Note: *In answering the questions other than Question 2/b), if a student has used some method/s different from the one/s being preferred and used in this solution key then the respective instructor should award the student appropriately.*

Solution of Question 1: a) $|A^{-1}| = 1$ so that $|A| = 1$. [1 mark]

Hence, $adj\big(adj(A)\big) = |A|^{3-2}A = A$ [1 mark]

$$= (A^{-1})^{-1} = |A|\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 0 & -1\end{bmatrix} = \begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 0 & -1\end{bmatrix}.$$ [1 mark]

b) By the symmetricity, $\begin{bmatrix}\mathbf{2} & \mathbf{3k-2}\\ \mathbf{k^2} & \mathbf{-1}\end{bmatrix} = \begin{bmatrix}2 & k^2\\ 3k-2 & -1\end{bmatrix}$. [1 mark]

So, $k^2 - 3k + 2 = 0$. Thus, $k = 1, 2$. [1 mark]

c) $|B| = -30 \neq 0 \Leftrightarrow B$ is invertible $\Leftrightarrow$ $\boldsymbol{B}$ is a product of elementary matrices. [0.5+0.5+1 mark]

Solution of Question 2: a) $[A\!:B] = \left[\begin{array}{cccc|c}1 & -1 & 0 & 1 & 1\\ -1 & 2 & 0 & -2 & 2\\ 3 & 1 & 0 & 3 & -1\end{array}\right] \sim \left[\begin{array}{cccc|c}1 & -1 & 0 & 1 & 1\\ 0 & 1 & 0 & -1 & 3\\ 0 & 0 & 0 & 1 & -4\end{array}\right]$. [1.5 marks]

Hence, solution set of the system $= \{(4, -1, t, -4)\!: t \in \mathbb{R}\}$**.** [1.5 marks]

b) Let $\boldsymbol{A}$ denote the matrix of coefficients. Then:

$|\boldsymbol{A}| = \begin{vmatrix}1 & -1 & 0\\ -2 & 3 & -4\\ -2 & 3 & -3\end{vmatrix} = 1,\ |\boldsymbol{A_x}| = \begin{vmatrix}1 & -1 & 0\\ 0 & 3 & -4\\ 1 & 3 & -3\end{vmatrix} = 7,\ |\boldsymbol{A_y}| = \begin{vmatrix}1 & 1 & 0\\ -2 & 0 & -4\\ -2 & 1 & -3\end{vmatrix} = 6,\ |\boldsymbol{A_z}| = \begin{vmatrix}1 & -1 & 1\\ -2 & 3 & 0\\ -2 & 3 & 1\end{vmatrix} = 1.$ [2 marks]

Hence, $x = \frac{|\boldsymbol{A_x}|}{|\boldsymbol{A}|} = 7,\ y = \frac{|\boldsymbol{A_y}|}{|\boldsymbol{A}|} = 6,\ z = \frac{|\boldsymbol{A_z}|}{|\boldsymbol{A}|} = 1.$ [1 mark]

Solution of Question 3:

a) The zero polynomial $= ax - 2ax^4 + (a-b)x^6 + (3a+2b)x^7 \in \boldsymbol{E}$ with $a = b = 0 \in \mathbb{R}$. [0.5 mark]

Next, for any $a_1, a_2, b_1, b_2, \lambda, \delta \in \mathbb{R}$, we have: [2.5 marks]

$$\lambda(a_1x - 2a_1x^4 + (a_1 - b_1)x^6 + (3a_1 + 2b_1)x^7) + \delta(a_2x - 2a_2x^4 + (a_2 - b_2)x^6 + (3a_2 + 2b_2)x^7)$$

$$= (\lambda a_1 + \delta a_2)\,x - 2(\lambda a_1 + \delta a_2)x^4 + ((\lambda a_1 + \delta a_2) - (\lambda b_1 + \delta b_2))x^6 + (3(\lambda a_1 + \delta a_2) + 2(\lambda b_1 + \delta b_2))x^7 \in \boldsymbol{E}.$$

Hence, $\boldsymbol{E}$ is a vector subspace of $\boldsymbol{P_7}$.

b) $ax - 2ax^4 + (a-b)x^6 + (3a+2b)x^7 = a(x - 2x^4 + x^6 + 3x^7) + b(-x^6 + 2x^7)$ for all $a, b \in \mathbb{R}$. [0.5mark]

Moreover, $x - 2x^4 + x^6 + 3x^7$ and $-x^6 + 2x^7$ are linearly independent polynomials in $\boldsymbol{E}$. [0.5 mark]

Hence, $\{x - 2x^4 + x^6 + 3x^7,\ -x^6 + 2x^7\}$ is a basis of $\boldsymbol{E}$ and $dim(\boldsymbol{E}) = 2$. [0.5+0.5 mark]

c) For any $(x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3) \in \mathbb{R}^3$; $\alpha \in \mathbb{R}$, we have

$$< (x_1, x_2, x_3), (y_1, y_2, y_3) > = x_1y_1 + 2x_2y_2 + 3x_3y_3 = < (y_1, y_2, y_3), (x_1, x_2, x_3) >;$$

$$< (x_1, x_2, x_3) + (y_1, y_2, y_3), (z_1, z_2, z_3) > = < (x_1 + y_1, x_2 + y_2, x_3 + y_3), (z_1, z_2, z_3) >$$

$$= (x_1 + y_1) z_1 + 2(x_2 + y_2) z_2 + 3(x_3 + y_3) z_3 = x_1 z_1 + 2x_2 z_2 + 3x_3 z_3 + y_1 z_1 + 2y_2 z_2 + 3y_3 z_3$$

$$= < (x_1, x_2, x_3), (z_1, z_2, z_3) > + < (y_1, y_2, y_3), (z_1, z_2, z_3) > ;$$

$$< \alpha(x_1, x_2, x_3), (y_1, y_2, y_3) > = < (\alpha x_1, \alpha x_2, \alpha x_3), (y_1, y_2, y_3) > = (\alpha x_1) y_1 + 2(\alpha x_2) y_2 + 3(\alpha x_3) y_3$$

$$= \alpha(x_1 y_1 + 2x_2 y_2 + 3x_3 y_3 = \alpha < (x_1, x_2, x_3), (y_1, y_2, y_3) >;$$

$$< (x_1, x_2, x_3), (x_1, x_2, x_3) > = x_1{}^2 + 2x_2{}^2 + 3x_3{}^2 \geq 0.$$

Finally, $< (x_1, x_2, x_3), (x_1, x_2, x_3) > = 0 \Leftrightarrow x_1{}^2 + 2x_2{}^2 + 3x_3{}^2 = 0 \Leftrightarrow (x_1, x_2, x_3) = (0,0,0)$. Q.E.D. [2 marks]

d) $v_1 = u_1 = (1,1,1)$; $v_2 = u_2 - \frac{<u_2\,,\; v_1>}{||v_1||^2} v_1 = (\frac{1}{2}, \frac{1}{2}, \frac{-1}{2})$; [0.5+1 mark]

$$v_3 = u_3 - \frac{<u_3\,,\; v_1>}{||v_1||^2} v_1 - \frac{<u_3\,,\; v_2>}{||v_2||^2} v_2 = (\frac{-2}{3}, \frac{1}{3}, 0).$$ [1.5 marks]

Solution of Question 4: a) $(1,0) = \alpha(1,-1) + \beta(1,1) \Longrightarrow \alpha + \beta = 1,\; -\alpha + \beta = 0 \Longrightarrow \alpha = \beta = \frac{1}{2}$.
So that $(1,0) = \frac{1}{2}(1,-1) + \frac{1}{2}(1,1)$. Similarly, $(0,1) = -\frac{1}{2}(1,-1) + \frac{1}{2}(1,1)$. [1 mark]

Hence, $\mathbf{T}(1,0) = \mathbf{T}\left(\frac{1}{2}(1,-1) + \frac{1}{2}(1,1)\right) = \frac{1}{2}\mathbf{T}(1,-1) + \frac{1}{2}\mathbf{T}(1,1) = \frac{1}{2}\big((3,5,2) + (2,-1,-3)\big) = \frac{1}{2}(5,4,-1)$ and
$\mathbf{T}(0,1) = \frac{1}{2}(-1,-6,-5)$. [2 marks]

b) $(3,5,2) = \mathbf{T}(1,-1) = \alpha(1,1,0)) + \beta(1,0,1) + \gamma(1,1,1) \Longrightarrow \alpha + \beta + \gamma = 3,\; \alpha + \gamma = 5, \beta + \gamma = 2$

$\Longrightarrow \alpha = 1, \beta = -2, \gamma = 4$. So, $[\mathbf{T}(u_1)]_C = [\mathrm{T}(1,-1)]_C = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$. Similarly, $[\mathbf{T}(u_1)]_C = \begin{bmatrix} 5 \\ 3 \\ -6 \end{bmatrix}$. [2 marks]

Hence, $[\mathbf{T}]_B^C = [\,[\mathbf{T}(u_1)]_C \;\; [\mathbf{T}(u_1)]_C] = \begin{bmatrix} 1 & 5 \\ -2 & 3 \\ 4 & -6 \end{bmatrix}$. [1 mark]

c) $[\mathbf{T}(0,1)]_C = [\mathbf{T}]_B^C\,[(0,1)]_B = \begin{bmatrix} 1 & 5 \\ -2 & 3 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{5}{2} \\ -5 \end{bmatrix}$. [1.5+1+0.5 marks]

Solution of Question 5: a) $\boldsymbol{q_A(\lambda) = (\lambda - 0)^3(\lambda - 1)^2(\lambda + 1)^3 = \lambda^3(\lambda - 1)^2(\lambda + 1)^3}$. [2 marks]

b) Because algebraic multiplicity of eigenvalue λ_j = the geometric multiplicity of $\lambda_j, \forall j = 1,2,3$. [2 marks]

c) $$\boldsymbol{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$ [2 marks]

d) $\boldsymbol{A^{11} = (PDP^{-1})^{11} = PD^{11}P^{-1} = PDP^{-1} = A}$. [3 marks]

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