

[Solution Key]

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS
Semester 471 / MATH-244 (Linear Algebra) / Mid-term Exam 2

Max. Marks: 25

Max. Time: $1\frac{1}{2}$ hr**Question 2:** [Marks: 3 + 2]

I. Select the correct choice:

- (i) If $W = \{(1,1), (1,2)\}$, then W is:
(a) a vector space (b) a subspace of $\mathbb{R}^2$ (c) ✓ a basis of $\mathbb{R}^2$ (d) linearly dependent. [Mark 1]
- (ii) If $B = \{u, v, w\}$ is an orthogonal set of non-zero vectors in an inner product space E and $\alpha u + \beta v + \gamma w = 0$ for some scalars α, β and γ , then:
(a) ✓ $\alpha = \beta = \gamma = 0$ (b) $u = v = w$ (c) $\alpha = \gamma, v = 0, u = w$ (d) B is linearly dependent. [Mark 1]
- (iii) If A is an invertible matrix of order 3, then $\text{rank}(A)$ is equal to:
(a) 0 (b) 1 (c) 2 (d) ✓ 3. [Mark 1]

II. Give an example for each of the following. You don't have to prove your answers.

- (i) A linearly dependent subset of $\mathbb{R}^3$ consisting of three vectors in which every two vectors are linearly independent.

Example: $\{(1, 0, 0), (0, 1, 0), (1, 1, 0)\}$. [Mark 1]

- (ii) A matrix having rank 3 and nullity 4.

Example:
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

[Mark 1]

Question 2: [Marks: 4 + 3]

Let $A = \begin{bmatrix} 1 & 1 & 2 & -3 & 5 \\ 2 & 3 & 0 & -1 & -6 \\ 3 & 4 & 2 & -4 & -1 \end{bmatrix}$. Then:

- (i) Find bases for $\text{row}(A)$ and $\text{col}(A)$.
(ii) Express the last three columns of the matrix A as linear combinations of the first two.

Solution: (i) From $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 6 & -8 & 21 \\ 0 & 1 & -4 & 5 & -16 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$:

[Marks 2]

Basis for $\text{row}(A) = \{(1, 0, 6, -8, 21), (0, 1, -4, 5, -16)\}$, Basis for $\text{col}(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$.

[Marks 1+1]

(ii) $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$, $\begin{bmatrix} -3 \\ -1 \\ -4 \end{bmatrix} = -8 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ -6 \\ -1 \end{bmatrix} = 21 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 16 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$.

[Marks 1+1+1]

Question 3: [Marks: 3 + 4]

- (i) Let $F = \{p = 2 - x, q = 5 + 3x - x^2 + 2x^3, r = 1 + 7x + x^2 + 4x^3\}$. Find the real numbers α, β, γ so that $-3 + 5x^2 + 2x^3 = \alpha p + \beta q + \gamma r$. Also, show that the set F is linearly independent in the vector space P_3 .
(ii) Let $G = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ be a basis of the vector space $\mathbb{R}^6$. Let A be a 4×6 matrix such that $A(v_1)$ and $A(v_2)$ are linearly independent vectors in $\mathbb{R}^4$ and $A(v_i) = 0$, for all $i = 3, 4, 5, 6$. Show that the null space $N(A) = \text{span}(\{v_3, v_4, v_5, v_6\})$. Also, find $\text{nullity}(A^T)$.

Solution: (i) Substituting the values of polynomials p, q, r in $-3 + 5x^2 + 2x^3 = \alpha p + \beta q + \gamma r$, and then equating the coefficients of the same degree terms, we get that: $-3 = 2\alpha + 5\beta + \gamma$, $0 = -\alpha + 3\beta + 7\gamma$, $5 = 0\alpha - \beta + \gamma$, $2 = 0\alpha + 2\beta + 4\gamma$.

Solving this linear system, we obtain: $\alpha = 5, \beta = -3$ and $\gamma = 2$. [Marks 2]Also, $\alpha p + \beta q + \gamma r = 0 \Rightarrow \alpha = \beta = \gamma = 0$. Hence, the set F is linearly independent in the vector space P_3 . [Mark 1]

- (ii) $u \in N(A) \Rightarrow 0 = A(u) = A(\sum_{i=1}^6 \alpha_i v_i) = \sum_{i=1}^6 \alpha_i A(v_i) = \alpha_1 A(v_1) + \alpha_2 A(v_2) \Rightarrow \alpha_1 = \alpha_2 = 0$. So, $u = \sum_{i=3}^6 \alpha_i v_i$.

Hence, $N(A) \subseteq \text{span}(\{v_3, v_4, v_5, v_6\}) \subseteq N(A)$. Thus, $N(A) = \text{span}(\{v_3, v_4, v_5, v_6\})$, and so: [Marks 2] $\text{nullity}(A^T) = 4 - \text{rank}(A^T) = 4 - \text{rank}(A) = 4 - (6 - \text{nullity}(A)) = 4 - (6 - 4) = 4 - 2 = 2$. [Marks 2]**Question 4:** [Marks: 3 + 3]

- (i) Consider a vector space E of dimension 3. Let $B = \{u_1, u_2, u_3\}$ and $C = \{v_1, v_2, v_3\}$ be two ordered bases for E such that the transition matrix ${}_C P_B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ from B to C . Compute the coordinate vector $[v_1 - 2v_2 + 3v_3]_B$.
(ii) Consider the matrices $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ in the inner product space $M_2(\mathbb{R})$, consisting of all real matrices of size 2, with inner product $\langle A, B \rangle = \text{trace}(AB^T)$. Find the angle between the matrices A and B . Also, verify the Pythagorean theorem for the same matrices.

Solution: (i) ${}_B P_C = ({}_C P_B)^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$ and $[v_1 - 2v_2 + 3v_3]_C = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$. Hence: [Marks 2]

$[v_1 - 2v_2 + 3v_3]_B = {}_B P_C [v_1 - 2v_2 + 3v_3]_C = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$. [Mark 1]

- (ii) $\langle A, B \rangle = \text{trace}(AB^T) = \text{trace}\left(\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\right) = 0$, so that the angle θ between the matrices A and B :

$\theta = \cos^{-1} \frac{\langle A, B \rangle}{\|A\| \|B\|} = \cos^{-1} 0 = \frac{\pi}{2}$. Hence, A and B are orthogonal matrices. [Mark 1]

Next, $\|A\| = \sqrt{\langle A, A \rangle} = \sqrt{\text{trace}(AA^T)} = \sqrt{\text{trace}\left(\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\right)} = \sqrt{5}$. Similarly, $\|B\| = \sqrt{\text{trace}\left(\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\right)} = \sqrt{5}$ and

$\|A + B\| = \left\| \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \right\| = \sqrt{\text{trace}\left(\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}\right)} = \sqrt{10}$. So, $\|A + B\|^2 = 10 = 5 + 5 = \|A\|^2 + \|B\|^2$; as required. [Marks 2]

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