King Saud University,
College of Sciences,
Department of Mathematics.

M203 Final Examination/ Second Sem. 1445

Max. Marks: **40** Marks: [Q1) 4+4+4, Q2) 4+4+4, Q3) 4+4+4+4] Time: **3** hs.

Q 1. **(a)** Determine whether the series $\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{1+2n}$ is absolutely convergent, conditionally convergent or divergent.

(b) Find the interval of convergence and the radius of convergence for the power series: $\sum_{n=1}^{\infty}(e^{\frac{1}{n}}-1)x^{2n}$.

(c) Find the Taylor Series for $f(x)=e^{-3x}$ about $x=-2$.

Q 2. **(a)** Evaluate the double integral: $\int_0^3\int_{y^2}^9 ye^{-x^2}\,dxdy$.

(b) Find the surface area of the portion of the hemisphere $z=\sqrt{25-x^2-y^2}$ that lies above the region bounded by the circle $x^2+y^2=9$ in the xy-plane.

(c) Find the triple integral $\iiint_Q \frac{3}{1+(x^2+y^2+z^2)^{\frac{3}{2}}}dV$, where Q is the solid bounded above by the sphere $x^2+y^2+z^2=1$ and below by the cone $z=\sqrt{3x^2+3y^2}$.

Q 3. **(a)** If C is a path joining the points $(1,0)$ to $(2,2)$, show that the line integral $\int_C xdy+ydx$ is independent of path and evaluate the integral.

(b) Use Green's theorem to evaluate the integral $\oint_C \frac{y^2}{2}dx+2yxdy$, where C is the triangle with vertices $(1,0)$, $(0,1)$, $(-1,0)$.

(c) Verify the divergence theorem by evaluating both surface integral and the triple integral for the vector field $\overrightarrow{F}(x,y,z)=(x,y,z)$ and the surface S, where S is the cone $z=4-\sqrt{x^2+y^2}$ cut off by $z=0$.

(d) If S is the surface that is portion of the paraboloid $2z=x^2+y^2$ cut off by the plane $z=y$ with closed boundary C having parametric equations $x=\cos(t), y=1+\sin(t), z=1+\sin(t)$, $0\le t\le 2\pi$ and $\overrightarrow{F}=(x,y,z)$, verify the stoke's theorem.

King Saud University, College of Science,
Department of Mathematics
M-203, Final Examination (II Semester 1445, 2023/2024)
Max. Marks: 40 Solutions to the Questions. Time: 3 Hours

Q#1(a) Determine whether the series $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{1+2n}$ is absolutely convergent, conditionally convergent or divergent. [Marks: 4]

Soln. We have $\sum_{n=1}^{\infty}\left|(-1)^n \frac{\sqrt{n}}{1+2n}\right| = \sum_{n=1}^{\infty}\frac{\sqrt{n}}{1+2n}$ and by comparing with the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ we see that the series is divergent. (1)

Now, $\lim_{n\to\infty}\frac{\sqrt{n}}{1+2n}=0$ and if $f(x)=\frac{x^{\frac{1}{2}}}{1+2x}$, we have

$f'(x)=\frac{\frac{1}{2}x^{-\frac{1}{2}}(1+2x)-2x^{\frac{1}{2}}}{(1+2x)^2}<0$, when $x\geq 1$

Hence by AST, it is convergent. (2) Thus $\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{1+2n}$ is conditionally convergent. (1)

Q#1(b) Find the interval of convergence and the radius of convergence for the power series $\sum_{n=1}^{\infty}(e^{\frac{1}{n}}-1)x^{2n}$. [Marks: 4]

Soln. By absolute Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(e^{\frac{1}{n+1}}-1)}{e^{\frac{1}{n}}-1}\cdot x^2$

$=\lim_{n\to\infty}\frac{e^{\frac{1}{n+1}}}{e^{\frac{1}{n}}}\cdot\frac{n^2}{(n+1)^2}x^2 = x^2$ (2)

Hence, radius of convg. is $r=1$. (1)

At $|x|=1$, $\sum_{n=1}^{\infty}(e^{\frac{1}{n}}-1)x^{2n}=\sum_{n=1}^{\infty}(e^{\frac{1}{n}}-1)$

Since, $\lim_{n\to\infty}\frac{e^{\frac{1}{n}}-1}{\frac{1}{n}}=\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{x\to 0}e^x=1\neq 0$

Hence by LCT, $\sum_{n=1}^{\infty}(e^{\frac{1}{n}}-1)$ is divergent.

Therefore, the interval of convergence is $(-1,1)$ (1)

Q# 1(c) Find the Taylor series for $f(x)=e^{-3x}$ about $x=-$ [Mark: 4]

Soln. $f(x)=e^{-3x} \Rightarrow f(-2)=e^{6}$

$f'(x)=e^{-3x}(-3) \Rightarrow f'(-2)=3e^{6}$

$f''(x)=(-3)e^{-3x}(-3) \Rightarrow f''(-2)=9e^{6}$

$f'''(x)=9e^{-3x}(-3) \Rightarrow f'''(-2)=-27e^{6}$ (2)

$\vdots$ (2)

$f(x)\Big]_{x=-2}=e^{-3x}\Big]_{x=-2}=e^{6}-e^{6}\,3(x+2)+e^{6}\cdot 3^{2}\frac{(x+2)^{2}}{2!}-3^{3}e^{6}\frac{(x\ldots}{3!} \ldots +(-1)^{n}e^{6}3^{n}\frac{(\ldots}{n!}$

Q# 2(a) Evaluate the double integral $\int_0^3\int_{y^2}^9 y e^{-x^2}\,dx\,dy$

Soln. Reversing the order, we have $\int_0^9\int_0^{\sqrt{x}} y e^{-x^2}\,dy\,dx$ (3)

$$=\frac{1}{2}\int_0^9 x e^{-x^2}\,dx=\frac{1}{4}\left(1-e^{-81}\right)$$ (1)

Q# 2(c) Find the surface area of the portion of the hemisphere $z=\sqrt{25-x^2-y^2}$ that lies above the region bounded by the circle $x^2+y^2=9$ in the xy-plane. [Mark: 4]

Soln. We have $z=\sqrt{25-x^2-y^2}=g(x,y)$

$\therefore g_x=\dfrac{-x}{\sqrt{25-x^2-y^2}};\ g_y=\dfrac{-y}{\sqrt{25-x^2-y^2}}$ (1)

$\therefore dS=\sqrt{1+g_x^2+g_y^2}=\dfrac{5}{\sqrt{25-x^2-y^2}}$

$\therefore S.A.=\iint_R \dfrac{5}{\sqrt{25-x^2-y^2}}\,dA=\int_0^{2\pi}\int_0^3 \dfrac{5}{\sqrt{25-r^2}}\,r\,dr\,d\theta$ (2)

$=10\pi$ (1)

Q # 2(c) Find the triple integral $\iiint_Q \frac{3}{1+(x^2+y^2+z^2)^{3/2}}\, dV$

where Q is the solid bounded above by the sphere $x^2+y^2+z^2=1$ and below by the cone $z=\sqrt{3x^2+3y^2}$ [Marks: 4]

Soln. $\int_0^{2\pi}\int_0^{\pi/6}\int_0^1 \frac{3}{1+\rho^3}\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$ 3

$= \ln(2)\int_0^{2\pi}\int_0^{\pi/6} \sin\varphi\, d\varphi\, d\theta$

$= \ln(2)(2\pi)\left[-\cos\varphi\right]_0^{\pi/6}$

$= \ln(2)(2\pi)\left[-\cos\pi/6 + \cos 0\right]$

$= \ln(2)(2\pi)\left[-\frac{\sqrt{3}}{2}+1\right]$

$= \ln(2)(2-\sqrt{3})\pi$ 1

Put $1+\rho^3 = t$ so
$3\rho^2 d\rho = dt$
$\therefore \int \frac{1}{t}\, dt$
$= \ln(t)$
$= \ln(1+\rho^3)\Big]_0^1$
$= \ln(2) - \ln(1) =$

Q # 3(a) If C is a path joining the points (1,0) to (2,2), show that the line integral $\int x\,dy + y\,dx$ is independent of path and evaluate the integral. [Marks. 4]

Soln. Here $M = y$ and $N = x$: $\frac{\partial N}{\partial x} = 1$ and $\frac{\partial M}{\partial y} = 1$

Hence $\int_C x\,dy + y\,dx$ is independent of path. 1

We find a function f such that $\vec{F} = \nabla f$

$\vec{F} = \nabla f = f_x(x,y)\vec{i} + f_y(x,y)\vec{j} = y\vec{i} + x\vec{j}$

$\Rightarrow f_x(x,y) = y$ — (1) and $f_y(x,y) = x$ — (2)

Integrating (1) w.r.t. x and (2) w.r.t. y we get

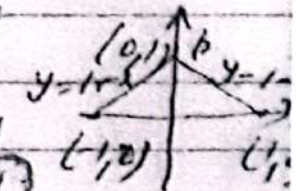
$f(x,y) = xy + g(y)$ — (3) and $f(x,y) = xy + h(x)$ — (4)

Comparing, we get $g(y) = h(x) = 0$ and hence
$f(x,y) = xy + c$ 2

$$\therefore \int_{(1,0)}^{(2,2)} \vec{F}\cdot d\vec{r} = \left[xy + c\right]_{(1,0)}^{(2,2)} = 4 - 0 = 4 \quad 1$$

Q#3(b) Use Green's theorem to evaluate $\oint_C \frac{y^2}{2}dx + 2yx\,dy$ where C is the triangle with vertices (1,0) (0,1), (-1,0). [Marks: 4]

Soln: By Green's theorem $\int_C \underbrace{\frac{y^2}{2}}_{=M} dx + \underbrace{2yx}_{=N} dy$



$0 \le y \le 1$, $y-1 \le x \le 1-y$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \quad 1$$

$$= \iint_R (2y - y)\,dA = \iint_R y\,dA$$

$$\therefore \int_0^1\int_{y-1}^{1-y} y\,dx\,dy = \int_0^1 [xy]_{y-1}^{1-y}\,dy \quad 2$$

$$= \int_0^1 y((1-y)-(y-1))\,dy$$

$$= \int_0^1 y(2-2y)\,dA = \int_0^1 (2y - 2y^2)\,dy$$

$$\left[2\frac{y^2}{2} - 2\frac{y^3}{3}\right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3} \quad 1$$

Q#3(c) Verify the divergence theorem by evaluating both surface integral and the triple integral for the vector field $\vec{F}(x,y,z) = \langle x, y, z\rangle$ and the surface S where S is the cone $z = 4 - \sqrt{x^2+y^2}$ cut off by $z = 0$. [Marks: 4]

Soln. We have the divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_Q \operatorname{div}(\vec{F})\,dV = \iiint_Q \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) \quad 1 \quad \text{--- (1)}$$

We find L.H.S. of (1). That is, $\iint_S \vec{F}\cdot\vec{n}\, ds$

We have $z = 4-\sqrt{x^2+y^2} = g(x,y)$

$$\therefore g_x = \frac{-\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}};\quad g_y = \frac{-\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}$$

$$\therefore \iint_S \vec{F}\cdot\vec{n}\, ds = \iint_R (-Mg_x - Ng_y + P)\, dA$$

$$= \iint_R \left(\frac{x^2}{\sqrt{x^2+y^2}} + \frac{y^2}{\sqrt{x^2+y^2}} + 4-\sqrt{x^2+y^2}\right) dA$$

$$= \iint_R \frac{\cancel{x^2+y^2} + 4\sqrt{x^2+y^2} - \cancel{(x^2+y^2)}}{\sqrt{x^2+y^2}}\, dA$$

$$= \iint_R \frac{4}{\cancel{\sqrt{x^2+y^2}}}\, dA$$

$$= \int_0^{2\pi}\int_0^4 4\, r\, dr\, d\theta = \int_0^{2\pi}\left[4\cdot\frac{r^2}{2}\right]_0^4 d\theta$$

$$= 2\pi(32) = 64\pi \quad 1$$

Now, we find R.H.S. of (1). That is, $\iiint_Q \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}\right)dV$

$$= \int_0^{2\pi}\int_0^4\int_0^{4-r} 3\, r\, dz\, dr\, d\theta = 3\int_0^{2\pi}\int_0^4 [z]_0^{4-r}\, r\, dr\, d\theta$$

1

$$= 3(2\pi)\int_0^4 r(4-r)\, dr$$

$$= 6\pi\left[\frac{\overset{2}{\cancel{4}}r^2}{\cancel{2}} - \frac{r^3}{3}\right]_0^4 = 6\pi\left(32-\frac{64}{3}\right)$$

$$= 64\pi \quad 1$$

Hence the L.H.S. of (1) = R.H.S. of (1).

Q#3(a) If S is the surface that is portion of the paraboloid $2z = x^2 + y^2$ cut off by the plane $z = y$ with closed boundary C having parametric equations:
$x = \cos(t)$, $y = 1 + \sin(t)$, $z = 1 + \sin(t)$; $0 \le t \le 2\pi$
and $\vec{F} = \langle x, y, z \rangle$, verify the Stoke's theorem.
[Mark: 4]

Soln: We have the Stoke's theorem;

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F}) \cdot \vec{n}\, ds \quad \text{---} (1)$$

(1)

First, we evaluate $\iint_S (\text{curl } \vec{F}) \cdot \vec{n}\, ds$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}($$

Hence $\iint_S (\text{curl } \vec{F}) \cdot \vec{n}\, ds = 0$ (1)

Now, we evaluate $\oint_C \vec{F} \cdot d\vec{r} = \oint_C x\,dx + y\,dy + z\,dz$

The parametric equations of C are

$C:\ x = \cos(t),\ y = 1 + \sin(t),\ z = 1 + \sin(t);\ 0 \le t \le 2$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \oint_C x\,dx + y\,dy + z\,dz$$

$$= \int_0^{2\pi} \cos(t)(-\sin t)\,dt + (1+\sin t)\cos t\,dt + (1+\sin(t))\cos t\,dt$$

(1)

$$= \int_0^{2\pi} [1 + (1+\sin t)\cos t]\,dt$$

$$= \int_0^{2\pi} (1 + \cos t + \sin t \cos t)\,dt$$

$$= \int_0^{2\pi} (1 + \cos t + \tfrac{1}{2}\sin 2t)\,dt = 0 \quad (1)$$

Hence L.H.S = R.H.S.