

King Saud University,
College of Sciences,
Department of Mathematics.

M203 Final Examination/ Second Sem. 1445

Max. Marks: 40 Marks: [Q1) 4+4+4, Q2) 4+4+4, Q3) 4+4+4+4] Time: 3 hs.

Q 1. (a) Determine whether the series $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+2n}$ is absolutely convergent, conditionally convergent or divergent.

(b) Find the interval of convergence and the radius of convergence for the power series: $\sum_{n=1}^{\infty} (e^{\frac{1}{n}} - 1)x^{2n}$.

(c) Find the Taylor Series for $f(x) = e^{-3x}$ about $x = -2$.

Q 2. (a) Evaluate the double integral: $\int_0^3 \int_{y^2}^9 ye^{-x^2} dx dy$.

(b) Find the surface area of the portion of the hemisphere $z = \sqrt{25 - x^2 - y^2}$ that lies above the region bounded by the circle $x^2 + y^2 = 9$ in the xy -plane.

(c) Find the triple integral $\iiint_Q \frac{3}{1 + (x^2 + y^2 + z^2)^{\frac{3}{2}}} dV$, where Q is the solid bounded above by the sphere $x^2 + y^2 + z^2 = 1$ and below by the cone $z = \sqrt{3x^2 + 3y^2}$.

Q 3. (a) If C is a path joining the points $(1, 0)$ to $(2, 2)$, show that the line integral $\int_C xdy + ydx$ is independent of path and evaluate the integral.

(b) Use Green's theorem to evaluate the integral $\oint_C \frac{y^2}{2} dx + 2yxdy$, where C is the triangle with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$.

(c) Verify the divergence theorem by evaluating both surface integral and the triple integral for the vector field $\vec{F}(x, y, z) = (x, y, z)$ and the surface S , where S is the cone $z = 4 - \sqrt{x^2 + y^2}$ cut off by $z = 0$.

(d) If S is the surface that is portion of the paraboloid $2z = x^2 + y^2$ cut off by the plane $z = y$ with closed boundary C having parametric equations $x = \cos(t)$, $y = 1 + \sin(t)$, $z = 1 + \sin(t)$, $0 \leq t \leq 2\pi$ and $\vec{F} = (x, y, z)$, verify the stoke's theorem.

King Saud University, College of Science,
Department of Mathematics

M-203, Final Examination (II semester 1443
2023/2024)

Max. Marks: 40 Solutions to the Questions. Time: 3 Hours

Q#1(a) Determine whether the series $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+2n}$ is absolutely convergent, conditionally convergent or divergent. [Marks: 4]

Soln. We have $\sum_{n=1}^{\infty} \left| (-1)^n \frac{\sqrt{n}}{1+2n} \right| = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+2n}$ and by comparing

with the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ we see that the series is divergent. ①

Now, $\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{1+2n} = 0$ and if $f(x) = \frac{x^2}{1+2x}$, we have

$$f'(x) = \frac{\frac{1}{2} x^{-\frac{1}{2}} (1+2x) - 2x^{\frac{1}{2}}}{(1+2x)^2} < 0, \text{ when } x \geq 1$$

Hence by AST, it is convergent. Thus $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+2n}$ is conditionally convergent. ②

Q#1(b) Find the interval of convergence and the radius of convergence for the power series $\sum_{n=1}^{\infty} (e^{\frac{1}{n}} - 1) x^{2n}$. [Marks: 4]

Soln. By absolute Ratio test $\left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(e^{\frac{1}{n+1}} - 1) x^{2n+2}}{(e^{\frac{1}{n}} - 1) x^{2n}}$

$$= \lim_{n \rightarrow \infty} \frac{e^{\frac{1}{n+1}}}{e^{\frac{1}{n}}} \cdot \frac{n^2}{(n+1)^2} x^2 = x^2 \quad \text{②}$$

Hence, radius of conv. is $r = 1$. ①

At $|x| = 1$, $\sum_{n=1}^{\infty} (e^{\frac{1}{n}} - 1) x^{2n} = \sum_{n=1}^{\infty} (e^{\frac{1}{n}} - 1)$

Since $\lim_{n \rightarrow \infty} \frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}} = \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \lim_{x \rightarrow 0} e^x = 1 \neq 0$

Hence by LCT, $\sum_{n=1}^{\infty} (e^{\frac{1}{n}} - 1)$ is divergent.

Therefore, the interval of convergence is $(-1, 1)$ ①

Q# 1(c) Find the Taylor series for $f(x) = e^{-3x}$ about $x = -2$ [Mark: 4]

Soln. $f(x) = e^{-3x} \Rightarrow f(-2) = e^6$
 $f'(x) = e^{-3x} (-3) \Rightarrow f'(-2) = -3e^6$
 $f''(x) = (-3)e^{-3x} (-3) \Rightarrow f''(-2) = 9e^6$
 $f'''(x) = 9e^{-3x} (-3) \Rightarrow f'''(-2) = -27e^6$
 \vdots

$f(x) \Big|_{x=-2} = e^{-3x} \Big|_{x=-2} = e^6 - e^6 3(x+2) + e^6 \frac{3^2(x+2)^2}{2!} - \frac{3^3 e^6}{3!} (x+2)^3 + \dots$

Q# 2(a) Evaluate the double integral $\int_0^3 \int_{y^2}^9 y e^{-x^2} dx dy$

Soln. Reversing the order, we have $\int_0^3 \int_0^{\sqrt{y}} y e^{-x^2} dy dx$ ③
 $= \frac{1}{2} \int_0^9 x e^{-x^2} dx = \frac{1}{4} (1 - e^{-81})$ ①

Q# 2(c) Find the surface area of the portion of the hemisphere $z = \sqrt{25 - x^2 - y^2}$ that lies above the region bounded by the circle $x^2 + y^2 = 9$ in the xy -plane.

[Mark: 4]

Soln. We have $z = \sqrt{25 - x^2 - y^2} = g(x, y)$

$\therefore f_x = \frac{-x}{\sqrt{25 - x^2 - y^2}}$; $f_y = \frac{-y}{\sqrt{25 - x^2 - y^2}}$ ①

$\therefore ds = \sqrt{1 + f_x^2 + f_y^2} = \frac{5}{\sqrt{25 - x^2 - y^2}}$

$\therefore S.A. = \iint_R \frac{5}{\sqrt{25 - x^2 - y^2}} dA = \int_0^{2\pi} \int_0^3 \frac{5}{\sqrt{25 - r^2}} r dr d\theta$
 $= 10\pi$ ①

Q # 2(c) Find the triple integral $\iiint_Q \frac{3}{1+(x^2+y^2+z^2)^{3/2}} dV$

where Q is the solid bounded above by the sphere $x^2+y^2+z^2=1$ and below by the cone $z=\sqrt{3x^2+3y^2}$ [Marks: 4]

Soln. $\int_0^{2\pi} \int_0^{\pi/6} \int_0^1 \frac{3}{1+\rho^3} \rho^2 \sin\phi d\rho d\phi d\theta$

$= \ln(2) \int_0^{2\pi} \int_0^{\pi/6} \sin\phi d\phi d\theta$

$= \ln(2)(2\pi) [-\cos\phi]_0^{\pi/6}$

$= \ln(2)(2\pi) [-\cos\pi/6 + \cos 0]$

$= \ln(2)(2\pi) [-\frac{\sqrt{3}}{2} + 1]$

$= \ln(2)(2-\sqrt{3})\pi$

(3)

Put $1+\rho^3 = t \Rightarrow$

$3\rho^2 d\rho = dt$

$\therefore \int \frac{1}{t} dt$

$= \ln(t)$

$= \ln(1+\rho^3)$

$= \ln(2) - \ln(1)$

Q # 3(a) If C is a path joining the points $(1,0)$ to $(2,2)$, Show that the line integral $\int_C (2y dx + x dy)$ is independent of path and evaluate the integral. [Marks: 4]

Soln. Here $M = y$ and $N = x$: $\frac{\partial N}{\partial x} = 1$ and $\frac{\partial M}{\partial y} = 1$
Hence $\int_C (2y dx + x dy)$ is independent of path. (1)

We find a function f such that $\vec{F} = \nabla f$

$\vec{F} = \nabla f = f_x(x,y)\vec{i} + f_y(x,y)\vec{j} = y\vec{i} + x\vec{j}$

$\Rightarrow f_x(x,y) = y \quad (1)$ and $f_y(x,y) = x \quad (2)$

Integrating (1) w.r. to x and (2) w.r. to y we get

$f(x,y) = xy + g(y) \quad (3)$ and $f(x,y) = xy + h(x) \quad (4)$

Comparing, we get $g(y) = h(x) = 0$ and hence

$$f(x, y) = 2y + c \quad (2)$$

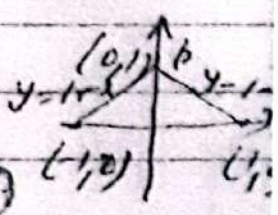
$$\therefore \int_{(1,0)}^{(2,2)} F \cdot d\vec{i} = [2y + c]_{(1,0)}^{(2,2)} = 4 - 0 = 4 \quad (1) =$$

Q#3(b) Use Green's theorem to evaluate $\int_C \frac{y^2}{2} dx + 2xy dy$, where C is the triangle with vertices $(1,0), (0,1), (-1,0)$

Soln. By Green's theorem $\int_C \frac{y^2}{2} dx + 2xy dy$ [Marks: 4]

$$\left. \begin{aligned} 0 \leq x \leq 1 \\ y-1 \leq x \leq 1-y \end{aligned} \right\}$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA \quad (1)$$



$$= \iint_R (2y - y) dA = \iint_R y dA$$

$$\therefore \int_0^1 \int_{y-1}^{1-y} y dx dy = \int_0^1 [xy]_{y-1}^{1-y} dy \quad (2)$$

$$= \int_0^1 y((1-y) - (y-1)) dy$$

$$= \int_0^1 y(2-2y) dx = \int_0^1 (2y - 2y^2) dy$$

$$\left[2 \frac{y^2}{2} - 2 \frac{y^3}{3} \right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3} \quad (1)$$

Q#3(c) Verify the divergence theorem by evaluating both surface integral and the triple integral of a the vector field $F(x, y, z) = (x, y, z)$ and the surface S where S is the cone $z = 4 - \sqrt{x^2 + y^2}$ cut off by $z = 0$. [Mark: 4]

Soln. we have the divergence theorem

$$\iint_S F \cdot d\vec{s} = \iiint_Q \text{div}(F) dV = \iiint_Q \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right) dV \quad (1)$$

We find L.H.S. of (1). That is, $\iint_S \vec{F} \cdot \vec{n}' ds$

We have $z = 4 - \sqrt{x^2 + y^2} = g(x, y)$

$$\therefore q_x = \frac{-2x}{2\sqrt{x^2+y^2}}; q_y = \frac{-2y}{2\sqrt{x^2+y^2}}$$

$$\therefore \iint_S \vec{F} \cdot \vec{n}' ds = \iint_R (Mq_x - Nq_y + P) dA$$

$$= \iint_R \left(\frac{z^2}{\sqrt{x^2+y^2}} + \frac{y^2}{\sqrt{x^2+y^2}} + 4 - \sqrt{x^2+y^2} \right) dA$$

$$= \iint_R \frac{x^2 + y^2 + 4\sqrt{x^2+y^2} - (x^2+y^2)}{\sqrt{x^2+y^2}} dA$$

$$= \iint_R \frac{4}{\sqrt{x^2+y^2}} dA$$

$$= \int_0^{2\pi} \int_0^4 4 r dr d\theta = \int_0^{2\pi} \left[\frac{4 \cdot r^2}{2} \right]_0^4 d\theta$$

$$= 2\pi (32) = 64\pi \text{ (1)}$$

Now, we find R.H.S. of (1). That is, $\iiint_Q \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right) dz$

$$= \int_0^{2\pi} \int_0^4 \int_0^{4-r} 3 r dz dr d\theta = 3 \int_0^{2\pi} \int_0^4 [3]^{4-r} r dr d\theta$$

$$\text{(1)} = 3(2\pi) \int_0^4 r(4-r) dr$$

$$= 6\pi \left[\frac{4^2 r^2}{2} - \frac{r^3}{3} \right]_0^4 = 6\pi \left(32 - \frac{64}{3} \right)$$

$$= 64\pi \text{ (1)}$$

Hence the L.H.S. of (1) = R.H.S. of (1).

Q. 3(a) If S is the surface that is portion of the paraboloid $2z = x^2 + y^2$ cut off by the plane $z = 1$ with closed boundary C having parametric equations:

$$x = \cos(t), y = 1 + \sin(t), z = 1 + \sin(t); 0 \leq t \leq 2\pi$$

and $\vec{F} = \langle x, y, z \rangle$, Verify the Stoke's theorem. [Mark: 4]

Soln. We have the Stoke's theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F}) \cdot \vec{n} \, dS \quad \text{--- (1)}$$

First, we evaluate $\iint_S (\text{curl } \vec{F}) \cdot \vec{n} \, dS$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$\text{Hence } \iint_S (\text{curl } \vec{F}) \cdot \vec{n} \, dS = 0 \quad \text{--- (2)}$$

$$\text{Now, we evaluate } \oint_C \vec{F} \cdot d\vec{r} = \int_C x dx + y dy + z dz$$

The parametric equations of C are:

$$C: x = \cos(t), y = 1 + \sin(t), z = 1 + \sin(t); 0 \leq t \leq 2\pi$$

$$\begin{aligned} \therefore \oint_C \vec{F} \cdot d\vec{r} &= \int_C x dx + y dy + z dz \\ &= \int_0^{2\pi} \cos(t)(-\sin t) dt + (1 + \sin t) \cos t dt \\ &\quad + (1 + \sin t) \cos t dt \\ &= \int_0^{2\pi} [1 + (1 + \sin t) \cos t] dt \quad \text{--- (3)} \end{aligned}$$

$$= \int_0^{2\pi} (1 + \cos t + \sin t \cos t) dt$$

$$= \int_0^{2\pi} (1 + \cos t + \frac{1}{2} \sin 2t) dt = 0 \quad \text{--- (4)}$$

Hence L.H.S = R.H.S.