King Saud University - College of Science - Department of Mathematics - First Midterm Exam M201, October 13, 2025, 5:45-7:15 PM

Calculators are not allowed.

Question 1 [3] Find and sketch the domain of the function $f(x, y) = Ln(x) + Ln(y - 1)$.

Question 2 [4] Does the following limits exist ? Justify.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}, \quad \lim_{(x,y)\to(0,0)} \frac{y(x^2+x)}{x^2+y^2}$$

Question 3 [6] Study the continuity and differentiability at $(0, 0)$ for the function

$$f(x,y) = \begin{cases} \frac{x^2y+xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Question 4 [6]

1. If $x + y + Ln(x) + Ln(y) = 2$ then find $\frac{dy}{dx}(1)$.

2. $$\text{If} \quad W = \frac{2x-y}{x+3y}, \; x = e^{2u}\cos(3v), \; y = e^{2u}\sin(3v)$$

   then find $\frac{\partial W}{\partial u}$ and $\frac{\partial W}{\partial v}$

Question 5 [6]

1. Find local extrema and saddle points (if any) for the function

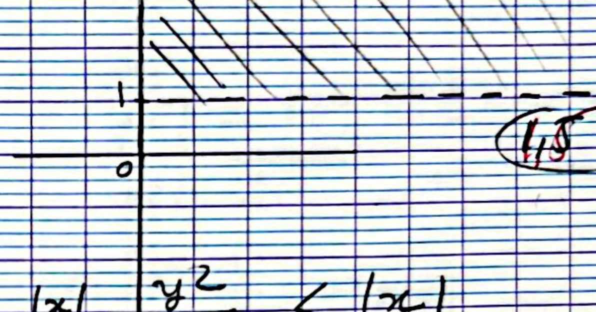
$$f(x,y) = 2x^3 + 2y^3 - 6xy + 1 .$$

2. Does there exists $(a, b) \in \mathbb{R}^2$ such that $f(a, b) < -1$? Justify.

Q1/ $f(x,y) = \ln(x) + \ln(y-1)$

$D_f = \{(x,y) : x > 0 \text{ and } y > 1\}$

1,5



1,5

Q2/ ./ $0 \leq \left|\frac{xy^2}{x^2+y^2}\right| \leq |x| \frac{y^2}{x^2+y^2} \leq |x|$

$\lim_{x \to 0} |x| = 0$ so by sandwich theorem $\lim_{(x,y)\to 0} \frac{xy^2}{x^2+y^2} = 0$

2

../ Path $x = 0$ $\quad \frac{y(0^2+0)}{0^2+y} = 0 \xrightarrow[y \to 0]{} 0$

Path $y = x$ $\quad \frac{x(x^2+x)}{x^2+x^2} = \frac{x^3+x^2}{2x^2} = \frac{x+1}{2}$

$\xrightarrow[x \to 0]{} \frac{1}{2}$

2

Paths theorem $\frac{1}{2} \neq 0$ so $\lim_{(x,y)\to 0} \frac{y(x^2+x)}{x^2+y^2}$

doesn't exist

Q3/ $f(x,y) = \begin{cases} \frac{x^2y + xy^3}{x^2+y^2} & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = 0 \end{cases}$

If $(x,y) \neq 0$ $\quad 0 \leq |f(x,y)| \leq |y| \dfrac{x^2}{x^2+y^2} + |xy| \dfrac{y^2}{x^2+y^2}$

$$\leq |y| + |xy|$$

$\lim |y| + |xy| = 0$

1,5 Sandwich theorem $\Rightarrow \lim f(x,y) = 0 = f(0,0)$

thus f is continuous at $(0,0)$.

1,5 $f_x(0,0) = \lim\limits_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim \dfrac{0-0}{h} = 0$

1,5 $f_y(0,0) = \lim\limits_{h \to 0} \dfrac{f(0,h) - f(0,0)}{h} = 0$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{|f(\Delta x, \Delta y) - f(0,0) - f_x(0,0)\Delta x - f_y(0,0)\Delta y|}{\sqrt{\Delta x^2 + \Delta y^2}}$$

1,5 $$= \lim \frac{|\Delta x^2 \Delta y + \Delta x \Delta y^3|}{(\Delta x^2 + \Delta y^2)\sqrt{\Delta x^2 + \Delta y^2}}$$

$$\frac{|\Delta x \Delta y^3|}{(\Delta x^2 + \Delta y^2)\sqrt{\Delta x^2 + \Delta y^2}} \leq \frac{|\Delta x|}{\sqrt{\Delta x^2 + \Delta y^2}} \frac{|\Delta y^2|}{(\Delta x^2 + \Delta y^2)} |\Delta y| \leq |\Delta y|$$

$$\Rightarrow \lim \frac{\Delta x \Delta y^3}{(\Delta x^2 + \Delta y^2)\sqrt{\Delta x^2 + \Delta y^2}} = 0$$

Path $\Delta y = 0$ $\quad \dfrac{\Delta x^2 \Delta y}{(\Delta x^2 + \Delta y^2)\sqrt{\Delta x^2 + \Delta y^2}} = 0 \longrightarrow 0$

Path $\Delta x = \Delta y$ $\quad \dfrac{\Delta x^2 \Delta y}{(\Delta x^2 + \Delta y^2)\sqrt{\Delta x^2 + \Delta y^2}} = \dfrac{\Delta x^3}{2\Delta x^2 \sqrt{2} \Delta x} = \dfrac{1}{2\sqrt{2}} \neq 0$

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Path theorem $\Rightarrow$ $\lim \dfrac{\Delta x^2 \Delta y}{(\Delta x^2+\Delta y^2)\sqrt{\Delta x^2+\Delta y^2}}$ does not exist

Hence f is not differentiable at $(0,0)$

Q4/

1/ $F(x,y) = x + y + \ln x + \ln y - 2 = 0$

2 $\quad y' = -\dfrac{F_x}{F_y} = -\dfrac{\left(1+\frac{1}{x}\right)}{\left(1+\frac{1}{y}\right)} \Rightarrow y'(1) = -\dfrac{2}{1+\frac{1}{y(1)}} = -\dfrac{2y(1)}{1+y(1)}$

$y(1) = 1 \Rightarrow y'(1) = -1$

2/ $W = \dfrac{2x-y}{x+3y}$, $x = e^{2u}\cos(3v)$, $y = e^{2u}\sin(3v)$

1st method: $W = \dfrac{2\cos(3v) - \sin(3v)}{\cos(3v) + 3\sin(3v)}$

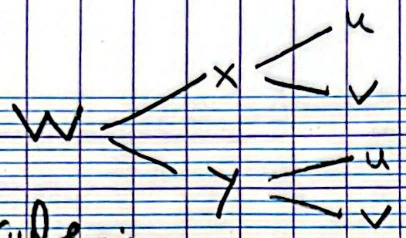
2 $\quad \dfrac{\partial w}{\partial u} = 0$ and

$$\frac{\partial w}{\partial v} = \frac{(-6\sin 3v - 3\cos 3v)(\cos 3v + 3\sin 3v) - (2\cos 3v - \sin 3v)(-3\sin 3v + 9\cos 3v)}{(\cos 3v + 3\sin 3v)^2}$$

2

$$= -\frac{21}{(\cos 3v + 3\sin 3v)^2}$$

2nd method



Chain rule:

$$\frac{\partial W}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial w}{\partial x} = \frac{2(x+3y)-(2x-y)}{(x+3y)^2} = \frac{7y}{(x+3y)^2}$$

$$\frac{\partial w}{\partial y} = \frac{-(x+3y)-3(2x-y)}{(x+3y)^2} = -\frac{7x}{(x+3y)^2}$$

$$\frac{\partial x}{\partial u} = 2x \qquad \frac{\partial y}{\partial u} = 2y$$

$$\frac{\partial x}{\partial v} = -3y \qquad \frac{\partial y}{\partial v} = 3x$$

So $\frac{\partial w}{\partial u} = \frac{0}{(x+3y)^2} = 0$ 2

$$\frac{\partial w}{\partial v} = \frac{-21y^2-21x^2}{(x+3y)^2} = -\frac{21}{(\cos 3v + 3\sin 3v)^2}$$ 2

$$x^2 = e^{4u}\cos^2 3v \qquad y^2 = e^{4u}\sin^2 3v$$

$$x^2+y^2 = e^{4u}$$

Q5/ 1. $f(x,y) = 2x^3 + 2y^3 - 6xy + 1$

$f_x = 6(x^2 - y)$ 0,5 $f_y = 6(y^2 - x)$ 0,5

Critical points: $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 = y & (1) \\ x = y^2 & (2) \end{cases}$

$(1) \& (2) \Rightarrow x = x^4 \Leftrightarrow x(x^3 - 1) = 0$

$\Leftrightarrow x = 0$ ou $x = 1$

critical points $(0,0)$ 1 and $(1,1)$ 1

$f_{xx} = 12x \quad f_{yy} = 12y \quad f_{xy} = -6$

$D(x,y) = 12^2 xy - 36$

$D(0,0) = -36 < 0 \Rightarrow (0,0,1 = f(0,0))$ saddle point 1

$D(1,1) = 12^2 - 36 > 0 \quad f_{xx}(1,1) = 12 > 0$

$\Rightarrow f(1,1) = -1$ is a local minimum. 1

2. $f(x,0) = 2x^3 + 1$

$f(-2,0) = -15 < -1$ 1

Answer: yes.