Summary of Integration

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How to Choose the Right Method for Integration

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Integration Table

Integration Table (28 rules)

$$\int 1\,dx = x + c$$

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + c \quad (r \neq -1)$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int \csc^2 x\,dx = -\cot x + c$$

$$\int \sec x \tan x\,dx = \sec x + c$$

$$\int \csc x \cot x\,dx = -\csc x + c$$

Integration Table

$$\int \frac{1}{x}\, dx = \ln|x| + c$$

$$\int e^x\, dx = e^x + c$$

$$\int a^x\, dx = \frac{1}{\ln a}\, a^x + c$$

Integration Table

$$\int \tan x \, dx = \ln|\sec x| + c$$

$$\int \cot x \, dx = \ln|\sin x| + c$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + c$$

$$\int \csc x \, dx = \ln|\csc x - \cot x| + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c,$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\left(\tfrac{x}{a}\right) + C$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\tfrac{x}{a}\right) + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

$$\int \frac{dx}{a^2 - x^2} = \tfrac{1}{a} \tanh^{-1}\left(\tfrac{x}{a}\right) + C$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + c$$

$$\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\tfrac{1}{a}\, \mathrm{sech}^{-1} \left(\tfrac{|x|}{a}\right) + C$$

Integration Table

$$\int \sinh x \, dx = \cosh x + C$$

$$\int \cosh x \, dx = \sinh x + C$$

$$\int \operatorname{sech}^2 x \, dx = \tanh x + C$$

$$\int \operatorname{csch}^2 x \, dx = -\coth x + C$$

$$\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$$

$$\int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + C$$

Methods of Integration

Integration by Substitution

Theorem

If F is an antiderivative of f then

$$\int f(g(x))g'(x)dx = F(g(x)) + c$$

If $u = g(x)$ and $du = g'(x)dx$, then

$$\int f(u)du = F(u) + c$$

Example

$$\int 3x^2(x^3+3)^4\,dx$$

First Step for Integration

1. Fits the integration table (integration rule)

$$\int \sin x \, dx = -\cos x + c, \qquad \int \frac{1}{x^2+4} \, dx = \frac{1}{2} \tan^{-1} \frac{x}{2} + c$$

2. Simplify or split

$$\int \frac{x^2+2x}{\sqrt{x}} \, dx = \int (x^{\frac{3}{2}} + 2x^{\frac{1}{2}}) \, dx$$

$$\int \frac{x+2}{\sqrt{4-x^2}} \, dx = \int \frac{x}{\sqrt{4-x^2}} \, dx + \int \frac{2}{\sqrt{4-x^2}} \, dx$$

3. Make it fit the integration table (integration by substitution)

$$\int \sin(x^2+x)(2x+1) \, dx = \int \sin u \, du$$

$$\int \frac{e^x}{e^{2x}+4} \, dx = \int \frac{1}{u^2+4} \, du$$

Integration By Parts

Theorem

If $u = f(x)$ *and* $v = g(x)$ *and if* f' *and* g' *are continuous then*

$$\boxed{\int u dv = uv - \int v du}$$

Example

$$\int x^2 \sin x dx, \qquad \int x e^x dx$$

$$\int \sin^{-1} x dx, \qquad \int \ln x \, dx$$

LIATE Rule

Chose u as the function that comes earliest in LIATE

- L – Logarithmic functions $(\ln x, \log_a x)$
- I – Inverse trigonometric functions $(\sin^{-1} x, \tan^{-1} x$, etc.)
- A – Algebraic functions $(x^2, x, \sqrt{x}$, polynomials)
- T – Trigonometric functions $(\sin x, \cos x, \tan x$, etc.)
- E – Exponential functions (e^x, a^x)

.

Trigonometric Integrals

To evaluate

$$\boxed{\int \sin^m x \, \cos^n x \, dx}$$

1. If m is an odd integer

$$\int \sin^m x \, \cos^n x \, dx = \int \sin^{m-1} x \, \cos^n x \, \sin x \, dx$$

Then use the identity $\sin^2 x = 1 - \cos^2 x$, and the substitution $u = \cos x$

2. If n is an odd integer

$$\int \sin^m x \, \cos^n x \, dx = \int \sin^m x \, \cos^{n-1} x \, \cos x \, dx$$

Then use the identity $\cos^2 x = 1 - \sin^2 x$, and the substitution $u = \sin x$

3. If m and n is are even integer, use the half-angle formula

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \text{and} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

To evaluate

$$\boxed{\int \tan^m x \, \sec^n x \, dx}$$

1. If m is an odd integer

$$\int \tan^m x \, \sec^n x \, dx = \int \tan^{m-1} x \, \sec^{n-1} x \, \sec x \tan x \, dx$$

Then use the identity $\tan^2 x = \sec^2 x - 1$, and the substitution $u = \sec x$

Trigonometric Integrals

2. If n is an even integer

$$\int \tan^m x \, \sec^n x \, dx = \int \tan^m x \, \sec^{n-2} x \, \sec^2 x \, dx$$

Then use the identity $\sec^2 x = 1 + \tan^2 x$, and the substitution $u = \sec x$

3. If m is even and n is odd integers, use integration by parts.

Trigonometric Integrals

$$\boxed{\int \sin mx \, \cos nx \, dx} \qquad \boxed{\int \sin mx \, \sin nx \, dx}$$

$$\boxed{\int \cos mx \, \cos nx \, dx}$$

Use the following formulas

$$\sin nx \, \cos mx = \frac{1}{2}[\sin(n-m)x + \sin(n+m)x]$$

$$\cos nx \, \cos mx = \frac{1}{2}[\cos(n-m)x + \cos(n+m)x]$$

$$\sin nx \, \sin mx = \frac{1}{2}[\cos(n-m)x - \cos(n+m)x]$$

Trigonometric Substitutions

$$\boxed{\int \sqrt{a^2 - x^2}\, dx}$$

$$\boxed{\int (a^2 + x^2)^{3/2}\, dx}$$

$$\boxed{\int \frac{1}{(x^2 - a^2)^2}\, dx}$$

$$\vdots$$

Trigonometric Substitutions

1. For $\sqrt{a^2 - x^2}$, use

$$x = a\sin\theta, \qquad \theta \in [\pi/2, \pi/2]$$
$$dx = a\cos\theta\, d\theta$$
$$\sqrt{a^2 - x^2} = a\cos\theta$$

2. For $\sqrt{a^2 + x^2}$, use

$$x = a\tan\theta, \qquad \theta \in (\pi/2, \pi/2)$$
$$dx = a\sec^2\theta\, d\theta$$
$$\sqrt{a^2 + x^2} = a\sec\theta$$

3. For $\sqrt{x^2 - a^2}$, use

$$x = a\sec\theta, \qquad \theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$$
$$dx = a\sec\theta\tan\theta\, d\theta$$
$$\sqrt{x^2 - a^2} = a\tan\theta$$

Trigonometric Substitutions

Expression	Substitution	Simplifies to
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$a\sec\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$a\tan\theta$

It is also used for

$$(a^2 - x^2)^{\frac{3}{2}} \quad \text{or} \quad \frac{1}{(a^2 - x^2)^2}$$

$$\ldots$$

Example

$$\int \frac{1}{x^2\sqrt{16 - x^2}}\, dx$$

$$\int \frac{1}{(4 + x^2)^2}\, dx$$

Partial Fraction Decomposition

To find the integral

$$\boxed{\int \frac{f(x)}{g(x)}\, dx}$$

where $f(x)$ and $g(x)$ are polynomials

1. If $\deg f(x) \geq \deg g(x)$, use long division to get

$$\frac{f(x)}{g(x)} = h(x) + \frac{r(x)}{g(x)}$$

   where $\deg r(x) < \deg g(x)$
2. If $\deg f(x) < \deg g(x)$, use partial fraction.

Partial Fraction

We write

$$\frac{f(x)}{g(x)} = F_1(x) + F_2(x) + \dots + F_r(x)$$

where each $F_k(x)$ has one of the forms

$$\frac{A}{(ax+b)^n} \qquad \text{or} \qquad \frac{Ax+B}{(ax^2+bx+c)^n}$$

Example

$$\int \frac{x^2+2}{x^2-5x-6}\,dx$$

$$\int \frac{4x^2+4x+5}{(x+1)\left(x^2+4\right)}dx$$

Partial Fraction Decomposition

Case 1: Distinct Linear Factors

$$g(x) = (x - a_1)(x - a_2) \cdots (x - a_n),$$

$$\frac{f(x)}{g(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \cdots + \frac{A_n}{x - a_n}.$$

Case 2: Repeated Linear Factors

If $(x - a)^m$ is a factor of $g(x)$, then include terms of the form:

$$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_m}{(x - a)^m}.$$

Case 3: Irreducible Quadratic Factors

If $(ax^2 + bx + c)$ is an irreducible quadratic factor, include:

$$\frac{Bx + C}{ax^2 + bx + c}$$

Partial Fraction Decomposition

If $(ax^2+bx+c)^m$ is a factor of $g(x)$, then include terms of the form:

$$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\cdots+\frac{B_mx+C_m}{(ax^2+bx+c)^m}.$$

Integrals Involving Quadratic Forms

If the integral contains irreducible quadratic expression

$$ax^2 + bx + c$$

where $b \neq 0$.
Transform the quadratic into a perfect square by completing the square

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Example

$$\int \frac{2x-1}{x^2-6x+13}\,dx$$

$$\int \frac{1}{(x^2+8x+25)^2}\,dx$$

$$\int \frac{1}{\sqrt{2x-x^2}}\,dx$$

Integrals Involving Fractional Powers

$$\boxed{\int \frac{1}{x^{\frac{1}{k}} + x^{\frac{1}{m}}}\, dx}$$

substitute

$$\boxed{u = x^{\frac{1}{n}}}$$

where n is the least common multiple of m and k.

$$u^n = x$$

$$nu^{n-1}du = dx$$

$$x^{\frac{1}{k}} = u^{\frac{n}{k}} \qquad\qquad x^{\frac{1}{m}} = u^{\frac{n}{m}}$$

Example

$$\int \frac{1}{\sqrt{x} + \sqrt[3]{x}}\, dx$$

Integrals of the Form $\sqrt[n]{f(x)}$

If integral contains an expression of the form $\sqrt[n]{f(x)}$ we use the substitution

$$u = \sqrt[n]{f(x)}$$

$$u^n = f(x)$$

$$nu^{n-1}du = f'(x)dx$$

Example

$$\int \frac{x^3}{\sqrt[3]{x^2+4}}\,dx$$

$$\int \sqrt{e^x+1}\,dx$$

Fractional Functions in $\sin x$ and $\cos x$

$$\boxed{\int \frac{1}{a\sin x + b\cos x + c}\,dx}$$

If the integral consists of rational expression in $\sin x$ and $\cos x$ then we use tangent half-angle substitution

$$\boxed{u = \tan\frac{x}{2}} \qquad -\pi < x < \pi$$

then we have

$$\sin x = \frac{2u}{u^2+1}$$

$$\cos x = \frac{1-u^2}{u^2+1}$$

$$dx = \frac{2}{u^2+1}du$$

Fractional Functions in $\sin x$ and $\cos x$

Example

$$\int \frac{1}{1+\sin x}\,dx$$

$$\int \frac{1}{4\sin x - 3\cos x}\,dx$$