Department of Mathematics, College of Sciences
King Saud University, Riyadh.
M-203 (Differential and Integral Calculus)
2^{nd} MidTerm Examination (2^{nd} semester 1445) (2023/2024),
Time: 90 Minutes Max. Marks: 25.
Note: **All questions carry equal marks.**

Q 1. Evaluate the double integral:

$$\int_0^1 \int_{y^2}^1 y \sin(x^2) dx dy.$$

Q 2. Find the volume of the solid in the first octant bounded by $z = y; x^2 + y^2 = 4; x = 0; y = 0$.

Q 3. Find the surface area of the portion of the cone $z = \sqrt{x^2 + y^2}$ that lies inside the cylinder $x^2+y^2 = 1$ in the first octant.

Q 4. Using triple integral, find the volume of the solid bounded by the graphs of the equations: $z = 2-x^2-y^2$ and $z = x^2+y^2$.

Q 5. Using cylindrical coordinates evaluate the integral:

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \int_0^{x^2+y^2} \sqrt{x^2+y^2}\, dz dy dx.$$

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M-203 II Mid-term Examination
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Solutions to the questions

Time: 90 Minutes Max. Marks: 25

Q#1) Evaluate the double integral $\int_0^1 \int_{y^2}^1 y\sin(x^2)$ [Marks:

Soln. Given: $y^2 \le x \le 1$ and $0 \le y \le 1$

Changing to $0 \le y \le \sqrt{x}$ and $0 \le x \le 1$

Hence, we have $\int_0^1 \int_0^{\sqrt{x}} y\sin(x^2)\,dy\,dx$ (3)

$= \int_0^1 \sin(x^2)\left[\frac{y^2}{2}\right]_0^{\sqrt{x}} dx$

$= \frac{1}{2}\int_0^1 x\sin x^2\,dx = \frac{1}{4}\left[-\cos x^2\right]_0^1 = \frac{1-\cos}{4}$ (2)

Q#2) Find the volume of the solid in the first octant bounded by $z = y$; $x^2+y^2=4$; $x=0, y=$
[Marks: 5]

Soln. Volume $V = \iint_R y\,dA = \int_0^2 \int_0^{\sqrt{4-x^2}} y\,dy\,dx$ (3)

$= \frac{1}{2}\int_0^2 (\sqrt{4-x^2})^2\,dx = \frac{1}{2}\int_0^2 (4-x^2)\,dx$

$= \frac{1}{2}\left[4x - \frac{x^3}{3}\right]_0^2 = \frac{1}{2}\left(8-\frac{8}{3}\right) = \frac{16}{2\times 3} = \frac{8}{3}$ (2)

Q #3) Find the surface area of the portion of the cone $z=\sqrt{x^2+y^2}$ that lies inside the cylinder $x^2+y^2=1$ in the first octant. [Marks: 5]

Soln. We have $z=\sqrt{x^2+y^2}=g(x,y)$

$$g_x=\frac{x}{\sqrt{x^2+y^2}}\;;\quad g_y=\frac{y}{\sqrt{x^2+y^2}} \quad (1)$$

Hence,

$$S.A=\iint_R\sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1}\,dA$$

$$\sqrt{\frac{x^2+y^2+x^2+y^2}{x^2+y^2}}=\sqrt{2}$$

$$=\int_0^{\pi/2}\int_0^1\sqrt{2}\,r\,dr\,d\theta=\sqrt{2}\times\frac{\pi}{2}\left[\frac{r^2}{2}\right]_0^1$$

$2+1=3$

$$\sqrt{2}\int_0^1\int_0^{\sqrt{1-x^2}}dy\,dx=\sqrt{2}\int_0^1\sqrt{1-x^2}\,dx$$

$$=\frac{\sqrt{2}\,\pi}{4} \quad (1)$$

Q#4) Using triple integral, find the volume of the solid bounded by the graphs of the equations: $z=2-x^2-y^2$ and $z=x^2+y^2$ [Marks: 5]

Soln. We have $2-x^2-y^2=x^2+y^2$

or $2(x^2+y^2)=2 \;\therefore\; x^2+y^2=1$

$$\text{Volume } V=\int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2}r\,dz\,dr\,d\theta \quad (3)$$

$$=\int_0^{2\pi}\int_0^1 r\cdot(2-r^2-r^2)\,dr\,d\theta$$

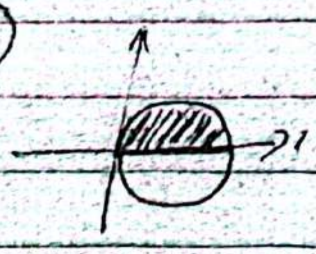
$$=\int_0^{2\pi}\int_0^1(2r-2r^3)\,dr\,d\theta=2\pi\left[2\frac{r^2}{2}\right]_0^1$$

$$=2\pi\times\left[\frac{2r^2}{2}-\frac{2r^4}{4}\right]_0^1=2\pi\left[1-\frac{1}{2}\right]$$

Q#5) Using cylindrical Coordinates evaluate the integral: $\int_0^2\int_0^{\sqrt{2x-x^2}}\int_0^{x^2+y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$

[Marks: 5]

Sol. $\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2} r\cdot r\,dz\,dr\,d\theta$ 3



$= \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,[z]_0^{r^2}\,dr\,d\theta$

$= \int_0^{\pi/2}\int_0^{2\cos\theta} r^4\,dr\,d\theta = \int_0^{\pi/2} \frac{r^5}{5}\Big]_0^{2\cos\theta} d\theta$

$= \int_0^{\pi/2} \frac{2^5}{5}\cos^5\theta\,d\theta$

$= \frac{2^5}{5}\int_0^{\pi/2}(1-\sin^2\theta)^2\cos\theta\,d\theta$

$= \frac{2^5}{5}\times\frac{2^3}{15}$

$= \frac{2^8}{75}$ 2

Put $\sin\theta = t$

$\cos\theta\,d\theta = dt$

$\int(1-t^2)^2\,dt$

$= (1-2t^2+t^4)\,dt$

$= t-\frac{2}{3}t^3+\frac{t^5}{5}$

$= \sin\theta-\frac{2}{3}\sin^3\theta+\frac{1}{5}$

$= 1-\frac{2}{3}+\frac{1}{5}$

$= \frac{15-10+3}{15} = \frac{8}{15}$