Surface Integrals

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1 Surface Integrals
2 Flux Integrals

Theorem (Evaluation Theorem)

Consider a surface S in $\mathbb{R}^3$ defined by $z = g(x, y)$ for (x, y) on a region $R_{x,y} \subset \mathbb{R}^2$, where g has continuous first partial derivatives, then

$$\iint_S f(x, y, z)dS = \iint_{R_{x,y}} f(x, y, g(x, y))\sqrt{1 + g_x^2 + g_y^2}dA,$$

where $g_x = \dfrac{\partial g}{\partial x}$ and $g_y = \dfrac{\partial g}{\partial y}$.

Example

Evaluate the integral $\iint_S f(x, y, z)dS$, where $f(x, y, z) = x^2 + yz$ and S the upper half sphere $x^2 + y^2 + z^2 = R^2$.

$$\begin{aligned}
\iint_S f(x,y,z)dS &= \iint_{D(0,R)} \left(x^2 + y\sqrt{R^2 - x^2 - y^2}\right) \sqrt{1 + \frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y}} \\
&= \int_0^{2\pi} \int_0^R \left(r^2 \cos^2\theta + r\sin\theta\sqrt{R^2 - r^2}\right) \frac{Rr}{\sqrt{R^2 - r^2}} drd\theta \\
&= R\int_0^{2\pi} \int_0^R \frac{r^3}{\sqrt{R^2 - r^2}} \cos^2\theta drd\theta = \frac{2\pi}{3} R^4.
\end{aligned}$$

Definition

A surface S is called orientable if a unit normal vector $\mathbf{n}$ can be defined at every non boundary point of S and $\mathbf{n}$ is continuous over the surface.
For a surface defined by $f(x, y, z) = c$,

$$\mathbf{n} = \pm \frac{\nabla f}{\|\nabla f\|}.$$

In particular if the surface is defined by $z = g(x, y)$,
$\nabla f = (-g_x, -g_y, 1)$, $dS = \sqrt{1 + g_x^2 + g_y^2}$, $\mathbf{n} dS = \nabla f dA$.

Flux of a Vector Field

Consider **F** a vector field which can represents the velocity of some fluid in the space. The flux of the fluid across S measures how much fluid is passing through the surface S.
Consider the unit normal vector **n** to the surface at a point, the number $\mathbf{F}.\mathbf{n}$ represents the scalar projection of F onto the direction of **n**. So it measures how fast the fluid is moving across the surface. Thus, the total flux across S is $\displaystyle\int_S \mathbf{F}.\mathbf{n}dS$.

Theorem

Let $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a continuous vector field defined on an oriented surface S defined by $z = g(x, y)$ on a region $R_{x,y}$. The surface integral of F over S (or the flux of F over S) is:

$$\int_S F.ndS = \iint_{R_{x,y}} (-Mg_x - Ng_y + P)dA$$

if the surface is oriented upward and

$$\int_S \mathbf{F}.\mathbf{n}dS = \iint_{R_{x,y}} (Mg_x + Ng_y - P)dA$$

if the surface is oriented downward.

Example

Compute the flux of the vector field $\mathbf{F}(x, y, z) = (x, y, 0)$ over the portion of the paraboloid $z = x^2 + y^2$ below $z = 4$ (oriented with upward-pointing normal vectors).

Solution First, observe that at any given point, the normal vectors for the paraboloid $z = x^2 + y^2$ are $\pm(2x, 2y, -1)$. For the normal vector to point upward, we need a positive $z-$component. In this case,

$$u = -(2x, 2y, -1) = (-2x, -2y, 1)$$

is such a normal vector. A unit vector pointing in the same direction as u is then

$$\mathbf{n} = \frac{1}{\sqrt{4x^2 + 4y^2 + 1}}(-2x, -2y, 1).$$

We have $dS = \|u\| dA = \sqrt{4x^2 + 4y^2 + 1} dA$. Then

$$\begin{aligned}
\iint_S \mathbf{F}.\mathbf{n}dS &= \iint_R (x,y,0).\frac{(-2x,-2y,1)}{\sqrt{4x^2+4y^2+1}}\sqrt{4x^2+4y^2+1}dA \\
&= \iint_R (x,y,0).(-2x,-2y,1)dA = \iint_R (-2x^2-2y^2)dA.
\end{aligned}$$

The region $R_{x,y}$ is the disc $D(0,2)$, then

$$\iint_S \mathbf{F}.\mathbf{n}dS = \int_0^{2\pi}\int_0^2 -2r^3 dr d\theta = -16\pi.$$

Exercises

Exercise 1 :
Evaluate $\displaystyle\int_D (2,-3,4).\mathbf{n}dS$, where D is given by $z = x^2 + y^2$, $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, oriented up.

Exercise 2 :
Evaluate $\displaystyle\int_D (x, y, 3).\mathbf{n}dS$, where D is given by $z = 3x - 5y$, $1 \leq x \leq 2, 0 \leq y \leq 2$, oriented up.

Exercise 3 :
Evaluate $\displaystyle\int_D (x, y, -2).\mathbf{n}dS$, where D is given by $z = 1 - x^2 - y^2$, $x^2 + y^2 \leq 1$, oriented up.

Exercise 4 :
Evaluate $\displaystyle\int_D (xy, yz, zx).\mathbf{n}dS$, where D is given by $z = x + y^2 + 2$, $0 \leq x \leq 1, x \leq y \leq 1$, oriented up.

Exercise 5 :
Evaluate $\displaystyle\int_D (e^x, e^y, z).\mathbf{n}dS$, where D is given by $z = xy, 0 \leq x \leq 1, -x \leq y \leq x$, oriented up.

Exercise 6 :
Evaluate $\displaystyle\int_D (xz, yz, z).\mathbf{n}dS$, where D is given by $z = a^2 - x^2 - y^2$, $x^2 + y^2 \leq b^2$, oriented up.

Example

Compute the flux of $F = (x, y, z^4)$ across the cone $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 1$, in the downward direction.
We write the cone as a vector function: $\gamma = (v \cos u, v \sin u, v)$, $0 \leq u \leq 2\pi$ and $0 \leq v \leq 1$. Then $\gamma_u = (-v \sin u, v \cos u, 0)$, $\gamma_v = (\cos u, \sin u, 1)$, and $\gamma_u \times \gamma_v = (v \cos u, v \sin u, -v)$. The third coordinate $-v$ is negative, which is exactly what we desire, that is, the normal vector points down through the surface.

Then

$$
\begin{aligned}
& \int_0^{2\pi}\int_0^1 \langle (x, y, z^4), (v\cos u, v\sin u, -v)\rangle \, dv\, du \\
= \;& \int_0^{2\pi}\int_0^1 xv\cos u + yv\sin u - z^4 v \, dv\, du \\
= \;& \int_0^{2\pi}\int_0^1 v^2\cos^2 u + v^2\sin^2 u - v^5 \, dv\, du \\
= \;& \int_0^{2\pi}\int_0^1 v^2 - v^5 \, dv\, du = \frac{\pi}{3}.
\end{aligned}
$$