[Solution Key]

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

Semester 462 / MATH-244 (Linear Algebra) / Mid-term Exam 2

Max. Marks: 25 **Max. Time: $1\frac{1}{2}$ hr**

Solution of Question 1: **Correct choices:**

(i)   Let P_2 be the vector space of all real polynomials in one variable of degree ≤ 2 and $S = \{u, v\} \subseteq P_2$. If E denotes the set of all linear combinations of the vectors in S, then the set E is equal to:

   (a) $\{\alpha u + v \mid \alpha \epsilon \mathbb{R}\}$ (b) $\{u + \beta v \mid \beta \epsilon \mathbb{R}\}$ (c) P_2 (d) ✓ a vector space.

(ii)   If $W = \{w_1, w_2, w_3, w_4\}$ spans the vector space V, then:

   (a) $dim(V) = 3$ (b) $dim(V) = 4$ (c) $dim(V) > 4$ (d) ✓ $dim(V) \leq 4$.

(iii)   Consider the vector space $\mathbb{R}^2$ with ordered basis $B = \{(1,0), (1,2)\}$. If $v \epsilon \mathbb{R}^2$ with the coordinate vector $[v]_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then:

   (a) $v = (1,0)$ (b) ✓ $v = (3,4)$ (c) $v = (1,2)$ (d) $v = (2,2)$.

(iv)   Which of the following matrices cannot be a transition matrix?

   (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (c) ✓ $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

(v)   If A is an invertible matrix of order 3, then $rank(A)$ is equal to:

   (a) 0   (b) 1   (c) 2   (d) ✓ 3.

Question 2:

   Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 2 \end{bmatrix}$.

(a)   Find a basis B_1 for the null space $N(A)$.

Solution: $B_1 = \{(-1, -1, 1)\}$ is a basis for $N(A)$.

(b)   Find a basis B_2 for the column space $col(A)$.

Solution: $B_2 = \{(1, 2, 1), (2, 4, 1)\}$ is a basis for $col(A)$.

(c)   Find nullity and rank of the matrix A.

Solution: From Part (a), nullity(A) = 1. From Part (b), rank(A) = 2.

(d)   Show that $B_1 \cup B_2$ is a basis for the vector space $\mathbb{R}^3$.

Solution: Since $B_1 \cup B_2 = \{(-1,-1,1), (1,2,1), (2,4,1)\}$ is linearly independent and $dim(\mathbb{R}^3) = 3$, $B_1 \cup B_2$ is a basis for $\mathbb{R}^3$.

Question 3:

   Consider a vector space E of dimension 3. Let $B = \{u_1, u_2, u_3\}$ and $C = \{v_1, v_2, v_3\}$ be two ordered bases for E such that the transition matrix $_C P_B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ from B to C. Then, compute:

(a)   Transition matrix $_B P_C$ from C to B.

Solution: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \Rightarrow {}_B P_C = {}_C P^{-1}{}_B = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$.

(b)   Coordinate vectors $[v_1 - v_2]_C$, $[v_1 - v_2]_B$ and $[v]_C$, where $v = v_1 - 2u_2 + v_3$.

Solution: $v_1 - v_2 = v_1 + (-1)v_2 + 0v_3 \Rightarrow [v_1 - v_2]_C = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

$[v_1 - v_2]_B = {}_B P_C [v_1 - v_2]_C = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

$_C P_B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ gives $u_2 = v_1 + v_3$. Then, $v = v_1 - 2u_2 + v_3 = -v_1 - v_3$. Hence, $[v]_C = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$.

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