Complete solution of the Mid term
Exam, Semester 1442/1443

Question 1: (a) $\begin{cases} (x-3)y' + y\ln x = 2x \\ y(1) = 2 \end{cases}$

$$\frac{dy}{dx} = \frac{2x - y\ln x}{x-3} = f(x,y)$$

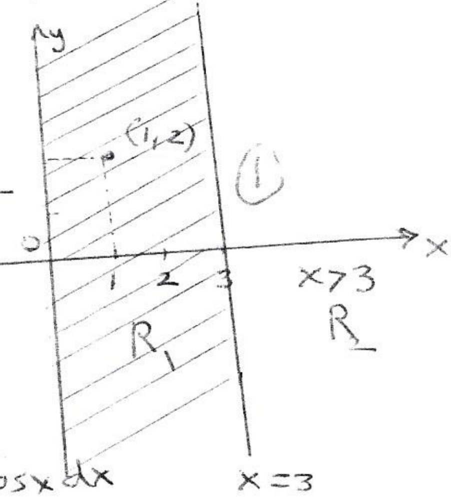
$\frac{\partial f}{\partial y} = -\frac{\ln x}{x-3}$. Both f and $\frac{\partial f}{\partial y}$ are continuous (1)

on $R = \{(x,y) \in \mathbb{R}^2;\ x > 0 \text{ and } x \neq 3\}$

$= \{(x,y) \in \mathbb{R}^2,\ 0 < x < 3\} \cup \{(x,y) \in \mathbb{R}^2,\ x > 3\}$

$= R_1 \cup R_2$ (2)

But $(1, 2) \in R_1 = \{(x,y) \in \mathbb{R}^2,\ 0 < x < 3,\ y \in \mathbb{R}\}$ is the desired region (1)



(b) $\frac{dy}{dx} = \frac{\sin x + x\cos x}{y(2\ln y + 1)}$, $y > 0$

Solution $\int y(2\ln y + 1)\,dy = \int(\sin x + x\cos x)\,dx$ (2)

$$2\int y\ln y + \int y\,dy = \int \sin x\,dx + \int x\cos x\,dx$$

By using integrating by parts, we have

$$y^2\ln y - \cancel{\tfrac{1}{2}y^2} + \cancel{\tfrac{1}{2}y^2} = -\cancel{\cos x} + [x\sin x + \cancel{\cos x}] + C$$ (2)

$$\boxed{y^2\ln y = x\sin x + C}$$

Question 2: (a) $\underbrace{(x + y^2 + \sin^{-1}(y))}_{M}dx + \underbrace{\left(2xy + \frac{x}{\sqrt{1-y^2}}\right)}_{N}dy = 0$

$\frac{\partial M}{\partial y} = 2y + \frac{1}{\sqrt{1-y^2}}$, $\frac{\partial N}{\partial x} = 2y + \frac{1}{\sqrt{1-y^2}}$ (1)

Then the D.E. is an exact equation, so there exists F of x and y such that: $\frac{\partial F}{\partial x} = M = x + y^2 + \sin^{-1}(y)$, $\frac{\partial F}{\partial y} = 2xy + \frac{x}{\sqrt{1-y^2}}$ (1)

$$F(x,y) = \int (x + y^2 + \sin^{-1}(y))\,dx = \frac{1}{2}x^2 + y^2 x + (\sin^{-1} y)x + \phi(y)$$

$$\frac{\partial F}{\partial y} = \cancel{2yx} + \cancel{\frac{x}{\sqrt{1-y^2}}} + \phi'(y) = \cancel{2xy} + \cancel{\frac{x}{\sqrt{1-y^2}}} \Rightarrow \phi'(y) = 0, \boxed{\phi(y) = C}$$

Then the solution of the D.E is

2

$$\boxed{F(x,y) = \frac{1}{2}x^2 + y^2 x + x\sin^{-1}(y) + C = 0}$$

(b) $\begin{cases} (x-y)\,dx + (3x+y)\,dy = 0 \\ y(3) = -2 \end{cases}$ (homogeneous D.E.)

We put $\frac{y}{x} = u$, $x > 0$, $y = u \cdot x$, $dy = u\,dx + x\,du$

$$\left(1 - \frac{y}{x}\right)dx + \left(3 + \frac{y}{x}\right)dy = 0$$

1

$$(1-u)\,dx + (3+u)(u\,dx + x\,du) = 0$$

$$(1 - u + 3u + u^2)\,dx + x(3+u)\,du = 0$$

$$(u^2 + 2u + 1)\,dx + x(3+u)\,du = 0$$

$$\frac{dx}{x} + \frac{3+u}{(u+1)^2}\,du = 0 \Rightarrow \frac{dx}{x} + 3\frac{1}{(u+1)^2}\,du + \frac{u+1-1}{(u+1)^2}\,du = 0$$

$$\frac{dx}{x} + 3\frac{1}{(u+1)^2}\,du + \frac{1}{(u+1)}\,du - \frac{1}{(u+1)^2}\,du = 0$$

1

$$\int \frac{dx}{x} + 2\int \frac{1}{(u+1)^2}\,du + \int \frac{1}{u+1}\,du = 0$$

$$\boxed{\ln x - \frac{2}{u+1} + \ln|u+1| = C}$$

1

$$\ln x - \frac{2x}{y+x} + \ln\left|\frac{y+x}{x}\right| = C$$

For $y(3) = 2 \Rightarrow \cancel{\ln 3} - 6 - \cancel{\ln 3} = C \Rightarrow \boxed{C = -6}$

1

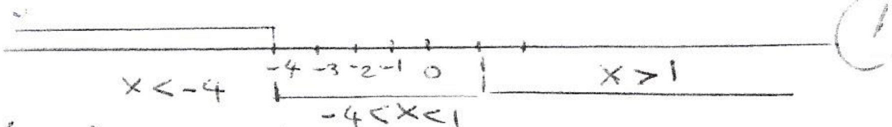
$\boxed{\ln|y+x| - \frac{2x}{y+x} + 6 = 0}$ is the solution of the IVP.

Question 3:

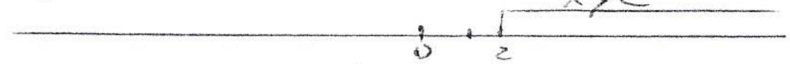
(a) $$\begin{cases} (x-1)(x+4)\,y''' + \dfrac{\ln(x-2)}{x^2+1}\,y' + e^x y = 4x^2+1 \\ y(4) = -1, \quad y'(4) = 2 \end{cases}$$

Solution: $a_2(x) = (x-1)(x+4)$ is continuous on $\mathbb{R}$, and

$a_2(x) \neq 0$ if $x<-4$, $-4<x<1$ or $x>1$ 1



$a_1(x) = \dfrac{\ln(x-2)}{x^2+1}$ is continuous on $x>2$ 1



$a_0(x) = e^x$ and $R(x) = 4x^2+1$ are continuous on $\mathbb{R}$

But $x=4>2 \Rightarrow I=(2,\infty)$ is the largest interval 2

for which the initial value problem has a unique solution.

(b) $f_1 = 1+x$, $f_2 = x$, $f_3 = 2x+3$

Solution: We see that $W(x, f_1, f_2, f_3) = \begin{vmatrix} 1+x, & x, & 2x+3 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$ for all $x \in \mathbb{R}$

So we have to use the Definition: 1

Let c_1, c_2 and $c_3 \in \mathbb{R}$ such that

$$c_1(1+x) + c_2 x + c_3(2x+3) = 0 \text{ for all } x \in \mathbb{R}$$

hence $(c_1 + c_2 + 2c_3)x + c_1 + 3c_3 = 0$ for all $x \in \mathbb{R}$. 1

Then $c_1 + c_2 + 2c_3 = 0$ and $c_1 + 3c_3 = 0$

If we take $c_1 = 1$, then $c_3 = \frac{-1}{3}$, $c_2 = -c_1 - 2c_3 = -1 + \frac{2}{3} =$ $c_2 = -1/3$

So we have $(c_1 = 1,\ c_2 = -1/3,\ c_3 = -1/3)$ such that 2

$(1)(1+x) - \frac{1}{3}x - \frac{1}{3}(2x+3) = 0$ for all $x \in \mathbb{R}$, hence

f_1, f_2 and f_3 are linearly dependent on $\mathbb{R}$.

Question 4: $\frac{dy}{dt} = ky$, $y(0) = y_0$, $y(1) = \frac{5}{2} y_0$

$\frac{dy}{y} = k\,dt$, $\ln y = kt + c$ (5)

$y = e^{kt} \cdot e^{c} = c_1 e^{kt}$; $e^{c} = c_1$

$y(0) = y_0 \Rightarrow y_0 = c_1$, $y(t) = y_0 e^{kt}$

At $t = 1$ (hour) we have $y(1) = \frac{5}{2} y_0 = y_0 e^{k}$

hence $\boxed{k = \ln(5/2)}$ (2)

$y(t) = y_0 e^{t \ln(5/2)}$

Now we have to find t s.t $y(t) = 4 y_0$. For (1)

$y(t) = 4y_0 = \cancel{y_0} e^{t \ln(5/2)}$

$\ln 4 = t \ln(5/2)$ or $t = \frac{\ln 4}{\ln(5/2)}$

$\boxed{t \simeq 1.5129 \simeq 1.5 \text{ hour}}$ (2)