King Saud University, Department of Mathematics
Final Exam, 2023/24, Course: M-205
Marks: 40, Time: 3 hours

Question 1 [3+3+5+3+3 Marks]

1. Discuss the convergence of the sequence $\left\{\frac{3+\cos^2 n}{2^n}\right\}_{n=1}^{\infty}$.
2. Determine whether the series $\sum_{n=0}^{\infty}(-1)^n e^{-n}$ is absolutely convergent, conditionally convergent, or divergent
3. Find the radius and the interval of convergence of the power series $\sum_{n=0}^{\infty}\frac{(2x-5)^n}{n^2}$.
4. Find the Taylor series for $f(x) = \ln(x)$ at x=1 and use it to find the Taylor series for $(x-1)\ln(x)$ at x=1.
5. Evaluate the double integral $\int_0^1\int_y^1 \frac{1}{1+x^4}\,dxdy$

Question 2 [3+3 Marks]

a) Given three points $P(1,-1,-1)$, $Q(1,1,-1)$ and $R(2,-1,1)$, find
 1) A unit vector perpendicular to the plane determined by P, Q and R
 2) The area of the triangle PQR.
 3) The distance from R to the line through P and Q.

b) Let $A(0,1,2)$, $B(1,-1,0)$, $C(1,1,2)$, and $D(0,1,1)$ be a four point
 1) Find an equation of the plane determined by the points A, B and C.
 2) Find the distance of point D from the plane $P: y - z + 1 = 0$.
 3) Find the parametric equations for the line L through D that is orthogonal to the plane $y - z + 1 = 0$.

Question 3 [3+3+5 Marks]

a) Use chain rule to find $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ where $w = r^2 + s^2, r = x - y$ and $s = x + y$.

b) Find the points on the paraboloid $z = x^2 + y^2$ at which the normal line is parallel to the line through the points $A(1,-1,0)$ and $B(0,1,1)$.

c) Find the directional derivative of $f(x,y,z) = xze^y + \cos(xy)$ at the point $P(2,0,1)$ In the direction of the line $x = -1 + 3t, y = 2 - 4t, z = 1 - 5t$. In which direction it incrases most rapidly? What is the maximum rate of increase of f at P?

Question 4 [3+3 Marks]

(a) Find local extrema and saddle points, if any, on the surface

$$f(x,y) = x^3 - y^2 + 3x + 2y.$$

(b) Use method of Lagrange multiplier to find extrema of the function $f(x,y) = x + 2y$, subject to $x^2 + y^2 = 1$.

Q1) 1) $a_n = \dfrac{3+\cos^2 n}{2^n}$, $0 \le \cos^2 n \le 1 \Rightarrow \dfrac{3}{2^n} < \dfrac{3+\cos^2 n}{2^n} \le \dfrac{4}{2^n}$

$\lim\limits_{n\to\infty} \dfrac{3}{2^n} = \lim \dfrac{4}{2^n} = 0 \Rightarrow \lim\limits_{n\to\infty} \dfrac{3+\cos^2 n}{2^n} = 0 \Rightarrow \left\{\dfrac{3+\cos^2 n}{2^n}\right\}$ CV to 0

2) $\sum\limits_{n=0}^{\infty} |(-1)^n e^{-n}| = \sum\limits_{n=0}^{\infty} e^{-n} \Rightarrow \sum (-1)^n e^{-n}$ is absolutely convergent.

$r = \frac{1}{e} < 1$ geo series.

3) $\sum \dfrac{(2x-5)^n}{n^2} = \sum u_n$

$\left|\dfrac{u_{n+1}}{u_n}\right| = \left|\dfrac{(2x-5)^{n+1}}{(n+1)^2} \cdot \dfrac{n^2}{(2x-5)^n}\right| = \dfrac{n^2}{n^2+2n+1}|2x-5|$

$\lim\limits_{n\to\infty} \dfrac{n^2}{n^2+2n+1}|2x-5| = |2x-5|$

if $|2x-5| < 1 \Rightarrow -1 < 2x-5 < 1 \Rightarrow 2 < x < 3$ the series AC

If $x = 2 \Rightarrow \sum \dfrac{[2(2)-5]^n}{n^2} = \sum \dfrac{(-1)^n}{n^2}$

series is CV, Alternating series being $\lim \frac{1}{n^2} = 0$

If $x = 3$ $\sum \dfrac{[2(3)-5]^n}{n^2} = \sum \dfrac{1}{n^2}$ is $p = 2$, convergent.

$\Rightarrow$ interval of conv $[2, 3] \Rightarrow r = \dfrac{3-2}{2} = 1/2$ radius

4) $f(x) = \ln x$ at $x = 1$ and $(x-1)\ln x$ at $x = 1$

taylor $f(x) = f(1) + (x-1)f'(1) + \dfrac{(x-1)^2}{2!}f''(1) + \dots + \dfrac{(x-1)^n}{n!}f^{(n)}(1) + \dots$

$f(x) = \ln x \Rightarrow f(1) = 0$, $f'(x) = \dfrac{1}{x} \Rightarrow f'(1) = 1$, $f''(x) = -\dfrac{1}{x^2}$, $f''(1) = -1 = -(1)!$

$f'''(x) = \dfrac{2}{x^3} \Rightarrow f'''(1) = 2 = 2!$, $f^{(IV)} = -\dfrac{2.3}{x^4}$, $f^{(IV)}(1) = -(3!)$

$\Rightarrow f^{(n)}(x) = -\dfrac{2 \cdot 3 \dots (n-1)}{x^n} \Rightarrow f^{(n)}(1) = \pm[(n-1)!]$.

$\ln(x) = 0 + (x-1) + \dfrac{(x-1)^2}{2!}(-1!) + \dfrac{(x-1)^3}{3!}(2!) + \dots + \dfrac{(x-1)^n}{n!}(-1)^{n-1}((n-1)!)$

$\ln(x) = (x-1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3} + \dots + (-1)^{n-1}\dfrac{(x-1)^n}{n} + \dots$

$\Rightarrow (x-1)\ln x = (x-1)^2 - \dfrac{(x-1)^3}{2} + \dfrac{(x-1)^4}{3} + \dots + (-1)^{n-1}\dfrac{(x-1)^{n+1}}{n} + \dots$

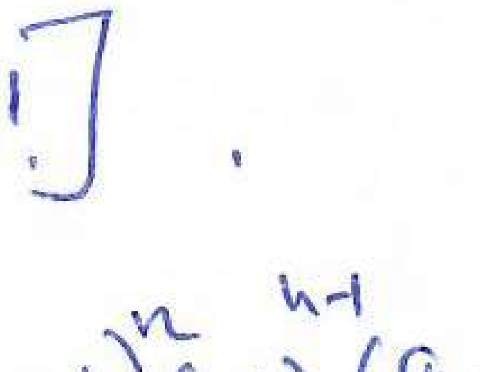
5) $\int_0^1\int_y^1 \dfrac{1}{1+x^4}\,dx\,dy = \int_0^1\int_0^x \dfrac{1}{1+x^4}\,dy\,dx = \int_0^1 \dfrac{x}{1+x^4}dx = \dfrac{1}{2}\int_0^1 \dfrac{1}{1+u^2}du$

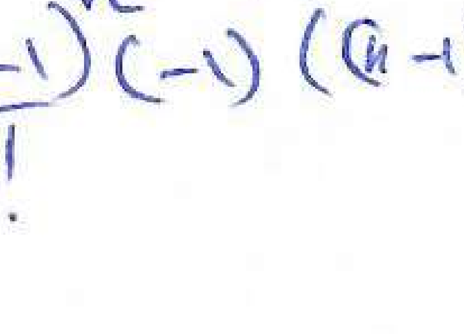
$u = x^2 \Rightarrow \frac{1}{2}du = x\,dx$

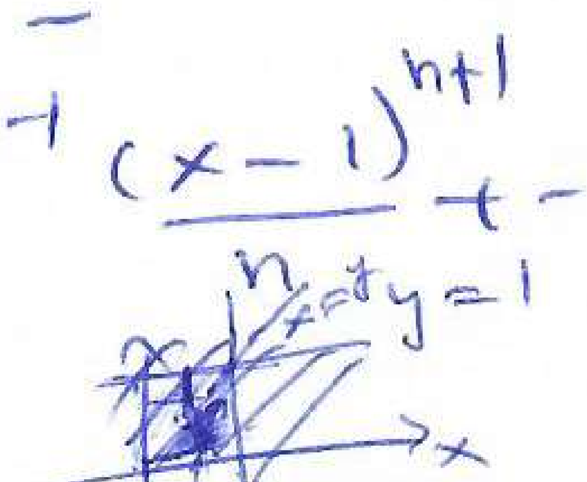
$0 \le y \le 1$, $y \le x \le 1$ $\rightarrow$ $0 \le x \le 1$, $0 \le y \le x$

$= \dfrac{1}{2}\left[\tan^{-1}(u)\right]_0^1 = \dfrac{1}{2}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$

$= \pi/8$.





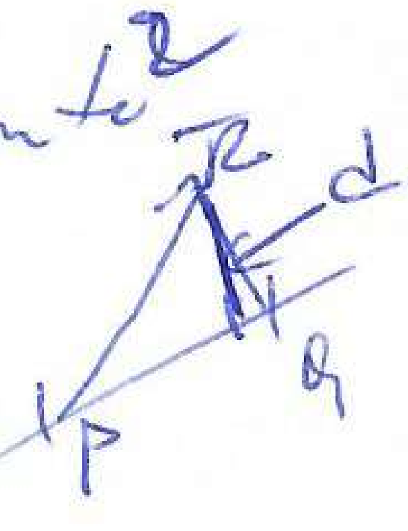


Q3/ a/ 1/ $\vec{PQ} = \langle 0,2,0\rangle$, $\vec{PR} = \langle 1,2,2\rangle$

$$n = \vec{PQ}\times\vec{PR} = \begin{vmatrix} i & j & k \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 4i - 2k$$

$$u = \frac{\vec{PQ}\times\vec{PR}}{\|\vec{PQ}\times\vec{PR}\|} = \frac{1}{\sqrt{16+4}}(4i - 2k) = \frac{4}{\sqrt{20}}i - \frac{2}{\sqrt{20}}k = \frac{2}{\sqrt5}i - \frac{1}{\sqrt5}k$$

2/ $\text{Area} = \frac{1}{2}\|\vec{PQ}\times\vec{PR}\| = \frac{1}{2}\sqrt{20} = \sqrt5$ unite^2



3/ $$d = \frac{\|\vec{PQ}\times\vec{PR}\|}{\|\vec{PQ}\|} = \frac{\sqrt{16+4}}{\sqrt4} = \frac{\sqrt{20}}{\sqrt4} = \sqrt5 \text{ unite}$$

b/ 1/ $\vec{AB} = \langle 1,-2,-2\rangle$, $\vec{AC} = \langle 1,0,0\rangle$

vector normal to the plan $\hat{n} = \begin{vmatrix} i & j & k \\ 1 & -2 & -2 \\ 1 & 0 & 0 \end{vmatrix} = 0i - 2j + 2k$

let $A(0,1,2)$ be other point eq of plane is

$$0(x-0) - 2(y-1) + 2(z-2) = 0$$

$$-2y + 2z - 2 = 0 \Rightarrow \boxed{y - z + 1 = 0}$$

2/ $$d = \frac{|1 - 1 + 1|}{\sqrt{1+1}} = \frac{1}{\sqrt2}$$

3/ $D(0,1,1)$, $n = \langle 0,2,2\rangle$ vector parallel to line L

$$\begin{cases} x = 0 \\ y = 1 - 2t \\ z = 1 + 2t, \ t \in \mathbb{R} \end{cases}$$

Q5/ a/ $$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} = (2r)(1) + (2s)(1) = 2r + 2s = 2x - 2y + 2x + 2y = 4x$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} = 2r(-1) + (2s)(1) = -2r + 2s = -2(x-y) + 2(x+y) = -2x + 2y + 2x + 2y = 4y$$

b/ $F(x,y,z) = x^2 + y^2 - z = 0$, $\nabla F = \langle 2x, 2y, -1\rangle$, $\vec{AB} = \langle 1,2,1\rangle$

∇F is parallel to $\vec{AB} \Rightarrow \frac{2x}{1} = \frac{2y}{2} = \frac{-1}{1} \Rightarrow x = \frac{1}{2}, y = -1$

$z = x^2 + y^2 = \frac{1}{4} + 1 = 5/4$ req point is $(\frac{1}{2}, -1, 5/4)$

c/ $\nabla f = \langle ze^y - y\sin(xy), xze^y - x\sin(xy), xe^y\rangle$, $\nabla f(2,0,1) = \langle 1,2,2\rangle$

in Direction of vector $a = \langle 3,-4,5\rangle$, $\|a\| = \sqrt{9+16+25} = \sqrt{50}$

$$u = \frac{1}{\sqrt{50}}\langle 3,-4,5\rangle$$

$$\frac{5}{5\sqrt2}$$

- $D_uf(2,0,1) = \langle 1,2,2\rangle \cdot \frac{1}{\sqrt{50}}\langle 3,-4,5\rangle = \frac{1}{\sqrt{50}}(3 - 8 + 10) = \frac{1}{\sqrt2}$

- $f(2,0,1)$ increase rapidly in the direction $\nabla f(2,0,1) = \langle 1,2,2\rangle$

- maxi rate of increase is $\|\nabla f\| = \sqrt{1+4+4} = \sqrt9 = 3$

Q4

a/ $f(x,y) = x^3 - y^2 + 3x + 2y$, $f_x = 3x^2 - 3$, $f_{xx} = 6x$

$f_y = -2y + 2$, $f_{xy} = 0$, $f_{yy} = -2$

Critical points $3x^2 - 3 = 0 \Rightarrow x = \pm 1$

$-2y + 2 = 0 \quad y = 1$

points are $(1,1)$, $(-1,1)$

$D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = (6x)(-2) - 0 = -12x$

→ at point $(1,1)$

$D(1,1) = -12 < 0$ → saddle point

$f(1,1) = -1$

→ at $(-1,1)$ $D(-1,1) = 12 > 0$ relative extreme

$f_{yy}(-1,1) = -2 < 0 \Rightarrow$ relative maximum

$f(-1,1) = 3$

5/ $f(x,y) = x + 2y$, $g(x,y) = x^2 + y^2 - 1 = 0$

$\nabla f = \langle 1, 2 \rangle$, $\nabla g = \langle 2x, 2y \rangle$, $\nabla f = \lambda \nabla g$

$\Rightarrow 1 = 2\lambda x$ $(x \neq 0)$ $\Rightarrow \lambda = \frac{1}{2x}$ $x \neq 0$

$2 = 2\lambda y \qquad 2 = 2 \frac{1}{2x} y \Rightarrow y = 2x$

$x^2 + y^2 = 1 \Rightarrow x^2 + 4x^2 = 1 \Rightarrow 5x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{5}}$

$y = \pm \frac{2}{\sqrt{5}}$

points are $(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$; $\cancel{(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}})}$; $\cancel{(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})}$; $(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}})$

$f(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}) = \sqrt{5}$, → rela

$\cancel{f(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}) = -\frac{3}{\sqrt{3}}}$

$\cancel{f(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}) = \frac{3}{\sqrt{5}}}$

$f(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}) = -\sqrt{5}$ → Minim.