

Department of Mathematics  
College of Sciences  
King Saud University, Riyadh.

M-203 (Differential and Integral Calculus)

1<sup>st</sup> MidTerm Examination (2<sup>nd</sup> semester 1445) (2023/2024),

Time: 90 Minutes

Max. Marks: 25.

Note: All questions carry equal marks.

③ Q1. Determine whether the sequence  $\left\{ \left( \frac{n^2 - 2}{n^2 + 3} \right)^n \right\}$  converges or diverges, and if it converges find its limit.

Q2. Find the sum of the series:

③ 
$$\sum_{n=1}^{\infty} \left[ \cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+3}\right) \right].$$

⑤ ~~⑤~~ Q3. Test the convergence of the series  $\sum_{n=1}^{\infty} \frac{3 + \cos(n)}{e^n}$ .

⑥ Q4. Find the interval of convergence and the radius of convergence of the power series:  $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n(-5)^n}$ .

⑥ Q5. Find the power series representation for the function  $f(x) = \frac{x}{(1+x)^2}$  and by using its first three nonzero terms approximate the integral  $\int_0^1 \frac{x^2}{(1+x^2)^2} dx$ .

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①

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Solutions to the Exam Questions and  
Marking Schemes.

Q-#1) Determine whether the sequence  $\left\{ \left( \frac{n^2-2}{n^2+3} \right)^n \right\}$   
converges or diverges, and if it converges, find its limit.  
[Marks: 5]

Soln. Let  $y = \left( \frac{n^2-2}{n^2+3} \right)^n$

$$\lim_{n \rightarrow \infty} \ln y = \lim_{n \rightarrow \infty} n \ln \left( \frac{n^2-2}{n^2+3} \right) \quad \text{①}$$

$$= \lim_{n \rightarrow \infty} n \ln(n^2-2) - \ln(n^2+3)$$

$$= \lim_{n \rightarrow \infty} \frac{\ln(n^2-2) - \ln(n^2+3)}{\frac{1}{n}} \quad \text{①}$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{1}{2n} - \frac{1}{2n}}{\frac{1}{n^2} - \frac{1}{n^2+3}}$$

$$= \lim_{n \rightarrow \infty} \frac{(-n^2)(2n^3+6n-2n^3+4n)}{(n^2-2)(n^2+3)} \quad \text{①}$$

$$= \lim_{n \rightarrow \infty} \frac{-10n^3}{(n^2-2)(n^2+3)}$$

$$\therefore \lim_{n \rightarrow \infty} y = \lim_{n \rightarrow \infty} \left( \frac{n^2-2}{n^2+3} \right)^n = e^0 = 1; \text{ Cony.} \quad \text{②}$$

Q # 2) Find the sum of the series:  $\sum_{n=1}^{\infty} [\cos(\frac{1}{n}) - \cos(\frac{1}{n+3})]$  [Marks: 5]

Soln. we have

$$\begin{aligned}
 u_1 &= \cos(1) - \cos(\frac{1}{4}) & \therefore S_n &= u_1 + u_2 + \dots + u_n \\
 u_2 &= \cos(\frac{1}{2}) - \cos(\frac{1}{5}) & &= \cos(1) + \cos(\frac{1}{2}) + \cos(\frac{1}{3}) \\
 & & & - \cos(\frac{1}{n+1}) - \cos(\frac{1}{n+2}) \\
 u_3 &= \cos(\frac{1}{3}) - \cos(\frac{1}{6}) \\
 u_4 &= \cos(\frac{1}{4}) - \cos(\frac{1}{7}) & & - \cos(\frac{1}{n+3}) \quad (1) \\
 u_{n-3} &= \cos(\frac{1}{n-3}) - \cos(\frac{1}{n}) & \lim_{n \rightarrow \infty} S_n &= \cos(1) + \cos(\frac{1}{2}) \\
 & & & + \cos(\frac{1}{3}) \\
 u_{n-2} &= \cos(\frac{1}{n-2}) - \cos(\frac{1}{n+1}) \\
 u_{n-1} &= \cos(\frac{1}{n-1}) - \cos(\frac{1}{n+2}) \\
 u_n &= \cos(\frac{1}{n}) - \cos(\frac{1}{n+3}) \quad (3) & \therefore \sum_{n=1}^{\infty} [\cos(\frac{1}{n}) - \cos(\frac{1}{n+3})] &= \cos(1) + \cos(\frac{1}{2}) + \cos(\frac{1}{3}) \quad (1)
 \end{aligned}$$

Q # 3) Test the convergence of the series:  $\sum_{n=1}^{\infty} \frac{3 + \cos(n)}{e^n}$  [Marks: 5]

Soln. Let  $\sum_{n=1}^{\infty} \frac{3 + \cos(n)}{e^n} = \sum_{n=1}^{\infty} a_n$

Choose  $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{4}{e^n}$  which is conv. by Geom. series test. (3)

We see  $\frac{3 + \cos(n)}{e^n} \leq \frac{4}{e^n} \quad \forall n$  (1)

Hence by Basic comparison test  $\sum_{n=1}^{\infty} \frac{3 + \cos(n)}{e^n}$  is also convergent. (1)

Q # 4) Find the interval of convergence and the radius of convergence for the power series  $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n(-5)^n}$  [Marks: 5]

Soln. we apply Absolute ratio test

$$\lim_{n \rightarrow \infty} \left| \frac{(x+2)^{n+1}}{(n+1)(-5)^{n+1}} \times \frac{n(-5)^n}{(x+2)^n} \right|$$

(3)

$$= \frac{1}{5} |x+2|$$

For abs. cong:  $\frac{1}{5} |x+2| < 1 \Rightarrow |x+2| < 5$

$$\Rightarrow -5 < x+2 < 5 \Rightarrow -7 < x < 3 \quad (2)$$

At  $x = -7$ , we have  $\sum_{n=1}^{\infty} \frac{(-5)^n}{n(-5)^n} = \sum_{n=1}^{\infty} \frac{1}{n}$  which diverges by p-series test (1)

At  $x = 3$ , we have  $\sum_{n=1}^{\infty} \frac{5^n}{n(-5)^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  which is also by integral test (1)

Convergent by AST. Hence interval of cong.  $(-7, 3]$  and radius of cong.  $\frac{3 - (-7)}{2} = 5 \quad (1)$

Q #5) Find the power series representation for the function  $f(x) = \frac{x}{(1+x)^2}$  and by using its first

three non-zero terms approximate the integral

$$\int_0^1 \frac{x^2}{(1+x^2)^2} dx. \quad [\text{Marks: 5}]$$

Soln. We know  $\frac{1}{1+x} = 1 - x + x^2 - \dots + (-1)^n x^n + \dots$  for  $|x| < 1$  (1)

Differentiating w.r. to  $x$ , we get

$$-\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 + \dots + (-1)^n n x^{n-1} + \dots$$

$$\Rightarrow \frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - \dots + (-1)^{n+1} n x^{n-1} + \dots$$

$$\therefore f(x) = \frac{x}{(1+x)^2} = x - 2x^2 + 3x^3 - \dots + (-1)^{n+1} n x^n + \dots \quad (2)$$

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Replace in (2)  $x$  by  $x^2$ . we have

$$\frac{x^2}{(1+x^2)^2} = x^2 - 2x^4 + 3x^6 + \dots + (-1)^n n x^{2n} + \dots$$

(1)

$$\therefore \int_0^1 \frac{x^2}{(1+x^2)^2} dx = \int_0^1 (x^2 - 2x^4 + 3x^6 - \dots) dx$$

$$= \int_0^1 \left[ \frac{x^3}{3} - 2 \frac{x^5}{5} + 3 \frac{x^7}{7} - \dots \right] dx$$

$$= \frac{1}{3} - \frac{2}{5} + \frac{3}{7} = \frac{38}{105} \approx 0.72187$$

(1)