Surface Area

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March 25, 2024

1 Surface Area

Consider a surface S defined by $z = f(x, y)$, for (x, y) in a closed region $R \in \mathbb{R}^2$. We assume that $f(x, y) \geq 0$ and f is continuously differentiable. We assume also that no normal vector to S is parallel to the $xy-$plane. $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \neq (0, 0)$ for all $(x, y) \in R$. The surface area of S is

$$A = \iint_R \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2} dxdy.$$

Example

Consider the surface $z = 1 - x^2 - y^2$, for $z \geq 0$.
We have $\dfrac{\partial f}{\partial x} = -2x$ and $\dfrac{\partial f}{\partial y} = -2y$. Then

$$\begin{aligned} A &= \iint_{x^2+y^2<1} \sqrt{1+4x^2+4y^2}\,dxdy \\ &= \int_0^{2\pi}\int_0^1 r\sqrt{1+4r^2}\,drd\theta \\ &= 2\pi\frac{1}{12}\left[(1+4r^2)^{\frac{3}{2}}\right]_0^1 = \frac{\pi.(5^{\frac{3}{2}}-1)}{6} \end{aligned}$$

Example

Consider the sphere $S = \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 = R^2\}$.
The area of S is the double of the area of the surface of the upper half sphere $S' = \{(x, y, z) \in \mathbb{R}^3 : \ z = \sqrt{1 - x^2 - y^2}\}$.
We have $\dfrac{\partial z}{\partial x} = -\dfrac{x}{\sqrt{R^2 - x^2 - y^2}}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{y}{\sqrt{R^2 - x^2 - y^2}}$.
The area is

$$
\begin{aligned}
A &= 2\iint_{x^2+y^2<R^2} \sqrt{1 + \frac{x^2 + y^2}{R^2 - x^2 - y^2}}\,dxdy \\
&= 2R\int_0^{2\pi}\int_0^R \frac{r}{\sqrt{R^2 - r^2}}\,drd\theta = 4\pi R^2.
\end{aligned}
$$

Example

Consider the surface $S = \{(x, y, z) \in \mathbb{R}^3 :\ z = 13 - 4x^2 - 4y^2\}$ on the domain $z = 1$, $x > 0$ and $y < 0$.

We have $\dfrac{\partial z}{\partial x} = -8x$ and $\dfrac{\partial z}{\partial y} = -8y$. Then the area of S is

$$\begin{aligned}
A &= \iint_{x^2+y^2<3, x>0, y<0} \sqrt{1 + 16(x^2 + y^2)}dxdy \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{3}} \sqrt{1 + 16r^2} rdrd\theta \\
&= \frac{\pi}{2} \frac{1}{48} \left[(1 + 16r^2)^{\frac{3}{2}}\right]_0^{\sqrt{3}} = \frac{\pi}{2}.
\end{aligned}$$

Exercises

Exercise 1 :
Find the area of the surface S if S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that is inside the cylinder $x^2 + y^2 = 2y$.