int_0^{2\pi}\int_0^h dzr^2drd\theta = \frac{2}{3}\pi h R^3.$$

Exercise 66 :

An object delimited by the unit sphere and has density equal to the distance from the $x-$axis. Compute the mass.

Solution to Exercise 57:

$$M = \int_0^1 \int_0^{2\pi} \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} dz r^2 |\cos\theta| dr d\theta = 4 \int_0^1 r^2 \sqrt{1-r^2} dr = \frac{\pi}{2}.$$

Exercise 67 :
Compute the volume above the $x-y$ plane, under the surface $r^2 = 2z$, and inside $r = 2$.

Solution to Exercise 58:

Exercise 68 :
Compute the volume inside both $r = 1$ and $r^2 + z^2 = 4$.

Solution to Exercise 59:

Exercise 69 :
Compute the volume below $z = \sqrt{1-r^2}$ and above the top half of the cone $z = r$.

Solution to Exercise 60:

Exercise 70 :
Compute the volume below $z = r$, above the $x-y$ plane, and inside $r = \cos\theta$.

Solution to Exercise 61:

Exercise 71 :
Compute the volume below z=r, above the $x-y$plane, and inside $r = 1+\cos\theta$.

Solution to Exercise 62:

Exercise 72 :
Compute the volume between $x^2 + y^2 = z^2$ and $x^2 + y^2 = z$.

Solution to Exercise 63:

Exercise 73 :
Compute the area inside $r = 1 + \sin\theta$ and outside $r = 2\sin\theta$.

Solution to Exercise 64:

Exercise 74 :
Compute the area inside both $r = 2\sin\theta$ and $r = 2\cos\theta$.

Solution to Exercise 65:

Exercise 75 :
Compute the area inside the four-leaf rose $r = \cos(2\theta)$ and outside $r = \frac{1}{2}$.

Solution to Exercise 66:

Exercise 76 :

Compute the area inside the cardioid $r = 2(1 + \cos\theta)$ and outside $r = 2$.

Solution to Exercise 67:

Exercise 77 :
Compute the area of one loop of the three-leaf rose $r = \cos(3\theta)$.

Solution to Exercise 68:

Exercise 78 :
Compute $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin(x^2 + y^2)\, dy\, dx$ by converting to cylindrical coordinates.

Solution to Exercise 69:

Exercise 79 :
Compute $\displaystyle\int_{0}^{a}\int_{-\sqrt{a^2-x^2}}^{0} x^2 y\, dy\, dx$ by converting to cylindrical coordinates.

Solution to Exercise 70:

Exercise 80 :
Compute the volume under $z = y^2 + x + 2$ above the region $x^2 + y^2 \leq 4$

Solution to Exercise 71:

Exercise 81 :
Compute the volume between $z = x^2y^3$ and $z = 1$ above the region $x^2 + y^2 \leq 1$

Solution to Exercise 72:

Exercise 82 :
Compute the volume inside $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

Solution to Exercise 73:

Exercise 83 :
Compute the volume under $z = r$ above $r = 3 + \cos\theta$.

Solution to Exercise 74:

Exercise 84 :
Sketch and describe the cylindrical surface of the given equation. $x^2 + z^2 = 1$,

$x^2 + y^2 = 9$,
$z = \cos\left(\frac{\pi}{2} + x\right)$,
$z = 9 - y^2$,
$z = e^x$.

Solution to Exercise 75:

Exercise 85 :

Let E be the region bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the paraboloid $z = 2-x^2-y^2$. Set up a triple integral in cylindrical coordinates to find the volume of the region.

The cone is of radius 1. Since $z = 2-x^2-y^2 = 2-r^2$ and $z = \sqrt{x^2+y^2} = r$, we have $2-r^2 = r$, then $r = 1$. Therefore $z = 1$. So the intersection of these two surfaces is a circle of radius 1 in the plane $z = 1$. Thus, the region $E = \{(r,\theta,z) : 0 \le \theta \le 2\pi,\ \theta \le r \le 1,\ r \le z \le 2-r^2\}$. Hence the integral for the volume is

$$V = \int_0^{2\pi}\int_0^1\int_r^{2-r^2} r\,dz\,dr\,d\theta = \frac{5\pi}{6}.$$

Solution to Exercise 76:

We can also see that

$$E = \{(r,\theta,z)|0 \le \theta \le 2\pi,\ 0 \le z \le 1,\ 0 \le r \le z\} \cup \{(r,\theta,z)|0 \le \theta \le 2\pi,\ 1 \le z \le 2,\ 0 \le r \le \sqrt{2-z}\}.$$

Then

$$V = \int_0^{2\pi}\int_0^1\int_0^z r\,dr\,dz\,d\theta + \int_0^{2\pi}\int_1^2\int_0^{\sqrt{2-z}} r\,dr\,dz\,d\theta = \frac{\pi}{3} + \frac{\pi}{2} = \frac{5\pi}{6}.$$

Exercise 86 :

Let E be the region bounded below by the $xy-$plane, above by the sphere $x^2+y^2+z^2 = 4$, and on the sides by the cylinder $x^2+y^2 = 1$. Set up a triple integral in cylindrical coordinates to find the volume of the region.
The equation for the sphere is: $x^2+y^2+z^2 = 4$ or $r^2+z^2 = 4$. The equation for the cylinder is $x^2+y^2 = 1$ or $r^2 = 1$. Then the region E is defined as follows: $E = \{(r,\theta,z)|0 \le z \le \sqrt{4-r^2},\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$. Then the volume of E is

Solution to Exercise 77:

$$\begin{aligned} V &= \int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 \left(r\sqrt{4-r^2}\right) dr\,d\theta \\ &= \int_0^{2\pi}\left(\frac{8}{3}-\sqrt{3}\right) d\theta = 2\pi\left(\frac{8}{3}-\sqrt{3}\right). \end{aligned}$$

Exercise 87 :

Evaluate $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta}\int_0^{r\cos\theta} r^2\,dz\,dr\,d\theta.$

Solution to Exercise 78:

Exercise 88 :

Evaluate $\displaystyle\int_0^{\pi}\int_0^{\sin\theta}\int_0^{r\sin\theta} r\cos^2\theta\,dz\,dr\,d\theta.$

Solution to Exercise 79:

Exercise 89 :

Compute $\displaystyle\iiint_D f(x,y,z)dxdydz$, for $D=\left\{x^2+y^2\le R^2, 0\le z\le a\right\}$, $f(x,y,z)=x^3+y^3+z^3-3z\left(x^2+y^2\right)$,

Solution to Exercise 80:

Exercise 90 :

Compute $\displaystyle\iiint_D f(x,y,z)dxdydz$, for $D=\left\{x^2+y^2\le z^2, 0\le z\le 1\right\}$, $f(x,y,z)=\dfrac{z}{(x^2+y^2+1)^2}$,

Solution to Exercise 81:

Exercise 91 :

Evaluate the triple integrals $\displaystyle\iiint_E f(x,y,z)dxdydz.$

1. $f(x,y,z)=z, E=\{(x,y,z):\ x^2+y^2\le 9, x\le 0, y\le 0, 0\le z\le 1\}$,
2. $f(x,y,z)=xz^2, E=\{(x,y,z):\ x^2+y^2\le 16, x\ge 0, y\le 0, -1\le z\le 1\}$,
3. $f(x,y,z)=xy, E=\{(x,y,z):\ x^2+y^2\le 1, x\ge 0, x\ge y, -1\le z\le 1\}$,
4. $f(x,y,z)=x^2+y^2, E=\{(x,y,z):\ x^2+y^2\le 4, x\ge 0, x\le y, 0\le z\le 3\}$,
5. $f(x,y,z)=e^{\sqrt{x^2+y^2}}, E=\{(x,y,z):\ 1\le x^2+y^2\le 4, y\le 0, x\le y\sqrt{3}, 2\le z\le 3\}$,
6. $f(x,y,z)=\sqrt{x^2+y^2}, E=\{(x,y,z):\ 1\le x^2+y^2\le 9, y\le 0, 0\le z\le 1\}$,

Solution to Exercise 82:

1. Using the cylindrical coordinates, we get:
$$\iiint_E f(x,y,z)dxdydz = \int_0^3\int_{\pi}^{\frac{3\pi}{2}}\int_0^1 zdzd\theta dr=\frac{3\pi}{4}.$$
2. Using the cylindrical coordinates, we get:
$$\iiint_E f(x,y,z)dxdydz = \int_0^4\int_{-\frac{\pi}{2}}^{0}\int_{-1}^1 r\cos\theta z^2dzd\theta dr=\frac{8}{3}.$$

3. Using the cylindrical coordinates, we get:

$$\iiint_E f(x,y,z)dxdydz = \int_0^1 \int_0^{\frac{\pi}{4}} \int_{-1}^{1} r^2 \cos\theta \sin\theta dz d\theta dr = \frac{1}{3}.$$

4. $f(x,y,z) = x^2+y^2, E = \{(x,y,z): x^2+y^2 \leq 4, x \geq 0, x \leq y, 0 \leq z \leq 3\}$, Using the cylindrical coordinates, we get:

$$\iiint_E f(x,y,z)dxdydz = \int_0^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^3 r^2 dz d\theta dr = \pi.$$

5. Using the cylindrical coordinates, we get:

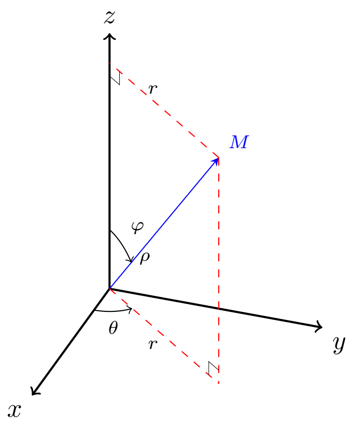
$$\iiint_E f(x,y,z)dxdydz = \int_1^2 \int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} \int_2^3 e^r dz d\theta dr = \frac{\pi}{3}(e^2-1).$$

6. Using the cylindrical coordinates, we get:

$$\iiint_E f(x,y,z)dxdydz = \int_1^3 \int_{\pi}^{2\pi} \int_0^1 r dz d\theta dr = \pi.$$

7 Spherical Coordinates

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases}$$



$r = \rho\sin\varphi$, $\rho \geq 0$, $\varphi \in [0,\pi]$ and $\theta \in [0,2\pi]$.

Example 41 :
Consider the points M and N of coordinates $(10, \frac{\pi}{2}, \frac{\pi}{6})$ and $(6, \frac{\pi}{3}, \frac{\pi}{6})$ respectively in spherical coordinates:

1. The rectangular coordinates of M are: $(0, 5, 5\sqrt{3})$.
   The rectangular coordinates of N are: $(\dfrac{3}{2}, \dfrac{3\sqrt{3}}{2}, 3)$.

2. The cylindrical coordinates of M are: $(5, \frac{\pi}{2}, 5\sqrt{3})$.
   The cylindrical coordinates of N are: $(3, \frac{\pi}{2}, 3)$.

Example 42 :
Change the rectangular coordinates$(1, \sqrt{3}, 0)$ to

1. the spherical coordinates

2. the cylindrical coordinates.

$\rho^2 = x^2 + y^2 + z^2 = 4$, then $\rho = 2$.
$\theta = \tan^{-1}(\dfrac{y}{x}) = \tan^{-1}\sqrt{3}$, $\theta = \frac{\pi}{3}$.
$\varphi = \cos^{-1}(\frac{z}{\rho}) = \frac{\pi}{2}$.
Thus the spherical coordinates are: $(2, \frac{\pi}{3}, \frac{\pi}{2})$.
for the cylindrical coordinates, we have $r^2 = x^2 + y^2 = 4$, then $r = 2$.
The cylindrical coordinates are $(2, \frac{\pi}{3}, 0)$.

Example 43 :
Change the cylindrical coordinates$(4, -\frac{\pi}{2}, 6)$ to

1. the rectangular coordinates

2. the spherical coordinates.

For the rectangular coordinates, we have $x = r\cos\theta = 0$.
$y = r\sin\theta = -2$ and $z = 6$.
For the spherical coordinates, we have $\rho^2 = x^2 + y^2 + z^2 = 4 + 36$, then $\rho = 2\sqrt{10}$.
$\theta = -\dfrac{\pi}{2}$, $z = 6 = \rho\cos\varphi$, then $\varphi = \cos^{-1}\dfrac{3}{\sqrt{10}}$.
Thus the spherical coordinates are: $(2\sqrt{10}, -\frac{\pi}{2}, \cos^{-1}\dfrac{3}{\sqrt{10}})$.

Example 44 :

1. Describe the graph of the equation $\rho = \sec\varphi = 6$.
   $\rho\sec\varphi = 6 \iff \rho = 6\cos\varphi \iff \rho^2 = 6z$. Hence $x^2 + y^2 + z^2 = \rho^2 \iff x^2 + y^2 + (z-3)^2 = 9$. This is the equation of the sphere of radius 3 and center $(0, 0, 3)$.

2. Describe the graph of the equation $\rho = 6\sin\varphi\cos\theta$.
   We know that $x = \rho\sin\varphi\cos\theta$, then $\frac{x}{\rho} = \sin\varphi\cos\theta$. Then $\rho^2 = 6x$. This is the equation of the sphere of radius 3 and center $(3, 0, 0)$.

Example 45 :
Change the equations to spherical coordinates:

1. $x^2+y^2=4z$
2. $x^2-4z^2+y^2=0$

1. $x^2+y^2=4z \iff \rho^2\sin^2\varphi=4\rho\cos\varphi$, which is equivalent to: $\rho=4\dfrac{\cos\varphi}{\sin^2\varphi}$.
2. $x^2-4z^2+y^2=0 \iff \rho^2\sin^2\varphi=4\rho^2\cos^2\varphi \iff \tan^2\varphi=4$.

7.1 Triple Integrals In Spherical Coordinates

Consider a domain $\Omega\subset\mathbb{R}^3$ defined in spherical coordinates by:
$\Omega=\{(\rho,\theta,\varphi)\in]0,+\infty[\times]0,2\pi[\times]0,\pi[:\ a\le\rho\le b,\ \theta_1(\rho)\le\theta\le\theta_2(\rho),\ \varphi_1(\theta,\rho)\le\varphi\le\varphi_1(\theta,\rho)\}$.

Theorem 7.1

Let $f:\Omega\subset\mathbb{R}^3\longrightarrow\mathbb{R}$ be a continuous function, then the triple integral of function f in the region Ω can be expressed in spherical coordinates as follows,

$$\iiint_\Omega f(x,y,z)\,dV=\int_a^b\left(\int_{\theta_1(\rho)}^{\theta_2(\rho)}\left(\int_{\varphi_1(\theta,\rho)}^{\varphi_2(\theta,\rho)}\rho^2\sin\varphi f(\rho\sin\varphi\cos\theta,\rho\sin\varphi\sin\theta,\rho\cos\varphi)d\varphi\right)d\theta\right)d\rho.$$

Then the volume of a domain Ω is equal to:

$$V=\iiint_\Omega dV=\int_a^b\int_{\theta_1(\rho)}^{\theta_2(\rho)}\int_{\varphi_1(\theta,\rho)}^{\varphi_2(\theta,\rho)}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

Example 46 :
Evaluate the integral by using the spherical coordinates.

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{1}{x^2+y^2+z^2}\,dzdydx$$

Solution:
We have: $0\le z\le\sqrt{1-x^2-y^2}$, then $0\le\rho\le1$.
$0\le y\le\sqrt{1-x^2},\quad 0\le x\le1$, then $0\le\varphi\le\frac{\pi}{2}$ and $0\le\theta\le\frac{\pi}{2}$.

We get

$$\begin{aligned}
\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{dzdydx}{x^2+y^2+z^2} &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1\frac{1}{\rho^2}\cdot\rho^2\sin\varphi\,d\rho d\varphi d\theta \\
&= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left(\int_0^1 d\rho\right)\sin\varphi\,d\varphi d\theta \\
&= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\Big[\rho\Big]_0^1\sin\varphi\,d\varphi d\theta = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin\varphi\,d\varphi d\theta \\
&= \int_0^{\frac{\pi}{2}}\left(\int_0^{\frac{\pi}{2}}\sin\varphi\,d\varphi\right)d\theta = \int_0^{\frac{\pi}{2}}\Big[-\cos\varphi\Big]_0^{\frac{\pi}{2}}d\theta \\
&= \int_0^{\frac{\pi}{2}}(0+1)d\theta = \Big[\theta\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{2}.
\end{aligned}$$

7.2 Exercises

Exercise 92 :
Set up an integral for the volume of the region bounded by the cone $z = \sqrt{3(x^2+y^2)}$ and the hemisphere $z = \sqrt{4-x^2-y^2}$.

Solution to Exercise 83:
Using the conversion formulas from rectangular coordinates to spherical coordinates, we have:

For the cone: $z = \sqrt{3(x^2+y^2)}$ or ρ cos $\varphi = \sqrt{3}\rho$ sin φ, or tan $\varphi = \frac{1}{\sqrt{3}}$. the $\varphi = \frac{\pi}{6}$.
For the sphere: $z = \sqrt{4-x^2-y^2}$ or $\rho = 2$, then

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^2\rho^2\sin\,\varphi\,d\rho\,d\varphi\,d\theta = \frac{16\pi}{3}(1-\frac{\sqrt{3}}{2}).$$

Exercise 93 :
Let E be the region bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the sphere $z = x^2+y^2+z^2$.

Solution to Exercise 84:
Use the conversion formulas to write the equations of the sphere and cone in spherical coordinates.
For the sphere: $x^2+y^2+z^2 = z \iff \rho^2 = \rho\cos\,\varphi \iff \rho = \cos\,\varphi$.
For the cone:

$$\begin{aligned}
z = \sqrt{x^2+y^2} &\iff \rho\,\cos\,\varphi = \sqrt{\rho^2\sin^2\varphi\,(\cos^2\phi+\sin^2\phi)} \\
&\iff \rho\,\cos\,\varphi = \rho\,\sin\,\varphi \\
&\iff \cos\,\varphi = \sin\,\varphi.
\end{aligned}$$

Then $\varphi = \frac{\pi}{4}$. Then the integral for the volume of the solid region E becomes:

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\cos\,\varphi} \rho^2 \sin\,\varphi\, d\rho\, d\varphi\, d\theta = \frac{\pi}{8}.$$

Exercise 94 :
Evaluate the following triple integrals $\iiint_E f(x,y,z)dxdydz$.

1. $f(x,y,z) = 1 - \sqrt{x^2+y^2+z^2}, E = \{(x,y,z) : \ x^2+y^2+z^2 \leq 9, y \geq 0, z \geq 0\}$,
2. $f(\rho,\theta,\varphi) = \rho\sin\varphi(\cos\theta + \sin\theta), E = \{(\rho,\theta,\varphi) : \ 1 \leq \rho \leq 2, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq \dfrac{\pi}{2}\}$,
3. $f(x,y,z) = 2xy;\ E = \{(x,y,z) : \ \sqrt{x^2+y^2} \leq z \leq \sqrt{1-x^2-y^2}, x \geq 0, y \geq 0\}$,
4. $f(\rho,\theta,\varphi) = \rho\cos\varphi;\ E = \{(\rho,\theta,\varphi) : \ 0 \leq \rho \leq 2\cos\varphi, 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \varphi \leq \frac{\pi}{4}\}$,

Solution to Exercise 85:

1. Using the spherical coordinates, we get:

$$\iiint_E f(x,y,z)dxdydz = \int_0^3 \int_0^{\frac{\pi}{2}} \int_0^{\pi} (1-\rho)d\theta d\varphi d\rho = -\frac{3\pi^2}{4}.$$

2.

$$\iiint_E f(\rho,\theta,\varphi)d\rho d\theta d\varphi = \int_1^2 \int_0^{\pi} \int_0^{\frac{\pi}{2}} \rho\sin\varphi(\cos\theta + \sin\theta)d\varphi d\theta d\rho = 3.$$

Exercise 95 :
Compute the volume of the solids E and F defined in rectangular coordinates by:
$E = \{(x,y,z) : \ \sqrt{x^2+y^2} \leq z \leq \sqrt{16-x^2-y^2}, x \geq 0, y \geq 0\}$,
$F = \{(x,y,z) : \ x^2+y^2+z^2-2z \leq 0, \sqrt{x^2+y^2} \leq z\}$,

Exercise 96 :
Convert the integral into an integral in spherical coordinates:

1. $\displaystyle\int_{-4}^{4} \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} (x^2+y^2+z^2)^2 dzdydx$,

2. $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} dzdydx.$

Solution to Exercise 86:

1.

2.

Exercise 97 :

Let B_R the ball of center $(0,0,0)$ and radius R.

$$\begin{aligned}\iiint_{B_R}\sqrt{x^2+y^2+z^2}dxdydz &= \int_0^{2\pi}\int_0^{\pi}\int_0^{R}\rho^3\sin\varphi d\rho d\theta d\varphi\\ &= 4\pi\int_0^R\rho^3 d\rho = \pi R^4.\end{aligned}$$

Exercise 98 :

Consider the solid delimited by the plane xoy, the cylinder $x^2+y^2=x$ and the sphere $x^2+y^2+z^2=1$.

The solid is defined by: $\begin{cases} x^2+y^2+z^2\leq 1\\ x^2+y^2\leq x\\ z\geq 0.\end{cases}$

In cylindrical coordinates, we have $\begin{cases}\rho^2+z^2\leq 1\\ \rho^2\leq\rho\cos\theta\\ z\geq 0.\end{cases}$

Then $\rho\in[0,1]$ and if ρ is fixed, θ is in $[-\cos^{-1}\rho,\cos^{-1}\rho]$ and $z\in[0,\sqrt{1-\rho^2}]$. Then the volume of the solid is given by:

$$\begin{aligned}V &= \int_0^1\int_{-\cos^{-1}\rho}^{\cos^{-1}\rho}\int_0^{\sqrt{1-\rho^2}}\rho dzd\theta d\rho\\ &= \int_0^1 2\rho\sqrt{1-\rho^2}cos^{-1}\rho d\rho = \frac{2}{3}(\frac{\pi}{2}-\frac{2}{3}).\end{aligned}$$

Exercise 99 :

Compute $\iiint_D f(x,y,z)dxdydz$, for $D=\left\{x^2+y^2+z^2\leq R^2\right\}$, $f(x,y,z)=\dfrac{1}{\sqrt{a^2-x^2-y^2-z^2}}$ $(a>R>0)$,

Exercise 100 :

Compute the center of mass of an object that occupies the upper hemisphere of $x^2+y^2+z^2=1$ and has density $\rho(x,y)=x^2+y^2$.

Exercise 101 :
Compute the center of mass of an object that occupies the surface $z = xy$, $0 \leq x \leq 1$, $0 \leq y \leq 1$ and has density $\rho(x, y) = \sqrt{1 + x^2 + y^2}$.

Exercise 102 :
Compute the center of mass of an object that occupies the surface $z = \sqrt{x^2 + y^2}$, $1 \leq z \leq 4$ and has density $\rho(x, y) = x^2 z$.

Exercise 103 :
Compute the centroid of the surface of a right circular cone of height h and base radius R, not including the base.

CHAPTER III

VECTOR CALCULUS

In this chapter, we introduce the differential calculus in several variables.

1 Vectors in $\mathbb{R}^n$

1.1 Representation of Vectors in $\mathbb{R}^n$

We focus in this book on two and three-dimensional problems.
A vector is a magnitude in a certain direction. Vectors are used to represent forces, velocity, acceleration, and many other quantities. For example, the position of a particle in a the plane or in space can be given by a position vector $\overrightarrow{x}$. The velocity is by definition $\dfrac{d}{dt}\overrightarrow{x}$. Another example, the temperature at each point in the plane or the space, and we study the change of the temperature with position.

- an arrow starting from the origin.
- an arrow pointing in a certain direction with a certain length (magnitude).
- an arrow with a certain length and direction.
- the coordinates of a point relative to a reference point.
- a line segment oriented between two points.

Hence a vector in $\mathbb{R}^n$ is interpreted as an $n-$tuple, i.e. an $1 \times n$ matrix.
Consider the cartesian coordinate system in $\mathbb{R}^n$, M any point in $\mathbb{R}^n$ and O the origin, then the vector $\overrightarrow{OM}$ is called the position vector of the point M.

If $A = (x_1, \ldots, x_n)$ and $B = (y_1, \ldots, y_n)$ are two points, then the vector from A to B (represented by $\overrightarrow{AB}$) is defined by: $\overrightarrow{AB} = (y_1 - x_1, \ldots, y_n - x_n)$.
There is another standard way of representing vectors used in this book. In $\mathbb{R}^2$ we define

$$\overrightarrow{\mathbf{i}} = (1, 0), \quad \overrightarrow{\mathbf{j}} = (0, 1).$$

Then any vector in $\mathbb{R}^2$ can be written as

$$\overrightarrow{u} = (x, y) = x\,\overrightarrow{\mathbf{i}} + y\,\overrightarrow{\mathbf{j}}.$$

Similarly in $\mathbb{R}^3$ we define

$$\overrightarrow{\mathbf{i}} = (1, 0, 0), \quad \overrightarrow{\mathbf{j}} = (0, 1, 0), \quad \overrightarrow{\mathbf{k}} = (0, 0, 1).$$

Then any vector in $\mathbb{R}^3$ can be written as

$$\overrightarrow{u} = (x, y, z) = x\,\overrightarrow{\mathbf{i}} + y\,\overrightarrow{\mathbf{j}} + z\,\overrightarrow{\mathbf{k}}.$$

In $\mathbb{R}^2$, the vectors $\{\overrightarrow{\mathbf{i}}, \overrightarrow{\mathbf{j}}\}$ define the standard basis of $\mathbb{R}^2$. Similarly in $\mathbb{R}^3$ the vectors $\{\overrightarrow{\mathbf{i}}, \overrightarrow{\mathbf{j}}, \overrightarrow{\mathbf{k}}\}$ are the standard basis of $\mathbb{R}^3$.

1.2 The Dot (or the Inner) Product

Definition 1.1

1. In $\mathbb{R}^n$, if $\overrightarrow{u} = (x_1, \ldots, x_n)$ and $\overrightarrow{v} = (y_1, \ldots, y_n)$, the dot product of $\overrightarrow{u}$ and $\overrightarrow{v}$ is the number denoted by

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle = \sum_{k=1}^{n} x_k y_k.$$

   The dot product is also denoted by: $\overrightarrow{u}.\overrightarrow{v}$.
   For $n = 2$, $\overrightarrow{u} = (x, y)$ and $\overrightarrow{v} = (x', y')$, $\langle \overrightarrow{u}, \overrightarrow{v} \rangle = xx' + yy'$.
   For $n = 3$, $\overrightarrow{u} = (x, y, z)$ and $\overrightarrow{v} = (x', y', z')$, $\langle \overrightarrow{u}, \overrightarrow{v} \rangle = xx' + yy' + zz'$.

2. If $\overrightarrow{u} = (x_1, \ldots, x_n)$, then $\langle \overrightarrow{u}, \overrightarrow{u} \rangle = \sum_{k=1}^{n} x_k^2 \geq 0$. The number $\sqrt{\sum_{k=1}^{n} x_k^2}$ is denoted by $\|\overrightarrow{u}\|$ and called the magnitude (or the length or the norm) of $\overrightarrow{u}$.

3. If $M = (x_1, \ldots, x_n)$ and $N = (y_1, \ldots, y_n)$ are two points in $\mathbb{R}^n$. The distance from M to N is the norm of the vector $\overrightarrow{MN} = \sum_{k=1}^{n}(y_k - x_k)^2$.

Example 47 :

1. The distance from the point $A = (1, -1, 2)$ and $N = (-1, 1, 0)$ is $\sqrt{(-1-1)^2 + (1+1)^2 + (0-2)^2} = 2\sqrt{3}$.
2. A sphere of radius r centered at (a, b, c) is defined as all points (x, y, z) which are distance r from the center, we get the equation of a sphere is $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$.
3. The equation $x^2+y^2+z^2+2x-3y = 0$ can be rewritten by completing the square as $(x+1)^2-1+(y-\frac{3}{2})^2-\frac{9}{4}+z^2 = 0$ or $(x+1)^2+(y-\frac{3}{2})^2+z^2 = \frac{13}{4}$, so it is a sphere of radius $\frac{\sqrt{13}}{2}$ centered at $(-1, \frac{3}{2}, 0)$.

Properties 1.2: (Properties of the Dot Product)

1. $\langle \overrightarrow{u}, \overrightarrow{v} \rangle = \langle \overrightarrow{v}, \overrightarrow{u} \rangle$, for all vectors $\overrightarrow{u}, \overrightarrow{v}$.
2. For all vectors $\overrightarrow{u}, \overrightarrow{v}$,

$$|\langle \overrightarrow{u}, \overrightarrow{v} \rangle| \leq \|\overrightarrow{u}\|.\|\overrightarrow{v}\|.$$

This inequality is called the Cauchy-Schwarz inequality.

Definition 1.3: The Cosine Law

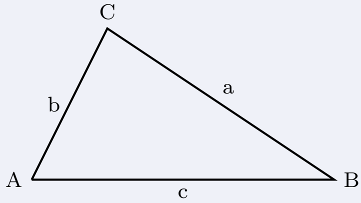
The angle θ between the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ is defined as follows,

$$\overrightarrow{u}.\overrightarrow{v} = \langle \overrightarrow{u}, \overrightarrow{v} \rangle = \|\overrightarrow{u}\| \, \|\overrightarrow{v}\| \cos\theta.$$

Corollary 1.4

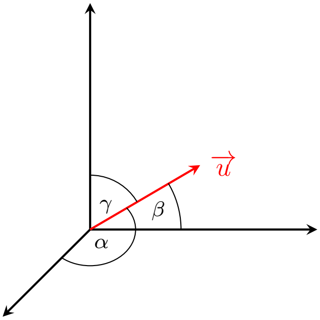
If ABC is a triangle in the plane, $a = \|\overrightarrow{BC}\|$, $b = \|\overrightarrow{AC}\|$, $c = \|\overrightarrow{AB}\|$ and $\theta = m\angle ACB$, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



Example 48 :

Consider the cartesian coordinate system in $\mathbb{R}^3$. A vector $\overrightarrow{u}$ makes angles with three axes



The direction angles associated to a vector $\overrightarrow{u}$ are given by: $\cos\alpha = \frac{\langle \overrightarrow{u}, \overrightarrow{\mathbf{i}} \rangle}{\|\overrightarrow{u}\|}$,
$\cos\beta = \frac{\langle \overrightarrow{u}, \overrightarrow{\mathbf{j}} \rangle}{\|\overrightarrow{u}\|}$, $\cos\gamma = \frac{\langle \overrightarrow{u}, \overrightarrow{\mathbf{k}} \rangle}{\|\overrightarrow{u}\|}$.

Definition 1.5

Two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ in $\mathbb{R}^n$ are called orthogonal if $\langle \overrightarrow{u}, \overrightarrow{v} \rangle = 0$.

Theorem 1.6: Pythagorean Theorem

If $\overrightarrow{u}$ and $\overrightarrow{v}$ are orthogonal in $\mathbb{R}^n$, then

$$\|\overrightarrow{u} + \overrightarrow{v}\|^2 = \|\overrightarrow{u}\|^2 + \|\overrightarrow{v}\|^2.$$

Solved Exercise 1 :

Let $\overrightarrow{u}$ and $\overrightarrow{v}$ two vectors with the same norm.
Prove that the vectors $\overrightarrow{u} + \overrightarrow{v}$ and $\overrightarrow{u} - \overrightarrow{v}$ are orthogonal.
Solution
$\langle \overrightarrow{u} + \overrightarrow{v}, \overrightarrow{u} - \overrightarrow{v} \rangle = \|u\|^2 - \|v\|^2 + \langle \overrightarrow{u}, \overrightarrow{v} \rangle - \langle \overrightarrow{u}, \overrightarrow{v} \rangle = 0.$

1.3 Projection and Component Along a Vector

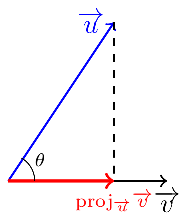
Definition 1.7

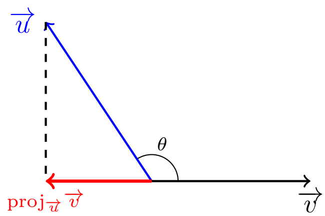
Let $\overrightarrow{u}$ and $\overrightarrow{v}$ be two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, the component of $\overrightarrow{u}$ along $\overrightarrow{v}$ is

$$\text{comp}_{\overrightarrow{v}}\overrightarrow{u} = \frac{\langle \overrightarrow{u}, \overrightarrow{v} \rangle}{\|\overrightarrow{v}\|} = \|\overrightarrow{u}\| \cos\theta,$$

where θ is the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$.
The projection of the vector $\overrightarrow{u}$ on the vector $\overrightarrow{v}$ is

$$(\text{comp}_{\overrightarrow{v}}\overrightarrow{u})\frac{1}{\|\overrightarrow{v}\|}\overrightarrow{v} = \frac{\langle \overrightarrow{u}, \overrightarrow{v} \rangle}{\|\overrightarrow{v}\|^2}\overrightarrow{v}.$$





Example 49 :

1. Consider the vectors $\overrightarrow{u} = (2, -1)$ and $\overrightarrow{v} = (1, 3)$. The projection of $\overrightarrow{u}$ onto $\overrightarrow{v}$ is the vector $\overrightarrow{w} = \frac{\langle \overrightarrow{u}, \overrightarrow{v} \rangle}{\|\overrightarrow{v}\|^2}\overrightarrow{v} = -\frac{1}{10}\overrightarrow{v}$.

2. Consider the vectors $\overrightarrow{u} = (-1, 1, 3)$ and $\overrightarrow{v} = (1, -1, -2)$. The projection of $\overrightarrow{u}$ onto $\overrightarrow{v}$ is the vector $\overrightarrow{w} = \frac{\langle \overrightarrow{u}, \overrightarrow{v} \rangle}{\|\overrightarrow{v}\|^2}\overrightarrow{v} = -\frac{4}{3}\overrightarrow{v}$.

1.4 The Cross Product

Definition 1.8

If $\overrightarrow{u}_1 = (x_1, y_1, z_1)$ and $\overrightarrow{u}_2 = (x_2, y_2, z_2)$, then the cross product of $\overrightarrow{u}_1$ and $\overrightarrow{u}_2$ is the vector

$$\overrightarrow{u}_1 \wedge \overrightarrow{u}_2 = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \overrightarrow{\mathbf{i}} + \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \overrightarrow{\mathbf{j}} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \overrightarrow{\mathbf{k}}.$$

Remark 4 :

1. The vector $\overrightarrow{u}_1 \wedge \overrightarrow{u}_2$ is orthogonal to the vectors $\overrightarrow{u}_1$ and $\overrightarrow{u}_2$ and its direction is given by the right-hand rule i.e. the determinant $|\overrightarrow{u}_1, \overrightarrow{u}_2, \overrightarrow{u}_1 \wedge \overrightarrow{u}_2|$ is non negative.

2. $\|\overrightarrow{u}_1 \wedge \overrightarrow{u}_2\|$ is the area of the parallelogram spanned by $\overrightarrow{u}_1$ and $\overrightarrow{u}_2$, i.e.,
$$\|\overrightarrow{u}_1 \wedge \overrightarrow{u}_2\| = \|\overrightarrow{u}_1\| \, \|\overrightarrow{u}_2\| \, \sin\theta.$$

3. Two vectors $\overrightarrow{u}_1$ and $\overrightarrow{u}_2$ are parallel if and only if $\overrightarrow{u}_1 \wedge \overrightarrow{u}_2 = 0$.

4. $\|\overrightarrow{u}_1 \wedge \overrightarrow{u}_2\|^2 = \|\overrightarrow{u}_1\|^2 \|\overrightarrow{u}_1\|^2 - \langle u_1, v_2\rangle^2$. Indeed $\|\overrightarrow{u}_1 \wedge \overrightarrow{u}_2\|^2 = \|\overrightarrow{u}_1\|^2 \|\overrightarrow{u}_2\|^2 \sin^2\theta = \|\overrightarrow{u}_1\|^2 \|\overrightarrow{u}_2\|^2 (1 - \cos^2\theta)$.

Solved Exercise 2 :
Compute the area of the triangle with vertices $(2, 3, -1)$, $(1, 3, 2)$, $(3, 0, -2)$.
Solution
Two sides are: $\overrightarrow{u} = (-1, 0, 3)$, $\overrightarrow{v} = (1, -3, -1)$,
$\overrightarrow{u} \wedge \overrightarrow{v} = (9, 2, 3)$, $\|\overrightarrow{u} \wedge \overrightarrow{v}\|^2 = 81 + 4 + 9 = 104 = 8.13$. The area of the triangle is $\sqrt{26}$.

Theorem 1.9: (Cross Product Properties)

Let $\overrightarrow{u}_1$, $\overrightarrow{u}_2$, and $\overrightarrow{u}_3$ be vectors and let c be a constant:

1. $\overrightarrow{u}_1 \wedge \overrightarrow{u}_2 = -\overrightarrow{u}_2 \wedge \overrightarrow{u}_1$;
2. $(c\overrightarrow{u}_1) \wedge \overrightarrow{u}_2 = c(\overrightarrow{u}_1 \wedge \overrightarrow{u}_2) = \overrightarrow{u}_1 \wedge (c\overrightarrow{u}_2)$;
3. $\overrightarrow{u}_1 \wedge (\overrightarrow{u}_2 + \overrightarrow{u}_3) = \overrightarrow{u}_1 \wedge \overrightarrow{u}_2 + \overrightarrow{u}_1 \wedge \overrightarrow{u}_3$;
4. $(\overrightarrow{u}_1 + \overrightarrow{u}_2) \wedge \overrightarrow{u}_3 = \overrightarrow{u}_1 \wedge \overrightarrow{u}_3 + \overrightarrow{u}_2 \wedge \overrightarrow{u}_3$;
5. $\langle \overrightarrow{u}_1, (\overrightarrow{u}_2 \wedge \overrightarrow{u}_3)\rangle = \langle (\overrightarrow{u}_1 \wedge \overrightarrow{u}_2), \overrightarrow{u}_3\rangle$;
6. $\overrightarrow{u}_1 \wedge (\overrightarrow{u}_2 \wedge \overrightarrow{u}_3) = (\langle \overrightarrow{u}_1, \overrightarrow{u}_3\rangle)\overrightarrow{u}_2 - (\langle \overrightarrow{u}_1, \overrightarrow{u}_2\rangle)\overrightarrow{u}_3$.

1.5 Scalar Triple Product

The scalar triple product of three vectors $\overrightarrow{u}_1$, $\overrightarrow{u}_2$, and $\overrightarrow{u}_3$ is the determinant
$$\langle \overrightarrow{u}_1, (\overrightarrow{u}_2 \wedge \overrightarrow{u}_3)\rangle = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

The volume of the parallelepiped formed by the vectors $\overrightarrow{u}_1$, $\overrightarrow{u}_2$, and $\overrightarrow{u}_3$ is

$$|\langle \overrightarrow{u}_1, (\overrightarrow{u}_2 \wedge \overrightarrow{u}_3)\rangle|.$$

The vectors $\overrightarrow{u}_1$, $\overrightarrow{u}_2$ and $\overrightarrow{u}_1$ are in the same plane if the scalar triple product $\langle \overrightarrow{u}_1, (\overrightarrow{u}_2 \wedge \overrightarrow{u}_3)\rangle$ is 0.

Example 50 :
Compute the volume of the parallelepiped spanned by the 3 vectors
$\overrightarrow{u}_1 = (2, 3, -1)$, $\overrightarrow{u}_2 = (1, 3, 2)$ and $\overrightarrow{u}_3 = (3, 0, -2)$.
$\overrightarrow{u}_2 \wedge \overrightarrow{u}_3 = (-6, 8, -9)$, $\langle \overrightarrow{u}_1, (\overrightarrow{u}_2 \wedge \overrightarrow{u}_3)\rangle = 21$.

Remark 5 :
$\langle \overrightarrow{u}_1, (\overrightarrow{u}_2 \wedge \overrightarrow{u}_3)\rangle = \langle (\overrightarrow{u}_1 \wedge \overrightarrow{u}_2), \overrightarrow{u}_3\rangle$.

1.6 Exercises

2 Line and Plane Parametrization

2.1 Lines

A line L in three-dimensional space is determined by
• A point $M_0 = (x_0, y_0, z_0)$ on the line
• A vector $\overrightarrow{v} = (a, b, c)$ that gives the direction of the line.
Any point M on the line can be expressed as $M_0 + t\overrightarrow{v}$ for some real number t called the parameter.

Line-Vector Equation
The parametrization $t \longmapsto M_0 + t\overrightarrow{v}$ is called the vector equation of a line L, where M_0 is a point on the line and $\overrightarrow{v}$ is the direction of the line.

Line-Parametric Equation
If $M_0 = (x_0, y_0, z_0)$ and $\overrightarrow{v} = (a, b, c)$, the equations

$$\begin{cases} x = x_0 + at, \\ y = y_0 + bt, \\ z = z_0 + ct \end{cases}$$

give the parametric equations for the line passing through M_0 and in direction of the vector $\overrightarrow{v}$.

Line Symmetric Equation
If we begin with the parametric equations of a line:

$$\begin{cases} x = x_0 + at, \\ y = y_0 + bt, \\ z = z_0 + ct \end{cases}$$

we can eliminate the parameter to get the symmetric equation of a line;

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Let $M_1 = (x_1, y_1, z_1), M_2 = (x_2, y_2, z_2)$ be two points on the space. The parametric equation of the line passing through M_1 to M_2 is the parametric equation of the line with M_1 on the line and the direction $\overrightarrow{M_1M_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.
The vector equation of the line is: $M(t) = M_1 + t\overrightarrow{M_1M_2}$. If $t \in [0,1]$, this equation in the is segment which goes from M_1 to M_2.

Theorem 2.1: [Distance from a Point to a Line]

The distance from a point M to a line passing through a point M_0 and parallel to a vector $\overrightarrow{u}$ is

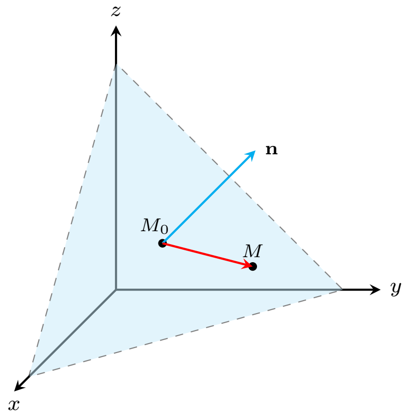
$$\frac{\|\overrightarrow{M_0M} \wedge \overrightarrow{u}\rangle\|}{\|\overrightarrow{u}\|}.$$

Solved Exercise 3 :
Find the distance from $M = (2, -3, 1)$ to the line containing $M_1 = (1, 3, -1)$ and $M_2 = (2, -1, 1)$.
Solution
$\overrightarrow{M_1M_2} = (1, -4, 2)$, $\overrightarrow{MM_1} = (1, -6, 2)$, $\overrightarrow{M_1M_2} \wedge \overrightarrow{MM_1} = (4, 0, -2)$. The distance is $\dfrac{\sqrt{20}}{\sqrt{21}}$.

2.2 Planes

In order to find the equation of a plane, we need:
• a point on the plane $M_0 = (x_0, y_0, z_0)$
• a vector that is orthogonal to the plane $\overrightarrow{n} = (a, b, c)$. This vector is called the normal vector the to plane.

Plane - Vector Equation
Any point M of the plane verifies $\langle\overrightarrow{M_0M}, \overrightarrow{u}\rangle = 0$. This is the vector equation of the plane.



Plane-Scalar Equation

The scalar (or component) equation of the plane is $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$.

Solved Exercise 4 :

Determine the equation of the plane that passes through the points $M_1 = (1,2,3)$, $M_2 = (3,2,1)$ and $M_3 = (-1,-2,2)$.

Solution

$\overrightarrow{M_1M_2} = (2,0,-2)$, $\overrightarrow{M_1M_3} = (-2,-4,-1)$, $\overrightarrow{M_1M_2} \wedge \overrightarrow{M_1M_3} = (-8,6,-8)$. The scalar equation of the plane is $-8(x-1)+6(y-2)-8(z-3) = 0 \iff 4x-3y+4z = 10$.

Remark 6 :

Two planes are parallel if and only if the normal vectors are parallel i.e. $\overrightarrow{n_1} \wedge \overrightarrow{n_2} = 0$.

Theorem 2.2: [Distance from a Point to a plane]

The distance between a point M and the plane passing through a point M_0 and normal to a vector $\overrightarrow{n}$ is

$$\frac{\left|\langle \overrightarrow{M_0M}, \overrightarrow{n} \rangle\right|}{\|\overrightarrow{n}\|}.$$

If $M_0 = (x_0,y_0,z_0)$, $M = (x,y,z)$ and $\overrightarrow{n} = (a,b,c)$, then

$$d(M,P) = \frac{|a(x-x_0)+b(y-y_0)+c(z-z_0)|}{\sqrt{a^2+b^2+c^2}}.$$

Solved Exercise 5 :

Find the distance from $M = (1, 2, 0)$ to the plane $2x - 3y + 2z = 1$. $M_1 = (-1, -1, 0)$, $\overrightarrow{n} = (2, -3, 2)$, $\overrightarrow{M_1M} = (2, 3, 0)$.

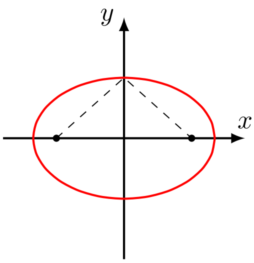
$$d(M, P) = \frac{5}{\sqrt{17}}.$$

2.3 Exercises

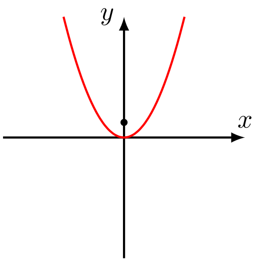
3 Curves and Surfaces

3.1 Quadratic Curves in $\mathbb{R}^2$

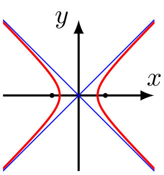
A quadratic curve is the graph of a second-degree equation in two variables taking one of the forms

1. 

The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with foci $(\pm c, 0)$, where $a^2 = b^2 + c^2$.

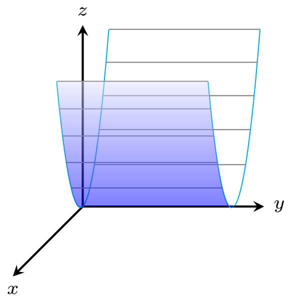
2. 

The parabola $x^2 = 4py$ with focus at $(0, p)$ and directrix at $y = -p$

3. 

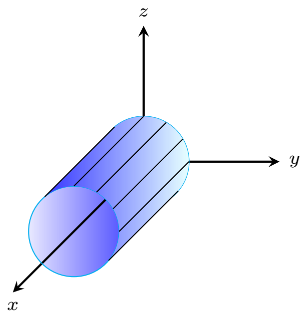
The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with foci at $(\pm c, 0)$ where $c^2 = a^2 + b^2$.

3.2 Surfaces in Space

1. 

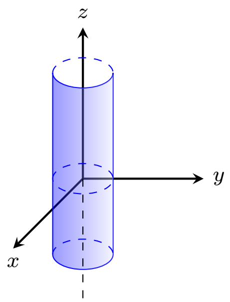
Cylinders which consist of all lines (called rulings) that are parallel to a given line and pass through a given plane curve

Example 51 :

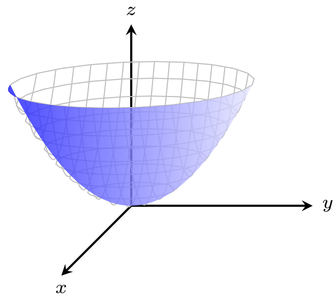


$$y^2 + z^2 = 1$$

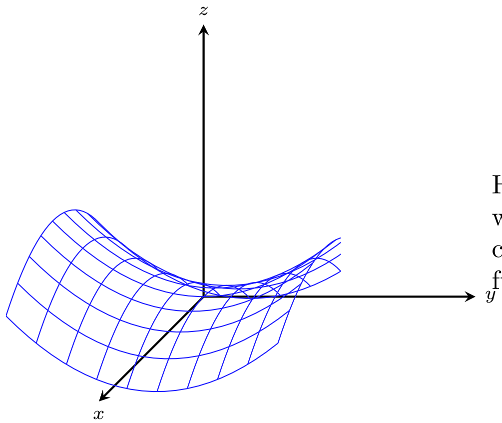
Example 52 :



The set of points (x, y, z) that satisfy the equation $x^2 + y^2 = 1$ is the cylinder of radius 1 centered at $(0, 0, 0)$ whose axis of symmetry is the z-axis.

2. 

Elliptic Paraboloids which will model functions with local maxima or minima

3. 

Hyperbolic Paraboloids ("saddles") which model a new kind of critical point, called a *saddle point*, for functions of two variables

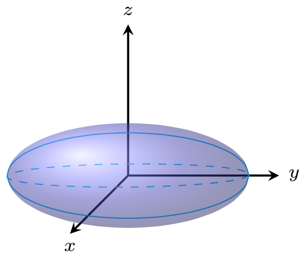
3.3 Quadric Surfaces in Space

A quadric surface is the graph of a second-degree equation in x, y, and z taking one of the standard forms

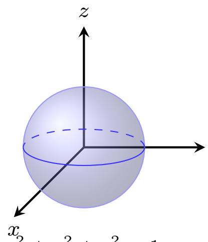
$$Ax^2 + By^2 + Cz^2 + D = 0, \quad Ax^2 + By^2 + Cz = 0.$$

We can graph a quadric surface by studying its traces in planes parallel to the x, y, and z axes. The traces are always quadratic curves.

1. The ellipsoid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$



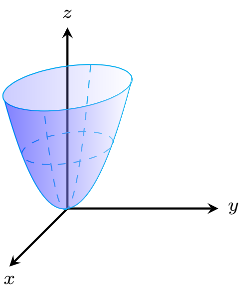
$x^2 + \frac{y^2}{4} + \frac{z^2}{3} = 1$



$x^2 + y^2 + z^2 = 1$

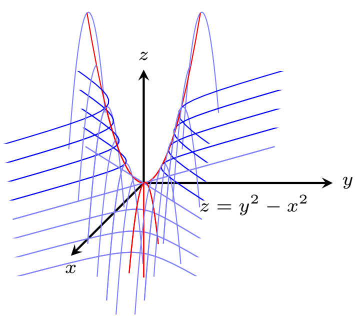
The set of points (x, y, z) that satisfy the equation $x^2 + y^2 + z^2 = 1$ is the sphere of radius 1 centered at $(0, 0, 0)$.

2. Elliptic Paraboloid $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

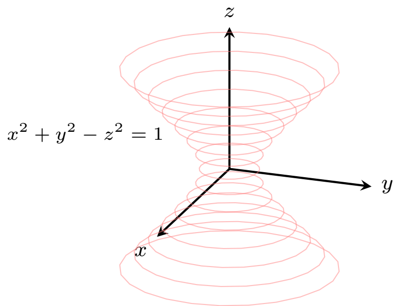


$z = x^2 + y^2$

3. Hyperbolic Paraboloid (Saddle) $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$



4. Hyperboloid of one sheet $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$



5. Cone: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = z^2$

   You can have similar equations with x, y, z permuted or with the origin shifted.

3.4 Exercises

4 Vector Functions and Space Curves

4.1 Vector-Valued Functions

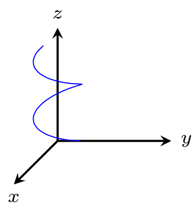
4.1.1 Continuity and Differentiability of Vector-Valued Functions

Definition 4.1

A vector-valued function is a function $\mathbf{r}(t)$ whose domain is a set of real numbers and whose range is a set of vectors in two- or three-dimensional space. We can specify $\mathbf{r}(t)$ through its component functions:

$$\mathbf{r}(t) = (f(t), g(t), h(t)) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

Example 53 :
$\mathbf{r}(t) = (\cos t, t, \sin t)$.



Definition 4.2

1. The limit of a vector-valued function is the limit of the component functions:
$$\lim_{t\to a}(x(t), y(t), z(t)) = \left(\lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t)\right).$$

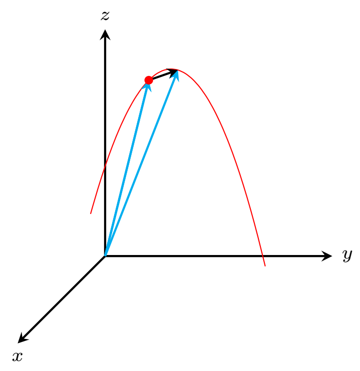
2. A vector-valued function $\mathbf{r}(t) = (x(t), y(t), z(t))$ is continuous if each of the component functions $x(t)$, $y(t)$, $z(t)$ is continuous.

3. A vector-valued function $\mathbf{r}(t) = (x(t), y(t), z(t))$ is differentiable if each of the component functions $x(t)$, $y(t)$, $z(t)$ is differentiable and we have $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$.
$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{h\to 0}\frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}.$$

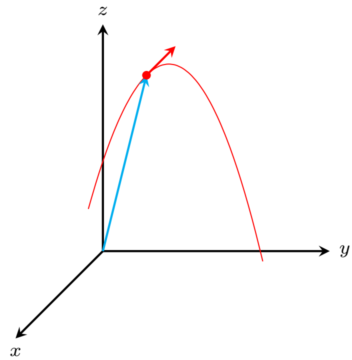
4. The integral of a vector-valued function $\mathbf{r}(t) = (x(t), y(t), z(t))$ on an interval $[a, b]$ is defined by:
$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt\right).$$

Remark 7 :

The vector $\dfrac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$ measures the displacement from t to $t+h$.



The vector $\mathbf{r}'(t)$ gives the instantaneous change in displacement The magnitude $|\mathbf{r}'(t)|$ gives instantaneous speed.



4.1.2 Tangent Lines

Consider the curve $\mathbf{r}(t) = (2t, e^{-t}, \cos t - t^2)$. $\mathbf{r}'(t) = (2, -e^{-t}, -\sin t - 2t)$ and $\mathbf{r}'(0) = (2, -1, 0)$. The parametric equations for the tangent line to the curve at $(0, 1, 1)$ is

$$\begin{cases} x = 2t, \\ y = 1 - t, \\ z = 1 \end{cases}$$

Definition 4.3

The unit tangent to $\mathbf{r}(t)$ is the vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

4.1.3 The Arc Length

Definition 4.4

The arc length of a plane curve $\mathbf{r}(t) = (x(t), y(t))$, $t \in [a, b]$ is

$$L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} dt = \int_a^b \|\mathbf{r}'(t)\| dt.$$

The arc length of a plane curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ $t \in [a, b]$ is

$$L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} dt = \int_a^b \|\mathbf{r}'(t)\| dt.$$

If $\mathbf{r}(t)$ is the space curve of a moving body and if t is time:

1. $\mathbf{r}'(t)$ is the velocity of the moving body
2. $\|\mathbf{r}'(t)'\|$ is the speed of the moving body
3. $\mathbf{r}''(t)$ is the acceleration of the moving body

Definition 4.5: (The Arc Length Function)

Let $\mathscr{C}$ be a space curve given by a vector function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

for $t \in [a, b]$.
the arc length function for $\mathscr{C}$ is defined by:

$$s(t) = \int_a^t \|\mathbf{r}'(u)\| du.$$

By the Fundamental Theorem of Calculus,

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|.$$

4.1.4 The Partial Derivatives

Let f be a function defined on a domain $D \subset \mathbb{R}^2$. For $(x_0, y_0) \in D$, the partial derivatives of f with respect to x and y if they exist are defined by:

$$\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = f_y(x_0, , y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Consider a smooth function $f\colon D \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$, the partial derivatives of f with respect to x, y and z if they exist are defined by:

$$\frac{\partial f}{\partial x}(x, y, z) = f_x(x, y, z) = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h};$$
$$\frac{\partial f}{\partial y}(x, y, z) = f_y(x, y, z) = \lim_{h \to 0} \frac{f(x, y+h, z) - f(x, y, z)}{h};$$
$$\frac{\partial f}{\partial z}(x, y, z) = f_z(x, y, z) = \lim_{h \to 0} \frac{f(x, y, z+h) - f(x, y, z)}{h}.$$

Theorem 4.6: (Schwarz's Theorem)

Let f be a function defined on a domain D that contains the point (a, b). If the functions $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ are both continuous on D, then

$$\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b).$$

4.1.5 The Directional Derivative

Let f be a function defined on a domain $D \subset \mathbb{R}^2$. For $(x_0, y_0) \in D$ and $\overrightarrow{u} = (a, b)$ a unit vector in $\mathbb{R}^2$. The directional derivative of f in the direction of $\overrightarrow{u}$ at (x_0, y_0) if it exists is

$$\begin{aligned} D_u f(x_0, y_0) &= \lim_{h \to 0} \frac{f((x_0, y_0) + hu) - f(x_0, y_0)}{h} \\ &= \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}. \end{aligned}$$

If f is a function defined on a domain $D \subset \mathbb{R}^3$. For $(x_0, y_0, z_0) \in D$ and $\overrightarrow{u} = (a, b, c)$ a unit vector in $\mathbb{R}^3$. The directional derivative of f in the direction of $\overrightarrow{u}$ at (x_0, y_0, z_0) if it exists is

$$\begin{aligned} D_u f(x_0, y_0, z_0) &= \lim_{h \to 0} \frac{f((x_0, y_0, z_0) + hu) - f(x_0, y_0, z_0)}{h} \\ &= \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh, z_0 + ch) - f(x_0, y_0, z_0)}{h}. \end{aligned}$$

Example 54 :

1. If $\overrightarrow{u} = (a, b)$, $D_u f(x_0, y_0)$ is the same as the derivative of $f(x_0+at, y_0+bt)$ at $t = 0$. We can compute this by the chain rule and get
$$D_u f(x_0, y_0) = af_x(x_0, y_0) + bf_y(x_0, y_0).$$
2. Find the directional derivative of $f(x, y) = xy^3 - x^2$ at $(1, 2)$ in the direction $\overrightarrow{u} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$
3. Find the directional derivative of $f(x, y) = x^2 \ln y$ at $(3, 1)$ in the direction of $\overrightarrow{u} = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$.

4.1.6 The Tangent Plane

The derivatives $\frac{\partial f}{\partial x}(a, b)$ and $\frac{\partial f}{\partial y}(a, b)$ define a tangent plane to the graph of f at $(a, b, f(a, b))$.
The differential of $z = f(x, y)$ is
$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Definition 4.7

If f has continuous partial derivatives, the tangent plane to $z = f(x, y)$ at $(a, b, f(a, b))$ is
$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

Chain Rule for Functions of One Variable If $y = f(u)$ and $u = u(x)$, then
$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx} = f'(u) . \frac{du}{dx}.$$

The Chain Rule for Two Variables If $z = f(x, y)$, $x = g(t)$, and $y = h(t)$, then
$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

If
$$z = f(x, y), x = g(s, t), y = h(s, t),$$
then
$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}, \qquad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t},$$

4.2 Exercises

5 Vector Fields

Definition 5.1

1. A two-dimensional transformation is a function f that maps each point (x,y) in a domain $\Omega \subset \mathbb{R}^2$ to a point $f(x,y) = (u(x,y), v(x,y))$ in $\mathbb{R}^2$.
2. A two-dimensional vector field is a function f that maps each point (x,y) in a domain $\Omega \subset \mathbb{R}^2$ to a two-dimensional vector $f(x,y) = u(x,y)\overrightarrow{\mathbf{i}} + v(x,y)\overrightarrow{\mathbf{j}}$, where $\overrightarrow{\mathbf{i}} = (1,0)$ and $\overrightarrow{\mathbf{j}} = (0,1)$.
3. A three-dimensional transformation is a function f that maps each point (x,y,z) in a domain $\Omega \subset \mathbb{R}^3$ to a point $f(x,y,z) = (u(x,y,z), v(x,y,z), w(x,y,z))$ in $\mathbb{R}^3$.
4. A three-dimensional vector field maps (x,y,z) in a domain $\Omega \subset \mathbb{R}^3$ to a three-dimensional vector $f(x,y,z) = u(x,y,z)\overrightarrow{\mathbf{i}} + v(x,y,z)\overrightarrow{\mathbf{j}} + w(x,y,z)\overrightarrow{\mathbf{k}}$, where $\overrightarrow{\mathbf{i}} = (1,0,0)$, $\overrightarrow{\mathbf{j}} = (0,1,0)$ and $\overrightarrow{\mathbf{k}} = (0,0,1)$.

Vector fields have many important applications, as they can be used to represent many physical quantities:

- Mechanics: the classical example is a gravitational field.
- Electricity and Magnetism: electric and magnetic fields.
- Fluid Mechanics: wind speed or the velocity of some other fluid.

If $\mathbf{r}(t) = x(t)\overrightarrow{\mathbf{i}} + y(t)\overrightarrow{\mathbf{j}} + z(t)\overrightarrow{\mathbf{k}}$ is the position vector field of an object. We can define various physical quantities associated with the object as follows:
velocity: $v(t) = \mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = x'(t)\overrightarrow{\mathbf{i}} + y'(t)\overrightarrow{\mathbf{j}} + z'(t)\overrightarrow{\mathbf{k}}$,
acceleration: $a(t) = v'(t) = \frac{dv}{dt} = \mathbf{r}''(t) = \frac{d^2\mathbf{r}}{dt^2} = x''(t)\overrightarrow{\mathbf{i}} + y''(t)\overrightarrow{\mathbf{j}} + z''(t)\overrightarrow{\mathbf{k}}$,
The norm $\|v(t)\|$ of the velocity vector is called the speed of the object.

Example 55 :

1. The gravitational force field between the Earth with mass M and a point particle with mass m is given by:

$$F(x,y,z) = -\frac{GmM}{(x^2+y^2+z^2)^{\frac{3}{2}}}(x\overrightarrow{\mathbf{i}} + y\overrightarrow{\mathbf{j}} + z\overrightarrow{\mathbf{k}}),$$

where G is the gravitational constant, and the (x, y, z) coordinates are chosen so that $(0, 0, 0)$ is the center of the Earth.

2. **The Electrostatic fields**:
In $3D$, $E = -\dfrac{q}{4\pi\varepsilon_0 \|\mathbf{r}\|^3}\mathbf{r}$.
In $2D$, $E = \dfrac{\rho}{2\pi\varepsilon_0 \|\mathbf{r}\|^2}\mathbf{r}$.

5.1 Gradient Fields

Let f be a scalar function of two variables, the gradient of f is defined by

$$\nabla f(x, y) = (\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y)).$$

If f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = (\frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z)).$$

The operator ∇ will be denoted by:
$\nabla = \frac{\partial}{\partial x}\overrightarrow{\mathbf{i}} + \frac{\partial}{\partial y}\overrightarrow{\mathbf{j}} + \frac{\partial}{\partial z}\overrightarrow{\mathbf{k}}$ or $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ as a vector.

Remark 8 :
Let f be a function. The vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface of f $S = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = C\}$ that contains (x_0, y_0, z_0).

Theorem 5.2

Consider f and g two smooth scalar functions $\mathbf{F} = (f_1, f_2, f_3)$ and $\mathbf{G} = (g_1, g_2, g_3)$ two smooth vector fields defined on a domain $\Omega \subset \mathbb{R}^3$. We have:

$$\begin{aligned}\nabla(fg) &= (\frac{\partial(fg)}{\partial x}, \frac{\partial(fg)}{\partial y}, \frac{\partial(fg)}{\partial z}) \\ &= f\nabla(g) + g\nabla(f).\end{aligned}$$

$$\begin{aligned}\nabla(\mathbf{F}.\mathbf{G}) &= \nabla(f_1g_1 + f_2g_2 + f_3g_3) \\ &= \nabla(f_1g_1) + \nabla(f_2g_2) + \nabla(f_3g_3) \\ &= f_1\nabla(g_1) + g_1\nabla(f_1) + f_2\nabla(g_2) + g_2\nabla(f_2) \\ &\quad +f_3\nabla(g_3) + g_3\nabla(f_3).\end{aligned}$$

Example 56 :

$$\mathbf{F} = \left(\frac{-x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right) = \nabla \frac{1}{\sqrt{x^2+y^2+z^2}}.$$

Definition 5.3: (Inverse square field)

Let $\mathbf{r}(x,y,z) = x\,\overrightarrow{\mathbf{i}} + y\,\overrightarrow{\mathbf{j}} + z\,\overrightarrow{\mathbf{k}}$ be the position vector of the point $M(x,y,z)$. The vector field $\mathbf{F}(x,y,z) = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r}(x,y,z)$ is called an inverse square field, where $c \in \mathbb{R}$.

5.2 The Divergence

Definition 5.4

The divergence of a vector field $F = (P, Q, R)$ is

$$\langle \nabla, F \rangle = \left\langle \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right), (P, Q, R) \right\rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

5.3 The Curl of a Vector Field

The curl of $F = (P, Q, R)$ is

$$\nabla \wedge F = \begin{vmatrix} \overrightarrow{\mathbf{i}} & \overrightarrow{\mathbf{j}} & \overrightarrow{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

If $F = P\,\overrightarrow{\mathbf{i}} + Q\,\overrightarrow{\mathbf{j}}$ is a two dimensional vector field, the curl $\nabla \wedge F$ can also be defined by regarding the $k-$component to be zero, i.e. $F = P\,\overrightarrow{\mathbf{i}} + Q\,\overrightarrow{\mathbf{j}} + 0\,\overrightarrow{\mathbf{k}}$, then $\mathrm{curl}F = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \overrightarrow{\mathbf{k}}$.

Here are two simple but useful facts about divergence and curl.

Theorem 5.5

$\langle \nabla, (\nabla \wedge F) \rangle = 0$. In other words, the divergence of the curl is zero.

Theorem 5.6

$\nabla \wedge (\nabla f) = 0$. That is, the curl of a gradient is the zero vector.

5.4 Exercises

6 Line Integral

6.1 Line Integral in Plane

Consider a plane curve given by the parametric equations

$$\gamma(t) = (x(t), y(t)), \quad t \in [a, b].$$

Definition 6.1

Let f be a continuous function on $\mathbb{R}^2$. If γ is continuously differentiable, the line integral of f on γ with respect to the arc length is defined by:

$$\int_a^b f \circ \gamma(t)\sqrt{(x'(t))^2 + (y'(t))^2}dt = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}dt.$$

Remark 9 :

1. If $f = 1$, $\displaystyle\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt$ is the length of γ.
   Note that $\sqrt{(x'(t))^2 + (y'(t))^2} = \|\gamma'(t)\|$. We denote $ds = \sqrt{(x'(t))^2 + (y'(t))^2}dt$.
2. The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

Example 57 : (Integrating along an arc of circle)
Consider the arc of circle C parametrized by $(\cos t, \sin t)$, with $t \in [0, \frac{\pi}{2}]$. In this case $ds = \sqrt{\cos^2 t + \sin^2 t dt} = dt$

$$\begin{aligned}
\int_C (x + 4xy^2)ds &= \int_0^{\frac{\pi}{2}} (\cos t + 4\cos t \sin^2 t)dt \\
&= \int_0^{\frac{\pi}{2}} \cos t(1 + 4\sin^2 t)dt \\
&\overset{u=\cos t}{=} \int_0^1 (1 + 4u^2)du = \frac{7}{3}.
\end{aligned}$$

Definition 6.2

Let f be a continuous function on $\mathbb{R}^2$ and let γ be piecewise-smooth curve, that is, γ is a union of a finite number of smooth curves $\gamma_1, \ldots, \gamma_k$, such that the initial point of γ_{j+1} is the terminal point of γ_j. Then we define the integral of a continuous function f along γ with respect to the arc length by:

$$\int_\gamma f(x,y)ds = \sum_{j=1}^{k} \int_{\gamma_j} f(x,y)ds.$$

Definition 6.3: [Center of mass of a wire]

If $\rho(x,y)$ is the linear density at a point (x,y) of a thin wire shaped like a curve $\gamma\colon [a,b] \longrightarrow \mathbb{R}^2$. The mass of the thin is

$$m = \int_a^b \rho(\gamma(t))\|\gamma'(t)\|dt$$

and the center of mass of the thin

$$(x_0, y_0) = \left(\int_a^b x(t)\rho(\gamma(t))\|\gamma'(t)\|dt, \int_a^b y(t)\rho(\gamma(t))\|\gamma'(t)\|dt \right).$$

Example 58 :
A wire takes the shape of an arc of circle $(\cos t, \sin t)$, with $t \in [0, \pi]$. If the density of the thin is $\rho(x,y) = x^2 + y^2$. Then the mass of the thin is

$$m = \int_0^\pi dt = \pi$$

and the center of mass of the this $\left(\int_0^\pi \cos t dt, \int_0^\pi \sin t dt \right) = (0, 2)$.

6.2 Line Integral in Space

Consider a space curve given by the parametric equations

$$\gamma(t) = (x(t), y(t), z(t)), \quad t \in [a, b].$$

Definition 6.4

Let f be a continuous function on $\mathbb{R}^3$. If γ is continuously differentiable, the line integral of f on γ with respect to the arc length is defined by:

$$\int_a^b f\circ\gamma(t)\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt = \int_a^b f(x(t),y(t),z(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt.$$

Remark 10 :

1. If $f=1$, $\displaystyle\int_a^b \sqrt{(x'(t))^2+(y'(t))^2}dt$ is the length of γ.
   Note that $\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2} = \|\gamma'(t)\|$ and we denote $ds = \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}dt$.
2. The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

Example 59 :

Consider the curve γ parametrized by $\gamma(t)=(\cos t,\sin t,1)$, with $t\in[0,\frac{\pi}{2}]$. In this case $ds=\sqrt{\cos^2 t+\sin^2 t}dt = dt$

$$\begin{aligned}
\int_C (2xz+5xy^2+z)ds &= \int_0^{\frac{\pi}{2}} (2\cos t+5\cos t\sin^2 t+1)dt \\
&= \frac{\pi}{2}+\int_0^{\frac{\pi}{2}} \cos t(2+5\sin^2 t)dt \\
\tfrac{\pi}{2}+ &\overset{u=\cos t}{=} \int_0^1 (2+5u^2)du = \frac{\pi}{2}+\frac{11}{3}.
\end{aligned}$$

Definition 6.5

Let f be a continuous function on $\mathbb{R}^3$ and let γ be piecewise-smooth curve, that is, γ is a union of a finite number of smooth curves $\gamma_1,\dots,\gamma_k$, such that the initial point of γ_{j+1} is the terminal point of γ_j. Then we define the integral of a continuous function f along γ with respect to the arc length as

$$\int_\gamma f(x,y,z)ds = \sum_{j=1}^k \int_{\gamma_j} f(x,y,z)ds.$$

Definition 6.6

Let f be a continuous function on $D \subset \mathbb{R}^3$ and let C be piecewise-smooth curve on D parametrized by $(x(t), y(t), z(t)),\ t \in [a, b]$:

1. The line integral of $f(x, y, z)$ with respect to x along the oriented curve C is written $\displaystyle\int_C f(x, y, z)dx$ and defined by:

$$\int_C f(x, y, z)dx = \int_a^b f(x(t), y(t), z(t))x'(t)dt$$

2. The line integral of $f(x, y, z)$ with respect to y along the oriented curve C is written $\displaystyle\int_C f(x, y, z)dy$ and defined by:

$$\int_C f(x, y, z)dy = \int_a^b f(x(t), y(t), z(t))y'(t)dt$$

Definition 6.7

1. The line integral of $f(x, y, z)$ with respect to z along the oriented curve C is written $\displaystyle\int_C f(x, y, z)dz$ and defined by:

$$\int_C f(x, y, z)dz = \int_a^b f(x(t), y(t), z(t))z'(t)dt$$

6.3 Work of a Force Field

If $F = (f, g, h)$ is a force field defined on a domain $D \subset \mathbb{R}^3$ and let C be piecewise-smooth curve on D parametrized by $(x(t), y(t), z(t)),\ t \in [a, b]$: The work of F along the curve C is defined by:

$$\begin{aligned} W &= \int_a^b f(x(t),y(t),z(t))x'(t)dt + \int_a^b g(x(t),y(t),z(t))y'(t)dt \\ &\quad + \int_a^b h(x(t),y(t),z(t))z'(t)dt \\ &= \int_a^b \langle F \circ C(t), C'(t) \rangle dt. \end{aligned}$$

$\int_a^b \langle F \circ C(t), C'(t) \rangle dt$ is denoted also $\int_C F(x,y,z).dr$

7 Independence of Path and Conservative Vector Field

Definition 7.1

We say that the line integral $\int_C F.d\mathbf{r}$ is independent of path in the domain D if the integral is the same for every path contained in D that has the same beginning and ending points.

Theorem 7.2

Let $F = (f,g,h)$ be a continuous vector field defined on a connected region D and let C be a smooth parametric curve on D parameterized by $C(t) = (x(t),y(t),z(t))$, $t \in [a,b]$.
The integral

$$\begin{aligned} \int_C F.d\mathbf{r} &= \int_a^b f(x(t),y(t),z(t))x'(t)dt + \int_a^b f(x(t),y(t),z(t))x'(t)dt \\ &\quad \int_a^b f(x(t),y(t),z(t))x'(t)dt \end{aligned}$$

is independent of the path if and only if F is conservative.

7.1 Independence of Path

Theorem 7.3: (Fundamental Theorem of Line Integrals)

Consider a smooth parametric curve C parameterized by a smooth vector function $C(t) = (x(t), y(t), z(t))$, $t \in [a, b]$. If f is a continuously differentiable function on a domain containing the curve C, then $\displaystyle\int_C \nabla f.d\mathbf{r} = f(C(b)) - f(C(a))$.

In particular, if the curve is closed, (i.e. $C(b) = C(a)$), then $\displaystyle\int_C \nabla f.d\mathbf{r} = 0$.

Example 60 :
Consider the vector field $F(x, y) = (2xy - 3, x^2 + 4y^3 + 5)$.
The line integral $\displaystyle\int_C F.d\mathbf{r}$ is independent of path. Then, evaluate the line integral for any curve C with initial point at $(-1, 2)$ and terminal point at $(2, 3)$.
$F = \nabla f$, $\frac{\partial f}{\partial x} = 2xy - 3$, $f = x^2y - 3x + g(y)$, $\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 + 4y^3 + 5$.
Then $f = x^2y - 3x + y^4 + 5y$.
$\displaystyle\int_C F.d\mathbf{r} = f(2, 3) - f(-1, 2) = 102 - 31 = 71$.

7.2 Conservative Vector Fields

Let $F(x, y) = (M(x, y), N(x, y))$, where we assume that $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives on an open, simply-connected region $D \subset \mathbb{R}^2$. The following five statements are equivalent, meaning that for a given vector field, either all five statements are true or all five statements are false.

1. $F(x, y)$ is conservative on D.
2. $F(x, y)$ is a gradient field in D (i.e., $F(x, y) = \nabla f(x, y)$, for some potential function f, for all $(x, y) \in D$).
3. $\displaystyle\int_C F.d\mathbf{r}$ is independent of path in D.
4. $\displaystyle\int_C F.d\mathbf{r} = 0$ for every piecewise-smooth closed curve C lying in D.
5. $\dfrac{\partial M}{\partial y}(x, y) = \dfrac{\partial N}{\partial x}(x, y)$, for all $(x, y) \in D$.

Theorem 7.4

Consider a simple connected region D and let F be a vector field defined on D.
The following properties of a vector field F are equivalent:

1. F is conservative.
2. $\displaystyle\int_C F.d\mathbf{r}$ is path-independent, (i.e. meaning that it only depends on the endpoints of the curve C.
3. $\displaystyle\oint_C F.d\mathbf{r} = 0$ around any closed smooth curve C in D.

8 Green's Theorem

Theorem 8.1: (Green's Theorem)

Let γ be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by γ. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_\gamma P(x,y)dx + Q(x,y)dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

Remark 11 :
The notation $\oint_\gamma P(x,y)dx + Q(x,y)dy$ is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve. The Green's Theorem can be written as

$$\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{\partial D} P(x,y)dx + Q(x,y)dy$$

where ∂D is the positively oriented boundary curve of D.

Example 61 :
Consider the curve defined by the boudary of the triangle Δ of vertices $(0,0),(1,0),(0,1)$.
Use Green's Theorem to calculate a line integral $\displaystyle\int_\gamma x^2ydx + xy^2dy$.

$$\begin{aligned}\int_\gamma x^2ydx + xy^2dy &= \int_\Delta \left(y^2 - x^2\right) dxdy \\ &= \int_0^1 \left(\int_0^{1-x} (y^2 - x^2)dy\right) dx = 0.\end{aligned}$$

Example 62 :
Consider the curve defined by the circle C defined by $x^2 + y^2 = 9$. Use Green's Theorem to calculate a line integral $\int_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4 + 1})dy$.

$$\begin{aligned}\int_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4 + 1})dy &= \int_D (7 - 3)dxdy \\ &= 36\pi.\end{aligned}$$

Remark 12 :
Another application of Green's Theorem is in computing areas. Since the area of D is $\int\int_D dxdy$, we wish to choose P and Q so that $(\frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y}) = 1$. Hence the area of D id

$$A = \oint_{\partial D} xdy = -\oint_{\partial D} ydx = \frac{1}{2}\oint_{\partial D} (xdy - ydx).$$

For example the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. A paramatrization of the ellipse E is $x(t) = a\cos t$, $y(t) = b\sin t$.

$$A = \frac{1}{2}\oint_E (xdy - ydx) = \frac{1}{2}\int_0^{2\pi} ab\cos^2 t + ab\sin^2 tdt = \pi ab.$$

8.1 Exercises

Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

Exercise 104 :
$\int_C (xy^2dx + 2x^2ydy)$, where C is the triangle with vertices $(0,0)$, $(2,2)$, and $(2,4)$.

Solution to Exercise 87:
$\int_C (xy^2dx + 2x^2ydy) = \int_0^2 \int_x^{2x} (2xy)dydx = \int_0^2 3x^3dx = 12.$

Exercise 105 :

$\int_C (\cos y dx + x^2 \sin y dy)$, where C is the rectangle with vertices $(0,0)$, $(5,0)$, and $(5,2)$.

Solution to Exercise 88:

$\int_C (\cos y dx + x^2 \sin y dy) = \int_0^5 \int_0^2 (2x+1) \sin y dy dx = 30(1-\cos 2)$.

Exercise 106 :

$\int_C (xe^{-2x} dx + (x^4 + 2x^2y^2) dy)$, where C is the boundary of the region between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Solution to Exercise 89:

$$\begin{aligned}\int_C (xe^{-2x} dx + (x^4 + 2x^2y^2) dy) &= \int_1^2 \int_0^{2\pi} (4r^3 \cos^3\theta + 4r^3 \cos\theta \sin^2\theta) r dr d\theta \\ &= 4 \int_1^2 r^4 \int_0^{2\pi} \cos\theta dr d\theta = 0.\end{aligned}$$

9 Surface Integrals

Theorem 9.1: (Evaluation Theorem)

Consider a surface S in $\mathbb{R}^3$ defined by $z = g(x,y)$ for (x,y) on a region $R_{x,y} \subset \mathbb{R}^2$, where g has continuous first partial derivatives, then

$$\iint_S f(x,y,z) dS = \iint_{R_{x,y}} f(x,y,g(x,y)) \sqrt{1+g_x^2+g_y^2} dA,$$

where $g_x = \dfrac{\partial g}{\partial x}$ and $g_y = \dfrac{\partial g}{\partial y}$.

Example 63 :

Evaluate the integral $\iint_S f(x,y,z) dS$, where $f(x,y,z) = x^2 + yz$ and S the upper half sphere $x^2+y^2+z^2 = R^2$.

$$\begin{aligned}\iint_S f(x,y,z) dS &= \iint_{D(0,R)} \left(x^2 + y\sqrt{R^2-x^2-y^2}\right) \sqrt{1 + \frac{x^2}{R^2-x^2-y^2} + \frac{y^2}{R^2-x^2-y^2}} dA \\ &= \int_0^{2\pi} \int_0^R \left(r^2 \cos^2\theta + r \sin\theta \sqrt{R^2-r^2}\right) \frac{Rr}{\sqrt{R^2-r^2}} dr d\theta \\ &= R \int_0^{2\pi} \int_0^R \frac{r^3}{\sqrt{R^2-r^2}} \cos^2\theta dr d\theta = \frac{2\pi}{3} R^4.\end{aligned}$$

10 Flux Integrals

Definition 10.1

A surface S is called orientable if a unit normal vector $\mathbf{n}$ can be defined at every non boundary point of S and $\mathbf{n}$ is continuous over the surface. For a surface defined by $f(x,y,z)=c$,

$$\mathbf{n} = \pm\frac{\nabla f}{\|\nabla f\|}.$$

In particular if the surface is defined by $z=g(x,y)$, $\nabla f = (-g_x, -g_y, 1)$, $dS = \sqrt{1+g_x^2+g_y^2}$, $\mathbf{n}dS = \nabla f dA$.

10.1 Flux of a Vector Field

Consider $\mathbf{F}$ a vector field which can represents the velocity of some fluid in the space. The flux of the fluid across S measures how much fluid is passing through the surface S.

Consider the unit normal vector $\mathbf{n}$ to the surface at a point, the number $\mathbf{F}.\mathbf{n}$ represents the scalar projection of $\mathbf{F}$ onto the direction of $\mathbf{n}$. So it measures how fast the fluid is moving across the surface. Thus, the total flux across S is $\displaystyle\int_S \mathbf{F}.\mathbf{n}dS$.

Theorem 10.2

Let $\mathbf{F}(x,y,z) = M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$ be a continuous vector field defined on an oriented surface S defined by $z=g(x,y)$ on a region $R_{x,y}$. The surface integral of $\mathbf{F}$ over S (or the flux of $\mathbf{F}$ over S) is:

$$\int_S \mathbf{F}.\mathbf{n}dS = \iint_{R_{x,y}} (-Mg_x - Ng_y + P)dA$$

if the surface is oriented upward and

$$\int_S \mathbf{F}.\mathbf{n}dS = \iint_{R_{x,y}} (Mg_x + Ng_y - P)dA$$

if the surface is oriented downward.

Example 64 :
Compute the flux of the vector field $\mathbf{F}(x, y, z) = (x, y, 0)$ over the portion of the paraboloid $z = x^2 + y^2$ below $z = 4$ (oriented with upward-pointing normal vectors).
Solution First, observe that at any given point, the normal vectors for the paraboloid $z = x^2 + y^2$ are $\pm(2x, 2y, -1)$. For the normal vector to point upward, we need a positive $z-$component. In this case,

$$u = -(2x, 2y, -1) = (-2x, -2y, 1)$$

is such a normal vector. A unit vector pointing in the same direction as u is then

$$\mathbf{n} = \frac{1}{\sqrt{4x^2 + 4y^2 + 1}}(-2x, -2y, 1).$$

We have $dS = \|u\| dA = \sqrt{4x^2 + 4y^2 + 1}dA$. Then

$$\begin{aligned}
\iint_S \mathbf{F}.\mathbf{n} dS &= \iint_R (x, y, 0).\frac{(-2x, -2y, 1)}{\sqrt{4x^2 + 4y^2 + 1}}\sqrt{4x^2 + 4y^2 + 1}dA \\
&= \iint_R (x, y, 0).(-2x, -2y, 1)dA = \iint_R (-2x^2 - 2y^2)dA.
\end{aligned}$$

The region $R_{x,y}$ is the disc $D(0, 2)$, then

$$\iint_S \mathbf{F}.\mathbf{n} dS = \int_0^{2\pi}\int_0^2 -2r^3 dr d\theta = -16\pi.$$

10.2 Exercises

Exercise 107 :
Evaluate $\int_D (2, -3, 4).\mathbf{n} dS$, where D is given by $z = x^2 + y^2$, $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, oriented up.

Exercise 108 :
Evaluate $\int_D (x, y, 3).\mathbf{n} dS$, where D is given by $z = 3x - 5y$, $1 \leq x \leq 2, 0 \leq y \leq 2$, oriented up.

Exercise 109 :
Evaluate $\int_D (x, y, -2).\mathbf{n} dS$, where D is given by $z = 1 - x^2 - y^2$, $x^2 + y^2 \leq 1$, oriented up.

Exercise 110 :
Evaluate $\int_D (xy, yz, zx).\mathbf{n} dS$, where D is given by $z = x + y^2 + 2$, $0 \leq x \leq 1, x \leq y \leq 1$, oriented up.

Exercise 111 :
Evaluate $\int_D (e^x, e^y, z).\mathbf{n}dS$, where D is given by $z = xy, 0 \leq x \leq 1, -x \leq y \leq x$, oriented up.

Exercise 112 :
Evaluate $\int_D (xz, yz, z).\mathbf{n}dS$, where D is given by $z = a^2 - x^2 - y^2$, $x^2 + y^2 \leq b^2$, oriented up.

Example 65 :
Compute the flux of $\mathbf{F} = (x, y, z^4)$ across the cone $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 1$, in the downward direction.

We write the cone as a vector function: $\gamma = (v\cos u, v\sin u, v)$, $0 \leq u \leq 2\pi$ and $0 \leq v \leq 1$. Then $\gamma_u = (-v\sin u, v\cos u, 0)$, $\gamma_v = (\cos u, \sin u, 1)$, and $\gamma_u \wedge \gamma_v = (v\cos u, v\sin u, -v)$. The third coordinate $-v$ is negative, which is exactly what we desire, that is, the normal vector points down through the surface.

Then

$$\begin{aligned}
&\int_0^{2\pi}\int_0^1 \langle (x, y, z^4), (v\cos u, v\sin u, -v)\rangle \, dv\, du \\
= &\int_0^{2\pi}\int_0^1 xv\cos u + yv\sin u - z^4 v\, dv\, du \\
= &\int_0^{2\pi}\int_0^1 v^2\cos^2 u + v^2\sin^2 u - v^5\, dv\, du \\
= &\int_0^{2\pi}\int_0^1 v^2 - v^5\, dv\, du = \frac{\pi}{3}.
\end{aligned}$$

11 The Divergence Theorem

Theorem 11.1: The Divergence Theorem

Let Q be a solid region bounded by a closed surface S oriented by a normal vector directed outward and if $\mathbf{F}$ is vector field $\mathscr{C}^1$. Then

$$\underbrace{\iint_S \mathbf{F}.\mathbf{n}dS}_{\substack{\text{total outward flux} \\ \text{over the interior}}} = \underbrace{\iiint_Q \nabla.\mathbf{F}dV}_{\substack{\text{integral of local flux} \\ \text{through the surface } S}} = \iiint_Q \operatorname{div}\mathbf{F}dV. \quad (11.1)$$

Example 66 :

Compute the outward flux $\iint_S \mathbf{F}.\mathbf{n}dS$ of the vector field $\mathbf{F} = (yz-3x)\mathbf{i} + (x-2y)\mathbf{j} + (2+z^2)\mathbf{k}$ through S, which is the surface of the ellipsoid $2x^2+2y^2+z^2 = 8$ lying above the plane $z = 0$.
Solution The surface S is not closed (is not the boundary of the considered solid), so we cannot use divergence theorem.
Add a second surface S' so that $S \cup S'$ is a closed surface with interior D. We can take the surface S' the disc $x^2 + y^2 \leq 4$ in the $xy-$plane

$$\iint_S \mathbf{F}.\mathbf{n}dS + \iint_{S'} \mathbf{F}.\mathbf{n}'dS = \iiint_D \operatorname{div}\mathbf{F}dV.$$

Hence

$$\iint_S \mathbf{F}.\mathbf{n}dS = \iiint_D \operatorname{div}\mathbf{F}dV - \iint_{S'} \mathbf{F}.\mathbf{n}'dS.$$

$\operatorname{div}\mathbf{F} = -5 + 2z$,

$$\begin{aligned}
\iiint_D \operatorname{div}\mathbf{F}dV &= \iiint_D (-5+2z)dV \\
&= \int_0^{2\pi}\int_0^2\int_0^{\sqrt{8-2r^2}} (-5+2z)rdzdrd\theta \\
&= 2\pi\int_0^2 (-5(8-2r^2) + (8-2r^2)^2)rdr \\
&= 2\pi\int_0^2 (24r - 22r^3 + 4r^5)dr = 64\pi.
\end{aligned}$$

$$\begin{aligned}
-\iint_{S'} \mathbf{F}.\mathbf{n}'dS &= -\iint_{S'} \mathbf{F}.(-\mathbf{k})dS = \iint_{S'} (2+z^2)dS \\
&= \iint_{S'} 2dS = 8\pi.
\end{aligned}$$

Hence

$$\iint_S \mathbf{F}.\mathbf{n}dS = 64\pi + 8\pi = 72\pi.$$

Example 67 :

Use the Divergence Theorem to evaluate the surface integral $\iint\limits_S \mathbf{F}\cdot d\mathbf{S}$ of the vector field $\mathbf{F}(x,y,z) = \left(x^3, y^3, z^3\right)$, where S is the surface of a solid bounded by the cone $x^2+y^2-z^2 = 0$ and the plane $z = 1$.

Solution: Applying the Divergence Theorem, we can write:

$$\begin{aligned} I &= \iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \iiint\limits_G (\nabla\cdot\mathbf{F})\,dV \\ &= \iiint\limits_G \left[\frac{\partial}{\partial x}\left(x^3\right) + \frac{\partial}{\partial y}\left(y^3\right) + \frac{\partial}{\partial z}\left(z^3\right)\right] dxdydz \\ &= 3\iiint\limits_G \left(x^2+y^2+z^2\right) dxdydz. \end{aligned}$$

By changing to cylindrical coordinates, we have

$$\begin{aligned} I &= 3\iiint\limits_G \left(x^2+y^2+z^2\right) dxdydz \\ &= 3\int\limits_0^1 dz \int\limits_0^{2\pi} d\varphi \int\limits_0^z \left(r^2+z^2\right) rdr = 6\pi\int\limits_0^1 \left[\left(\frac{r^4}{4}+\frac{z^2r^2}{2}\right)\Bigg|_{r=0}^{z}\right] dz \\ &= 6\pi\int\limits_0^1 \frac{3z^4}{4} dz = \frac{9\pi}{2}\left[\left(\frac{z^5}{5}\right)\Bigg|_0^1\right] = \frac{9\pi}{10}. \end{aligned}$$

Example 68 :

Evaluate the surface integral $\iint\limits_S x^3dydz + y^3dxdz + z^3dxdy$, where S is the surface of the sphere $x^2+y^2+z^2=a^2$ that has upward orientation.

Solution: Using the Divergence Theorem, we can write:

$$\begin{aligned} I &= \iint\limits_S x^3dydz + y^3dxdz + z^3dxdy = \iiint\limits_G \left(3x^2+3y^2+3z^2\right) dxdydz \\ &= 3\iiint\limits_G \left(x^2+y^2+z^2\right) dxdydz. \end{aligned}$$

By changing to spherical coordinates, we have

$$\begin{aligned} I &= 3\iiint\limits_G \left(x^2+y^2+z^2\right) dxdydz = 3\iiint\limits_G r^2\cdot r^2\sin\theta drd\psi d\theta \\ &= 3\int\limits_0^{2\pi} d\psi \int\limits_0^{\pi} \sin\theta d\theta \int\limits_0^a r^4dr \\ &= 3\cdot 2\pi\cdot\left[\left(-\cos\theta\right)\big|_0^{\pi}\right]\cdot\left[\left(\frac{r^5}{5}\right)\Bigg|_0^a\right] = \frac{12\pi a^5}{5}. \end{aligned}$$

Example 69 :

Using the Divergence Theorem calculate the surface integral $\iint\limits_S \mathbf{F}\cdot d\mathbf{S}$ of the vector field $\mathbf{F}(x,y,z)=(2xy,8xz,4yz)$, where is the surface of tetrahedron with vertices $A=(0,0,0)$, $B=(1,0,0)$, $C=(0,1,0)$, $D=(0,0,1)$.

Solution: By Divergence Theorem,

$$\begin{aligned}
I &= \iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \iiint\limits_G (\nabla\cdot\mathbf{F})\,dV \\
&= \iiint\limits_G \left[\frac{\partial}{\partial x}(2xy)+\frac{\partial}{\partial y}(8xz)+\frac{\partial}{\partial z}(4yz)\right]dV \\
&= \iiint\limits_G (2y+0+4y)\,dxdydz = 6\iiint\limits_G ydxdydz.
\end{aligned}$$

$$\begin{aligned}
I &= 6\iiint\limits_G ydxdydz = 6\int_0^1 dx\int_0^{1-x} dy\int_0^{1-x-y} ydz \\
&= 6\int_0^1 dx\int_0^{1-x}(1-x-y)\,ydy = 6\int_0^1 dx\int_0^{1-x}\left[y(1-x)-y^2\right]dy \\
&= 6\int_0^1\left[\left((1-x)\frac{y^2}{2}-\frac{y^3}{3}\right)\Big|_{y=0}^{1-x}\right]dx \\
&= 6\int_0^1\left[\frac{(1-x)^3}{2}-\frac{(1-x)^3}{3}\right]dx \\
&= 6\cdot\frac{1}{6}\int_0^1(1-x)^3dx = \frac{1}{4}.
\end{aligned}$$

Example 70 :

Use the Divergence Theorem to evaluate the surface integral $\iint\limits_S \mathbf{F}\cdot d\mathbf{S}$ of the vector field $\mathbf{F}(x,y,z)=(x,y,z)$, where S is the surface of the solid bounded by the cylinder $x^2+y^2=a^2$ and the planes $z=-1$ and $z=1$.

Solution: Using the Divergence Theorem, we can have:

$$\begin{aligned}\iint\limits_S \mathbf{F}\cdot d\mathbf{S} &= \iiint\limits_G (\nabla\cdot\mathbf{F})\,dV \\ &= \iiint\limits_G \left[\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)\right] dxdydz \\ &= \iiint\limits_G (1+1+1)\,dxdydz = 3\iiint\limits_G dxdydz.\end{aligned}$$

By switching to cylindrical coordinates, we have

$$\begin{aligned} I &= 3\iiint\limits_G dxdydz = 3\int_{-1}^{1} dz\int_0^{2\pi} d\varphi\int_0^a r dr \\ &= 3\cdot 2\cdot 2\pi\cdot\left[\left(\frac{r^2}{2}\right)\Bigg|_0^a\right] = 6\pi a^2.\end{aligned}$$

11.1 Exercises

12 Stokes's Theorem

Theorem 12.1: (Stokes's Theorem)

Let S be an oriented, piecewise-smooth surface with unit normal vector $\mathbf{n}$, bounded by the simple closed, piecewise-smooth boundary curve $\mathscr{C}$ having positive orientation. Let $\mathbf{F}(x,y,z)$ be a vector field continuously differentiable in some open domain containing S. Then,

$$\oint_{\mathscr{C}} \mathbf{F}.d\mathbf{r} = \oint_{\mathscr{C}} \mathbf{F}.T ds = \iint_S \text{curl}F.\mathbf{n}dS.$$

$\mathbf{r}=(x,y,z)$ is the position vector, $d\mathbf{r}=(dx,dy,dz)$, the unit tangent vector to S at $\mathbf{r}=(x,y,z)$ is

$$T=\frac{dx}{ds}\vec{\mathrm{i}}+\frac{dy}{ds}\vec{\mathbf{j}}+\frac{dz}{ds}\vec{\mathbf{k}}.$$

Hence $d\mathbf{r}=dTds$.
If the surface S is defined by $z=g(x,y)$ on a region $R_{x,y}$, then $\iint_S \text{curl}F.\mathbf{n}dS =$

$\iint_{R_{x,y}} (-M_1 g_x - N_1 g_y + P_1) dA$, where $g_x = \dfrac{\partial g}{\partial x}$, $g_y = \dfrac{\partial g}{\partial y}$ and curl$F$ = (M_1, N_1, P_1).

Example 71 :

Use Stoke's Theorem to evaluate the line integral $\oint_{\mathscr{C}} (y+2z)\,dx + (x+2z)\,dy + (x+2y)\,dz$, where $\mathscr{C}$ is the curve formed by intersection of the sphere $x^2+y^2+z^2 = 1$ with the plane $x+2y+2z = 0$.

Solution: Let S be the circle cut by the sphere from the plane. Find the coordinates of the unit normal vector $\mathbf{n}$ to the surface S,

$\mathbf{n} = \dfrac{1\cdot\overrightarrow{\mathrm{i}} + 2\cdot\overrightarrow{\mathbf{j}} + 2\cdot\overrightarrow{\mathbf{k}}}{\sqrt{1^2+2^2+2^2}} = \dfrac{1}{3}\overrightarrow{\mathrm{i}} + \dfrac{2}{3}\overrightarrow{\mathbf{j}} + \dfrac{2}{3}\overrightarrow{\mathbf{k}}.$

In this case $P = y+2z$, $Q = x+2z$, $R = x+2y$. Hence, the curl of the vector $\mathbf{F}$ is

$$\begin{aligned} \nabla\wedge\mathbf{F} &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\overrightarrow{\mathrm{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\overrightarrow{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\overrightarrow{\mathbf{k}} \\ &= (2-2)\,\overrightarrow{\mathrm{i}} + (2-1)\,\overrightarrow{\mathbf{j}} + (1-1)\,\overrightarrow{\mathbf{k}} = \overrightarrow{\mathbf{j}}. \end{aligned}$$

Using Stoke's Theorem, we have

$$\begin{aligned} \oint_{\mathscr{C}} (y+2z)\,dx + (x+2z)\,dy + (x+2y)\,dz &= \iint_S (\nabla\wedge\mathbf{F})\cdot\mathbf{n}dS \\ &= \iint_S \overrightarrow{\mathbf{j}}\cdot\left(\frac{1}{3}\overrightarrow{\mathrm{i}} + \frac{2}{3}\overrightarrow{\mathbf{j}} + \frac{2}{3}\overrightarrow{\mathbf{k}}\right) dS \\ &= \frac{2}{3}\iint_S dS. \end{aligned}$$

As the sphere $x^2+y^2+z^2 = 1$ is centered at the origin and the plane $x+2y+2z = 0$ also passes through the origin, the cross section is the circle of radius 1. Hence the integral is

$$I = \frac{2}{3}\iint_S dS = \frac{2}{3}\cdot\pi\cdot 1^2 = \frac{2\pi}{3}.$$

Example 72 :

Use Stoke's Theorem to calculate the line integral

$$\oint_{\mathscr{C}} y^3dx - x^3dy + z^3dz.$$

The curve $\mathscr{C}$ is the intersection of the cylinder $x^2+y^2 = a^2$ and the plane $x+y+z = b$.

Solution

We suppose that S is the part of the plane cut by the cylinder. The curve $\mathscr{C}$ is oriented counterclockwise when viewed from the end of the normal vector $\mathbf{n}$ which has coordinates

$$\mathbf{n} = \frac{1 \cdot \overrightarrow{\mathrm{i}} + 1 \cdot \overrightarrow{\mathbf{j}} + 1 \cdot \overrightarrow{\mathbf{k}}}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}} \overrightarrow{\mathrm{i}} + \frac{1}{\sqrt{3}} \overrightarrow{\mathbf{j}} + \frac{1}{\sqrt{3}} \overrightarrow{\mathbf{k}}.$$

As $P = y^3$, $Q = -x^3$, $R = z^3$, we can write:

$$\begin{aligned}
\nabla \wedge \mathbf{F} &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \overrightarrow{\mathrm{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \overrightarrow{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \overrightarrow{\mathbf{k}} \\
&= -3\left(x^2 + y^2\right) \overrightarrow{\mathbf{k}}.
\end{aligned}$$

Applying Stoke's Theorem, we find:

$$\begin{aligned}
I &= \oint_{\mathscr{C}} y^3 dx - x^3 dy + z^3 dz \\
&= \iint_S (\nabla \wedge \mathbf{F}) \cdot \mathbf{n} dS = \iint_S (\nabla \wedge \mathbf{F}) \cdot \mathbf{n} dS \\
&= \iint_S \left(-3\left(x^2 + y^2\right) \overrightarrow{\mathbf{k}}\right) \cdot \left(\frac{1}{\sqrt{3}} \overrightarrow{\mathrm{i}} + \frac{1}{\sqrt{3}} \overrightarrow{\mathbf{j}} + \frac{1}{\sqrt{3}} \overrightarrow{\mathbf{k}}\right) dS \\
&= -\sqrt{3} \iint_S \left(x^2 + y^2\right) dS.
\end{aligned}$$

We can express the surface integral in terms of the double integral:

$$\begin{aligned}
I &= -\sqrt{3} \iint_S \left(x^2 + y^2\right) dS \\
&= -\sqrt{3} \iint_{D(0,a)} \left(x^2 + y^2\right) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy.
\end{aligned}$$

The equation of the plane is $z = b - x - y$, so the square root in the integrand is equal to

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + (-1)^2 + (-1)^2} = \sqrt{3}.$$

Hence,

$$I = -\sqrt{3} \iint_{D(0,a)} \left(x^2 + y^2\right) \sqrt{3} dxdy = -3 \iint_{D(x,y)} \left(x^2 + y^2\right) dxdy.$$

By changing to polar coordinates, we get

$$I = -3 \int_0^{2\pi} \int_0^a r^3 dr d\theta = -3 \cdot 2\pi \cdot \left.\frac{r^4}{4}\right|_0^a = -\frac{3\pi a^4}{2}.$$

Example 73 :
Use Stoke's Theorem to evaluate the line integral

$$\oint_{\mathscr{C}} (x+z)\,dx + (x-y)\,dy + xdz.$$

The curve $\mathscr{C}$ is the ellipse defined by the equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$, $z = 1$.
Solution:
Let the surface S be the part of the plane $z = 1$ bounded by the ellipse. Obviously that the unit normal vector is $\mathbf{n} = \mathbf{k}$. Since $P = x + z$, $Q = x - y$, $R = x$, then the curl of the vector field $\mathbf{F}$ is

$$\begin{aligned}
\nabla \wedge \mathbf{F} &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\overrightarrow{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\overrightarrow{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\overrightarrow{\mathbf{k}} \\
&= (1-0)\,\overrightarrow{\mathbf{k}} = \overrightarrow{\mathbf{k}}.
\end{aligned}$$

By Stoke's Theorem,

$$\begin{aligned}
\oint_{\mathscr{C}} (x+z)\,dx + (x-y)\,dy + xdz &= \iint_S (\nabla \wedge \mathbf{F}) \cdot \mathbf{n} dS \\
&= \iint_S (\nabla \wedge \mathbf{F}) \cdot \mathbf{n} dS \\
&= \iint_S \overrightarrow{\mathbf{k}} \cdot \overrightarrow{\mathbf{k}}\, dS = \iint_S dS.
\end{aligned}$$

The double integral in the latter formula is the area of the ellipse. Therefore, the integral is

$$\iint_S dS = \pi \cdot 2 \cdot 3 = 6\pi.$$

Example 74 :
Show that the line integral $\oint_{\mathscr{C}} yzdx + xzdy + xydz$ is zero along any closed contour $\mathscr{C}$.
Solution :
Let S be a surface bounded by a closed curve $\mathscr{C}$. Applying Stoke's formula, we identify that $P = yz$, $Q = xz$, $R = xy$.
Then

$$\begin{aligned}
\nabla \wedge \mathbf{F} &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\overrightarrow{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\overrightarrow{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\overrightarrow{\mathbf{k}} \\
&= (x-x)\,\overrightarrow{\mathbf{i}} + (y-y)\,\overrightarrow{\mathbf{j}} + (z-z)\,\overrightarrow{\mathbf{k}} = 0 \cdot \overrightarrow{\mathbf{i}} + 0 \cdot \overrightarrow{\mathbf{j}} + 0 \cdot \overrightarrow{\mathbf{k}} = \mathbf{0}.
\end{aligned}$$

Hence, the line integral:

$$\oint_{\mathscr{C}} yzdx + xzdy + xydz = \iint_S (\nabla \wedge \mathbf{F}) \cdot \mathbf{n} dS = \iint_S \mathbf{0} \cdot \mathbf{n} dS = 0.$$

12.1 Exercises

13 Curvature

Curvature measures the rate at which a space curve $\mathbf{r}(t)$ changes direction. The *direction* of curve is given by the unit tangent vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

which has length 1 and is tangent to $\mathbf{r}(t)$. The picture below shows the unit tangent vector $\mathbf{T}(t)$ to the curve $\mathbf{r}(t) = (2\cos(t), \sin(t))$ at several points.

Obviously, if $\mathbf{r}(t)$ is a straight line, the curvature is 0. Otherwise the curvature is non-zero. To be precise, curvature is defined to be the magnitude of the rate of change of the unit vector with respect to arc length:

$$\text{curvature} = \kappa = |\frac{dT}{ds}|$$

The reason that arc length comes into the definition is that arc length is independent of parameterization.

In most cases a curve is described by a particular parameterization and we have the unit tangent vector as a function of t: $\mathbf{T(t)}$. We can compute the curvature using the chain rule

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}$$

Recall that ds/dt=—**r**'(t)—. Hence, it follows that

$$\kappa = |\frac{d\mathbf{T}}{ds}| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

Example 75 :

What is the curvature of the ellipse $\mathbf{r}(t) = (2\cos(t), \sin(t))$? We have several computations to perform. First, we have

$$\mathbf{r}'(t) = (-2\sin t, \cos t),$$

and

$$|\mathbf{r}'(t)| = \sqrt{4\sin^2 t + \cos^2 t},$$

implying

$$\mathbf{T}(t) = (-\frac{2\sin t}{\sqrt{4\sin^2 t + 2\cos^2 t}}, \frac{\cos t}{\sqrt{4\sin^2 t + 2\cos^2 t}}).$$

Differentiating again, we have

$$\mathbf{T}'(t) = (-\frac{2\cos t}{(4\sin^2 t + \cos^2 t)^{3/2}}, -\frac{4\sin t}{(4\sin^2 t + \cos^2 t)^{3/2}})$$

Finally, we have

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{2}{(4\sin^2 t + \cos^2 t)^{3/2}}$$

When t=0 or pi, the denominator is at its minimum, implying the curvature is at a maximum. When t=pi or 3*pi/2, the curvature is at a maximum. This is verified in the plot of the ellipse above.

Curvature of a Circle of Radius r

A circle of radius r can be described by the parameterization $\mathbf{r}(t) = r(cost, sint)$. It can be shown that

$$\mathbf{r}'(t) = r(-\sin t, \cos t),$$

and

$$|\mathbf{r}'(t)| = r$$

implying

$$\mathbf{T}(t) = (-\sin t, \cos t),$$

Differentiating again, we have

$$\mathbf{T}'(t) = (-\cos t, -\sin t).$$

Finally, we have

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{a}$$

Alternative formula for the Curvature

It can be shown that the curvature is also given by the formula

$$\kappa = \frac{|\mathbf{r}'(t) \wedge \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$