Revision Sheet – Solutions

Math 481 – Riemann Integration and Sequences of Functions

Exercise 1: Riemann Integrability

Determine whether each function is Riemann integrable on $[0, 1]$.

1. $$f(x) = \begin{cases} 1, & x < 1/2, \\ 0, & x \geq 1/2 \end{cases}$$

   This function is bounded on $[0, 1]$ and has only one point of discontinuity, namely $x = \frac{1}{2}$. Therefore, it is Riemann integrable. Its integral is

   $$\int_0^1 f(x)\,dx = \int_0^{1/2} 1\,dx + \int_{1/2}^1 0\,dx = \frac{1}{2}.$$

2. $$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q} \end{cases}$$

   This function is not Riemann integrable. Indeed, every subinterval of $[0, 1]$ contains both rational and irrational numbers, so for every partition:

   $$\inf f = 0, \qquad \sup f = 1.$$

   Hence all lower sums are 0 and all upper sums are 1, so the lower and upper integrals are different. Therefore, f is not Riemann integrable.

3. $f(x) = x^2$

   The function $f(x) = x^2$ is continuous on $[0, 1]$, so it is Riemann integrable. Its integral is

   $$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

Exercise 2: Riemann Sums

Write the following limits as integrals.

1. $$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2}$$

   This is a Riemann sum for the function $f(x) = x$ on $[0, 1]$. Therefore,

   $$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\frac{k}{n} = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}.$$

2. $$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\frac{1}{1+(k/n)^2}$$

   This is a Riemann sum for the function

   $$f(x) = \frac{1}{1+x^2}$$

   on $[0, 1]$. Thus,

   $$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\frac{1}{1+(k/n)^2} = \int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}(x)\right]_0^1 = \frac{\pi}{4}.$$

Exercise 3:

Determine the pointwise limit and uniform convergence.

1. $f_n(x) = x^n$ on $[0, 1]$

   For $0 \le x < 1$, we have $x^n \to 0$ as $n \to \infty$, while for $x = 1$,

   $$f_n(1) = 1.$$

   Hence the pointwise limit is

   $$f(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x = 1. \end{cases}$$

   The convergence is not uniform on $[0, 1]$, because each f_n is continuous but the limit function is not continuous at $x = 1$.

2. $f_n(x) = \frac{x}{n}$ on $[0,1]$

   For every $x \in [0,1]$,
$$\frac{x}{n} \to 0.$$
   So the pointwise limit is $f(x) = 0$. Also,
$$\sup_{x\in[0,1]} \left|\frac{x}{n} - 0\right| = \sup_{x\in[0,1]} \frac{x}{n} = \frac{1}{n} \to 0.$$
   Therefore, the convergence is uniform.

3. $f_n(x) = \frac{1}{1+nx}$ on $[0,1]$

   For $x > 0$, we have
$$\frac{1}{1+nx} \to 0,$$
   while at $x = 0$,
$$f_n(0) = 1.$$
   Hence the pointwise limit is
$$f(x) = \begin{cases} 1, & x = 0, \\ 0, & 0 < x \le 1. \end{cases}$$
   Since the limit function is not continuous at $x = 0$, the convergence cannot be uniform on $[0,1]$.

Exercise 4:

Let
$$f_n(x) = \begin{cases} n, & 0 \le x \le \frac{1}{n}, \\ 0, & \text{otherwise.} \end{cases}$$

1. Find the pointwise limit.

   If $x = 0$, then for all n,
$$f_n(0) = n,$$
   so the sequence does not converge at $x = 0$.

   If $x > 0$, then for sufficiently large n we have $x > \frac{1}{n}$, so
$$f_n(x) = 0$$
   eventually. Hence for each $x > 0$,
$$f_n(x) \to 0.$$
   Therefore, f_n does not converge pointwise on all of $[0,1]$, because it fails to converge at $x = 0$.

2. Compute $\int_0^1 f_n(x)\,dx$.

   We have
   $$\int_0^1 f_n(x)\,dx = \int_0^{1/n} n\,dx = n \cdot \frac{1}{n} = 1.$$

3. Does $\int f_n \to \int f$?

   Since the sequence does not converge pointwise on all of $[0,1]$, this comparison is not directly applicable in the usual sense. However, note that
   $$\int_0^1 f_n(x)\,dx = 1$$
   for all n, so the integrals do not tend to 0.

4. Is the convergence uniform?

   No. In fact, the sequence is not even pointwise convergent on all of $[0,1]$, so it cannot converge uniformly.