

Question (1): If X is the number of "3"s that turn up when 50 ordinary dice are independently thrown,

a. The probability function of X .

$$n = 50 \\ p = \frac{1}{6}$$

$$X \sim B(n, p)$$

$$f(x) = \binom{50}{x} \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{50-x}, \quad x = 0, \dots, 50$$

b. The expected value of $(X-1)^2$.

$$E((X-1)^2) = \sum (x-1)^2 \cdot f(x)$$

$$E(X^2 - 2X + 1)$$

$$E(X^2) - 2E(X) + 1$$

$$\frac{11375}{18} - 2\left(\frac{25}{3}\right) + 1$$

$$\boxed{49.75}$$

$$E(X) = np = 50 \times \frac{1}{6} = \frac{25}{3}$$

$$Var(X) = npq$$

$$50 \times \frac{1}{6} \times \frac{5}{6} = \frac{125}{18}$$

$$E(X^2) = Var(X) + (E(X))^2 = \frac{125}{18} + \left(\frac{25}{3}\right)^2$$

$$E(X^2) = \frac{11375}{18}$$

c. $P(X > 1 | X \leq 3) = \frac{P(X > 1 \cap X \leq 3)}{P(X \leq 3)}$

$$P(X > 1 | X \leq 3)$$

$$\frac{P(X > 1 \cap X \leq 3)}{P(X \leq 3)}$$

$$\boxed{0.25}$$

$$\frac{P(1 < X \leq 3)}{P(X \leq 3)}$$

$$= \frac{P(X \leq 3) - P(X \leq 1)}{P(X \leq 3)}$$

$$= \frac{0.023 - 1.098 \times 10^{-4}}{0.023}$$

$$= 0.99$$

Question (2): An urn contains 10 blue and 4 red balls. 6 balls are chosen at random and without replacement from the urn. If X is the number of red balls chosen, find

a. The distribution of X

b. The standard deviation of $2X + 1$.

Question (3): The time to failure of a component in an electronic device has an exponential distribution with a median of four hours.

a. The parameter λ value is

$$P(X < 4) = 0.5$$

$$1 - e^{-\lambda 4} = 0.5$$

$$e^{-\lambda 4} = 0.5$$

$$-\lambda 4 = \ln(0.5)$$

$$\lambda = \frac{\ln(0.5)}{-4}$$

$$\lambda = 0.17$$

$$f(x) = \lambda e^{-\lambda x}$$
$$F(x) = 1 - e^{-\lambda x}$$

15. The probability that the component will work without failing for at least five hours.

$$P(X > 5) = e^{-\lambda x} = e^{-0.17 \times 5} = 0.42$$

c. $P(X > 6 | X > 4) = P(X > 2)$

$$\frac{P(X > 6 \cap X > 4)}{P(X > 4)} = \frac{e^{-\lambda x}}{e^{-\lambda x}} = \frac{e^{-0.17(6)}}{e^{-0.17(4)}} = 0.71$$

16. If X, Y, Z are independent exponential random variables with parameter λ , what is the variance of W , where $W = \min(X, Y, Z)$

$$\lambda = 0.17 + 0.17 + 0.17$$

$$\lambda = 0.51$$

$$\text{Var}(X) = \frac{1}{\lambda^2} = \frac{1}{(0.51)^2}$$

Question (4): On any given day, a certain machine has either no malfunctions or exactly one malfunction. The probability of malfunction on any given day is 0.40. Machine malfunctions on different days are mutually independent.

- a. The probability distribution of X represents the number of no malfunction days before the second malfunction occurs.

$$r = 2$$

$$X \sim \text{NB}(2, 0.4)$$

$$f(x) = \binom{r+x-1}{x} p^r q^x$$

$$f(x) = \binom{2+x-1}{x} (0.4)^2 (0.6)^x, x = 0, 1, \dots$$

- b. The expected number of no malfunction days before the 5th malfunction occurs.

$p = 5$

$$E(X) = \frac{qp}{p} = \frac{0.6 \times 5}{0.4} = 7.5$$

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- c. The probability of having seven no malfunction days before the first malfunction machine occurs, given that there were ^{more than} ~~already~~ four no malfunction days before the first malfunction machine occurs.

$p = 1$

$$P(X=7 \mid X > 4) = P(X=3)$$

$$P(X=3) = \binom{1+3-1}{3} (0.4)^3 (0.6) = 0.0864$$

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Question (5): Let X a discrete uniform random variable on points $1, \dots, 12$.

- a. The mean and the standard deviation of X .

$$f(x) = \frac{1}{N}$$

$$E(X) = \frac{N+1}{2} = \frac{12+1}{2} = \frac{13}{2} = 6.5$$

$$Var(X) = \frac{N^2-1}{12} = \frac{(12^2)-1}{12} = \frac{143}{12}$$

$$SD = \sqrt{\frac{143}{12}} = 3.4517$$

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- b. The approximate probability $P(3 \leq X \leq 10) =$

$$P(3 \leq X \leq 10)$$

Question (6): The random variable X has a Poisson distribution such that

$$P(X = 0) = P(X = 1)$$

- a. The moment generating function.

b. $P(X > 2) =$

$$1 - P(X \leq 2)$$

$$1 - \sum_{x=0}^2 \frac{e^{-\lambda} \lambda^x}{x!}$$

$$1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$1 - [0.367 + 0.367 + 0.183] = 0.083$$

Question (7): let X has a normal distribution with mean μ and variance σ^2 .

- a. The probability that X is between $\mu - 1.5\sigma$ and the mode.

$$P(\mu - 1.5\sigma < X < \mu)$$

$$P\left(\frac{\mu - 1.5\sigma - \mu}{\sigma} < Z < \frac{\mu - \mu}{\sigma}\right)$$

$$P(-1.5 < Z < 0)$$

$$P(Z < 0) - P(Z < -1.5)$$

$$0.5 - 0.0668 = 0.4332$$

- b. 75.17% percentile of X .

$$P(X < K) = 0.7517$$

$$\frac{K - \mu}{\sigma} = Z_{0.7517}$$

$$\frac{K - \mu}{\sigma} = 0.68$$

$$K = 0.68\sigma + \mu$$

$$E(X) = \mu$$

- c. If $\mu = 50$, and $\sigma^2 = 10$ and a random variable $Y = e^X$, then $E(Y) =$

Question (8): Losses covered by an insurance policy are modelled by a uniform distribution on the interval $[0, b]$ and the median of coverage losses is 10.

a. The value of b .

$$P(X < 10) = 0.5$$

$$\frac{10 - 0}{b - 0} = 0.5$$

$$\frac{10}{b} = 0.5$$

$$b = \frac{10}{0.5}$$

$$f(x) = \frac{1}{20}$$

$$b = 20$$

b. The probability that the coverage losses is more than 4.

$$1 - P(X \leq 4)$$

c. The standard deviation of the coverage losses.

$$Var(X) = \frac{(b-a)^2}{12} = \frac{100}{3}$$

$$SD = \sqrt{Var(X)} = \sqrt{\frac{100}{3}} = 5.77$$

$$\frac{e^{-\lambda} \lambda^x}{x!}$$

Question (9): Let X has the following probability function

$$f(x) = \frac{(0.2) 3^x}{x!} e^{-3} + \frac{(0.8) 8^x}{x!} e^{-8}$$

a. What is the distribution of X .

$$-0.25$$

Mixture Poisson

with

$$a_1 = 0.2 \quad a_2 = 0.8$$

$$\lambda = 3$$

$$\lambda = 8$$

b. $E(X^2) =$

$$E(X^2) = \text{Var}(X) + (E(X))^2$$

$$3 + (3)^2 = 12$$

$$E(X^2) = 9, E(X^2) + 9 E(X^2)$$

$$E(X^2) = 8 + (8)^2 = 72$$

$$0.2(12) + 0.8(72) = 60$$

$$\text{Var}(X) = 3$$

$$E(X) = 3$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$3 = E(X^2) - 9$$

$$E(X) = 12$$

Question (10): Let X be a random variable with moment generating function

$$M_X(t) = \left(\frac{3}{3-t}\right)^2, 0 < t < 3$$

$$B=3$$

$$Q=2$$

a. $E(X-2) =$

$$E(X) = \frac{Q}{B}$$

$$E(X) = 2$$

$$\frac{2}{3} = 2$$

$$= \frac{4}{3}$$

End of Questions