

# Spherical Coordinates

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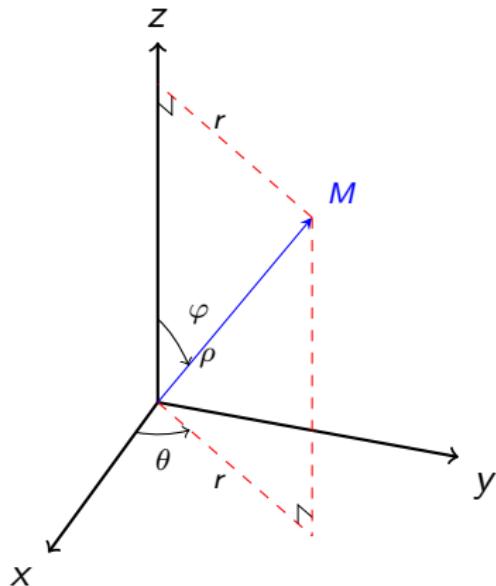
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# ① Spherical and Cartesian Coordinates

$$\begin{cases} x = \rho \cos \theta \sin \varphi \\ y = \rho \sin \theta \sin \varphi \\ z = \rho \cos \varphi \end{cases}$$

$r = \rho \sin \varphi$ ,  $\rho \geq 0$ ,  $\varphi \in [0, \pi]$  and  $\theta \in [0, 2\pi]$ .



# Example

Consider the points  $M$  and  $N$  of coordinates  $(10, \frac{\pi}{2}, \frac{\pi}{6})$  and  $(6, \frac{\pi}{3}, \frac{\pi}{6})$  respectively in spherical coordinates:

- ① The rectangular coordinates of  $M$  are:  $(0, 5, 5\sqrt{3})$ .

The rectangular coordinates of  $N$  are:  $(\frac{3}{2}, \frac{3\sqrt{3}}{2}, 3)$ .

- ② The cylindrical coordinates of  $M$  are:  $(5, \frac{\pi}{2}, 5\sqrt{3})$ .

The cylindrical coordinates of  $N$  are:  $(3, \frac{\pi}{2}, 3)$ .

# Example

Change the rectangular coordinates  $(1, \sqrt{3}, 0)$  to

- ① the spherical coordinates
- ② the cylindrical coordinates.

$$\rho^2 = x^2 + y^2 + z^2 = 4, \text{ then } \rho = 2.$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1} \sqrt{3}, \theta = \frac{\pi}{3}.$$

$$\varphi = \cos^{-1}\left(\frac{z}{\rho}\right) = \frac{\pi}{2}.$$

Thus the spherical coordinates are:  $(2, \frac{\pi}{3}, \frac{\pi}{2})$ .

for the cylindrical coordinates, we have  $r^2 = x^2 + y^2 = 4$ , then  
 $r = 2$ .

The cylindrical coordinates are  $(2, \frac{\pi}{3}, 0)$ .

# Example

Change the cylindrical coordinates  $(4, -\frac{\pi}{2}, 6)$  to

- ① the rectangular coordinates
- ② the spherical coordinates.

for the rectangular coordinates, we have  $x = r \cos \theta = 0$ .

$y = r \sin \theta = -2$  and  $z = 6$ .

For the spherical coordinates, we have  $\rho^2 = x^2 + y^2 + z^2 = 4 + 36$ ,  
then  $\rho = 2\sqrt{10}$ .

$$\theta = -\frac{\pi}{2}, z = 6 = \rho \cos \varphi, \text{ then } \varphi = \cos^{-1} \frac{3}{\sqrt{10}}.$$

Thus the spherical coordinates are:  $(2\sqrt{10}, -\frac{\pi}{2}, \cos^{-1} \frac{3}{\sqrt{10}})$ .

# Example

- ① Describe the graph of the equation  $\rho = \sec \varphi = 6$ .

$\rho \sec \varphi = 6 \iff \rho = 6 \cos \varphi \iff \rho^2 = 6z$ . Hence  
 $x^2 + y^2 + z^2 = \rho^2 \iff x^2 + y^2 + (z - 3)^2 = 9$ . This is the  
equation of the sphere of radius 3 and center  $(0, 0, 3)$ .

- ② Describe the graph of the equation  $\rho = 6 \sin \varphi \cos \theta$ .

We know that  $x = \rho \sin \varphi \cos \theta$ , then  $\frac{x}{\rho} = \sin \varphi \cos \theta$ . Then  
 $\rho^2 = 6x$ . This is the equation of the sphere of radius 3 and  
center  $(3, 0, 0)$ .

# Example

Change the equations to spherical coordinates:

①  $x^2 + y^2 = 4z$

②  $x^2 - 4z^2 + y^2 = 0$

①  $x^2 + y^2 = 4z \iff \rho^2 \sin^2 \varphi = 4\rho \cos \varphi$ , which is equivalent  
to:  $\rho = 4 \frac{\cos \varphi}{\sin^2 \varphi}$ .

②  $x^2 - 4z^2 + y^2 = 0 \iff \rho^2 \sin^2 \varphi = 4\rho^2 \cos^2 \varphi \iff \tan^2 \varphi = 4$ .

# Example

Let  $B_R$  the ball of center  $(0, 0, 0)$  and radius  $R$ .

$$\begin{aligned}\iiint_{B_R} \sqrt{x^2 + y^2 + z^2} dxdydz &= \int_0^{2\pi} \int_0^\pi \int_0^R \rho^3 \sin \varphi d\rho d\theta d\varphi \\ &= 4\pi \int_0^R \rho^3 d\rho = \pi R^4.\end{aligned}$$

# Example

Consider the solid delimited by the plane  $xoy$ , the cylinder  $x^2 + y^2 = x$  and the sphere  $x^2 + y^2 + z^2 = 1$ .

The solid is defined by:  $\begin{cases} x^2 + y^2 + z^2 \leq 1 \\ x^2 + y^2 \leq x \\ z \geq 0. \end{cases}$

In cylindrical coordinates, we have  $\begin{cases} \rho^2 + z^2 \leq 1 \\ \rho^2 \leq \rho \cos \theta \\ z \geq 0. \end{cases}$

Then  $\rho \in [0, 1]$  and if  $\rho$  is fixed,  $\theta$  is in  $[-\cos^{-1} \rho, \cos^{-1} \rho]$  and  $z \in [0, \sqrt{1 - \rho^2}]$ . Then the volume of the solid is given by:

$$\begin{aligned} V &= \int_0^1 \int_{-\cos^{-1} \rho}^{\cos^{-1} \rho} \int_0^{\sqrt{1-\rho^2}} \rho dz d\theta d\rho \\ &= \int_0^1 2\rho \sqrt{1-\rho^2} \cos^{-1} \rho d\rho = \frac{2}{3} \left( \frac{\pi}{2} - \frac{2}{3} \right). \end{aligned}$$

# Example

Compute  $\iiint_D f(x, y, z) dxdydz$ , for  $D = \{x^2 + y^2 + z^2 \leq R^2\}$ ,

$$f(x, y, z) = \frac{1}{\sqrt{a^2 - x^2 - y^2 - z^2}} \quad (a > R > 0),$$

# Exercises

## Exercise 1 :

Set up an integral for the volume of the region bounded by the cone  $z = \sqrt{3(x^2 + y^2)}$  and the hemisphere  $z = \sqrt{4 - x^2 - y^2}$ . Using the conversion formulas from rectangular coordinates to spherical coordinates, we have:

For the cone:  $z = \sqrt{3(x^2 + y^2)}$  or  $\rho \cos \varphi = \sqrt{3}\rho \sin \varphi$ , or  $\tan \varphi = \frac{1}{\sqrt{3}}$ . the  $\varphi = \frac{\pi}{6}$ .

For the sphere:  $z = \sqrt{4 - x^2 - y^2}$  or  $\rho = 2$ , then

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^2 \rho^2 \sin \varphi d\rho d\varphi d\theta = \frac{16\pi}{3} \left(1 - \frac{\sqrt{3}}{2}\right).$$

**Exercise 2 :**

Let  $E$  be the region bounded below by the cone  $z = \sqrt{x^2 + y^2}$  and above by the sphere  $z = x^2 + y^2 + z^2$ .

Use the conversion formulas to write the equations of the sphere and cone in spherical coordinates.

For the sphere:

$$x^2 + y^2 + z^2 = z \iff \rho^2 = \rho \cos \varphi \iff \rho = \cos \varphi.$$

For the cone:

$$\begin{aligned} z = \sqrt{x^2 + y^2} &\iff \rho \cos \varphi = \sqrt{\rho^2 \sin^2 \varphi (\cos^2 \phi + \sin^2 \phi)} \\ &\iff \rho \cos \varphi = \rho \sin \varphi \\ &\iff \cos \varphi = \sin \varphi. \end{aligned}$$

Then  $\varphi = \frac{\pi}{4}$ .

Then the integral for the volume of the solid region  $E$  becomes:

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\cos \varphi} \rho^2 \sin \varphi d\rho d\varphi d\theta = \frac{\pi}{8}.$$

Evaluate the triple integrals  $\iiint_E f(x, y, z) dx dy dz$ .

- ①  $f(x, y, z) = 1 - \sqrt{x^2 + y^2 + z^2}$ ,  $E = \{(x, y, z) : x^2 + y^2 + z^2 \leq 9, y \geq 0, z \geq 0\}$ ,

Using the spherical coordinates, we get:

$$\iiint_E f(x, y, z) dx dy dz = \int_0^3 \int_0^{\frac{\pi}{2}} \int_0^\pi (1 - \rho) d\theta d\varphi d\rho = -\frac{3\pi^2}{4}.$$

- ②  $f(\rho, \theta, \varphi) = \rho \sin \varphi (\cos \theta + \sin \theta)$ ,  $E = \{(\rho, \theta, \varphi) : 1 \leq \rho \leq 2, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq \frac{\pi}{2}\}$ ,

$$\iiint_E f(\rho, \theta, \varphi) d\rho d\theta d\varphi = \int_1^2 \int_0^\pi \int_0^{\frac{\pi}{2}} \rho \sin \varphi (\cos \theta + \sin \theta) d\varphi d\theta d\rho =$$

- ③  $f(x, y, z) = 2xy$ ;  $E = \{(x, y, z) : \sqrt{x^2 + y^2} \leq z \leq \sqrt{1 - x^2 - y^2}, x \geq 0, y \geq 0\}$ ,

- ④  $f(\rho, \theta, \varphi) = \rho \cos \varphi; E = \{(\rho, \theta, \varphi) : 0 \leq \rho \leq 2 \cos \varphi, 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \varphi \leq \frac{\pi}{4}\},$
- ⑤ In the following exercises, find the volume of the solid  $E$  whose boundaries are given in rectangular coordinates.  $E = \{(x, y, z) : \sqrt{x^2 + y^2} \leq z \leq \sqrt{16 - x^2 - y^2}, x \geq 0, y \geq 0\},$   $E = \{(x, y, z) : x^2 + y^2 + z^2 - 2z \leq 0, \sqrt{x^2 + y^2} \leq z\},$

- 6 Convert the integral into an integral in spherical coordinates:

$$\begin{aligned} \textcircled{1} \quad & \int_{-4}^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} (x^2 + y^2 + z^2)^2 dz dy dx, \\ \textcircled{2} \quad & \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} dz dy dx. \end{aligned}$$

**Exercise 3 :**

Compute the center of mass of an object that occupies the upper hemisphere of  $x^2 + y^2 + z^2 = 1$  and has density  $\rho(x, y) = x^2 + y^2$ .

**Exercise 4 :**

Compute the center of mass of an object that occupies the surface  $z = xy$ ,  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$  and has density  $\rho(x, y) = \sqrt{1 + x^2 + y^2}$ .

**Exercise 5 :**

Compute the center of mass of an object that occupies the surface  $z = \sqrt{x^2 + y^2}$ ,  $1 \leq z \leq 4$  and has density  $\rho(x, y) = x^2z$ .

**Exercise 6 :**

Compute the centroid of the surface of a right circular cone of height  $h$  and base radius  $R$ , not including the base.