King Saud University
College of Sciences
Department of Mathematics
[Solution Key] Math-244 (Linear Algebra); Mid-term Exam; Semester 2 (1442)

Max. Marks: 30 **Time: 2 hours**

Note: ***Attempt all the five questions***!

Question 1: [Marks: 3+3]

a) **Let** $\mathbf{A} = \begin{bmatrix} -1 & 1 & 3 & 0 \\ 1 & 2 & 3 & -2 \\ 0 & -1 & -2 & 7 \\ 2 & 1 & 0 & 6 \end{bmatrix}$. Then:

i) **Find** the **reduced row echelon form** of the **matrix A**.

ii) Use the reduced row echelon form to **show** that the **matrix A** is **not invertible**.

b) **Let** $\mathbf{X} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$. **Find** the value of λ such that $\mathbf{X}^8 - 4\lambda\mathbf{I} = \mathbf{O}$.

Solution: a) / i). $\mathbf{A} = \begin{bmatrix} -1 & 1 & 3 & 0 \\ 1 & 2 & 3 & -2 \\ 0 & -1 & -2 & 7 \\ 2 & 1 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (RREF).

ii). RREF of $\mathbf{A} \neq \mathbf{I} \Longrightarrow \mathbf{A}$ is not invertible.

b). $\mathbf{X}^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Longrightarrow \mathbf{X}^8 = (\mathbf{X}^2)^4 = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix} \Longrightarrow \begin{bmatrix} 16-4\lambda & 0 & 0 \\ 0 & 16-4\lambda & 0 \\ 0 & 0 & 16-4\lambda \end{bmatrix} = \mathbf{X}^8 - 4\lambda\mathbf{I} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Longrightarrow \lambda = 4.$

Question 2: [Marks: 3+3]

a) **Let** $\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. **Find** the **matrix Y** such that $(2\mathbf{X} + \mathbf{Y})^{-1} = \boldsymbol{adj}(\mathbf{X})$.

b) **Let** $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & bc & a \\ 1 & ca & b \\ 1 & ab & c \end{bmatrix}$. **Show** that $\boldsymbol{det}(\mathbf{B}) = -\boldsymbol{det}(\mathbf{A})$.

Solution: a) $(2\mathbf{X}+\mathbf{Y})^{-1} = \boldsymbol{adj}(\mathbf{X}) = |\mathbf{X}|\mathbf{X}^{-1} \Longrightarrow 2\mathbf{X}+\mathbf{Y} = (|\mathbf{X}|\mathbf{X}^{-1})^{-1} = -\mathbf{X} \Longrightarrow \mathbf{Y} = -3\mathbf{X} = \begin{bmatrix} -3 & -3 & -3 \\ -3 & 0 & -3 \\ 0 & -3 & -3 \end{bmatrix}$.

b). $\boldsymbol{det}(\mathbf{B}) = \begin{vmatrix} 1 & bc & a \\ 1 & ca & b \\ 1 & ab & c \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a & abc & a^2 \\ b & bca & b^2 \\ c & cab & c^2 \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a & b & c \\ abc & abc & abc \\ a^2 & b^2 & c^2 \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ a^2 & b^2 & c^2 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = -\boldsymbol{det}(\mathbf{A}).$

Question 3: [Marks: 3+3].

a) **Find** the value/s of α such that the following **linear system**

$$\begin{aligned} x + y + \frac{\alpha}{3}z &= 1 \\ x + y + z &= 1 \\ x + \alpha y + z &= 2 \end{aligned}$$

has: (i) **no solution** (ii) **unique solution** (iii) **infinitely many solutions**.

b) **Solve** the following **homogeneous linear system**. **Why** this system cannot be solved by Cramer's Rule?

$$\begin{aligned} x - 2y + 3z &= 0 \\ 3x + y - 2z &= 0 \\ 2x - 4y + 6z &= 0. \end{aligned}$$

Solution: a) Since $\begin{bmatrix} 1 & 1 & \frac{\alpha}{3} & \vdots & 1 \\ 1 & 1 & 1 & \vdots & 1 \\ 1 & \propto & 1 & & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & & 1 \\ 0 & \propto -1 & 0 & \vdots & 1 \\ 0 & 0 & \propto -3 & & 0 \end{bmatrix}$,

(i) the system has no solution for $\propto = 1$;
(ii) the system has unique solution for $\propto \in \mathbb{R} - \{1, 3\}$;
(iii) the system has infinitely many solutions for $\propto = 3$.

b). Matrix of coefficients = A = $\begin{bmatrix} 1 & -2 & 3 \\ 3 & 1 & -2 \\ 2 & -4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 7 & -11 \\ 0 & 0 & 0 \end{bmatrix}$. So, solution set of the given system $\left\{\left(\frac{1}{7}t, \frac{11}{7}t, t\right) \middle| t \in \mathbb{R}\right\}$. Since $det(A). = 0$, the Cramer's rule is not applicable.

Question 4: [Marks: 3+3]

a) **Show** that $\{1 - x, 1 - x^2, 1 + x + x^2\}$ is a **basis** of the **vector space** P_2 of all polynomials in real variable x with **degree** ≤ 2.

b) **Let S** $= \{(1,0,1,1), (1,-1,2,1), (1,-2,3,1)\}$ **generates** the **vector subspace F** of the Euclidean space $\mathbb{R}^4$. **Find** a **basis** of **F** contained in **S** and **show** that $(0,-2,7,6) \notin$ **F**.

Solution: a) Since $\{1, x\ x^2\}$ is a basis of the vector space P_2, dim $(P_2) = 3$. Next, $\{1- x, 1- x^2, 1+ x+ x^2\}$ is linearly independent because $\alpha(1-x) + \beta(1-x^2) + \gamma(1+x+x^2) = 0$ implies $\alpha = \beta = \gamma = 0$. So, $\{1- x, 1- x^2, 1+ x+x^2\}$ is a basis of the vector space P_2.

b). $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & -2 \\ 1 & 2 & 3 & 7 \\ 1 & 1 & 1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (REF). Hence, $\{(1,0,1,1), (1,-1,2,1)\}$ is a basis of **F** and $(0,-2,7,6) \notin$ **F**.

Question 5: [Marks: 3+3]

a) **Let B** = **{(2, 1), (1, 0)}** and **C** = **{(1, -2), (0, 1)}** be **bases** of the Euclidean space $\mathbb{R}^2$ and $v = (1,2)$. **Find** the **coordinate vector** $[v]_B$ and the **transition matrix** ${}_CP_B$. Then use the transition matrix to **find** $[v]_C$.

b) **Let A** = $\begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 2 & 1 & 1 & 1 & 0 \\ -1 & -2 & 1 & 0 & 1 \\ -2 & -2 & 0 & 1 & 4 \end{bmatrix}$. **Find**:

(i) a **basis** of ***col*** **(A)** (ii) ***rank*** **(A)** (iii) ***nullity*** **(A)**.

Solution: a) $[v]_B = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and ${}_CP_B = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$. Hence, $[v]_C = {}_CP_B\,[v]_B = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

b). A = $\begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 2 & 1 & 1 & 1 & 0 \\ -1 & -2 & 1 & 0 & 1 \\ -2 & -2 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -2 & -4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (REF). Hence:

(i) $\{(1,2,-1,-2), (1,1,-2,-2), (2,1,0,1)\}$ is a basis of *col* (A);
(ii) *rank* (A) = 3;
(iii) *nullity* (A) = 2.

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