King Saud University: Mathematics Department Math-254
Second Semester 1446 H First Midterm Exam.
Maximum Marks = 25 Time: 90 mins.

Questions: (6 + 6 + 6 + 7) Marks

Question 1:

One of the possible rearrangement of the nonlinear equation $e^x = x+2$, which has root in $[1,2]$ is

$$x_{n+1} = g(x_n) = \ln(x_n + 2); \qquad n = 0, 1, \ldots.$$

Show that $g(x)$ has a unique fixed-point in $[1,2]$. If $x_0 = 1.5$ then estimate the number of iterations n within accuracy 10^{-5}.

Solution. Since, we observe that $f(1)f(2) = (-0.2817)(3.3891) < 0$, then the solution we seek is in the interval $[1,2]$.
For continuous function $g(x) = \ln(x+2)$ in the interval $[1,2]$, we see that $g(x)$ is increasing function of x, and $g(1) = \ln(3) = 1.0986$ and $g(2) = \ln(4) = 1.3863$, both lie in the interval $[1,2]$. Thus $g(x) \in [1,2]$, for all $x \in [1,2]$. Also, for function $g(x) = \ln(x+2)$, we have its derivative $g'(x) = 1/(x+2) < 1$, for all x in the given interval $[1,2]$, so from fixed-point theorem the $g(x)$ has a unique fixed-point.
Using the given initial approximation $x_0 = 1.5$, we have the other approximations as

$$x_1 = g(x_0) = \ln(x_0 + 2) = \ln(1.5 + 2) = 1.2528.$$

Since $a = 1$ and $b = 2$, then the value of k can be found as follows

$$k_1 = |g'(1)| = |1/3| = 0.3333 \quad \text{and} \quad k_2 = |g'(2)| = |1/4| = 0.25,$$

which give $k = \max\{k_1, k_2\} = 0.3333$. From the error bound formula, we have

$$\frac{k^n}{1-k}|x_1 - x_0| \leq 10^{-5},$$

or

$$\frac{(0.3333)^n}{1-0.3333}|1.2528 - 1.5| \leq 10^{-5}, \quad n\ln(0.3333) \leq \ln(0.0270), \quad \text{gives,} \quad n \geq 9.5756.$$

So we need 10 (ten) approximations to get the desired accuracy for the given problem. •

Question 2:

Successive approximations x_n to the desired root are generated by the scheme

$$x_{n+1} = \frac{e^{x_n}(x_n + 1) + 2x_n^2}{e^{x_n} + 3x_n}, \qquad n \geq 0.$$

Find the nonlinear equation $f(x) = 0$ and then show that it has a zero in $[-1,0]$. Use the Newton's method to find the second approximation of the root $\alpha = -0.7035$, starting with

$x_0 = -0.5$. Compute the relative error.

Solution. Given

$$x_{n+1} = \frac{e^{x_n}(x_n+1) + 2x_n^2}{e^{x_n} + 3x_n} = g(x_n), \qquad n \geq 0.$$

$$x = \frac{e^{x}(x+1) + 2x^2}{e^{x} + 3x} = g(x),$$

$$g(x) - x = \frac{e^{x}(x+1) + 2x^2}{e^{x} + 3x} - x = 0,$$

$$g(x) - x = \frac{e^{x}(x+1) + 2x^2 - x(e^x + 3x)}{e^{x} + 3x} = 0,$$

and after simplifying, we obtained

$$g(x) - x = \frac{(xe^x + e^x + 2x^2 - xe^x - 3x^2)}{e^x + 3x} = \frac{(e^x - x^2)}{e^x + 3x} = -\frac{(x^2 - e^x)}{e^x + 3x} = x^2 - e^x = 0.$$

Thus

$$f(x) = g(x) - x = x^2 - e^x = 0,$$

and we can check

$$f(-1) = 0.6321, \qquad f(0) = -1, \qquad f(-1)f(0) = -0.6321 < 0,$$

so $f(x)$ has a zero in $[-1, 0]$. Applying Newton's iterative formula to find the approximation of this zero, we use the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n \geq 0,$$

where

$$f'(x) = 2x - e^x.$$

Thus

$$x_{n+1} = x_n - \frac{(x_n^2 - e^{x_n})}{(2x_n - e^{x_n})}, \qquad n \geq 0.$$

Finding the second approximation using the initial approximations $x_0 = -0.5$, we get

$$x_1 = x_0 - \frac{(x_0^2 - e^{x_0})}{(2x_0 - e^{x_0})} = -0.7219,$$

and

$$x_2 = x_1 - \frac{(x_1^2 - e^{x_1})}{(2x_1 - e^{x_1})} = -0.7036.$$

The relative error is,

$$\frac{|\alpha - x_2|}{|\alpha|} = \frac{|-0.7035 - (-0.7036)|}{|-0.7035|} = 0.00014215 = 1.4215e-04.$$

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Question 3:

Show that the order of convergence of the iterative scheme

$$x_{n+1} = \frac{x_n(x_n^2+3)}{3x_n^2+1}, \qquad n \geq 0,$$

at the fixed-point $\alpha = 1$ is at least cubically.

Solution. Since the given iterative scheme is

$$x_{n+1} = \frac{x_n(x_n^2+3)}{3x_n^2+1} = g(x_n),$$

which gives,

$$g(x) = \frac{x(x^2+3)}{3x^2+1} = \frac{x^3+3x}{3x^2+1}.$$

The first derivative of $g(x)$ can be found as

$$g'(x) = \frac{3x^4-6x^2+3}{(3x^2+1)^2},$$

and

$$g'(1) = \frac{3-6+3}{(4)^2} = 0.$$

Therefore, the order of convergence for the given iteration is at least quadratic. One can find the second derivative of $g(x)$ as

$$g''(x) = \frac{(3x^2+1)^2(12x^3-12x)-(3x^4-6x^2+3)(12x(3x^2+1))}{(3x^2+1)^4},$$

and

$$g''(1) = \frac{(4)^2(12-12)-(3-6+3)(12(4))}{(4)^4} = 0.$$

Thus, the order of convergence for the given iterative scheme is at least cubically. •

Question 4:

Show that the best iterative formula for computing the approximation of the root $\alpha = 0$ of the equation $x^3e^{2x} = 0$ is

$$x_{n+1} = x_n - \frac{3x_n}{(3+2x_n)}, \qquad n \geq 0.$$

Use it to find the absolute error $|\alpha - x_2|$ using $x_0 = 0.1$. Show that the given iterative formula is converges quadratically to a root $\alpha = 0$.

Solution. Since $\alpha = 0$ is a zero of $f(x)$, so

$$\begin{array}{rclrcl} f(x) & = & x^3e^{2x}, & f(0) & = & 0, \\ f'(x) & = & (3x^2+2x^3)e^{2x}, & f'(0) & = & 0, \\ f''(x) & = & (6x+12x^2+4x^3)e^{2x}, & f''(0) & = & 0, \\ f'''(x) & = & (6+36x+36x^2+8x^3)e^{2x}, & f'''(0) & = & 6 \neq 0, \end{array}$$

the function has zero of multiplicity 3. Using modified Newton's iterative formula, we get

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} = x_n - 3\frac{x^3 e^{2x}}{x^2(3+2x)e^{2x}} = x_n - \frac{3x_n}{(3+2x_n)}, \quad n \geq 0.$$

Now evaluating this at the give approximation $x_0 = 0.1$, gives

$$x_1 = x_0 - \frac{3x_0}{(3+2x_0)} = 0.0062 \quad \text{and} \quad x_2 = x_1 - \frac{3x_1}{(3+2x_1)} = 2.5934 \times 10^{-5}.$$

Thus the absolute error is

$$|\alpha - x_2| = |0 - 2.5934 \times 10^{-5}| = 2.5934 \times 10^{-5}.$$

Since the fixed point form of the given iterative (modified Newton's) formula is

$$x_{n+1} = g(x_n) = x_n - \frac{3x_n}{(3+2x_n)},$$

where

$$g(x) = x - \frac{3x}{(3+2x)}.$$

By taking the first derivative of $g(x)$, we have

$$g'(x) = 1 - \left(\frac{9}{(3+2x)^2}\right), \quad g'(0) = 1 - 1 = 0,$$

and the second derivative of $g(x)$, gives

$$g''(x) = \left(\frac{36}{(3+2x)^3}\right), \quad g''(0) = 4/3 \neq 0.$$

Thus the method converges quadratically to the given root. •