Summary of Math-107

(All material only in 12 pages)

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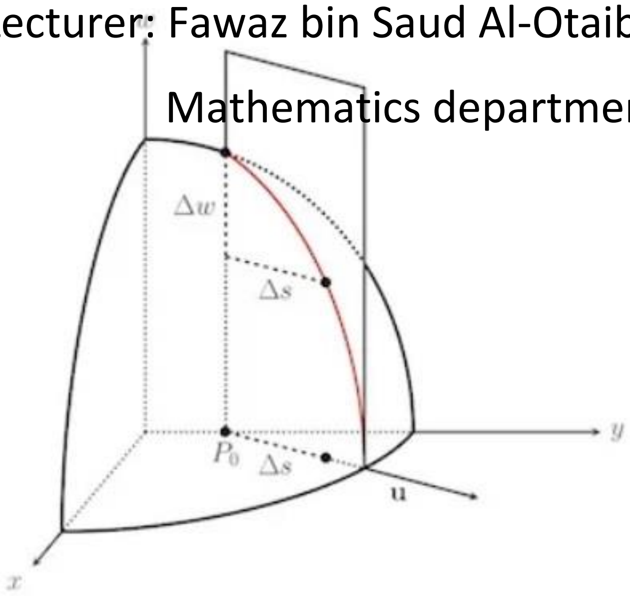


Directional derivatives for functions of two variables.

System of Linear equations:

Agumented matrix, Row Echelon, Reduced Row Echelon

Gauss-Gordan Elimination: فوق وتحت القطر أصفار

Gauss-Elimination: تحت القطر أصفار

trivial solution → كل المجاهيل تطلع أصفاراً

نظرية: A homogeneous linear system with more unknowns that equations has infinitely many solutions.

طريقة حل جاوس وجاوس-جوردان:

جاوس ① → ② → ...

$$\begin{bmatrix} 1 & & & \vdots \\ 0 & 1 & & \vdots \\ \vdots & 0 & & \vdots \\ 0 & 0 & & \vdots \end{bmatrix}$$

جاوس-جوردان

$$\begin{bmatrix} 1 & 0 & 0 & \vdots \\ 0 & 1 & 0 & \vdots \\ 0 & 0 & 1 & \vdots \\ \vdots & \vdots & 0 & \vdots \\ 0 & 0 & 0 & \vdots \end{bmatrix}$$

① → ② → ...

ثم نناقش طبيعة الحلول من خلال آخر صف ؛ المتغير الذي ليس في عموده "واحد متقدم" يكون هو الحر t or s , $t, s \in \mathbb{R}$

العمليات الصفية الأولية:

(i) $R_i \leftrightarrow R_j$

(ii) $aR_i + R_j$

(iii) aR_i

تستخدم في system: و المصفوفات والمحددات

Consistent system → النظام له حل سواءً وحيد أو عدد لانهائي من الحلول

inconsistent system → النظام ليس له حل

Homogenous system →
- فقط له الحل التافه الصفري
- له عدد لانهائي من الحلول

Matrices :

$$\underset{m\times r}{A}\ \underset{r\times n}{B}$$

$r = r \Rightarrow$ الضرب ممكن

الناتج من درجة $m\times n$

$tr(A) =$ مجموع عناصر قطر A

لـ مربعة.

$$I_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \quad I_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ldots$$

$$A\cdot I = A \quad , \quad I.A = A$$

If $A\cdot B = I$ or $BA = I \Rightarrow A$ is invertible (nonsingular) and $A^{-1} = B$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$A^n = \underbrace{A.A\ldots A}_{n \text{ مرة}} \quad , \quad A^{-n} = (A^{-1})^n = \underbrace{A^{-1}.A^{-1}\ldots A^{-1}}_{n \text{ مرة}}$$

$$(kA)^{-1} = \frac{1}{k}A^{-1}$$

لـ عدد حقيقي غير صفري

$$(A\pm B)^t = A^t \pm B^t, \ (AB)^t = B^tA^t, \ (kA)^t = kA^t$$

(k: عدد)

مثال $p(x) = x^2+3x+4 \Rightarrow p(A) = A^2+3A+4I$

لـ مصفوفة

$$(A^t)^{-1} = (A^{-1})^t$$

Row operations to find A^{-1} : $[A|I] \ldots [I|A^{-1}]$ (row operations)

Solution of system $Ax=b$ by A^{-1}: $x = A^{-1}b$ (لـ حل النظام) (A مصفوفة المعاملات في النظام، b عمود الثوابت في النظام)

If $A = A^t \Rightarrow A$ is symmetric (مثال: $\begin{bmatrix} 1 & 5 & -7 \\ 5 & 0 & 3 \\ -7 & 3 & -1 \end{bmatrix}$)

المحددة Determinant : ($|A|$ or $\det(A)$)

عدد حقيقي — A مصفوفة مربعة

طرق إيجاد المحددة

Arrows (انظر الصفحة المرفقة)

Direct multiplication

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Cofactors

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

cofactor $C_{ij} = (-1)^{i+j} M_{ij}$, $1 \le i,j \le n$

$M_{ij} = \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} =$ عدد

minor

الإشارات لـ cofactors: $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

(row reduction) row operations (row echelon form)

(i) $R_i \leftrightarrow R_j \Rightarrow$ تضرب المحددة بسالب

(ii) aR_i, $a \neq 0 \Rightarrow$ تضرب المحددة بـ $\frac{1}{a}$

(iii) $aR_i + R_j \Rightarrow$ لا تتغير شيئاً

الهدف تحويلها لمثلثية عليا (أي ما تحت القطر أصفار) (ليس ضرورية تحويل القطر إلى وحدات)

Note:

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc, \quad \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\cdot a_{22}\cdot a_{33}, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}\cdot a_{22}\cdot a_{33}$$

Properties of determinant:

1) $|KA| = K^n |A|$, k عدد ثابت , n is degree of A

2) $|A \pm B| \neq |A| \pm |B|$

3) $|AB| = |A|\cdot|B|$

4) $|A^{-1}| = \frac{1}{|A|}$

5) A is invertible $\Leftrightarrow |A| \neq 0$

6) $|A^t| = |A|$

7) إذا فيه صف أو عمود كله أصفار، فإن المحددة = صفر

8) إذا فيه صفان أو عمودان متساويان أو أحدهما مضاعفاً للآخر، فإن المحددة = صفر

9) $A\cdot adj(A) = adj(A)\cdot A = |A|\cdot I_n$

~~10) (KA)~~

10) $|A^n| = |A|^n$

Finding A^{-1} by cofactors:

$$A^{-1} = \frac{1}{|A|} adj(A)$$

$$adj(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^t$$

Cramer's Rule (طريقة لحل النظام)

$$x = \frac{|A_1|}{|A|}, \quad y = \frac{|A_2|}{|A|}, \quad z = \frac{|A_3|}{|A|}$$

$A \rightarrow$ مصفوفة المعاملات

$A_1 \rightarrow$ نستبدل العمود الاول في A بالثوابت

$A_2 \rightarrow$ 〃 الثاني في A 〃 〃

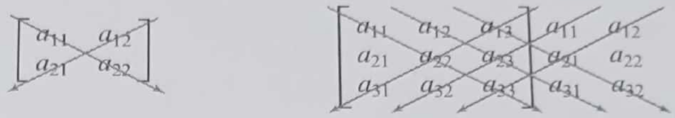
$A_3 \rightarrow$ 〃 الثالث في A 〃 〃

The method illustrated in Example 6 can be easily adapted to prove the following general result.

THEOREM 2.1.2 *If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then* $\det(A)$ *is the product of the entries on the main diagonal of the matrix; that is,* $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

A Useful Technique for Evaluating 2×2 and 3×3 Determinants

Determinants of 2×2 and 3×3 matrices can be evaluated very efficiently using the pattern suggested in Figure 2.1.1.



▶ Figure 2.1.1

تحذير

WARNING The arrow technique works only for determinants of 2×2 and 3×3 matrices. It *does not* work for matrices of size 4×4 or higher.

In the 2×2 case, the determinant can be computed by forming the product of the entries on the rightward arrow and subtracting the product of the entries on the leftward arrow. In the 3×3 case we first recopy the first and second columns as shown in the figure, after which we can compute the determinant by summing the products of the entries on the rightward arrows and subtracting the products on the leftward arrows. These procedures execute the computations

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{aligned}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}\end{aligned}$$

which agrees with the cofactor expansions along the first row.

▶ **EXAMPLE 7 A Technique for Evaluating 2 × 2 and 3 × 3 Determinants**

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = \begin{bmatrix} 3 & 1 \\ 4 & -2 \end{bmatrix} = (3)(-2) - (1)(4) = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 1 & 2 \\ -4 & 5 & 6 & -4 & 5 \\ 7 & -8 & 9 & 7 & -8 \end{vmatrix}$$

$$= [45 + 84 + 96] - [105 - 48 - 72] = 240 \blacktriangleleft$$

Vectors and surfaces — Ch 10

$P_1 \xrightarrow{\overrightarrow{P_1P_2}} P_2$

$\overrightarrow{P_1P_2} = P_2 - P_1$, P_1 & P_2 Points

Unite vector : $\vec{a}/\|\vec{a}\|$

$\vec{a} = c\,\vec{b}$ (عدد c)
- $c>0 \Rightarrow \vec{a}$ & $\vec{b}$ in same direction
- $c<0 \Rightarrow \vec{a}$ & $\vec{b}$ in opposite direction

P_1 & P_2 Points :

$d(P_1,P_2) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

midpoint : $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

$\vec{a} = \langle a_1, a_2, a_3\rangle$ & $\vec{b} = \langle b_1, b_2, b_3\rangle$

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 \rightarrow$ عدد

$\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}}$

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos(\theta)$, θ الزاوية بين $\vec{a}$ و $\vec{b}$

$\vec{a}$ & $\vec{b}$ are orthogonal $\Leftrightarrow \vec{a}\cdot\vec{b} = 0$

$\text{Com}_{\vec{b}}\vec{a} = \vec{a}\cdot\frac{\vec{b}}{\|\vec{b}\|}$ (Component of $\vec{a}$ along $\vec{b}$)

Work $= \overrightarrow{PQ}\cdot\overrightarrow{PR}$

$\overrightarrow{PQ}$ ← القوة (ثابتة) ; $\overrightarrow{PR}$ ← الإزاحة

direction angles : (α, β, γ)

$\cos\alpha = \frac{a_1}{\|\vec{a}\|}$, $\cos\beta = \frac{a_2}{\|\vec{a}\|}$, $\cos\gamma = \frac{a_3}{\|\vec{a}\|}$

Vector Product (Cross Product) : $\Rightarrow$ vector

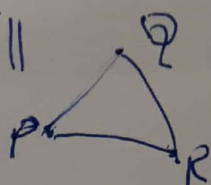
$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \leftarrow (\text{محددة})$$

$(\vec{a}\times\vec{b}) \perp \vec{a}$ and $(\vec{a}\times\vec{b}) \perp \vec{b}$

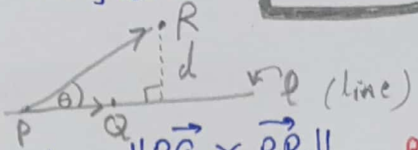
$\|\vec{a}\times\vec{b}\| = \|a\|\,\|b\|\sin(\theta)$

$\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a}\times\vec{b} = \langle 0,0,0\rangle$

مساحة المثلث $= \frac{1}{2}\|\overrightarrow{PQ}\times\overrightarrow{PR}\|$



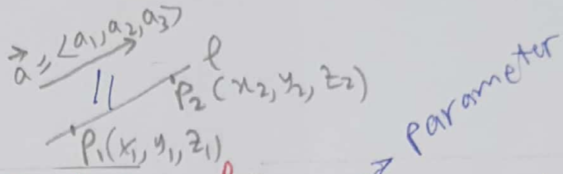
مساحة متوازي الأضلاع (Parallelogram) $= \|\overrightarrow{PQ}\times\overrightarrow{PR}\|$



$d = \frac{\|\overrightarrow{PQ}\times\overrightarrow{PR}\|}{\|\overrightarrow{PQ}\|}$ ← على الخط ; المسافة بين نقطة R و خط

Volume of box: $|(\vec{a}\times\vec{b})\cdot\vec{c}|$

(box أضلاعه $\vec{c}$, $\vec{b}$ و $\vec{a}$)



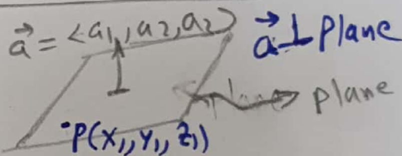
eq of line ℓ :

$x = x_1 + a_1t$, $y = y_1 + a_2t$, $z = z_1 + a_3t$, $t \in \mathbb{R}$

(Parametric equations)

Angles between two lines ℓ_1 & ℓ_2 : is the angle between $\vec{a}$ & $\vec{b}$, where $\vec{a} \parallel \ell_1$ & $\vec{b} \parallel \ell_2$

(θ and $\pi - \theta$)



[normal : نفس orthogonal

eq of Plane:

$a_1(x-x_1) + a_2(y-y_1) + a_3(z-z_1) = 0$

plane: $ax+by+cz+d=0 \rightarrow \langle a,b,c\rangle \perp$ Plane

Plane 1 & Plane 2
- Parallel if $\vec{a} \parallel \vec{b}$
- orthogonal if $\vec{a} \perp \vec{b}$

$\vec{a} \perp$ plane 1

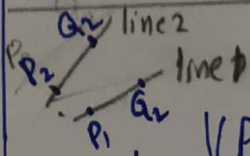
$\vec{b} \perp$ plane 2

Symmetric eq's of line :

$$\frac{x-x_0}{a_1} = \frac{y-y_0}{a_2} = \frac{z-z_0}{a_3}$$

distance from Point $P_0(x_0,y_0,z_0)$ to Plane $ax+by+cz+d=0$ is d :

$$d = \frac{|ax_0+by_0+cz_0+d|}{(a^2+b^2+c^2)^{\frac{1}{2}}}$$



$$d = \frac{|(\overrightarrow{P_1Q_1}\times\overrightarrow{P_2Q_2})\cdot\overrightarrow{P_1P_2}|}{\|\overrightarrow{P_1Q_1}\times\overrightarrow{P_2Q_2}\|}$$

إضافات من مذكرة د. خواجه

Ex If $a = \langle -6, -3, 6 \rangle$, find the vector that has

(i) the same direction as a and twice the magnitude of a,

(ii) the opposite direction of a and one-third magnitude of a,

(iii) the same direction of a and magnitude 2.

Soln.

(i) $b = 2a = 2 \langle -6, -3, 6 \rangle$

(ii) $b = -\frac{1}{3} a = -\frac{1}{3} \langle -6, -3, 6 \rangle$

(iii) $u = \frac{a}{\|a\|} = \frac{1}{\sqrt{36+9+36}} \langle -6, -3, 6 \rangle = \frac{1}{9} \langle -6, -3, -6 \rangle$

$b = 2u = \frac{2}{9} \langle -6, -3, -6 \rangle$

Note : 1 If $b = ka$, where k is scalar, then a and b are parallel.

Note : 2. Three points lie on same line, if two vectors from three points have

(i) Same initial point

(ii) and are parallel.

Exercise. Use vectors to determine whether the points lie on a straight line, points are $(1, -1, 5)$, $(0, -1, 6)$, $(3, -1, 3)$

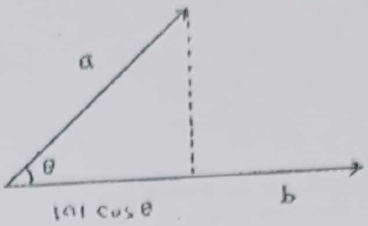
Note . 2. If $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$ are parallel, then $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

Component of a along b

Let a and b vectors in V_3 with $b \neq 0$.

The component of a along b

$$= \text{Comp}_b^a = \frac{a \cdot b}{\|b\|} = a \cdot \frac{b}{\|b\|}$$



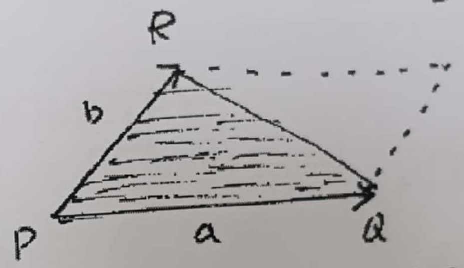
Projection of a on b

Vector projection of a onto b

$$= \left(\text{Comp}_b^a\right) \frac{b}{\|b\|} = \left(\frac{a \cdot b}{\|b\|}\right)\left(\frac{b}{\|b\|}\right)$$

$= \|a \times b\|$

Note.5



$$\text{Area of triangle } PQR = \frac{\|a \times b\|}{2}$$

Scalar Triple Product

$$(a \times b) \cdot c = a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

ALGEBRA

Arithmetic

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc}$$

Factoring

$$x^2 - y^2 = (x-y)(x+y)$$

$$x^3 - y^3 = (x-y)(x^2+xy+y^2)$$

$$x^3 + y^3 = (x+y)(x^2-xy+y^2)$$

$$x^4 - y^4 = (x-y)(x+y)(x^2+y^2)$$

Binomial

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Exponents

$$x^n x^m = x^{n+m}$$

$$\frac{x^n}{x^m} = x^{n-m}$$

$$(x^n)^m = x^{nm}$$

$$x^{-n} = \frac{1}{x^n}$$

$$(xy)^n = x^n y^n$$

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$x^{n/m} = \sqrt[m]{x^n}$$

$$\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$$

$$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

Lines

Slope m of line through (x_0, y_0) and (x_1, y_1)

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

Through (x_0, y_0), slope m

$$y - y_0 = m(x - x_0)$$

Slope m, y-intercept b

$$y = mx + b$$

Quadratic Formula

If $ax^2 + bx + c = 0$

then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

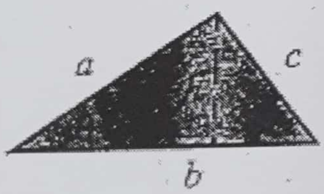
Distance

Distance d between (x_1, y_1) and (x_2, y_2)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

GEOMETRY

Triangle



Area $= \frac{1}{2}bh$

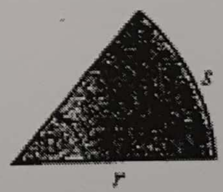
$c^2 = a^2 + b^2 - 2ab\cos\theta$

Circle

Area $= \pi r^2$

$C = 2\pi r$

Sector of a Circle

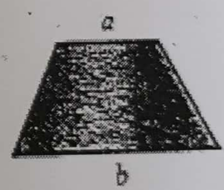


Area $= \frac{1}{2}r^2\theta$

$s = r\theta$

(for θ in radians only)

Trapezoid



Area $= \frac{1}{2}(a+b)h$

Sphere

Volume $= \frac{4}{3}\pi r^3$

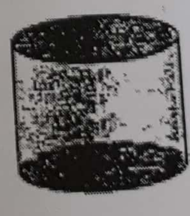
Surface Area $= 4\pi r^2$

Cone

Volume $= \frac{1}{3}\pi r^2 h$

Surface Area $= \pi r\sqrt{r^2 + h^2}$

Cylinder



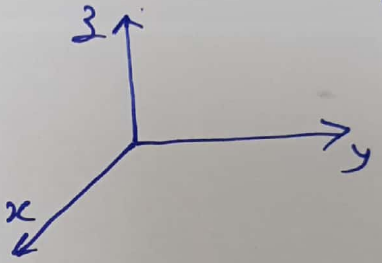
Volume $= \pi r^2 h$

Surface Area $= 2\pi rh$

10.6 SURFACES :

Trace	eq of trace	Kind	sketch of trace	identify (اسم السطح)
2D xy-trace ($z=0$)	$\cancel{z=0}$	—	—	
2D yz-trace ($x=0$)	$\cancel{x=0}$	—	—	
2D xz-trace ($y=0$)	$\cancel{y=0}$	—	—	
trace in one of the following plane: $x=k$ أو $y=k$ أو $z=k$ ، k عدد ثابت	$x=k$ أو $y=k$ أو $z=k$	—	—	Name of surface.

بعد الانتهاء من الجدول أعلاه نذهب إلى الرسمة في 3D :



(معلومات أساسية سابقة في 2D :

$x=k$, $z=k$, $y=k$ → line

$y-k=4p(x-h)^2$, parabola ↑↓ $V(h,k)$

$x-h=4p(y-k)^2$, Parabola ←→ $V(h,k)$

$\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$ ellipse

$a>b \Rightarrow C(h,k)$, $V_1(h+a,k)$, $V_2(h-a,k)$

$\dfrac{(y-k)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$ ellipse

$a>b \Rightarrow C(h,k)$, $V_1(h,k+a)$, $V_2(h,k-a)$

$\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$ hyperbola $C(h,k)$, $V_1=(h+a,k)$, $V_2=(h-a,k)$

$\dfrac{(y-k)^2}{a^2}-\dfrac{(x-h)^2}{b^2}=1$ hyperbola $C(h,k)$, $V_1(h,k+a)$, $V_2(h,k-a)$

$(x-h)^2+(y-k)^2=r^2$ circle $c(h,k)$, radius $=r$

$x^2+y^2=0$ origin Point

$x^2+y^2=$ عدد سالب no trace

Ch 11 :

Curve C: $x = f(t),\ y = g(t),\ z = h(t)$, $a \le t \le b$

Length of C: $L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$

Velocity: $v(t) = r'(t)$; $r(t)$ is position vector

Speed: $\upsilon(t) = \| r'(t) \|$

acceleration: $a(t) = r''(t) = v'(t)$

Force: $F = m \cdot a$, $m \to$ mass , $a \to$ acceleration

Unite tangent vector: $T(t) = \dfrac{r'(t)}{\| r'(t) \|}$

Principal Unite normal vector: $N(t) = \dfrac{T'(t)}{\| T'(t) \|}$

Curvature: if $y = f(x)$: $K = \dfrac{|y''|}{[1 + (y')^2]^{\frac{3}{2}}}$ ونعوض $x = ?$

curvature: $C: x = f(t),\ y = g(t)$ ~~$z = h$~~ in (2D) : $K = \dfrac{|f' \cdot g'' - g' \cdot f''|}{[(f')^2 + (g')^2]^{\frac{3}{2}}}$ ونعوض $t = ?$

$C: x = f(t),\ y = g(t),\ z = h(t)$ (in 3D) : $K = \| T'(t) \|$

Tangential Component of acceleration: $a_T = \dfrac{r'(t) \cdot r''(t)}{\| r'(t) \|}$

Normal component of acceleration: $a_N = \dfrac{\| r'(t) \times r''(t) \|}{\| r'(t) \|}$

Curvature: $C: x = f(t),\ y = g(t),\ z = h(t)$: $K = \dfrac{\| r'(t) \times r''(t) \|}{\| r'(t) \|^3}$

Parametric **eq of tangent line of $C: r(t)$ at $P(x_0, y_0, z_0)$**

$\langle a_1, a_2, a_3 \rangle = r'(t)$

$$\left.\begin{aligned} x &= x_0 + a_1 t \\ y &= y_0 + a_2 t \\ z &= z_0 + a_3 t \end{aligned}\right\} ,\ t \in \mathbb{R}$$

$(x_0, y_0, z_0) \longrightarrow t = ?$

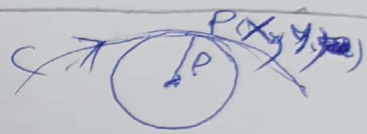
---

eq of normal Plane of $C: r(t)$ at $P(x_0, y_0, z_0)$

$a_1(x - x_0) + a_2(y - y_0) + a_3(z - z_0) = 0$

$\langle a_1, a_2, a_3 \rangle = r'(t) \perp$ Plane , $t = ?$

$(x_0, y_0, z_0) \to t = ?$



radius of curvature: $\rho = \dfrac{1}{K}$

Center of curvature: (h, k)

$h = x - \dfrac{y'(1 + (y')^2)}{y''}$

$K = y + \dfrac{1 + (y')^2}{y''}$

ملاحظة مهمة جداً

$P(x_0, y_0, z_0)$ on C

كيف نوجد t المقابلة لها؟

نأخذ واحدة من احداثيات النقطة ونساوي بما يقابلها من معادلة C

ونوجد $t =$

Ch 12:

Limit: $\lim_{(x,y)\to(a,b)} f(x,y)$

لإثبات عدم وجود النهاية:
(i) نختار مسارين يمران بالنقطة (a,b) النهاية عليهما مختلفة.
أو (ii) نختار مسار بدلالة m والنهاية بدلالة m
أو (iii) بالإحداثيات القطبية.

لإثبات وجود النهاية:
(i) بنظرية الساندويتش (الحصر)
أو (ii) تعريف النهاية

إذا كانت: $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$
فإن f متصلة عند النقطة (a,b)

Partial derivatives

$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}$

$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}$

$f_{yx} = (f_y)_x, \quad f_{xy} = (f_x)_y$

$\frac{\partial^3 f}{\partial x \partial z \partial y}$
x أولاً
z ثانياً
y ثالثاً

Increment of w:

$\Delta w = f(x+\Delta x, y+\Delta y) - f(x,y)$

$= f_x(x,y)\Delta x + f_y(x,y)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$

$\epsilon_1 \to 0, \ \epsilon_2 \to 0$

differential:

$dw = f_x(x,y)\,dx + f_y(x,y)\,dy$

Chain rule:

$w = f(u,v), \quad u = g(x,y), \quad v = h(x,y)$

$w \to u \to x, y$; $w \to v \to x, y$

$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x}$

$\partial w/\partial y = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial y}$

$F(x,y) = 0$

$\frac{dy}{dx} = -\frac{F_x}{F_y}$

$F(x,y,z) = 0$

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$

$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

Directional derivative of f

at $P(x,y)$ in the direction of u

$\vec{u} = u_1 \vec{i} + u_2 \vec{j}$ (unite vector)

$D_u f(x,y) = f_x(x,y)\,u_1 + f_y(x,y)\,u_2$

Gradient of f:

$\nabla f(x,y) = f_x \cdot \vec{i} + f_y \cdot \vec{j} + f_z \vec{k}$

max value of $D_u f$ is: $\|\nabla f\|$
max rate of increase of f in direction of ∇f

min value of $D_u f$ is: $-\|\nabla f\|$
min rate of increase of f in direction of $-\nabla f$.

equation of tangent plane: $z = f(x,y)$

$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

Local extrema of f:

1- Find Critical Point: $f_x = 0$ and $f_y = 0$
2- Find $D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$
3- D(النقطة الحرجة)
- $< 0 \Rightarrow$ Saddle Points
- $= 0 \Rightarrow$ يفشل الاختبار
- > 0:
  - $f_{xx} > 0$ local min
  - $f_{xx} < 0$ local max

ch 12 (تكملة)

Lagrange multipliers :

~~Local~~ extrema of $f(x,y)$ subject to constraint (قيد) $g(x,y)=0$

$$\nabla f = \lambda \nabla . g \Rightarrow \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = 0 \end{cases}$$

⇐ تطلع معنا نقاط ← نعوض بها في الجدول

(x,y)			...
$f(x,y)$			

ونقارن بين قيم $f(x,y)$

تعميم ما سبق ذكره في (3D) :

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g(x,y,z) = 0 \end{cases}$$

⇐ تطلع معنا نقاط ← نعوض بها في الجدول

(x,y,z)	...		...
$f(x,y,z)$			

ونقارن بين قيم $f(x,y,z)$

(والحمد لله الذي بنعمته تتم الصالحات)