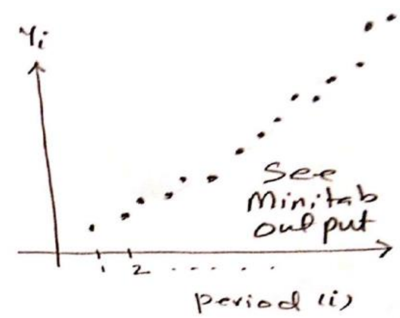
Chapter 4

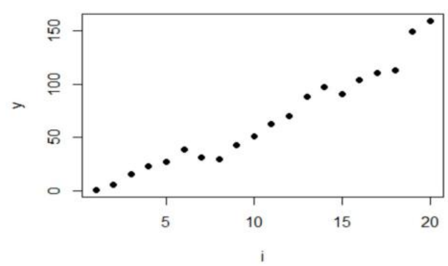
Exercise 4.1:

Period	y_i = # cases
1	1
2	6
3	16
4	23
5	27
⋮	⋮
20	159

(a)



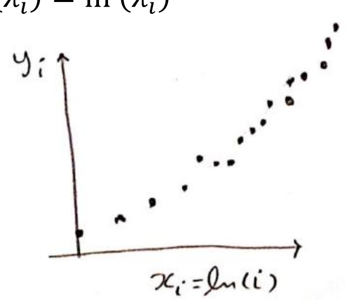
<u>or by R:</u>
plot(i,y,pch = 16)

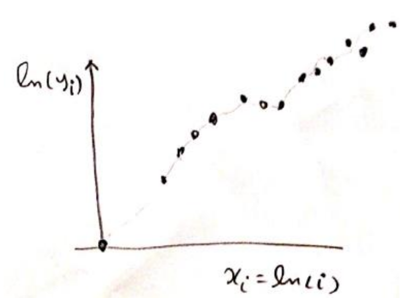


(b)

$$y_i \sim P_0(\lambda_i) \;\Rightarrow \mu_i = E(y_i) = \lambda_i \text{ and } \sigma_i^2 = \mathrm{Var}(y_i) = \lambda_i \quad \therefore E(y_i) = \mathrm{Van}(y_i)$$
$$\lambda_i = i^\theta \Leftrightarrow \ln(\lambda_i) = \theta \ln(i) = \theta x_i = \eta_i$$

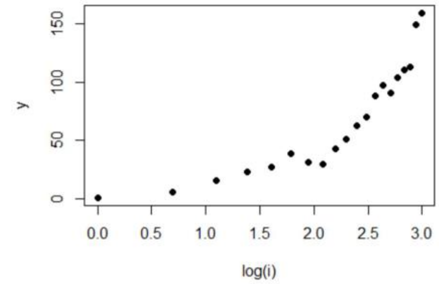
where $x_i = \ln(i)$ and $\eta_i = \eta_i(x) = \theta x_i$
Link function: $g(\lambda_i) = \ln(\lambda_i)$





(See Minitab output)
<u>or by R:</u>

plot(log(i),y, pch=16)



(c)

$$\mu_i = e^{\eta_i} = e^{\beta_1+\beta_2 x_i} = e^{\beta_1}e^{\beta_2 \ln(i)} = e^{\beta_1}e^{\ln(i^{\beta_2})} = e^{\beta_1} i^{\beta_2}$$

where $x_i = \ln(i)$.

$$\mu_i = e^{\eta_i} \Leftrightarrow \ln(\mu_i) = \eta_i \Rightarrow \begin{bmatrix} \dfrac{\partial \mu_i}{\partial \eta_i} = e^{\eta_i} \\ \dfrac{\partial \eta_i}{\partial \mu_i} = e^{-\eta_i} \end{bmatrix}$$

$$\underline{W} = \operatorname{diag}\left[\frac{1}{\operatorname{Var}(y_i)}\left(\frac{\partial \mu_i}{\partial \eta_i}\right)^2\right] = \left[\frac{1}{e^{\eta_i}}(e^{\eta_i})^2\right] = \operatorname{diag}[e^{\eta_i}] = \operatorname{diag}\left[e^{\beta_1} i^{\beta_2}\right] = e^{\beta_1}\operatorname{diag}\left[i^{\beta_2}\right]$$

$$= e^{\beta_1}\begin{bmatrix} 1^{\beta_2} & & & & 0 \\ & 2^{\beta_2} & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & N^{\beta_2} \end{bmatrix}_{N\times N}$$

$$\underset{\sim}{z} = \underset{\sim}{\eta} + \operatorname{diag}\left[\frac{\partial \eta_i}{\partial \mu_i}\right]\left(\underset{\sim}{y} - \underset{\sim}{\mu}\right) = \underset{\sim}{\eta} + \operatorname{diag}[e^{-\eta_i}]\left(\underset{\sim}{y} - \underset{\sim}{\mu}\right);\ y = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \text{ and } \underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_N \end{pmatrix}$$

$$\Rightarrow z_i = \eta_i + e^{-\eta_i}(y_i - \mu_i)\ ;\ i = 1,2,\dots,N$$
$$= \eta_i + y_i e^{-\eta_i} - e^{-\eta_i}e^{\eta_i} = \eta_i + y_i e^{-\eta_i} - 1 = (\beta_1 + \beta_2 x_i) + y_i e^{-(\beta_1+\beta_2 x_i)} - 1$$
$$= \beta_1 + \beta_2 \ln(i) + y_i e^{-\beta_1} e^{-\beta_2 \ln(i)} - 1 = \beta_1 + \beta_2 \ln(i) + y_i e^{-\beta_1} i^{-\beta_2} - 1$$
$$= \beta_1 + \beta_2 \ln(i) + \frac{y_i}{e^{\beta_1} i^{\beta_2}} - 1$$

Iterative Maximum Likelihood procedure:

The MLE of $\underset{\sim}{\beta} = \binom{\beta_1}{\beta_2}$ at iteration $(m+1)$ is:

$$\underset{\sim}{b}^{(m+1)} = \left(\underline{x}^t \underline{W}^{(\mathrm{m})} \underline{x}\right)^{-1} \underline{x}^t \underline{W}^{(\mathrm{m})}\ \underset{\sim}{z}\ ; m = 0,1,2,\dots.$$

Note: The Design matrix is $\underline{x} = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_N \end{bmatrix} = \begin{bmatrix} 1 & \ln(1) \\ \vdots & \vdots \\ 1 & \ln(N) \end{bmatrix}_{N\times 2}$

and the information matrix is $\underset{\sim}{J} = \underline{x}^t \underline{W} \underline{x} = e^{\beta_1}\begin{bmatrix} \sum_{i=1}^N i^{\beta_2} & \sum_{i=1}^N i^{\beta_2}\ln(i) \\ \sum_{i=1}^N i^{\beta_2}\ln(i) & \sum_{i=1}^N i^{\beta_2}[\ln(i)]^2 \end{bmatrix}$

- Step $(m = 0)$:

Let initial value of $\underset{\sim}{b}$ be $\underset{\sim}{b}^{(0)} = \begin{pmatrix} b_1^{(0)} \\ b_2^{(0)} \end{pmatrix} = \binom{1}{1}$.

$$b_1^{(0)} = 1, b_2^{(0)} = 1$$

For these initial values, we have:

$$\underline{W}^{(0)} = \text{diag}\left[e^{b_1^{(0)}} i^{b_2^{(0)}}\right] = \text{diag}[ei] = e\begin{bmatrix} 1 & & & & 0 \\ & 2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & N \end{bmatrix}_{N\times N}$$

$$\underset{\sim}{J}^{(0)} = \underline{x}^t \underline{W}^{(0)} \underline{x} = \begin{bmatrix} 1 & \cdots & 1 \\ \ln(1) & & \ln(N) \end{bmatrix} e \begin{bmatrix} 1 & & & & 0 \\ & 2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & N \end{bmatrix} \begin{bmatrix} 1 & \ln(1) \\ \vdots & \vdots \\ 1 & \ln(N) \end{bmatrix}$$

$$= e\begin{bmatrix} \sum_{i=1}^{N} i & \sum_{i=1}^{N} i\ln(i) \\ \sum_{i=1}^{N} i\ln(i) & \sum_{i=1}^{N} i[\ln(i)]^2 \end{bmatrix} = \begin{bmatrix} 570.839 & 1439.61 \\ 1439.61 & 3768.84 \end{bmatrix}$$

$$\left(\underline{x}^t \underline{W}^{(0)} \underline{x}\right)^{-1} = \begin{bmatrix} 0.0477528 & -0.0182405 \\ -0.0182405 & 0.0072328 \end{bmatrix}$$

$$z_i^{(0)} = b_1^{(0)} + b_2^{(0)}\ln(i) + \frac{y_i}{e^{b_1^{(0)}} i^{b_2^{(0)}}} - 1 = 1 + \ln(i) + \frac{y_i}{ei} - 1 = \ln(i) + \frac{y_i}{e_i}$$

$$\underset{\sim}{z}^{(0)} = (0.36788 \quad 1.79679 \quad \ldots .92037)$$

$$\underline{x}^t \underline{W}^{(0)} = \begin{bmatrix} 1 & \cdots & 1 \\ \ln(1) & \cdots & \ln(N) \end{bmatrix} e \begin{bmatrix} 1 & & & & 0 \\ & 2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & N \end{bmatrix} = e\begin{bmatrix} 1 & 2 & \cdots & N \\ 1\ln(1) & 2\ln(2) & & \cdots N\ln(N) \end{bmatrix}$$

$$\underline{x}^t \underline{W}^{(0)}\, \underset{\sim}{z}^{(0)} = e\begin{bmatrix} 1 & 2 & \cdots & N \\ 1\ln(1) & 2\ln(2) & . & N\ln(N) \end{bmatrix} \begin{bmatrix} z_1^{(0)} \\ z_2^{(0)} \\ \vdots \\ z_N^{(0)} \end{bmatrix} = e\begin{bmatrix} \sum_{i=1}^{N} i z_{(i)}^{(0)} \\ \sum_{i=1}^{N} i\ln(i) z_i^{(0)} \end{bmatrix} = \begin{bmatrix} 2750.61 \\ 7165.21 \end{bmatrix}$$

$$\underset{\sim}{b}^{(1)} = \left(\underline{x}^t \underline{W}^{(0)} \underline{x}\right)^{-1} \underline{x}^t \underline{W}^{(0)}\, \underset{\sim}{z}^{(0)} = \begin{bmatrix} 0.65254 \\ 1.65192 \end{bmatrix}$$

- Step $(m = 1)$

We repent step (0) using: $\begin{cases} b_1^{(1)} = 0.65254 \\ b_2^{(1)} = 1.65192 \end{cases}$,we will obtain: $\begin{cases} b_1^{(2)} = 0.8419 \\ b_2^{(2)} = 0.14296 \end{cases}$

The following table summarizes the result of the iteration process:

m	0	1	2	$\cdots$	5	6
$b_1^{(m)}$	1	0.6525	0.8419	$\cdots$	0.9960	0.9960
$b_2^{(m)}$	1	1.6519	1.4296	$\cdots$	1.32661	1.32661

we stop the iteration process at step $m = 6$.

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or by R:

We will use the iterative formula which is:

$$\mathbf{b}^{\mathbf{m}+1} = \mathbf{b}^{\mathbf{m}} + [\boldsymbol{\tau}(\mathbf{b}^{\mathbf{m}})]^{(-1)}\mathbf{U}(\mathbf{b}^{\mathbf{m}}), \mathbf{m} = \mathbf{0}, \mathbf{1}, 2 \ldots \ldots$$

We will start with an initial value $b^{(0)} = \begin{bmatrix} b_1^{(0)} \\ b_2^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

```
# Initial values b0

beta<-c(1,1)
beta

## [1] 1 1
```

$$\boxed{\begin{bmatrix} b_1^{(0)} \\ b_2^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Design matrix:

```
#Design matrix (X)

X=matrix(c(rep(1,20),log(i)),nrow=20,ncol = 2,byrow = F)
X

##       [,1]      [,2]
##  [1,]    1 0.0000000
##  [2,]    1 0.6931472
##  [3,]    1 1.0986123
##  [4,]    1 1.3862944
##  [5,]    1 1.6094379
##  [6,]    1 1.7917595
##  [7,]    1 1.9459101
##  [8,]    1 2.0794415
##  [9,]    1 2.1972246
## [10,]    1 2.3025851
## [11,]    1 2.3978953
## [12,]    1 2.4849066
## [13,]    1 2.5649494
## [14,]    1 2.6390573
## [15,]    1 2.7080502
## [16,]    1 2.7725887
## [17,]    1 2.8332133
## [18,]    1 2.8903718
## [19,]    1 2.9444390
## [20,]    1 2.9957323
```

$$\boxed{X = \begin{bmatrix} 1 & \ln(i) \\ \vdots & \vdots \\ 1 & \ln(N) \end{bmatrix}}$$

Transform of design matrix:

```
#Transform Design matrix (Xt)

Xt<-t(X)
Xt

##      [,1]      [,2]     [,3]     [,4]     [,5]     [,6]    [,7]     [,8]
## [1,]    1 1.0000000 1.000000 1.000000 1.000000 1.000000 1.00000 1.000000
## [2,]    0 0.6931472 1.098612 1.386294 1.609438 1.791759 1.94591 2.079442
##          [,9]    [,10]    [,11]    [,12]    [,13]    [,14]   [,15]    [,16]
## [1,] 1.000000 1.000000 1.000000 1.000000 1.000000 1.000000 1.00000 1.000000
## [2,] 2.197225 2.302585 2.397895 2.484907 2.564949 2.639057 2.70805 2.772589
##         [,17]    [,18]    [,19]    [,20]
## [1,] 1.000000 1.000000 1.000000 1.000000
## [2,] 2.833213 2.890372 2.944439 2.995732
```

$$\boxed{X^t = \begin{bmatrix} 1 & \cdots & 1 \\ \ln(1) & \cdots & \ln(N) \end{bmatrix}}$$

The matrix of working weights is:

$$W = \mathrm{diag}\left[\frac{1}{\mathrm{var}(y_i)}\left(\frac{\partial \mu_i}{\partial \eta_i}\right)\right]$$

```
#Working weights matrix (W )

W<-exp(beta[1])*diag(c(i^beta[2]),nrow = 20,ncol = 20)
W
```

$$\boxed{W = diag\left[e^{\beta_1}\, i^{\beta_2}\right]_{N\times N}}$$

The information matrix is:

$$\tau = x^t W x = \underset{\sim}{J}$$

```
#Information matrix, Tau= Xt*W*X
# Multiply the matrices.

tau<- Xt%*%W%*%X
tau
```

$$\boxed{\tau = X^t\, W\, X}$$

```
##           [,1]     [,2]
## [1,]  570.8392 1439.608
## [2,] 1439.6080 3768.835
```

Another way for find information matrix τ :

$$1-\quad \operatorname{Var}(U_1)=\operatorname{Var}\left[\sum_{i=1}^{N}\left(y_i-e^{\beta_1+\beta_2\ln(i)}\right)\right]=\sum_{i=1}^{N}\operatorname{Var}(y_i)=\sum_{i=1}^{N}e^{\beta_1+\beta_2\ln(i)}=\sum_{i=1}^{N}e^{\beta_1}i^{\beta_2}$$

$$2-\quad \operatorname{Var}(U_2)=\operatorname{Var}\left[\sum_{i=1}^{N}\left(y_i\ln(i)-\ln(i)e^{\beta_1+\beta_2\ln(i)}\right)\right]=\sum_{i=1}^{N}\operatorname{Var}(y_i\ln(i))=\sum_{i=1}^{N}[\ln(i)]^2\operatorname{Var}(y_i)$$

$$=\sum_{i=1}^{N}[\ln(i)]^2e^{\beta_1+\beta_2\ln(i)}=\sum_{i=1}^{N}[\ln(i)]^2e^{\beta_1}i^{\beta_2}$$

$$3-\operatorname{Cov}(U_1,U_2)=\operatorname{Cov}\left(\sum_{i=1}^{N}\left(y_i-e^{\beta_1+\beta_2\ln(i)}\right),\sum_{i=1}^{N}\left(y_i\ln(i)-\ln(i)e^{\beta_1+\beta_2\ln(i)}\right)\right)=\sum_{i=1}^{N}\operatorname{Cov}(y_i,y_i\ln(i))$$

$$=\sum_{i=1}^{N}\ln(i)\operatorname{Cov}(y_i,y_i)=\sum_{i=1}^{N}\ln(i)\operatorname{Var}(y_i)=\sum_{i=1}^{N}\ln(i)e^{\beta_1+\beta_2\ln(i)}=\sum_{i=1}^{N}\ln(i)e^{\beta_1}i^{\beta_2}$$

Information matrix τ is:

$$\tau=\begin{bmatrix}\operatorname{Var}(U_1) & \operatorname{Cov}(U_1,U_2)\\ \operatorname{Cov}(U_1,U_2) & \operatorname{Var}(U_2)\end{bmatrix}=\begin{bmatrix}\sum_{i=1}^{N}e^{\beta_1}i^{\beta_2} & \sum_{i=1}^{N}e^{\beta_1}i^{\beta_2}\ln(i)\\ \sum_{i=1}^{N}e^{\beta_1}i^{\beta_2}\ln(i) & \sum_{i=1}^{N}e^{\beta_1}i^{\beta_2}[\ln(i)]^2\end{bmatrix}_{2\times 2}$$

The vector of scores is:

```
#The score statistics are: (U1 and U2)

U1<-sum(y-exp(beta[1]+beta[2]*log(i)))
U2<-sum(y*log(i)-log(i)*exp(beta[1]+beta[2]*log(i)))

#The vector of scores (U)

U<-matrix(c(U1,U2))
U

##           [,1]
## [1,]  740.1608
## [2,] 1956.7710
```

$$U=\begin{bmatrix}U1\\U2\end{bmatrix}=\begin{bmatrix}\sum(y_i-e^{\beta_1+\beta_2\ln(i)})\\ \sum(y_i\ln(i)-\ln(i)\,e^{\beta_1+\beta_2\ln(i)})\end{bmatrix}$$

finding $U=$ Sore statistics:

- log -likelihood function is:

$$\ell(\beta)=\sum_{i=1}^{N}[y_i ln(\lambda_i)-\lambda_i-ln(y_i!)]=\sum_{i=1}^{N}\left[y_i ln\left(e^{\beta_1+\beta_2 lni}\right)-e^{\beta_1+\beta_2 lni}-ln(y_i!)\right]$$

$$=\sum_{i=1}^{N}\left[y_i(\beta_1+\beta_2 lni)-e^{\beta_1+\beta_2 lni}-ln(y_i!)\right]$$

where $\lambda_i=\mu_i=e^{\beta_1+\beta_2\ln i}$.

- Score Statistics are:

$$U_1 = \frac{\partial \ell}{\partial \beta_1} = \sum_{i=1}^{N} \left[y_i - e^{\beta_1 + \beta_2 \ln(i)} \right]$$

$$U_2 = \frac{\partial \ell}{\partial \beta_2} = \sum_{i=1}^{N} \left[y_i \ln(i) - \ln(i) e^{\beta_1 + \beta_2 \ln(1)} \right]$$

The vector of scare Statistics U is:

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{N} \left[y_i - e^{\beta_1 + \beta_2 \ln(i)} \right] \\ \sum_{i=1}^{N} \left[y_i \ln(i) - \ln(i) e^{\beta_1 + \beta_2 \ln(i)} \right] \end{pmatrix}$$

Calculation:

$$b^{\mathrm{m}+1} = \mathrm{b}^{\mathrm{m}} + [\tau(\mathrm{b}^{\mathrm{m}})]^{(-1)} \mathrm{U}(\mathrm{b}^{\mathrm{m}})$$

```
#iterative equation to find an approximate estimate of beta: b1, b2

b<-beta+solve(tau)%*%U
b

##          [,1]
## [1,] 0.652354
## [2,] 1.651991
```

$$\boxed{\mathbf{b^1 = b^0 + [\tau^0]^{-1}\, U^0}}$$

Repeat the steps but change the value of initial value to the last value of beta you get it, and stop the iteration process if you get the same value of Beta (in two successive steps)

The following table summarizes the results of the iterative procedure:

m	**0**	**1**	**2**	**3**	**4**	**5**	**6**
b_1^m	1	0.652354	0.841856	0.98454	0.995952	0.995998	0.995998
b_2^m	1	1.651991	1.429568	1.33373	1.326639	1.32661	1.32661

Since $b^{(5)} = b^{(6)} = \begin{bmatrix} 0.995998 \\ 1.32661 \end{bmatrix}$, the iteration algorithm was terminated at step (5).

The final approximated of MLE of $\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$ is $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0.995998 \\ 1.32661 \end{bmatrix}$

<u>or by using loop in R:</u>

```
################## loop function #################################

# Initial values
beta <- matrix(c(1, 1))
tolerance <- 1e-6
max_iter <- 100

#Design matrix (X)
X=matrix(c(rep(1,20),log(i)),nrow=20,ncol = 2,byrow = F)
X
#Transform Design matrix (Xt)
Xt<-t(X)
Xt

# Iterative process
for (iter in 1:max_iter) {
#Working weights matrix , Diagonal matrix having 20 rows and 20 columns :
W<- exp(beta[1])*diag((i)^(beta[2]), nrow=20, ncol=20)
#Information matrix , Tau= Xt*W*X
# Multiply the matrices.
Tau<-Xt %*% W %*% X
# to calculate the inverse of Tau:
Tau_inver<-solve(Tau)
U1=sum(y-exp(beta[1]+beta[2]*log(i)))
U2=sum(y*log(i)-log(i)*exp(beta[1]+beta[2]*log(i)))
U<-matrix(c(U1,U2))
U
```

```
#iterative equation to find an approximate estimate of beta: b1,b2
b<- beta+(Tau_inver %*% U)
b
# Check for convergence
if (max(abs(b - beta)) < tolerance) {
  break
}

# Update beta for the next iteration
beta <- b

# Print iteration results (optional)
cat("Iteration", iter, ": beta =", t(b), "\n")
}

## Iteration 1 : beta = 0.652354 1.651991
## Iteration 2 : beta = 0.8418567 1.429548
## Iteration 3 : beta = 0.9845403 1.33373
## Iteration 4 : beta = 0.9959522 1.326639
## Iteration 5 : beta = 0.995998 1.32661
```

(d) Using Minitab Software:
stat → Reg → Poisson Reg→ Fit Poisson Model
option: Link function→ log
We obtain:

$$b_1 = 0.996$$
$$b_2 = 1.3266$$

(See Minitab output)
or by R:

```
model<-glm(y~log(i) ,family =poisson(link = "log"))
summary(model)

##
## Call:
## glm(formula = y ~ log(i), family = poisson(link = "log"))
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept)  0.99600    0.16971   5.869 4.39e-09 ***
## log(i)       1.32661    0.06463  20.525  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for poisson family taken to be 1)
##
##     Null deviance: 677.264  on 19  degrees of freedom
## Residual deviance:  21.755  on 18  degrees of freedom
## AIC: 138.05
##
## Number of Fisher Scoring iterations: 4
```

$var(b_1) = 0.16971^2 = 0.02880148$
$var(b_2) = 0.06463^2 = 0.004177037$

For example, an approximate 95% confidence interval for $\boldsymbol{\beta_1}$ and $\boldsymbol{\beta_2}$ is:

$$b_1 \pm 1.96\ \mathrm{se}(b_1)$$
$$b_2 \pm 1.96\ \mathrm{se}(b_2)$$

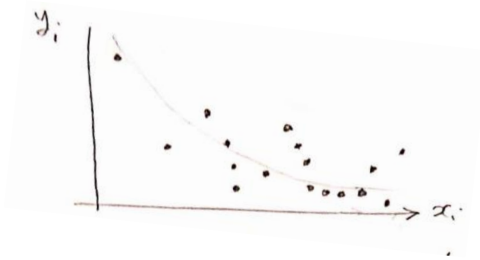
```
CI_of_beta <- confint.default(model,level=0.95)
CI_of_beta

##                 2.5 %   97.5 %
## (Intercept) 0.6633773 1.328619
## log(i)      1.1999299 1.453289
```

For obtaining the estimated variance-covariance matrix of parameter estimates in a fitted model:
The variance-covariance matrix of the MLE $\boldsymbol{b}$ ($\mathrm{cov}(\mathbf{b}) = \boldsymbol{\tau}^{-1}$) is

```
Tauinver<-vcov(model)
Tauinver

##             (Intercept)       log(i)


## (Intercept)  0.02880067 -0.010822607
## log(i)      -0.01082261  0.004177519
```

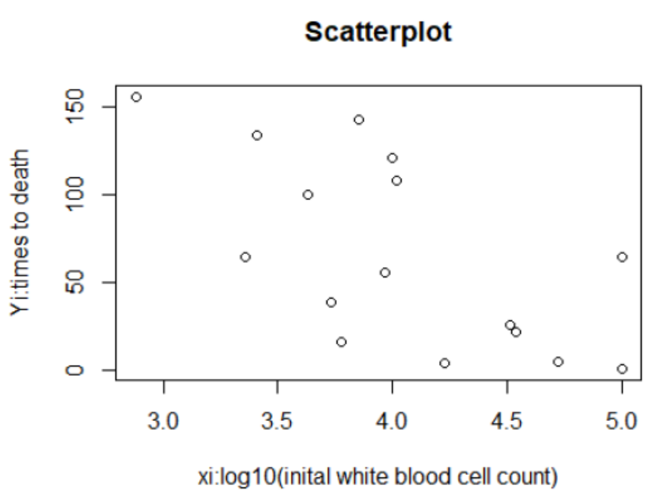
$$\mathrm{cov}(\hat{\beta}) = \mathrm{cov}(b) = \begin{bmatrix} \mathrm{var}(b_1) & \mathrm{cov}(b_1, b_2) \\ \mathrm{cov}(b_1, b_2) & \mathrm{var}(b_2) \end{bmatrix} = \begin{bmatrix} 0.02880067 & -0.010822607 \\ -0.01082261 & 0.004177519 \end{bmatrix}$$

Exercise 4.2
(a)

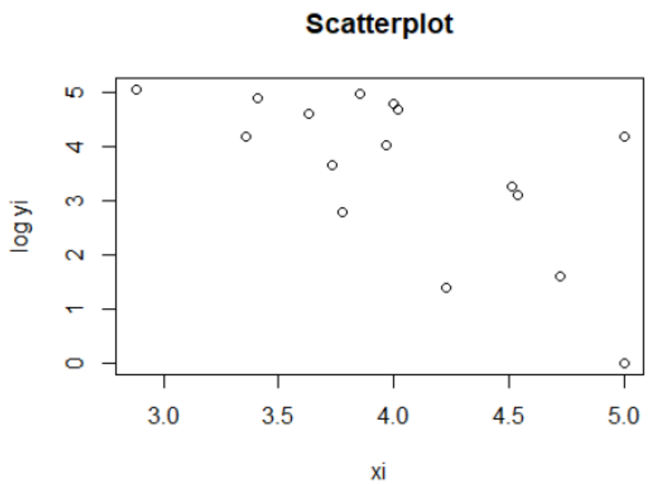


(See Minitab output)
This graph shows that (y_i) exponentially dereases with (x_i).
or by R:

```
# yi:times to death .
yi<-c(65,156,100,134,16,108,121,4,39,143,56,26,22,1,1,5,65)
yi

##  [1]  65 156 100 134  16 108 121   4  39 143  56  26  22   1   1   5  65

#xi:log10(inital white blood cell count).
xi<-c(3.36,2.88,3.63,3.41,3.78,4.02,4,4.23,3.73,3.85,3.97,4.51,4.54,5,5,4.72,5)
xi

##  [1] 3.36 2.88 3.63 3.41 3.78 4.02 4.00 4.23 3.73 3.85 3.97 4.51 4.54 5.00 5.00
## [16] 4.72 5.00
#   Plot the times to death yi against log10(inital white blood cell count) xi.
plot(xi,yi,main = "Scatterplot", xlab = "xi:log10(inital white blood cell count)",
ylab = "Yi:times to death")
```



```r
#Plot  log yi against log(i)  to examine this model.
plot(xi,log(yi), main = "Scatterplot", xlab = "xi", ylab = "log yi")
```



(b)

$$E(y_i) = \mu_i$$
$$\Rightarrow \mu_i = e^{\beta_1+\beta_2 x_i} = e^{\eta_i};\ \eta_i = \eta(x_i) = \beta_1 + \beta_2 x_i$$
$$\Leftrightarrow \ln(\mu_i) = \eta_i = \beta_1 + \beta_2 x_i$$

The link function is $\mathrm{g}(\mu_i) = \ln(\mu_i)$

(c) $f(y;\theta) = \theta e^{-\theta y}; y > 0, \theta > 0$

we need to show that $E(y) = \frac{1}{\theta}$ & $\mathrm{Var}(y) = \frac{1}{\theta^2}$.

- First Method:

$$E(y) = \int_0^\infty y f(y;\theta)dy = \cdots = \frac{1}{\theta} \text{ (using integration by parts)}$$
$$E(y^2) = \int_0^\infty y^2 f(y;\theta)dy = \cdots = \frac{2}{\theta^2} \text{ (using integration by parts)}$$
$$\mathrm{Var}(y) = E(y^2) - [E(y)]^2 = E(y^2) - \left(\frac{1}{\theta}\right)^2 = \cdots = \frac{1}{\theta^2}$$

- Second Method:

The exponential dist. belongs to the EF.

$$f(y;\theta) = \theta e^{-\theta y} = e^{\ln\theta} e^{\ln[e^{-\theta y}]} = e^{\ln\theta} e^{-\theta y} = e^{y(-\theta)+\ln(\theta)}$$
$$\Rightarrow a(y) = y \text{ (canonical form)}$$
$$b(\theta) = -\theta \text{ (natural parameter)}$$
$$c(\theta) = \ln(\theta)$$
$$d(y) = 1$$
$$\Rightarrow c'(\theta) = \frac{1}{\theta}, c''(\theta) = -\frac{1}{\theta^2}$$
$$b'(\theta) = -1, b''(\theta) = 0$$
$$\Rightarrow E[a(y)] = -\frac{c'(\theta)}{b'(\theta)} = -\frac{\frac{1}{\theta}}{-1} = \frac{1}{\theta}$$
$$\mathrm{Var}[a(y)] = \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{[b'(\theta)]^3} = \frac{0 - \left(-\frac{1}{\theta^2}\right)(-1)}{[-1]^3} = \frac{1}{\theta^2}$$

(d)

$$f(y_{i'}\theta_i) = \theta_i e^{-\theta_i y_i}; y_i > 0, \theta_i > 0$$
$$E(y_i) = \mu_i = \frac{1}{\theta_i}$$
$$\text{Var}(y_i) = \sigma_i^2 = \frac{1}{\theta_i^2}$$
$$\Rightarrow \text{Var}(y_i) = [E(y_i)]^2$$
$$\mu_i = e^{\beta_1+\beta_2 x_i} = e^{\eta_i}; \eta_i = \beta_1 + \beta_2 x_i \Leftrightarrow ln(\mu_i) = \beta_1 + \beta_2 x_i$$

Link function is $g(\mu_i) = \ln(\mu_i)$.

$$\begin{cases} \dfrac{\partial \mu_i}{\partial \eta_i} = e^{\eta_i} = e^{\beta_1+\beta_2 x_i} = \mu_i \\ \dfrac{\partial \eta_i}{\partial \mu_i} = \dfrac{1}{\mu_i} = \dfrac{1}{e^{\eta_i}} = e^{-\eta_i} = e^{-(\beta_1+\beta_2 x_i)} \end{cases}$$

Matrix of working weights is:

$$\underline{W} = \text{diag}\left[\frac{1}{\text{Var}(y_i)}\left(\frac{\partial \mu_i}{\partial \eta_i}\right)^2\right] = \text{diag}\left[\frac{1}{\mu_i^2}(e^{\eta_i})^2\right] = \text{diag}\left[\frac{1}{(e^{\eta_i})^2}(e^{\eta_i})^2\right] = \text{diag}[1] = \underline{I} = \begin{bmatrix} 1 & & & \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}_{N\times N}$$

The Vector of working responses is:

$$\underset{\sim}{z} = \underset{\sim}{\eta} + \text{diag}\left[\frac{\partial \eta_i}{\partial \mu_i}\right](\underset{\sim}{y} - \underset{\sim}{\mu})$$
$$\begin{aligned} \Rightarrow z_i &= \eta_i + \left(\frac{\partial \eta_i}{\partial \mu_i}\right)(y_i - \mu_i); i = 1,2,\dots,N \\ &= \eta_i + e^{-\eta_i}(y_i - e^{\eta_i}) = \eta_i + y_i e^{-\eta_i} - e^{-\eta_i}e^{\eta_i} = \eta_i + y_i e^{-\eta_i} - 1 \\ &= \beta_1 + \beta_2 x_i + y_i e^{-\beta_1 - \beta_i x_i} - 1 \end{aligned}$$

The MLE of $\beta_0 = \binom{\beta 1}{\beta}$, Combe obtain iteratively by:

$$\underset{\sim}{b}^{(m+1)} = \left[\underline{x}^t \underline{W}^{(m)} \underline{x}\right]^{-1} \underline{x}^t \underline{W}^{(m)} \underset{\sim}{z}^{(m)} = \left(\underline{x}^t \underline{x}\right)^{-1} \underline{x}^t \underset{\sim}{z}^{(m)}; m = 0,1,2,\dots$$

The information matrix is:

$$\underline{x}^t \underline{W} \quad \underline{x} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \dots & x_N \end{bmatrix}_{2\times N} I_{N\times N} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_N \end{bmatrix}_{N\times 2} = \begin{bmatrix} N & \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 \end{bmatrix}_{2\times 2}$$

$$\therefore \underline{x}^t \underline{W}^{(m)} \underline{x} = \begin{bmatrix} N & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix} \text{ for all } m \text{ (i.e. } \underline{W} \text{ is fixed matrix)}$$

$$\underline{x}^t \underline{W} = \underline{x}^t I = \underline{x}^t$$
$$\therefore \underline{x}^t \underline{W}^{(m)} = \underline{x}^t = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_N \end{bmatrix} \text{ for all } m \text{ (i.e. fixed)}$$

$$\underline{x}^t \underline{W}^{(m)} \underset{\sim}{z}^{(m)} = \underline{x}^t \underset{\sim}{z}^{(m)} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_N \end{bmatrix} \begin{bmatrix} z_1^{(\mathrm{m})} \\ z_2^{(\mathrm{m})} \\ \vdots \\ z_N^{(\mathrm{m})} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} z_i^{(m)} \\ \sum_{i=1}^{N} x_i z_i^{(m)} \end{bmatrix}$$

i	1	2	$\cdots$	17
x_i	3,36	2,88	$\cdots$	5,00
y_i	65	156	$\cdots$	65

$$N = 17, \sum x_i = 69.63, \sum x_i^2 = 291.457$$

$$\underset{\sim}{x}^t \underline{W} \underline{x} = \begin{bmatrix} 17 & 69.63 \\ 69.63 & 291.457 \end{bmatrix}, \left(\underline{x}^t \underline{W} \underline{x}\right)^{-1} = \begin{bmatrix} 2.73843 & -0.65422 \\ -0.65422 & 0.159726 \end{bmatrix}$$

Iterative ML procedure:

- Step ($m = 0$)

Let $\underset{\sim}{b}^{(0)} = \begin{pmatrix} b_1^{(0)} \\ b_2^{(0)} \end{pmatrix} = \begin{pmatrix} 11 \\ -2 \end{pmatrix}$ i. e. $\left\{ \begin{matrix} b_1^{(0)} = 11 \\ b_2^{(0)} = -2 \end{matrix} \right\}$ initial value:

$$z_i^{(0)} = b_1^{(0)} + b_2^{(0)} x_i + y_i e^{-b_1^{(0)} - b_2^{(0)} x_i} - 1 = 11 - 2x_i + y_i e^{-11+2x_i} - 1$$
$$= 10 - 2x_i + y_i e^{-11+2x_i}, i = 1,2, \dots, N$$
$$\Rightarrow \underset{\sim}{z}^{(0)} = [4.1798 \quad 5.0668 \quad \cdots \quad 23.9122]$$

$$\underset{\sim}{x}^t \underline{W}^{(0)} \underset{\sim}{z}^{(0)} = \underset{\sim}{x}^t \underset{\sim}{z}^{(0)} = \begin{bmatrix} \sum z_i^{(0)} \\ \sum x_i z_i^{(0)} \end{bmatrix} = \begin{bmatrix} 90.8865 \\ 378.255 \end{bmatrix}$$

$$\underset{\sim}{b}^{(1)} = \left(\underset{\sim}{x}^t \underline{W}^{(0)} \underline{x}\right)^{-1} \underset{\sim}{x}^t \underline{W}^{(0)} \underset{\sim}{z}^{(0)} = \begin{bmatrix} 2.73843 & -0.65422 \\ -0.65422 & 0.159726 \end{bmatrix} \begin{bmatrix} 90.8865 \\ 378.255 \end{bmatrix} = \begin{bmatrix} 1.42449 \\ 0.95749 \end{bmatrix}$$

- Stop $m = 1$:

We repeat Steep $m = 0$, using $\left\{ \begin{matrix} b_1^{(1)} = 1.42449 \\ b_2^{(1)} = 0.95749 \end{matrix} \right.$

The following table summarizes the results of the iterative procedure:

m	0	1		8	9
$b_1^{(m)}$	11	1.4245	...	8.4775	8.4775
$b_2^{(m)}$	-2	0.9575		-1.1093	-1.1093

sam

We stop the procedure at step $m = 9$

<u>or by R:</u>

We will use the iterative formula which is :

$$\mathbf{b}^{\mathbf{m}+1} = \mathbf{b}^{\mathbf{m}} + [\tau(\mathbf{b}^{\mathbf{m}})]^{(-1)} \mathbf{U}(\mathbf{b}^{\mathbf{m}}), \mathbf{m} = 0,1,2 \dots \dots$$

We will start with an initial value $b^{(0)} = \begin{bmatrix} b_1^{(0)} \\ b_2^{(0)} \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \end{bmatrix}$

```
# Initial values b0:
beta<-c(11,-2)
beta

## [1] 11 -2
```

Design matrix (X) :

```r
#Design matrix (X) :
X=matrix(c(rep(1,17),xi),nrow=17,ncol = 2,byrow = F)
X
```

```
##       [,1] [,2]
##  [1,]    1 3.36
##  [2,]    1 2.88
##  [3,]    1 3.63
##  [4,]    1 3.41
##  [5,]    1 3.78
##  [6,]    1 4.02
##  [7,]    1 4.00
##  [8,]    1 4.23
##  [9,]    1 3.73
## [10,]    1 3.85
## [11,]    1 3.97
## [12,]    1 4.51
## [13,]    1 4.54
## [14,]    1 5.00
## [15,]    1 5.00
## [16,]    1 4.72
## [17,]    1 5.00
```

Transpose of design matrix (X^T) :

```r
#Transpose of design matrix :
Xt<-t(X)
Xt
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12] [,13] [,14]
## [1,] 1.00 1.00 1.00 1.00 1.00 1.00    1 1.00 1.00  1.00  1.00  1.00  1.00     1
## [2,] 3.36 2.88 3.63 3.41 3.78 4.02    4 4.23 3.73  3.85  3.97  4.51  4.54     5
##      [,15] [,16] [,17]
## [1,]     1  1.00     1
## [2,]     5  4.72     5
```

The matrix of working weights $W = \text{diag}\left[\frac{1}{\text{var}(y_i)}\left(\frac{\partial\mu}{\partial\eta}\right)^2\right]$ is :

```r
#The matrix of working weights (W) is :
W<-diag(c(rep(1,17)),nrow = 17,ncol = 17)
W
```

```
##       [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12] [,13]
##  [1,]    1    0    0    0    0    0    0    0    0     0     0     0     0
##  [2,]    0    1    0    0    0    0    0    0    0     0     0     0     0
##  [3,]    0    0    1    0    0    0    0    0    0     0     0     0     0
##  [4,]    0    0    0    1    0    0    0    0    0     0     0     0     0
##  [5,]    0    0    0    0    1    0    0    0    0     0     0     0     0
##  [6,]    0    0    0    0    0    1    0    0    0     0     0     0     0
##  [7,]    0    0    0    0    0    0    1    0    0     0     0     0     0
##  [8,]    0    0    0    0    0    0    0    1    0     0     0     0     0
##  [9,]    0    0    0    0    0    0    0    0    1     0     0     0     0
## [10,]    0    0    0    0    0    0    0    0    0     1     0     0     0
## [11,]    0    0    0    0    0    0    0    0    0     0     1     0     0
## [12,]    0    0    0    0    0    0    0    0    0     0     0     1     0
## [13,]    0    0    0    0    0    0    0    0    0     0     0     0     1
## [14,]    0    0    0    0    0    0    0    0    0     0     0     0     0
## [15,]    0    0    0    0    0    0    0    0    0     0     0     0     0
## [16,]    0    0    0    0    0    0    0    0    0     0     0     0     0
## [17,]    0    0    0    0    0    0    0    0    0     0     0     0     0
##       [,14] [,15] [,16] [,17]
##  [1,]     0     0     0     0
##  [2,]     0     0     0     0
##  [3,]     0     0     0     0
##  [4,]     0     0     0     0
##  [5,]     0     0     0     0
##  [6,]     0     0     0     0
##  [7,]     0     0     0     0
##  [8,]     0     0     0     0
##  [9,]     0     0     0     0
## [10,]     0     0     0     0
## [11,]     0     0     0     0
## [12,]     0     0     0     0
## [13,]     0     0     0     0
## [14,]     1     0     0     0
## [15,]     0     1     0     0
## [16,]     0     0     1     0
## [17,]     0     0     0     1
```

The information matrix $\tau = X^t * W * X$ is:

```
##Information matrix, Tau= Xt*W*X (Multiply the matrices):
tau<- Xt%*%W%*%X
tau

##        [,1]     [,2]
## [1,] 17.00  69.6300
## [2,] 69.63 291.4571
```

The score statistics are: (U1 and U2):

```
#The score statistics are: (U1 and U2
U1<-sum(-1+(yi/exp(beta[1]+beta[2]*xi)))
U2<-sum(-xi+(yi*xi/exp(beta[1]+beta[2]*xi)))
```

The vector of scores $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$

```
U<-matrix(c(U1,U2))
U

##           [,1]
## [1,]  43.14648
## [2,] 195.23871
```

find Score Statistics (U):

- log. likelihood function is:

$$l(\beta) = \sum_{i=1}^{N} [\ln(\theta_i) - y_i\theta_i] = \sum_{i=1}^{N} \left[\ln\left(\frac{1}{\mu_i}\right) - \frac{y_i}{\mu_i}\right] = \sum_{i=1}^{N} \left[\ln((\mu_i)^{-1}) - \frac{y_i}{\mu_i}\right]$$

$$= \sum_{i=1}^{N} \left[-\beta_1 - \beta_2 x_i - \frac{y_i}{e^{\beta_{1+} + \beta_2 x_i}}\right]$$

- The Score statistics we:

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$u_1 = \frac{\partial l}{\partial \beta_1} = \sum_{i=1}^{N} \left[-1 + \frac{y_i}{e^{\beta_1 + \beta_2 x_i}}\right]$$

$$u_2 = \frac{\partial l}{\partial \beta_2} = \sum_{i=1}^{N} \left[-x_i + \frac{x_i y_i}{e^{\beta_1 + \beta_2 x_i}}\right]$$

The vector of U :

$$U = \begin{bmatrix} \sum_{i=1}^{N} \left[-1 + \frac{y_i}{e^{\beta_1 + \beta_2} x_i}\right] \\ \sum_{i=1}^{N} \left[-x_i + \frac{x_i y_i}{e^{\beta_1 + \beta_2 x_i}}\right] \end{bmatrix}$$

Another way for find information matrix $\tau = J$:

$$1 - \mathrm{Var}(u_1) = \mathrm{Var}\left[\sum_{i=1}^{N} \left[-1 + \frac{y_i}{e^{\beta_1 + \beta_2 x_i}}\right]\right] = \sum_{i=1}^{N} \frac{\mathrm{Var}(y_i)}{(e^{\beta_1 + \beta_2 x_i})^2} = \sum_{i=1}^{N} \frac{\left(e^{\beta_1 + \beta_2 x_i}\right)^2}{(e^{\beta_1 + \beta_2 x_i})^2} = \sum_{i=1}^{N} 1 = N$$

$$2 - \mathrm{Var}(u_2) = \mathrm{Var}\left[\sum_{i=1}^{N} \left(-x_i + \frac{x_i y_i}{e^{\beta_1 + \beta_2 x_i}}\right)\right] = \sum_{i=1}^{N} \frac{x_i^2 \mathrm{Var}(y_i)}{(e^{\beta_1 + \beta_2 x_i})^2} = \sum_{i=1}^{N} \frac{x_i^2 \left(e^{\beta_1 + \beta_2 x_i}\right)^2}{(e^{\beta_1 + \beta_2 x_i})^2} = \sum_{i=1}^{N} x_i^2$$

$$3 - \mathrm{Cov}(u_1, u_2) = \mathrm{Cov}\left(\sum_{i=1}^{N} \left(-1 + \frac{y_i}{e^{\beta_1 + \beta_2 x_i}}\right), \sum_{i=1}^{N} \left(-x_i + \frac{x_i y_i}{e^{\beta_1 + \beta_2 x_i}}\right)\right)$$

$$= \sum_{i=1}^{N} \text{Cov}\left(\left(-1 + \frac{y_i}{e^{\beta_1+\beta_2 x_i}}\right), \left(-x_i + \frac{x_i y_i}{e^{\beta_1+\beta_2 x_i}}\right)\right)$$
$$= \sum_{i=1}^{N} \text{Cov}\left(\frac{y_i}{e^{\beta_1+\beta_2 x_i}}, \frac{x_i y_i}{e^{\beta_1+\beta_2 x_i}}\right) = \sum_{i=1}^{N} \frac{x_i}{(e^{\beta_1+\beta_2 x_i})^2} \text{Cov}(y_i, y_i) = \sum_{i=1}^{N} \frac{x_i}{(e^{\beta_1+\beta_2 x_i})^2} \text{Var}(y_i)$$
$$= \sum_{i=1}^{N} \frac{x_i}{(e^{\beta_1+\beta_2 x_i})^2} \left(e^{\beta_1+\beta_2 x_i}\right)^2 = \sum_{i=1}^{N} x_i$$

3 −Information matrix J is:

$$J = \begin{bmatrix} \text{Var}(u_1) & \text{Cov}(u_1, u_2) \\ \text{Cov}(u_1, u_2) & \text{Var}(u_2) \end{bmatrix} = \begin{bmatrix} N & \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 17 & 69.63 \\ 69.63 & 291.4571 \end{bmatrix}$$

$$\Rightarrow J^{-1} = \begin{bmatrix} 2.73838856 & -0.65420947 \\ -0.65420947 & 0.159723697 \end{bmatrix}$$

where $N = 17$ $\sum_{i=1}^{12} x_i = 69.63$ $\sum_{i=1}^{N} x_i^2 = 291.4571$.

We make the following Calculation:

$$\mathbf{b}^{\mathrm{m}+1} = \mathbf{b}^{\mathrm{m}} + [\tau(\mathbf{b}^{\mathrm{m}})]^{-1}\mathbf{U}(\mathbf{b}^{\mathrm{m}})$$

```
#We make the following Calculation:
b<-beta+solve(tau)%*%U
b

##           [,1]
## [1,] 1.4248231
## [2,] 0.9574105
```

Repeat the steps but change the initial value to the last value of beta you obtained. Stop the iteration process if you get the same value of beta in two successive steps.

The following table summarizes the results of the iterative procedure:

m	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
b_1^m	11	1.4245							8.4775	8.4775
b_2^m	-2	0.9575							-1.1093	-1.1093

Since $b^{(8)} = b^{(9)} = \begin{bmatrix} 8.4775 \\ -1.1093 \end{bmatrix}$, the iteration algorithm was terminated at step (8).

The final approximation of MLE of $\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$ is $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 8.4775 \\ -1.1093 \end{bmatrix}$

OR By using Loop

```r
# yi:times to death .
yi<-c(65,156,100,134,16,108,121,4,39,143,56,26,22,1,1,5,65)
yi
```

```
##  [1]  65 156 100 134  16 108 121   4  39 143  56  26  22   1   1   5  65
```

```r
#xi:log10(inital white blood cell count).
xi<-c(3.36,2.88,3.63,3.41,3.78,4.02,4,4.23,3.73,3.85,3.97,4.51,4.54,5,5,4.72,5)
xi
```

```
##  [1] 3.36 2.88 3.63 3.41 3.78 4.02 4.00 4.23 3.73 3.85 3.97 4.51 4.54 5.00 5.00
## [16] 4.72 5.00
```

```r
# Initial values
beta <- matrix(c(11, -2))
epsilon <- 1e-6
max_iter <- 100
```

By use Loop

```r
############### By use Loop ###############

# Iterative process
for (iter in 1:max_iter) {
##Design matrix (X)
X=matrix(c(rep(1,17),xi),nrow=17,ncol = 2,byrow = F)
#Transpose of design matrix :
Xt<-t(X)
#Working weights matrix (W)= Diagonal matrix having 17 rows and 17 columns :
W<-diag(c(rep(1,17)),nrow = 17,ncol = 17)
#Information matrix , Tau= Xt*W*X
# Multiply the matrices.
Tau<-Xt %*% W %*% X
# to calculate the inverse of Tau:
Tau_inver<-solve(Tau)
#The score statistics are: (U1 and U2)
U1<-sum(-1+(yi/exp(beta[1]+beta[2]*xi)))
U2<-sum(-xi+(yi*xi/exp(beta[1]+beta[2]*xi)))
#The vector of scores (U)
U<-matrix(c(U1,U2))
U
#iterative equation to find an approximate estimate of beta: b1,b2
b<- beta+(Tau_inver %*% U)
b
# Check for convergence
if (max(abs(b - beta)) < epsilon ) {
  break
}

# Update beta for the next iteration
beta <- b

# Print iteration results (optional)
cat("Iteration", iter, ": beta =", t(b), "\n")
}
```

```
## Iteration 1 : beta = 1.424823 0.9574105
## Iteration 2 : beta = 4.058176 0.1854641
## Iteration 3 : beta = 6.027011 -0.4071352
## Iteration 4 : beta = 7.550717 -0.8500827
## Iteration 5 : beta = 8.297601 -1.060796
## Iteration 6 : beta = 8.461092 -1.105153
## Iteration 7 : beta = 8.476365 -1.109021
## Iteration 8 : beta = 8.477422 -1.10928
## Iteration 9 : beta = 8.477493 -1.109297
## Iteration 10 : beta = 8.477497 -1.109298
```

Fit the model described in (c) using statistical software

```
model<-glm(yi~xi ,family =Gamma(link = "log"))
summary(model, dispersion =1)

##
## Call:
## glm(formula = yi ~ xi, family = Gamma(link = "log"))
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   8.4775     1.6034   5.287 9.13e-05 ***
## xi           -1.1093     0.3872  -2.865   0.0118 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for Gamma family taken to be 0.9388638)
##
##     Null deviance: 26.282  on 16  degrees of freedom
## Residual deviance: 19.457  on 15  degrees of freedom
## AIC: 173.97
##
## Number of Fisher Scoring iterations: 8
```

Find the 95% confidence interval for $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ is:

$$b_1 \pm 1.96\ \mathrm{se}(\,\mathrm{b}_1) \ggg \quad 8.4775 \pm 1.96 * 1.6034$$

$$\mathrm{b}_2 \pm 1.96\ \mathrm{se}(\,\mathrm{b}_2) \ggg -1.1093 \pm 1.96 * 0.3872$$

```
CI_of_beta <- confint.default(model,level=0.95)
CI_of_beta

##                 2.5 %     97.5 %
## (Intercept)  5.334837 11.6201502
## xi          -1.868283 -0.3503104
```

Find approximately the variance-covariance matrix of the MLE $\boldsymbol{b}$:

Final approximate of the inverse of the information matrix evaluated at $\boldsymbol{b}$ is:

For obtaining the estimated variance-covariance matrix of parameter estimate s in a fitted model:

The variance-covariance matrix of the MLE $\boldsymbol{b}(\mathbf{cov}(\mathbf{b}) = \boldsymbol{\tau}^{-1})$ is

```
Tauinver<-vcov(model, dispersion =1)
Tauinver

##              (Intercept)         xi
## (Intercept)   2.7383886 -0.6542095
## xi           -0.6542095  0.1597237
```

$$\tau^{-1} = \mathrm{cov}(\hat{\beta}) = \mathrm{cov}(b) = \begin{bmatrix} \mathrm{var}(\mathbf{b}_1) & \mathrm{cov}(\mathbf{b}_1, b_2) \\ \mathrm{cov}(b_1, b_2) & \mathrm{var}(b_2) \end{bmatrix} \mathrm{cov}(b) = \begin{bmatrix} 2.7383886 & -0.6542095 \\ -0.6542095 & 0.1597237 \end{bmatrix}$$

Find approximate of the information matrix evaluated at b

```
Tau_at_b<-solve(vcov(model, dispersion =1))
Tau_at_b

##              (Intercept)      xi
## (Intercept)  17.00          69.6300
## log(i)       69.63          291.4571
```

$$\tau = \mathrm{cov}(U) = \begin{bmatrix} \mathrm{var}(\mathrm{U}_1) & \mathrm{cov}(\mathrm{U}_1, U_2) \\ \mathrm{cov}(\mathrm{U}_1, U_2) & \mathrm{var}(\mathrm{U}_2) \end{bmatrix} = \begin{bmatrix} 17 & 69.63 \\ 69.63 & 291.4571 \end{bmatrix}$$

(e) The fitted values are:

$$\hat{y}_i = \hat{\mu}_i = e^{b_1 + b_2 x_i} = e^{8.4775 + -1.1093 x_i}$$

i					
y_i	65	156	100	....	65
$\hat{y}_i$	115.61	196.90	85.69	....	18.75

(See Minitab output)

Standardized Residuals: $r_i = \frac{y_i - \hat{y}_i}{\hat{y}_i}$

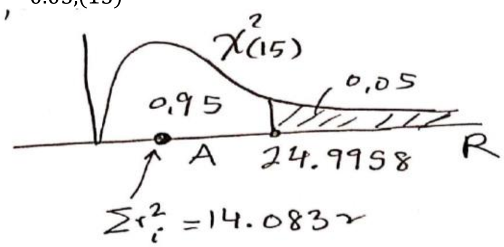
i	1	2	3	$\cdots$	17
r_i	-0.4378	-0.2077	0.1670	....	2.4673

$$\sum r_i^2 = (-0.4378)^2 + \cdots + (2.4673)^2 = 14.0832$$

If the model is correct, we have

$$\sum r_i^2 \sim \chi^2_{(N-p)} = \chi^2_{(15)} \text{ where } \begin{cases} N = 17 \\ p = 2 \end{cases}$$

$$X^2_{0.05,(15)} = 24.9958 \text{ where } \alpha = 0.05$$



Since $\sum r_i^2 < \chi^2_{0.05(15)} (\Leftrightarrow \sum r_i^2 \in A)$.we conclude that the model is adequate for describing the data.
(See Minitab output)
<u>or by R:</u>

```
yhat<-exp(8.4775-1.1093*xi)  #or yhat=fitted.values(m,dispersion =1)
yhat

##  [1] 115.61342 196.90394  85.69042 109.37551  72.55508  55.59615  56.84339
##  [8]  44.04276  76.69304  67.13429  58.76691  32.28352  31.22684  18.74637
## [15]  18.74637  25.57471  18.74637

y<-yi
y

##  [1]  65 156 100 134  16 108 121   4  39 143  56  26  22   1   1   5  65

ri<-(y-yhat)/yhat
ri

##  [1] -0.43778151 -0.20773551  0.16699164  0.22513715 -0.77947788  0.94258047
##  [7]  1.12865547 -0.90917917 -0.49147930  1.13005890 -0.04708284 -0.19463562
## [13] -0.29547788 -0.94665633 -0.94665633 -0.80449437  2.46733841

test_statistc<- sum(ri^2)
test_statistc

## [1] 14.08317

chi_table<-qchisq(1-0.05,17-2)
chi_table

## [1] 24.99579
```

1-Hypothesis:

$$H_0: Model\ fit\ data\ well \quad vs\ \ H_A: Model\ does\ not\ fit\ data\ well$$

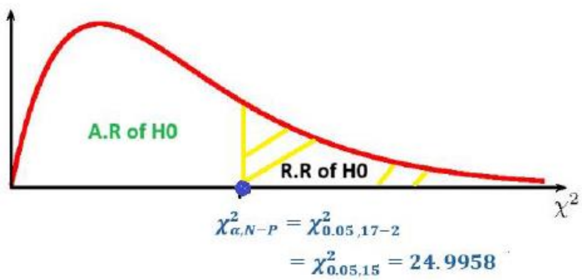
2- Test statistics:

$$\chi^2 = \sum_{i=1}^{17} r_i^2 = 14.0832$$

3- Rejection region of H_0:

$$N = \#\ of\ observation \quad and \quad P = \#\ of\ parameters$$

$$\chi^2_{\alpha,N-P} = \chi^2_{0.05,17-2} = \chi^2_{0.05,15} = 24.9958$$



4- Since $\sum_{i=1}^{17} r_i^2 < \chi^2$ >> ($\sum_{i=1}^{17} r_i^2 \in Accept\ Area$), we conclude that the model is adequate for describing the data .

Not:

We may use one of the following iterative equations to find an approximate estimate of

$\beta = \begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix}$:

$$b^{(m+1)} = b^{(m)} + [\mathcal{J}^{(m)}]^{-1}\, U^{(m)}$$

or

$$b^{(m+1)} = \left[X^t\, W^{(m)} X\right]^{-1} X^t\, W^{(m)} z^{(m)}$$

Exercise 4.3.:

$y_1, y_2, \dots, y_N$ are independent

$y_i \sim N(\ln(\beta), \sigma^2)$; σ^2 is known

$$f(y_i, \beta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}[y_i - \ln(\beta)]^2}$$

$$\Rightarrow l_i = l_i(\beta) = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}[y_i - \ln(\beta)]^2$$

$$\Rightarrow l = \ell(\beta) = \sum_{i=1}^{N} l_i = -\frac{N}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{N}[y_i - \ln(\beta)]^2$$

$$\Rightarrow \frac{\partial l}{\partial \beta} = -\frac{1}{2\sigma^2}\left[2\sum_{i=1}^{N}(y_i - \ln(\beta))\left(-\frac{1}{\beta}\right)\right] = \frac{1}{\sigma^2\beta}\sum_{i=1}^{N}[y_i - \ln(\beta)]$$

$$\frac{\partial l}{\partial \beta} = 0 \Leftrightarrow \sum_{i=1}^{N} [y_i - \ln(\beta)] = 0 \Leftrightarrow \sum y_i - N\ln(\beta) = 0 \Leftrightarrow \sum y_i = N\ln(\beta)$$

$$\Leftrightarrow \ln(\beta) = \frac{\sum y_i}{N} \Leftrightarrow \ln(\beta) = \bar{y} \Leftrightarrow \hat{\beta} = e^{\bar{y}}$$

$\hat{\beta} = e^{\bar{y}}$ is the MLE of β became:

$$\left.\frac{\partial^2 \ell}{\partial \beta^2}\right|_{\beta=\hat{\beta}} = \frac{1}{\sigma^2}\left[\left.\frac{\beta(-1/\beta) - \sum [y_i - \ln(\beta)]}{\beta^2}\right]\right|_{\beta=\hat{\beta}} = \frac{1}{\sigma^2\hat{\beta}^2}\left[-1 - \sum \left(y_i - \ln(\hat{\beta})\right]\right.$$

$$= \frac{1}{\sigma^2\hat{\beta}^2}\left[-1 - \sum_{i=1}^{N} (y_i - \bar{y})\right] = -\frac{1}{\sigma^2(e^{\bar{y}})^2} < 0$$