Se4

King Saud University, Department of Mathematics
First Midterm Exam, S2, 2023/24
M-205
All questions carry equal Marks (5 Marks)
Marks: 25, Time: 90 minutes

1. Determine whether the sequences (5)
$$\left\{n^2\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)\right\}_{n=1}^{\infty}$$
converges or not. If it converges, find its limit.

2. Find the sum of the series (5)
$$\sum_{n=1}^{\infty}\left(\frac{2}{n(n+1)}+\left(\frac{e}{4}\right)^n\right)$$

3. Determine whether the following series (5)
$$\sum_{n=3}^{\infty}(-1)^n\frac{1}{n\ln n}$$
is absolutely convergent, conditionally convergent, or divergent

4. Find the radius and the interval of convergence of the power series (5)
$$\sum_{n=0}^{\infty}\frac{2^n(x+1)^n}{n+1}$$

5. Find the Maclaurin series for the function (5)
$$f(x)=\tan^{-1}(x)$$
up to three non-zero terms. Approximate the value of the integral
$$\int_0^{0.1}\tan^{-1}(x^2)dx$$

4/

$$\lim_{n\to\infty}\left|\frac{2^{n+1}(x+1)^{n+1}}{n+2}\cdot\frac{n+1}{2^n(x+1)^n}\right| = 2|x+1| < 1$$ 2

$\cancel{\lim_{n\to\infty} 2|x+1|}$ $\Rightarrow |x+1| < 1/2$

$-3/2 < x < -\frac{1}{2}$

• If $x = -3/2$, $\sum_{n=0}^{\infty} 2^n \frac{(-3/2+1)^n}{n+1} = \sum_{n=0}^{\infty} 2^n \frac{(-1/2)^n}{n+1}$

1 $= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

converge using AST.

• $x = -1/2$, we have $\sum_{n=0}^{\infty} \frac{2^n}{n+1}(-1/2+1)^n = \sum_{n=0}^{\infty} \frac{1}{n+1}$

1

using integral test DV

$f(x) = \frac{1}{x+1}$ $\int_0^{\infty} \frac{1}{x+1}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{x+1}\,dx$

interval of converge $[-3/2, -1/2)$ $= \lim_{t\to\infty} (\ln(t+1) - \ln(1))$

Radius $\frac{-1/2-(-3/2)}{2} = 1/2$ 1 $= \infty$

5/

$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2}\,dt$ $|t| < 1$

$= \int_0^x (1 - t^2 + t^4 + \dots)\,dt = \left[t - \frac{t^3}{3} + \frac{t^5}{5} - \dots\right]_0^x$

2 $= x - \frac{x^3}{3} + \frac{x^5}{5} + \dots$

$\int_0^{0.1} \tan^{-1}(x^2)\,dx = \int_0^{0.1} \left(x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \dots\right)dx$

$= \left[\frac{x^3}{3} - \frac{x^7}{7(3)} + \frac{x^{11}}{11(5)} + \dots\right]_0^{0.1}$ 2

$= \frac{(0.1)^3}{3} - \frac{(0.1)^7}{21} + \frac{(0.1)^{11}}{55}$

$\simeq 0{,}000333 - \frac{0{,}0000001}{21}$

$\simeq 0{,}000333 - -$ 1

1/ $$a_n = n^2\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right) = n^2\,\frac{\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)\left(\sqrt{n^4+1}+\sqrt{n^4-1}\right)}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$ 2

$$= n^2\,\frac{(n^4+1)-(n^4-1)}{\sqrt{n^4+1}+\sqrt{n^4-1}} = \frac{2n^2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$ 1

$$= \frac{2n^2}{n^2\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}} = \frac{2}{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}}$$ 1

$$\lim_{n\to\infty} a_n = 1 \checkmark$$

a_n cv to $\underline{\underline{1}}$ 1

2/ $$\sum_{n=1}^{\infty} \frac{2}{n(n+1)} + \left(\frac{e}{4}\right)^n = \sum_{n=1}^{\infty}\left[\frac{2}{n}-\frac{2}{n+1}\right] + \frac{e/4}{1-e/4}$$ 1 2

$$|r| = \left|\frac{e}{4}\right| < 1$$

$$S_n = \left[\frac{2}{1}-\frac{2}{2}\right]+\left[\frac{2}{2}-\frac{2}{3}\right]+\dots+\left[\frac{2}{n}-\frac{2}{n+1}\right]$$

$$= 2-\frac{2}{n+1}, \quad \lim_{n\to\infty} S_n = 2$$ 2

$$\sum_{n=1}^{\infty} \frac{2}{n(n+1)} + \left(\frac{e}{4}\right)^n = 2+\frac{e}{4-e}$$

3/ $$\sum_{n=3}^{\infty} (-1)^n \frac{1}{n\ln n} = \sum_{n=3}^{\infty}(-1)^n a_n, \quad a_n = \frac{1}{n\ln n}$$

$$\sum_{n=3}^{\infty} \left|(-1)^n \frac{1}{n\ln n}\right| = \sum_{n=3}^{\infty}\frac{1}{n\ln n}$$, using integral test

(1) $$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n\ln n} = 0.$$ 2

(2) $$f(x) = \frac{1}{x\ln x}, \quad f'(x) = -\frac{1+\ln x}{(x\ln x)^2} < 0 \quad \forall x \geq 3$$

$$a_n > a_{n+1}, \quad \forall n \geq \underline{3}$$

$$\int_3^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_3^t \frac{1}{x\ln x}\,dx = \lim_{t\to\infty}\Big[\ln(\ln x)\Big]_3^t$$

$$= \lim_{t\to\infty}\left(\ln(\ln(t)) - \ln(\ln 3)\right)$$

$$= +\infty$$

$$\sum_{n=3}^{\infty}(-1)^n\frac{1}{n\ln n} \text{ is DV.}$$ 2

Now, AST, (1) $\lim \frac{1}{n\ln n} = 0$

(2) $a_n > a_{n+1}$

$$\Rightarrow \sum_{n=3}^{\infty}(-1)^n\frac{1}{n\ln n} \text{ CV}$$

$$\Rightarrow \sum(-1)^n\frac{1}{n\ln n} \text{ is CC}$$ 1