Department of Mathematics
College of Sciences
King Saud University, Riyadh.

M-203 (Differential and Integral Calculus)
1^{st} MidTerm Examination (2^{nd} semester 1445) (2023/2024),
Time: **90 Minutes** Max. Marks: 25.
Note: **All questions carry equal marks.**

5 **Q1.** Determine whether the sequence $\left\{\left(\frac{n^2-2}{n^2+3}\right)^n\right\}$ converges or diverges, and if it converges find its limit.

Q2. Find the sum of the series:
3

$$\sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n}\right)-\cos\left(\frac{1}{n+3}\right)\right].$$

5 ~~5~~ **Q3.** Test the convergence of the series $\sum_{n=1}^{\infty}\frac{3+\cos(n)}{e^n}$.

6 **Q4.** Find the interval of convergence and the radius of convergence of the power series: $\sum_{n=1}^{\infty}\frac{(x+2)^n}{n(-5)^n}$.

6 **Q5.** Find the power series representation for the function $f(x)=\frac{x}{(1+x)^2}$ and by using its first three nonzero terms approximate the integral $\int_0^1\frac{x^2}{(1+x^2)^2}dx$.

Department of Mathematics (1)

College of Science, KSU, Riyadh

M-203 I Mid-term Exam (II semester 1445 2023/2024)

Time: 90 Minutes Max. Marks: 25

Solutions to the Exam. Questions and Marking Schemes.

Q#1) Determine whether the sequence $\left\{\left(\frac{n^2-2}{n^2+3}\right)^n\right\}$ converges or diverges, and if it converges, find its limit. [Marks: 5]

Soln. Let $y=\left(\frac{n^2-2}{n^2+3}\right)^n$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} n \ln\left(\frac{n^2-2}{n^2+3}\right) \quad \text{(1)}$$

$$= \lim_{n\to\infty} n\left[\ln(n^2-2)-\ln(n^2+3)\right]$$

$$= \lim_{n\to\infty} \frac{\ln(n^2-2)-\ln(n^2+3)}{\frac{1}{n}} \quad \text{(1)}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^2-2}(2n)-\frac{1}{n^2+3}(2n)}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{(-n^2)(2n^3+6n-2n^3+4n)}{(n^2-2)(n^2+3)} \quad \text{(1)}$$

$$= \lim_{n\to\infty} \frac{-10n^3}{(n^2-2)(n^2+3)}$$

$$\therefore \lim_{n\to\infty} y = \lim_{n\to\infty}\left(\frac{n^2-2}{n^2+3}\right)^n = e^0 = 1; \text{ cong.} \quad \text{(2)}$$

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Q# 2) Find the Sum of the Series: $\sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n}\right)-\cos\left(\frac{1}{n+3}\right)\right]$

[Marks: 5]

Soln. we have

$u_1 = \cos(1) - \cos\left(\cancel{\frac{1}{4}}\right)$

$u_2 = \cos\left(\frac{1}{2}\right) - \cos\left(\frac{1}{5}\right)$

$u_3 = \cos\left(\frac{1}{3}\right) - \cos\left(\frac{1}{6}\right)$

$u_4 = \cos\left(\cancel{\frac{1}{4}}\right) - \cos\left(\frac{1}{7}\right)$

$u_{n-3} = \cos\left(\frac{1}{n-3}\right) - \cos\left(\cancel{\frac{1}{n}}\right)$

$u_{n-2} = \cos\left(\frac{1}{n-2}\right) - \cos\left(\frac{1}{n+1}\right)$

$u_{n-1} = \cos\left(\frac{1}{n-1}\right) - \cos\left(\frac{1}{n+2}\right)$

$u_n = \cos\left(\cancel{\frac{1}{n}}\right) - \cos\left(\frac{1}{n+3}\right)$ 3

$\therefore S_n = u_1 + u_2 + \cdots + u_n$

$= \cos(1) + \cos\left(\frac{1}{2}\right) + \cos\left(\frac{1}{3}\right) - \cos\left(\frac{1}{n+1}\right) - \cos\left(\frac{1}{n+2}\right) - \cos\left(\frac{1}{n+3}\right)$ 1

$\lim_{n\to\infty} S_n = \cos(1) + \cos\left(\frac{1}{2}\right) + \cos\left(\frac{1}{3}\right)$

$\therefore \sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n}\right)-\cos\left(\frac{1}{n+3}\right)\right] = \cos(1) + \cos\left(\frac{1}{2}\right) + \cos$ 1

Q#3) Test the convergence of the Series: $\sum_{n=1}^{\infty}\frac{3+\cos(n)}{e^n}$

[Marks: 5]

Soln. Let $\sum_{n=1}^{\infty}\frac{3+\cos(n)}{e^n} = \sum_{n=1}^{\infty} a_n$.

Choose $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{4}{e^n}$ which is conv. by Geom. Series test. 3

we see $\frac{3+\cos(n)}{e^n} \le \frac{4}{e^n}$ $\forall n$ 1

Hence by Basic Comparison test $\sum_{n=1}^{\infty}\frac{3+\cos(n)}{e^n}$ is also convergent. 1

Q#4) Find the Interval of convergence and the radius of convergence for the power Series $\sum_{n=1}^{\infty}\frac{(x+2)^n}{n(-5)^n}$

[Marks: 5]

Soln. we apply Absolute ratio test

$$\lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(n+1)(-5)^{n+1}} \times \frac{n(-5)^n}{(x+2)^n}\right|$$

$= \frac{1}{5}|x+2|$

For abs. conv: $\frac{1}{5}|x+2| < 1 \iff |x+2| < 5$

$\iff -5 < x+2 < 5 \iff -7 < x < 3$ 2

At $x=-7$, we have $\sum_{n=1}^{\infty} \frac{(-5)^n}{n(-5)^n} = \sum \frac{1}{n}$ which div by p-series test 1 (also by Integral test)

At $x=3$, we have $\sum_{n=1}^{\infty} \frac{5^n}{n(-5)^n} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ which is 1 convergent by AST. Hence interval of conv: $(-7, 3]$ and radius of conv: $\frac{3-(-7)}{2} = \underline{\underline{5}}$ 1

Q#5) Find the power series representation for the function $f(x) = \frac{x}{(1+x)^2}$ and by using its first three non-zero terms approximate the Integral $\int_0^1 \frac{x^2}{(1+x^2)^2}\,dx$. [Marks: 5]

Soln. We know $\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n + \cdots$ if $|x| <$ 1 —(1)

Differentiating w.r.to x, we get

$-\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 + \cdots + (-1)^n n x^{n-1} + \cdots$

$\Rightarrow \frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - \cdots + (-1)^{n+1} n x^{n-1} + \cdots$

$\therefore f(x) = \frac{x}{(1+x)^2} = x - 2x^2 + 3x^3 - \cdots + (-1)^{n+1} n x^n + \cdots$

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Replace in (2) x by x^2: we have

$$\frac{x^2}{(1+x^2)^2} = x^2 - 2x^4 + 3x^6 + \cdots + (-1)^n n x^{2n} + \cdots$$ 1

$$\therefore \int_0^1 \frac{x^2}{(1+x^2)^2}\,dx = \int_0^1 (x^2 - 2x^4 + 3x^6 - \cdots)\,dx$$

$$= \left[\frac{x^3}{3} - 2\frac{x^5}{5} + 3\frac{x^7}{7} \cdots \right]_0^1$$

$$= \frac{1}{3} - \frac{2}{5} + \frac{3}{7} = \frac{38}{105} \simeq 0.72187$$ 1