Stokes's Theorem

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1. Stokes's Theorem

Theorem (Stokes's Theorem)

Let S be an oriented, piecewise-smooth surface with unit normal vector $\mathbf{n}$, bounded by the simple closed, piecewise-smooth boundary curve C having positive orientation. Let $\mathbf{F}(x, y, z)$ be a vector field continuously differentiable in some open domain containing S. Then,

$$\oint_C \mathbf{F}.d\mathbf{r} = \oint_C \mathbf{F}.T ds = \iint_S \mathrm{curl} F.\mathbf{n} dS.$$

$\mathbf{r} = (x, y, z)$ is the position vector, $d\mathbf{r} = (dx, dy, dz)$, the unit tangent vector to S at $\mathbf{r} = (x, y, z)$ is

$$T = \frac{dx}{ds}\overrightarrow{\mathrm{i}} + \frac{dy}{ds}\overrightarrow{\mathbf{j}} + \frac{dz}{ds}\overrightarrow{\mathbf{k}}.$$

Hence $d\mathbf{r} = dTds$.
If the surface S is defined by $z = g(x, y)$ on a region $R_{x,y}$, then $\iint_S \mathrm{curl}F.\mathbf{n}dS = \iint_{R_{x,y}} (-M_1 g_x - N_1 g_y + P_1)dA$, where $g_x = \dfrac{\partial g}{\partial x}$, $g_y = \dfrac{\partial g}{\partial y}$ and $\mathrm{curl}F = (M_1, N_1, P_1)$.

Eample

Use Stoke's Theorem to evaluate the line integral $\oint_C (y+2z)\,dx + (x+2z)\,dy + (x+2y)\,dz$, where C is the curve formed by intersection of the sphere $x^2+y^2+z^2=1$ with the plane $x+2y+2z=0$.

Solution: Let S be the circle cut by the sphere from the plane. Find the coordinates of the unit normal vector $\mathbf{n}$ to the surface S,

$$\mathbf{n} = \frac{1\cdot\overrightarrow{\mathrm{i}} + 2\cdot\overrightarrow{\mathbf{j}} + 2\cdot\overrightarrow{\mathbf{k}}}{\sqrt{1^2+2^2+2^2}} = \frac{1}{3}\overrightarrow{\mathrm{i}} + \frac{2}{3}\overrightarrow{\mathbf{j}} + \frac{2}{3}\overrightarrow{\mathbf{k}}.$$

In this case $P=y+2z,\quad Q=x+2z,\quad R=x+2y$. Hence, the curl of the vector $\mathbf{F}$ is

$$
\begin{aligned}
\nabla \times \mathbf{F} &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\overrightarrow{\mathrm{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\overrightarrow{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\overrightarrow{\mathbf{k}} \\
&= (2-2)\,\overrightarrow{\mathrm{i}} + (2-1)\,\overrightarrow{\mathbf{j}} + (1-1)\,\overrightarrow{\mathbf{k}} = \overrightarrow{\mathbf{j}}\,.
\end{aligned}
$$

Using Stoke's Theorem, we have

$$\begin{aligned} \oint_C (y+2z)\,dx + (x+2z)\,dy + (x+2y)\,dz &= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS \\ &= \iint_S \overrightarrow{\mathbf{j}} \cdot \Big(\frac{1}{3}\overrightarrow{\mathrm{i}} + \frac{2}{3}\overrightarrow{\mathbf{j}} + \\ &= \frac{2}{3}\iint_S dS. \end{aligned}$$

As the sphere $x^2+y^2+z^2=1$ is centered at the origin and the plane $x+2y+2z=0$ also passes through the origin, the cross section is the circle of radius 1. Hence the integral is

$$I = \frac{2}{3}\iint_S dS = \frac{2}{3}\cdot\pi\cdot 1^2 = \frac{2\pi}{3}.$$

Eample

Use Stoke's Theorem to calculate the line integral

$$\oint_C y^3 dx - x^3 dy + z^3 dz.$$

The curve C is the intersection of the cylinder $x^2 + y^2 = a^2$ and the plane $x + y + z = b$.

Solution

We suppose that S is the part of the plane cut by the cylinder. The curve C is oriented counterclockwise when viewed from the end of the normal vector $\mathbf{n}$ which has coordinates

$$\mathbf{n} = \frac{1 \cdot \overrightarrow{\mathrm{i}} + 1 \cdot \overrightarrow{\mathbf{j}} + 1 \cdot \overrightarrow{\mathbf{k}}}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}} \overrightarrow{\mathrm{i}} + \frac{1}{\sqrt{3}} \overrightarrow{\mathbf{j}} + \frac{1}{\sqrt{3}} \overrightarrow{\mathbf{k}}.$$

As $P = y^3$, $Q = -x^3$, $R = z^3$, we can write:

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\overrightarrow{\mathrm{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\overrightarrow{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\overrightarrow{\mathbf{k}} \\ &= -3\left(x^2 + y^2\right)\overrightarrow{\mathbf{k}}.\end{aligned}$$

Applying Stoke's Theorem, we find:

$$\begin{aligned}
I &= \oint_C y^3 dx - x^3 dy + z^3 dz \\
&= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS \\
&= = \iint_S \left(-3\left(x^2+y^2\right)\overrightarrow{\mathbf{k}}\right) \cdot \left(\frac{1}{\sqrt{3}}\overrightarrow{\mathrm{i}} + \frac{1}{\sqrt{3}}\overrightarrow{\mathbf{j}} + \frac{1}{\sqrt{3}}\overrightarrow{\mathbf{k}}\right) dS \\
&= -\sqrt{3} \iint_S \left(x^2+y^2\right) dS.
\end{aligned}$$

We can express the surface integral in terms of the double integral:

$$\begin{aligned}
I &= -\sqrt{3} \iint_S \left(x^2+y^2\right) dS \\
&= -\sqrt{3} \iint_{D(0,a)} \left(x^2+y^2\right) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy.
\end{aligned}$$

The equation of the plane is $z = b - x - y$, so the square root in the integrand is equal to

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + (-1)^2 + (-1)^2} = \sqrt{3}.$$

Hence,

$$I = -\sqrt{3} \iint_{D(0,a)} \left(x^2 + y^2\right) \sqrt{3} dxdy = -3 \iint_{D(x,y)} \left(x^2 + y^2\right) dxdy.$$

By changing to polar coordinates, we get

$$I = -3 \int_0^{2\pi} \int_0^a r^3 dr d\theta = -3 \cdot 2\pi \cdot \left.\frac{r^4}{4}\right|_0^a = -\frac{3\pi a^4}{2}.$$

Eample

Use Stoke's Theorem to evaluate the line integral

$$\oint_C (x+z)\,dx + (x-y)\,dy + xdz.$$

The curve C is the ellipse defined by the equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$, $z = 1$.

Solution:

Let the surface S be the part of the plane $z = 1$ bounded by the ellipse. Obviously that the unit normal vector is $\mathbf{n} = \mathbf{k}$. Since $P = x + z,\ \ Q = x - y,\ \ R = x$, then the curl of the vector field $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\overrightarrow{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\overrightarrow{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\overrightarrow{\mathbf{k}}$$
$$= (1-0)\,\overrightarrow{\mathbf{k}} = \overrightarrow{\mathbf{k}}.$$

By Stoke's Theorem,

$$\oint_C (x+z)\,dx + (x-y)\,dy + xdz = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$
$$= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$
$$= \iint_S \overrightarrow{\mathbf{k}} \cdot \overrightarrow{\mathbf{k}}\, dS = \iint_S dS.$$

The double integral in the latter formula is the area of the ellipse. Therefore, the integral is

$$\iint_S dS = \pi \cdot 2 \cdot 3 = 6\pi.$$

Eample

Show that the line integral $\oint_C yzdx + xzdy + xydz$ is zero along any closed contour C.

Solution :

Let S be a surface bounded by a closed curve C. Applying Stoke's formula, we identify that $P = yz, \;\; Q = xz, \;\; R = xy$.

Then

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\overrightarrow{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\overrightarrow{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\overrightarrow{\mathbf{k}} \\ &= (x - x)\overrightarrow{\mathbf{i}} + (y - y)\overrightarrow{\mathbf{j}} + (z - z)\overrightarrow{\mathbf{k}} = 0 \cdot \overrightarrow{\mathbf{i}} + 0 \cdot \overrightarrow{\mathbf{j}} + 0 \cdot \overrightarrow{\mathbf{k}}\end{aligned}$$

Hence, the line integral:

$$\oint_C yzdx + xzdy + xydz = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}dS = \iint_S \mathbf{0} \cdot \mathbf{n}dS = 0.$$