Contents

31 Sum of Two Independent Random Variables

In this section we turn to the important question of determining the distribution of a sum of independent random variables in terms of the distributions of the individual constituents.

31.1 Discrete Case

In this subsection we consider only sums of discrete random variables, reserving the case of continuous random variables for the next subsection. We consider here only discrete random variables whose values are nonnegative integers. Their distribution mass functions are then defined on these integers.

Suppose X and Y are two independent discrete random variables with pmf $p_X(x)$ and $p_Y(y)$ respectively. We would like to determine the pmf of the random variable $X+Y$. To do this, we note first that for any nonnegative integer n we have

Find the distribution of $X+Y$

Or

Find the convolution

$$\{X+Y=n\}=\bigcup_{k=0}^{n}A_k$$

where $A_k=\{X=k\}\cap\{Y=n-k\}$. Note that $A_i\cap A_j=\emptyset$ for $i\neq j$. Since the A_i's are pairwise disjoint and X and Y are independent, we have

$$P(X+Y=n)=\sum_{k=0}^{n}P(X=k)P(Y=n-k).$$

☆

Thus,

$$p_{X+Y}(n)=p_X(n)*p_Y(n)$$

where $p_X(n)*p_Y(n)$ is called the **convolution** of p_X and p_Y.

Example 31.1

indep.

A die is rolled twice. Let X and Y be the outcomes, and let $Z=X+Y$ be the sum of these outcomes. Find the probability mass function of Z.

Solution. Note that X and Y have the common pmf :

x	1	2	3	4	5	6
p_X	1/6	1/6	1/6	1/6	1/6	1/6

The probability mass function of Z is then the convolution of p_X with itself. Thus,

$$\begin{aligned}
P(Z=2) =& p_X(1)p_X(1) = \frac{1}{36} \qquad = P(X=1, Y=1).\\
P(Z=3) =& p_X(1)p_X(2) + p_X(2)p_X(1) = \frac{2}{36} \qquad P(X=1, Y=2) \overset{\text{or}}{+} P(X=2, Y=1).\\
P(Z=4) =& p_X(1)p_X(3) + p_X(2)p_X(2) + p_X(3)p_X(1) = \frac{3}{36}
\end{aligned}$$

Continuing in this way we would find $P(Z=5) = 4/36, P(Z=6) = 5/36, P(Z=7) = 6/36, P(Z=8) = 5/36, P(Z=9) = 4/36, P(Z=10) = 3/36, P(Z=11) = 2/36$, and $P(Z=12) = 1/36$ ■

Example 31.2
Let X and Y be two independent Poisson random variables with respective parameters λ_1 and λ_2. Compute the pmf of $X+Y$.

Solution.
For every positive integer we have

$$\{X+Y=n\} = \bigcup_{k=0}^{n} A_k$$

where $A_k = \{X=k, Y=n-k\}$ for $0 \le k \le n$. Moreover, $A_i \cap A_j = \emptyset$ for $i \ne j$. Thus,

$$\begin{aligned}
p_{X+Y}(n) =& P(X+Y=n) = \sum_{k=0}^{n} P(X=k, Y=n-k)\\
=& \sum_{k=0}^{n} P(X=k)P(Y=n-k)\\
=& \sum_{k=0}^{n} e^{-\lambda_1}\frac{\lambda_1^k}{k!}e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!}\\
=& e^{-(\lambda_1+\lambda_2)}\sum_{k=0}^{n}\frac{\lambda_1^k\lambda_2^{n-k}}{k!(n-k)!}\\
=& \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\lambda_1^k\lambda_2^{n-k} \qquad = \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\underbrace{\sum_{k=0}^{n}\binom{n}{k}\lambda_1^k\lambda_2^{n-k}}_{\text{Binomial}}\\
=& \frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n
\end{aligned}$$

Thus, $X + Y$ is a Poisson random variable with parameter $\lambda_1 + \lambda_2$ ■

Example 31.3

Let X and Y be two independent binomial random variables with respective parameters (n, p) and (m, p). Compute the pmf of $X + Y$.

Solution.

X represents the number of successes in n independent trials, each of which results in a success with probability p; similarly, Y represents the number of successes in m independent trials, each of which results in a success with probability p. Hence, as X and Y are assumed to be independent, it follows that $X + Y$ represents the number of successes in $n + m$ independent trials, each of which results in a success with probability p. So $X + Y$ is a binomial random variable with parameters $(n + m, p)$ ■

Example 31.4

Alice and Bob flip bias coins independently. Alice's coin comes up heads with probability 1/4, while Bob's coin comes up head with probability 3/4. Each stop as soon as they get a head; that is, Alice stops when she gets a head while Bob stops when he gets a head. What is the pmf of the total amount of flips until both stop? (That is, what is the pmf of the combined total amount of flips for both Alice and Bob until they stop?)

Solution.

Let X and Y be the number of flips until Alice and Bob stop, respectively. Thus, $X+Y$ is the total number of flips until both stop. The random variables X and Y are independent geometric random variables with parameters 1/4 and 3/4, respectively. By convolution, we have

$$\begin{aligned} p_{X+Y}(n) =& \sum_{k=1}^{n-1} \frac{1}{4}\left(\frac{3}{4}\right)^{k-1} \frac{3}{4}\left(\frac{1}{4}\right)^{n-k-1} \\ =& \frac{1}{4^n} \sum_{k=1}^{n-1} 3^k = \frac{3}{2} \frac{3^{n-1} - 1}{4^n} \end{aligned}$$ ■

Practice Problems

Problem 31.1

Let X and Y be two independent discrete random variables with probability mass functions defined in the tables below. Find the probability mass function of $Z = X + Y$.

x	0	1	2	3
$p_X(x)$	0.10	0.20	0.30	0.40

y	0	1	2
$p_Y(y)$	0.25	0.40	0.35

Problem 31.2

Suppose X and Y are two independent binomial random variables with respective parameters $(20, 0.2)$ and $(10, 0.2)$. Find the pmf of $X + Y$.

Problem 31.3

Let X and Y be independent random variables each geometrically distributed with paramter p, i.e.

$$p_X(n) = p_Y(n) = \begin{cases} p(1-p)^{n-1} & n = 1, 2, \cdots \\ 0 & \text{otherwise} \end{cases}$$

Find the probability mass function of $X + Y$.

Problem 31.4

Consider the following two experiments: the first has outcome X taking on the values 0, 1, and 2 with equal probabilities; the second results in an (independent) outcome Y taking on the value 3 with probability 1/4 and 4 with probability 3/4. Find the probability mass function of $X + Y$.

Problem 31.5 ‡

An insurance company determines that N, the number of claims received in a week, is a random variable with $P[N = n] = \frac{1}{2^{n+1}}$, where $n \geq 0$. The company also determines that the number of claims received in a given week is independent of the number of claims received in any other week.

Determine the probability that exactly seven claims will be received during a given two-week period.

Problem 31.6

Suppose X and Y are independent, each having Poisson distribution with means 2 and 3, respectively. Let $Z = X + Y$. Find $P(X + Y = 1)$.

Problem 31.7

Suppose that X has Poisson distribution with parameter λ and that Y has geometric distribution with parameter p and is independent of X. Find simple formulas in terms of λ and p for the following probabilities. (The formulas should not involve an infinite sum.)

(a) $P(X+Y=2)$

(b) $P(Y>X)$

Problem 31.8

An insurance company has two clients. The random variables representing the claims filed by each client are X and Y. X and Y are indepedent with common pmf

x	0	1	2	y
$p_X(x)$	0.5	0.25	0.25	$p_Y(y)$

Find the probability mass function of $X+Y$.

Problem 31.9

Let X and Y be two independent random variables with pmfs given by

$$p_X(x)=\begin{cases} \frac{1}{3} & x=1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

$$p_Y(y)=\begin{cases} \frac{1}{2} & y=0 \\ \frac{1}{3} & y=1 \\ \frac{1}{6} & y=2 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability mass function of $X+Y$.

Problem 31.10

Let X and Y be two independent identically distributed geometric distributions with parameter p. Show that $X+Y$ is a negative binomial distribution with parameters $(2,p)$.

Problem 31.11

Let X,Y,Z be independent Poisson random variables with $E(X)=3, E(Y)=1$, and $E(Z)=4$. What is $P(X+Y+Z\le 1)$?

Problem 31.12

If the number of typographical errors per page type by a certain typist follows a Poisson distribution with a mean of λ, find the probability that the total number of errors in 10 randomly selected pages is 10.

31.2 Continuous Case

In this subsection we consider the continuous version of the problem posed in Section 31.1: How are sums of independent continuous random variables distributed?

Example 31.5
Let X and Y be two random variables with joint probability density

$$f_{XY}(x,y) = \begin{cases} 6e^{-3x-2y} & x > 0, y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the probability density of $Z = X + Y$.

Solution.
Integrating the joint probability density over the shaded region of Figure 31.1, we get

$$F_Z(a) = P(Z \le a) = \int_0^a \int_0^{a-y} 6e^{-3x-2y} dx dy = 1 + 2e^{-3a} - 3e^{-2a}$$

and differentiating with respect to a we find

$$f_Z(a) = 6(e^{-2a} - e^{-3a})$$

for $a > 0$ and 0 elsewhere ■

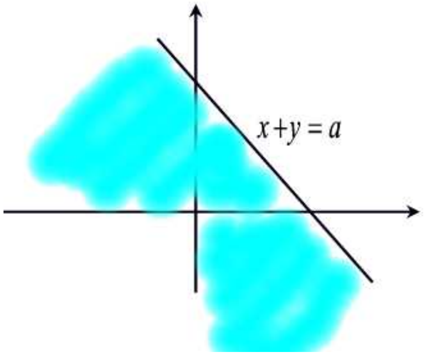

Figure 31.1

The above process can be generalized with the use of convolutions which we define next. Let X and Y be two continuous random variables with probability density functions $f_X(x)$ and $f_Y(y)$, respectively. Assume that both $f_X(x)$ and $f_Y(y)$ are defined for all real numbers. Then the **convolution** $f_X * f_Y$ of f_X and f_Y is the function given by

$$\begin{aligned}(f_X * f_Y)(a) &= \int_{-\infty}^{\infty} f_X(a-y)f_Y(y)dy \\ &= \int_{-\infty}^{\infty} f_Y(a-x)f_X(x)dx\end{aligned}$$

This definition is analogous to the definition, given for the discrete case, of the convolution of two probability mass functions. Thus it should not be surprising that if X and Y are independent, then the probability density function of their sum is the convolution of their densities.

Theorem 31.1
Let X and Y be two independent random variables with density functions $f_X(x)$ and $f_Y(y)$ defined for all x and y. Then the sum $X+Y$ is a random variable with density function $f_{X+Y}(a)$, where f_{X+Y} is the convolution of f_X and f_Y.

Proof.
The cumulative distribution function is obtained as follows:

$$\begin{aligned}F_{X+Y}(a) =& P(X+Y \le a) = \iint_{x+y\le a} f_X(x)f_Y(y)dxdy \\ =& \int_{-\infty}^{\infty}\int_{-\infty}^{a-y} f_X(x)f_Y(y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{a-y} f_X(x)dxf_Y(y)dy \\ =& \int_{-\infty}^{\infty} F_X(a-y)f_Y(y)dy\end{aligned}$$

Differentiating the previous equation with respect to a we find

$$\begin{aligned}f_{X+Y}(a) =& \frac{d}{da}\int_{-\infty}^{\infty} F_X(a-y)f_Y(y)dy \\ =& \int_{-\infty}^{\infty} \frac{d}{da}F_X(a-y)f_Y(y)dy \\ =& \int_{-\infty}^{\infty} f_X(a-y)f_Y(y)dy \\ =& (f_X * f_Y)(a) \blacksquare\end{aligned}$$

Example 31.6
Let X and Y be two independent random variables uniformly distributed on $[0, 1]$. Compute the distribution of $X + Y$.

Solution.
Since

$$f_X(a) = f_Y(a) = \begin{cases} 1 & 0 \leq a \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

by the previous theorem

$$f_{X+Y}(a) = \int_0^1 f_X(a-y)dy.$$

Now the integrand is 0 unless $0 \leq a - y \leq 1$(i.e. unless $a - 1 \leq y \leq a$) and then it is 1. So if $0 \leq a \leq 1$ then

$$f_{X+Y}(a) = \int_0^a dy = a.$$

If $1 < a < 2$ then

$$f_{X+Y}(a) = \int_{a-1}^1 dy = 2 - a.$$

Hence,

$$f_{X+Y}(a) = \begin{cases} a & 0 \leq a \leq 1 \\ 2-a & 1 < a < 2 \\ 0 & \text{otherwise} \ \blacksquare \end{cases}$$

Example 31.7
Let X and Y be two independent exponential random variables with common parameter λ. Compute $f_{X+Y}(a)$.

Solution.
We have

$$f_X(a) = f_Y(a) = \begin{cases} \lambda e^{-\lambda a} & 0 \leq a \\ 0 & \text{otherwise} \end{cases}$$

If $a \geq 0$ then

$$\begin{aligned} f_{X+Y}(a) =& \int_{-\infty}^{\infty} f_X(a-y)f_Y(y)dy \\ =& \lambda^2 \int_0^a e^{-\lambda a} dy = a\lambda^2 e^{-\lambda a} \end{aligned}$$

If $a < 0$ then $f_{X+Y}(a) = 0$. Hence,

$$f_{X+Y}(a) = \begin{cases} a\lambda^2 e^{-\lambda a} & 0 \le a \\ 0 & \text{otherwise} \end{cases} \blacksquare$$

Example 31.8

Let X and Y be two independent random variables, each with the standard normal density. Compute $f_{X+Y}(a)$. the convolution

Solution.

We have

$$f_X(a) = f_Y(a) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

By Theorem 31.1 we have $f_X(a-y)\, f_Y(y)$

$$\begin{aligned} f_{X+Y}(a) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{(a-y)^2}{2}} e^{-\frac{y^2}{2}} dy \\ =& \frac{1}{2\pi} e^{-\frac{a^2}{4}} \int_{-\infty}^{\infty} e^{-(y-\frac{a}{2})^2} dy \\ =& \frac{1}{2\pi} e^{-\frac{a^2}{4}} \sqrt{\pi}\left[\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-w^2} dw\right], \quad w = y - \frac{a}{2} \end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{(a^2-2ay+y^2+y^2)}{2}} dy$$

The expression in the brackets equals 1, since it is the integral of the normal density function with $\mu = 0$ and $\sigma = \frac{1}{\sqrt{2}}$. Hence,

$$f_{X+Y}(a) = \frac{1}{\sqrt{4\pi}} e^{-\frac{a^2}{4}} \blacksquare$$

Example 31.9

Let X and Y be two independent gamma random variables with respective parameters (s, λ) and (t, λ). Show that $X + Y$ is a gamma random variable with parameters $(s + t, \lambda)$.

Solution.

We have

$$f_X(a) = \frac{\lambda e^{-\lambda a}(\lambda a)^{s-1}}{\Gamma(s)} \text{ and } f_Y(a) = \frac{\lambda e^{-\lambda a}(\lambda a)^{t-1}}{\Gamma(t)}$$

By Theorem 31.1 we have

$$\begin{aligned}
f_{X+Y}(a) =&\frac{1}{\Gamma(s)\Gamma(t)}\int_0^a \lambda e^{-\lambda(a-y)}[\lambda(a-y)]^{s-1}\lambda e^{-\lambda y}(\lambda y)^{t-1}dy\\
=&\frac{\lambda^{s+t}e^{-\lambda a}}{\Gamma(s)\Gamma(t)}\int_0^a (a-y)^{s-1}y^{t-1}dy\\
=&\frac{\lambda^{s+t}e^{-\lambda a}a^{s+t-1}}{\Gamma(s)\Gamma(t)}\int_0^1 (1-x)^{s-1}x^{t-1}dx, \quad x=\frac{y}{a}
\end{aligned}$$

But we have from Theorem 29.8 that

$$\int_0^1 (1-x)^{s-1}x^{t-1}dx = B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$

Thus,

$$f_{X+Y}(a) = \frac{\lambda e^{-\lambda a}(\lambda a)^{s+t-1}}{\Gamma(s+t)} \blacksquare$$

2

Example 31.10

The percentages of copper and iron in a certain kind of ore are, respectively, X and Y. If the joint density of these two random variables is given by

$$f_{XY}(x,y) = \begin{cases} \frac{3}{11}(5x+y) & x,y>0, \quad x+2y<2\\ 0 & \text{elsewhere}\end{cases}$$

use the distribution function technique to find the probability density of $Z = X+Y$.

Solution.

Note first that the region of integration is the interior of the triangle with vertices at $(0,0),(0,1)$, and $(2,0)$. From the figure we see that $F(a)=0$ if $a<0$. Now, the two lines $x+y=a$ and $x+2y=2$ intersect at $(2a-2, 2-a)$. If $0\le a<1$ then

$$F_Z(a) = P(Z\le a) = \int_0^a\int_0^{a-y}\frac{3}{11}(5x+y)dxdy = \frac{3}{11}a^3$$

If $1\le a<2$ then

$$\begin{aligned}
F_Z(a) =&P(Z\le a)\\
=&\int_0^{2-a}\int_0^{a-y}\frac{3}{11}(5x+y)dxdy + \int_{2-a}^1\int_0^{2-2y}\frac{3}{11}(5x+y)dxdy\\
=&\frac{3}{11}\left(-\frac{7}{3}a^3+9a^2-8a+\frac{7}{3}\right)
\end{aligned}$$

If $a \geq 2$ then $F_Z(a) = 1$. Differentiating with respect to a we find

$$f_Z(a) = \begin{cases} \frac{9}{11}a^2 & 0 < a \leq 1 \\ \frac{3}{11}(-7a^2 + 18a - 8) & 1 < a < 2 \\ 0 & \text{elsewhere} \ \blacksquare \end{cases}$$

Practice Problems

Problem 31.13
Let X be an exponential random variable with parameter λ and Y be an exponential random variable with parameter 2λ independent of X. Find the probability density function of $X+Y$.

Problem 31.14
Let X be an exponential random variable with parameter λ and Y be a uniform random variable on [0,1] independent of X. Find the probability density function of $X+Y$.

Problem 31.15
Let X and Y be two independent random variables with probability density functions (p.d.f.) , f_X and f_Y respectively. Find the pdf of $X+2Y$.

Problem 31.16
Consider two independent random variables X and Y. Let $f_X(x)=1-\frac{x}{2}$ if $0\le x\le 2$ and 0 otherwise. Let $f_Y(y)=2-2y$ for $0\le y\le 1$ and 0 otherwise. Find the probability density function of $X+Y$.

Problem 31.17
Let X and Y be two independent and identically distributed random variables with common density function

$$f(x)=\begin{cases} 2x & 0<x<1\\ 0 & \text{otherwise}\end{cases}$$

Find the probability density function of $X+Y$.

Problem 31.18
Let X and Y be independent exponential random variables with pairwise distinct respective parameters α and β. Find the probability density function of $X+Y$.

Problem 31.19
A device containing two key components fails when and only when both components fail. The lifetime, T_1 and T_2, of these components are independent with a common density function given by

$$f_{T_1}(t)=f_{T_2}(t)=\begin{cases} e^{-t} & t>0\\ 0 & \text{otherwise}\end{cases}$$

The cost, X, of operating the device until failure is $2T_1+T_2$. Find the density function of X.

Problem 31.20
Let X and Y be independent random variables with density functions

$$f_X(x) = \begin{cases} \frac{1}{2} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{2} & 3 \le y \le 5 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of $X+Y$.

Problem 31.21
Let X and Y be independent random variables with density functions

$$f_X(x) = \begin{cases} \frac{1}{2} & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{y}{2} & 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of $X+Y$.

Problem 31.22
Let X and Y be independent random variables with density functions

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}$$

and

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}$$

Find the probability density function of $X+Y$.

Problem 31.23
Let X have a uniform distribution on the interval $(1,3)$. What is the probability that the sum of 2 independent observations of X is greater than 5?

Problem 31.24

The life (in days) of a certain machine has an exponential distribution with a mean of 1 day. The machine comes supplied with one spare. Find the density function (t measure in days) of the combined life of the machine and its spare if the life of the spare has the same distribution as the first machine, but is independent of the first machine.

Problem 31.25

X_1 and X_2 are independent exponential random variables each with a mean of 1. Find $P(X_1 + X_2 < 1)$.

(c) 0.15 (d) 0.875 (e) X and Y are independent

30.6 (a) $k = \frac{8}{7}$ (b) Yes (c) $\frac{16}{21}$

30.7 (a) We have

$$f_X(x) = \int_0^2 \frac{3x^2+2y}{24}dy = \frac{6x^2+4}{24}, 0 \leq x \leq 2, 0 \text{ otherwise}$$

and

$$f_Y(y) = \int_0^2 \frac{3x^2+2y}{24}dx = \frac{8+4y}{24}, 0 \leq y \leq 2, \quad 0 \text{ otherwise}$$

(b) X and Y are dependent. (c) 0.340

30.8 (a) We have

$$f_X(x) = \int_x^{3-x} \frac{4}{9}dy = \frac{4}{3} - \frac{8}{9}x, \quad 0 \leq x \leq \frac{3}{2}, \quad 0 \text{ otherwise}$$

and

$$f_Y(y) = \begin{cases} \frac{4}{9}y & 0 \leq y \leq \frac{3}{2} \\ \frac{4}{9}(3-y) & \frac{3}{2} \leq y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

(b) $\frac{2}{3}$ (c) X and Y are dependent

30.9 0.469

30.10 0.191

30.11 0.4

30.12 0.19

30.13 0.295

30.14 0.414

30.15 $f(z) = e^{-\frac{1}{2}z} - e^{-z}, \quad z > 0, \ 0$ otherwise

30.16 $f(x) = \frac{2}{(2x+1)^2}, \ x > 0, \quad 0$ otherwise

30.17 $\frac{3}{5}$

30.18 Suppose that X and Y are independent. Then $P(X = 0|Y = 1) = P(X = 0) = 0.6$ and $P(X = 1|Y = 0) = 0.7$. Since $P(X = 0) + P(X = 1) = 0.6 + 0.7 \neq 1$, it follows that X and Y can not be independent.

30.19 $\theta_1 = \frac{1}{4}$ and $\theta_2 = 0$

Section 31

31.1

$$\begin{aligned}
P(Z=0) =&P(X=0)P(Y=0) = (0.1)(0.25) = 0.025\\
P(Z=1) =&P(X=1)P(Y=0) + P(Y=1)P(X=0)\\
=&(0.2)(0.25) + (0.4)(0.1) = 0.09\\
P(Z=2) =&P(X=1)P(Y=1) + P(X=2)P(Y=0) + P(Y=2)P(X=0)\\
=&(0.2)(0.4) + (0.3)(0.25) + (0.35)(0.1) = 0.19\\
P(Z=3) =&P(X=2)P(Y=1) + P(Y=2)P(X=1) + P(X=3)P(Y=0)\\
=&(0.3)(0.4) + (0.35)(0.2) + (0.4)(0.25) = 0.29\\
P(Z=4) =&P(X=2)P(Y=2) + P(X=3)P(Y=1)\\
=&(0.3)(0.35) + (0.4)(0.4) = 0.265\\
P(Z=5) =&P(X=3)P(X=2) = (0.4)(0.35) = 0.14
\end{aligned}$$

and 0 otherwise

31.2 $p_{X+Y}(k) = \left(\begin{array}{c}30\\k\end{array}\right)0.2^k0.8^{30-k}$ for $0 \le k \le 30$ and 0 otherwise.

31.3 $p_{X+Y}(n) = (n-1)p^2(1-p)^{n-2}, \ \ n = 2, \cdots$ and $p_{X+Y}(a) = 0$ otherwise.

31.4

$$\begin{aligned}
p_{X+Y}(3) =&p_X(0)p_Y(3) = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}\\
p_{X+Y}(4) =&p_X(0)p_Y(4) + p_X(1)p_Y(3) = \frac{4}{12}\\
p_{X+Y}(5) =&p_X(1)p_Y(4) + p_X(2)p_Y(3) = \frac{4}{12}\\
p_{X+Y}(6) =&p_X(2)p_Y(4) = \frac{3}{12}
\end{aligned}$$

and 0 otherwise.

31.5 $\frac{1}{64}$

31.6 0.03368

31.7 $P(X+Y=2) = e^{-\lambda}p(1-p) + e^{-\lambda}\lambda p$ (b) $P(Y > X) = e^{-\lambda p}$

31.8

$$\begin{aligned}
p_{X+Y}(0) =& p_X(0)p_Y(0) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}\\
p_{X+Y}(1) =& p_X(0)p_Y(1) + p_X(1)p_Y(0) = \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{4}\\
p_{X+Y}(2) =& p_X(0)p_Y(2) + p_X(2)p_Y(0) + p_X(1)p_Y(1) = \frac{5}{16}\\
p_{X+Y}(3) =& p_X(1)p_Y(2) + p_X(2)p_Y(1) = \frac{1}{8}\\
p_{X+Y}(4) =& p_X(2)p_Y(2) = \frac{1}{16}
\end{aligned}$$

and 0 otherwise.

31.9

$$\begin{aligned}
p_{X+Y}(1) =& p_X(0)p_Y(1) + p_X(1)p_Y(0) = \frac{1}{6}\\
p_{X+Y}(2) =& p_X(0)p_Y(2) + p_X(2)p_Y(0) + p_X(1)p_Y(1) = \frac{5}{18}\\
p_{X+Y}(3) =& p_X(0)p_Y(3) + p_X(1)p_Y(2) + p_X(2)p_Y(1) + p_X(3)p_Y(0) = \frac{6}{18}\\
p_{X+Y}(4) =& p_X(0)p_Y(4) + p_X(1)p_Y(3) + p_X(2)p_Y(2) + p_X(3)p_Y(1) + p_X(4)p_Y(0) = \frac{3}{18}\\
p_{X+Y}(5) =& p_X(0)p_Y(5) + p_X(1)p_Y(4) + p_X(2)p_Y(3) + p_X(3)p_Y(2)\\
&+ p_X(4)p_Y(1) + p_X(4)p_Y(1) = \frac{1}{18}
\end{aligned}$$

and 0 otherwise.

31.10 We have

$$p_{X+Y}(a) = \sum_{n=0}^{a} p(1-p)^n p(1-p)^{a-n} = (a+1)p^2(1-p)^a = C(a+1,a)p^2(1-p)^a$$

Thus, $X+Y$ is a negative binomial with parameters $(2,p)$.

31.11 $9e^{-8}$

31.12 $e^{-10\lambda}\frac{(10\lambda)^{10}}{10!}$

31.13

$$f_{X+Y}(a) = \begin{cases} 2\lambda e^{-\lambda a}(1-e^{-\lambda a}) & 0 \le a \\ 0 & \text{otherwise} \end{cases}$$

31.14

$$f_{X+Y}(a) = \begin{cases} 1 - e^{-\lambda a} & 0 \le a \le 1 \\ e^{-\lambda a}(e^{\lambda} - 1) & a \ge 1 \\ 0 & \text{otherwise} \end{cases}$$

31.15 $f_{X+2Y}(a) = \int_{-\infty}^{\infty} f_X(a - 2y) f_Y(y) dy$
31.16 If $0 \le a \le 1$ then $f_{X+Y}(a) = 2a - \frac{3}{2}a^2 + \frac{a^3}{6}$. If $1 \le a \le 2$ then $f_{X+Y}(a) = \frac{7}{6} - \frac{a}{2}$. If $2 \le a \le 3$ then $f_{X+Y}(a) = \frac{9}{2} - \frac{9}{2}a + \frac{3}{2}a^2 - \frac{1}{6}a^3$. If $a > 3$ then $f_{X+Y}(a) = 0$.
31.17 If $0 \le a \le 1$ then $f_{X+Y}(a) = \frac{2}{3}a^3$. If $1 < a < 2$ then $f_{X+Y}(a) = -\frac{2}{3}a^3 + 4a - \frac{8}{3}$. If $a \ge 2$ then $f_{X+Y}(a) = 0$ and 0 otherwise.
31.18 $f_{X+Y}(a) = \frac{\alpha\beta}{\alpha - \beta}\left(e^{-\beta a} - e^{-\alpha a}\right)$ for $a > 0$ and 0 otherwise.
31.19 $f_{Z+T_2}(a) = e^{-\frac{a}{2}} - e^{-a}, \quad a > 0$ and 0 otherwise.
31.20 If $2 \le a \le 4$ then $f_{X+Y}(a) = \frac{a}{4} - \frac{1}{2}$. If $4 \le a \le 6$, then $f_{X+Y}(a) = \frac{3}{2} - \frac{a}{4}$ and $f_{X+Y}(a) = 0$ otherwise.
31.21 If $0 < a \le 2$ then $f_{X+Y}(a) = \frac{a^2}{8}$. If $2 < a < 4$ then $f_{X+Y}(a) = -\frac{a^2}{8} + \frac{a}{2}$ and 0 otherwise.
31.22 $f_{X+Y}(a) = \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} e^{-(a - (\mu_1 + \mu_2))^2/[2(\sigma_1^2 + \sigma_2^2)]}$.
31.23 $\frac{1}{8}$
31.24 $f_T(t) = \int_0^t e^{-t} ds = te^{-t}$ for $t > 0$ and 0 otherwise.
31.25 $1 - 2e^{-1}$

Section 32
32.1 $p_{X|Y}(0|1) = 0.25$ and $p_{X|Y}(1|1) = 0.75$ and 0 otherwise.
32.2 (a) For $1 \le j \le 5$ and $i = 1, \cdots, j$ then we have $p_{XY}(j, i) = \left(\frac{1}{5}\right)\left(\frac{1}{j}\right)$ and 0 otherwise.
(b) $p_{X|Y}(j|i) = \frac{\frac{1}{5j}}{\sum_{k=i}^{5}\left(\frac{1}{5k}\right)}$ and 0 otherwise.
(c) X and Y are dependent