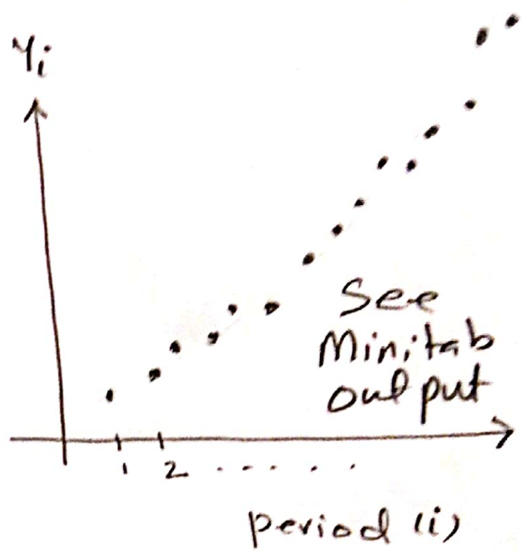
Chapter 4 : Exercises Stat 335

p. 69 Dobson 2008 :

Exercise 4.1 :

Period	Y_i = # cases
1	1
2	6
3	16
4	23
5	27
⋮	⋮
20	159

(a)



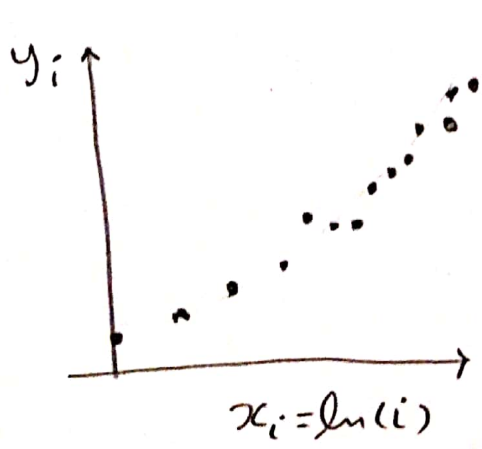
(b) $Y_i \sim Po(\lambda_i)$ $\left\{ \begin{array}{l} \mu_i = E(Y_i) = \lambda_i \\ \sigma_i^2 = Var(Y_i) = \lambda_i \end{array} \right\} E(Y_i) = Var(Y_i)$

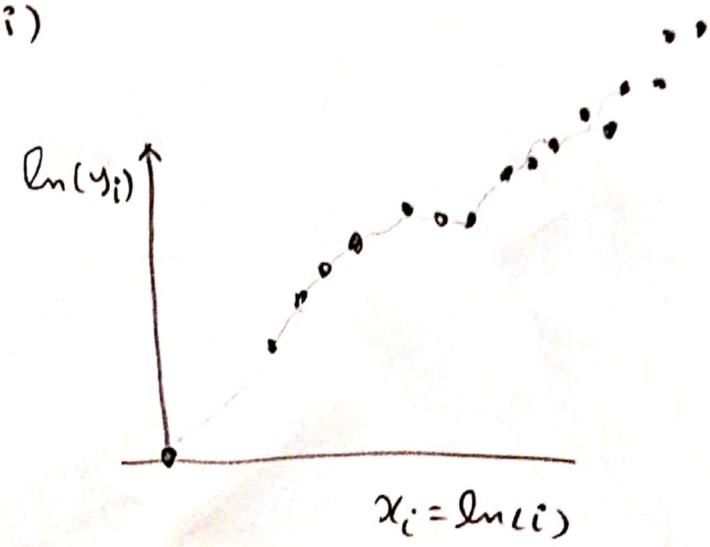
$$\lambda_i = i^\theta \iff \ln(\lambda_i) = \theta \ln(i)$$

$$\ln(\lambda_i) = \theta \ln(i)$$
$$= \theta x_i \quad (x_i = \ln(i))$$
$$= \eta_i \quad (\eta_i = \eta_i(x) = \theta x_i)$$

Link function

$$g(\lambda_i) = \ln(\lambda_i)$$





$E(Y_i) = \mu_i = \lambda_i$

$Var(Y_i) = \sigma_i^2 = \lambda_i$

(See Minitab output)

(c)

$$\mu_i = e^{\eta_i} = e^{\beta_1+\beta_2 x_i} \quad ; \quad x_i = \ln(i)$$

$$= e^{\beta_1} e^{\beta_2 \ln(i)} = e^{\beta_1} e^{\ln(i^{\beta_2})} = e^{\beta_1} i^{\beta_2}$$

$$\left.\begin{array}{l} \mu_i = e^{\eta_i} \\ \Leftrightarrow \ln(\mu_i) = \eta_i \end{array}\right\} \longrightarrow \left\{ \begin{array}{l} \dfrac{\partial \mu_i}{\partial \eta_i} = e^{\eta_i} \\ \dfrac{\partial \eta_i}{\partial \mu_i} = e^{-\eta_i} \end{array}\right.$$

$$\underline{W} = \mathrm{diag}\left[\frac{1}{Var(y_i)}\left(\frac{\partial \mu_i}{\partial \eta_i}\right)^2\right] = \left[\frac{1}{e^{\eta_i}}(e^{\eta_i})^2\right]$$

$$= \mathrm{diag}\left[e^{\eta_i}\right] = \mathrm{diag}\left[e^{\beta_1} i^{\beta_2}\right] = e^{\beta_1}\mathrm{diag}\left[i^{\beta_2}\right]$$

$$\underset{\sim}{Z} = \underset{\sim}{\eta} + \mathrm{diag}\left[\frac{\partial \eta_i}{\partial \mu_i}\right](\underset{\sim}{y} - \underset{\sim}{\mu}) \quad ; \quad \underset{\sim}{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}$$

$$= \underset{\sim}{\eta} + \mathrm{diag}\left[e^{-\eta_i}\right](\underset{\sim}{y} - \underset{\sim}{\mu}) \quad ; \quad \underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_N \end{pmatrix}$$

$$Z_i = \eta_i + e^{-\eta_i}(y_i - \mu_i) \quad ; \quad i = 1, 2, \ldots, N$$

$$= \eta_i + y_i e^{-\eta_i} - e^{-\eta_i} e^{\eta_i}$$

$$= \eta_i + y_i e^{-\eta_i} - 1$$

$$= (\beta_1 + \beta_2 x_i) + y_i e^{-(\beta_1+\beta_2 x_i)} - 1$$

$$= \beta_1 + \beta_2 \ln(i) + y_i e^{-\beta_1} e^{-\beta_2 \ln(i)} - 1$$

$$= \beta_1 + \beta_2 \ln(i) + y_i e^{-\beta_1} i^{-\beta_2} - 1$$

$$= \beta_1 + \beta_2 \ln(i) + \frac{y_i}{e^{\beta_1} i^{\beta_2}} - 1$$

Iterative Maximum Likelihood procedure:

* The MLE of $\underset{\sim}{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ at iteration $(m+1)$ is:

$$\underset{\sim}{b}^{(m+1)} = (\underline{X}^t \underline{W}^{(m)} \underline{X})^{-1} \underline{X}^t \underline{W}^{(m)} \underset{\sim}{z}^{(m)} \quad , m = 0, 1, 2, \ldots$$

Note:

the Design matrix is: $\underline{X} = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_N \end{bmatrix} = \begin{bmatrix} 1 & \ln(1) \\ \vdots & \vdots \\ 1 & \ln(N) \end{bmatrix}$

- Step $(m=0)$:

Let initial value of $\underset{\sim}{b}$ be $\underset{\sim}{b}^{(0)} = \begin{pmatrix} b_1^{(0)} \\ b_2^{(0)} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$b_1^{(0)} = 1 \quad , \quad b_2^{(0)} = 1$.

For this initial values, we have:

$$\underline{W}^{(0)} = diag\left[e^{b_1^{(0)}} i^{b_2^{(0)}} \right] = diag\left[e\, i \right] = e \begin{bmatrix} 1 & & 0 \\ & 2 & \\ & & \ddots \\ 0 & & & N \end{bmatrix}$$

$$\underline{J}^{(0)} = \underline{X}^t \underline{W}^{(0)} \underline{X}$$

$$= \begin{bmatrix} 1 & \cdots & 1 \\ \ln(1) & & \ln(N) \end{bmatrix} e \begin{bmatrix} 1 & & 0 \\ & 2 & \\ & & \ddots \\ 0 & & & N \end{bmatrix} \begin{bmatrix} 1 & \ln(1) \\ \vdots & \vdots \\ 1 & \ln(N) \end{bmatrix}$$

$$= e \begin{bmatrix} \sum_{i=1}^{N} i & \sum_{i=1}^{n} i \ln(i) \\ \sum_{i=1}^{N} i \ln(i) & \sum_{i=1}^{N} i [\ln(i)]^2 \end{bmatrix}$$

$$= \begin{bmatrix} 570.839 & 1439.61 \\ 1439.61 & 3768.84 \end{bmatrix}$$

$$(\underline{X}^t \underline{W}^{(0)} \underline{X})^{-1} = \begin{bmatrix} 0.0477528 & -0.0182405 \\ -0.0182405 & 0.0072328 \end{bmatrix}$$

$$z_i^{(0)} = b_1^{(0)} + b_2^{(0)} \ln(i) + \frac{y_i}{e^{b_1^{(0)}} i^{b_2^{(0)}}} - 1$$

$$= 1 + \ln(i) + \frac{y_i}{ei} - 1$$

$$= \ln(i) + \frac{y_i}{ei}$$

$$\underset{\sim}{z}^{(0)} = (0.36788 \quad 1.79679 \; \cdots\cdots \; 5.92037)$$

$$\underline{X}^t \underline{W}^{(0)} = \begin{bmatrix} 1 & \cdots & 1 \\ \ln(1) & \cdots & \ln(N) \end{bmatrix} e \begin{bmatrix} 1 & & 0 \\ & 2 \ddots & \\ 0 & & N \end{bmatrix}$$

$$= e \begin{bmatrix} 1 & 2 & \cdots & N \\ 1\ln(1) & 2\ln(2) & \cdots & N\ln(N) \end{bmatrix}$$

$$\underline{X}^t \underline{W}^{(0)} \underset{\sim}{z}^{(0)} = e \begin{bmatrix} 1 & 2 & \cdots & N \\ 1\ln(1) & 2\ln(2) & \cdots & N\ln(N) \end{bmatrix} \begin{bmatrix} z_1^{(0)} \\ z_2^{(0)} \\ \vdots \\ z_N^{(0)} \end{bmatrix}$$

$$= e \begin{bmatrix} \sum_{i=1}^{N} i z_{(i)}^{(0)} \\ \sum_{i=1}^{N} i \ln(i) z_i^{(0)} \end{bmatrix} = \begin{bmatrix} 2750.61 \\ 7165.21 \end{bmatrix}$$

$$\underset{\sim}{b}^{(1)} = (\underline{X}^t \underline{W}^{(0)} \underline{X})^{-1} \underline{X}^t \underline{W}^{(0)} \underset{\sim}{z}^{(0)} = \begin{bmatrix} 0.65254 \\ 1.65192 \end{bmatrix}$$

- Step $(m=1)$
  we repeat step(0) using $\begin{cases} b_1^{(1)} = 0.65254 \\ b_2^{(1)} = 1.65192 \end{cases}$

  we will obtain: $\begin{cases} b_1^{(2)} = 0.8419 \\ b_2^{(2)} = 0.14296 \end{cases}$

The following table summarizes the result of the iteration process:

m	0	1	2	---	5	6
$b_1^{(m)}$	1	0.6525	0.8419	---	0.9960	0.9960
$b_2^{(m)}$	1	1.6519	1.4296	---	1.32661	1.32661

Same

we Stop the iteration process at step $m=6$

[%D:\ PoissonExercise41p69.txt]

(d) Using Minitab Software:

Stat → Reg → Poisson Reg
→ Fit Poisson Model

option: Link function
→ log

we obtain:

$$b_1 = 0.996$$

$$b_2 = 1.3266$$

(See Minitab output)

Exercise 4.2

(a)

(See Minitab out put)

. This graph shows that (y_i) exponentially decreases with (x_i).

(b) $E(Y_i) = \mu_i$

$$\mu_i = e^{\beta_1+\beta_2 x_i} = e^{\eta_i} \quad ; \eta_i = \eta(x_i) = \beta_0+\beta_1 x_i$$

$$\Leftrightarrow \quad \ln(\mu_i) = \eta_i = \beta_1+\beta_2 x_i$$

The link function is

$$g(\mu_i) = \ln(\mu_i)$$

(c) $f(y;\theta) = \theta e^{-\theta y} \quad ; y>0 \;, \theta>0$

We need to show that $E(Y) = \frac{1}{\theta}$ & $Var(Y) = \frac{1}{\theta^2}$.

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- First Method:

$$E(Y) = \int_0^\infty y\, f(y;\theta)\, dy = \dots = \frac{1}{\theta}$$

(using integration by parts)

$$Var(Y) = E(Y^2) - [E(Y)]^2 = E(Y^2) - \left(\frac{1}{\theta}\right)^2 = \dots = \frac{1}{\theta^2}$$

$$E(Y^2) = \int_0^\infty y^2 f(y;\theta)\, dy = \dots = \frac{2}{\theta^2}$$

(using integration by parts)

- Second Method:

The exponential dist^{n} belongs to the EF.

$$f(y;\theta) = \theta e^{-\theta y} = e^{\ln\theta}\, e^{\ln[e^{-\theta y}]}$$

$$= e^{\ln\theta}\, e^{-\theta y} = e^{y(-\theta) + \ln(\theta)}$$

$$\begin{cases} a(y) = y & \text{(canonical form)} \\ b(\theta) = -\theta & \text{(natural parameter)} \\ c(\theta) = \ln(\theta) \\ d(y) = 1 \end{cases}$$

$$b'(\theta) = -1, \quad b''(\theta) = 0$$

$$c'(\theta) = \frac{1}{\theta}, \quad c''(\theta) = -\frac{1}{\theta^2}$$

$$E(Y) = -\frac{c'(\theta)}{b'(\theta)} = -\left[\frac{1/\theta}{-1}\right] = \frac{1}{\theta}.$$

$$Var(Y) = \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{[b'(\theta)]^3} = \frac{0 - (-1/\theta^2)(-1)}{(-1)^3}$$

$$= \frac{1}{\theta^2}.$$

(d)

$$f(y_i;\theta_i) = \theta_i e^{-\theta_i y_i} \quad ; \quad y_i > 0 \ , \ \theta_i > 0$$

$$\left.\begin{aligned} E(Y_i) &= \mu_i = \frac{1}{\theta_i} \\ Var(Y_i) &= \sigma_i^2 = \frac{1}{\theta_i^2} \end{aligned}\right\} \quad Var(Y_i) = [E(Y_i)]^2$$

$$\mu_i = e^{\beta_1 + \beta_2 x_i} = e^{\eta_i} \quad ; \quad \eta_i = \beta_1 + \beta_2 x_i$$

$$\Leftrightarrow \quad \ln(\mu_i) = \beta_1 + \beta_2 x_i = \eta_i$$

Link function is $g(\mu_i) = \ln(\mu_i)$

$$\left\{\begin{aligned} \frac{\partial \mu_i}{\partial \eta_i} &= e^{\eta_i} = e^{\beta_1 + \beta_2 x_i} \ (= \mu_i) \\ \frac{\partial \eta_i}{\partial \mu_i} &= \frac{1}{\mu_i} = \frac{1}{e^{\eta_i}} = e^{-\eta_i} = e^{-(\beta_1 + \beta_2 x_i)} \end{aligned}\right.$$

Matrix of working weights is:

$$\underline{W} = diag\left[\frac{1}{Var(Y_i)}\left(\frac{\partial \mu_i}{\partial \eta_i}\right)^2\right] = diag\left[\frac{1}{\mu_i^2}\left(e^{\eta_i}\right)^2\right]$$

$$= diag\left[\frac{1}{(e^{\eta_i})^2}\left(e^{\eta_i}\right)^2\right] = diag[1] = \underline{I} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

Vector of working responses is:

$$\underset{\sim}{z} = \hat{\eta} + diag\left[\frac{\partial \eta_i}{\partial \mu_i}\right](\underset{\sim}{y} - \underset{\sim}{\mu})$$

$$z_i = \eta_i + \left(\frac{\partial \eta_i}{\partial \mu_i}\right)(y_i - \mu_i) \quad ; \quad i = 1, 2, \dots, N$$

$$= \eta_i + e^{-\eta_i}(y_i - e^{\eta_i})$$

$$= \eta_i + y_i e^{-\eta_i} - e^{-\eta_i} e^{\eta_i}$$

$$= \eta_i + y_i e^{-\eta_i} - 1$$

$$z_i = \beta_1 + \beta_2 x_i + y_i e^{-\beta_1 - \beta_2 x_i} - 1$$

The MLE of $\underset{\sim}{\beta} = \begin{pmatrix}\beta_1 \\ \beta_2\end{pmatrix}$ can be obtain iteratively by:

$$\underset{\sim}{b}^{(m+1)} = \left(\underline{X}^t \underline{W}^{(m)} \underline{X}\right)^{-1} \underline{X}^t \underline{W}^{(m)} \underset{\sim}{Z}^{(m)} \quad ; \; m = 0, 1, 2, \cdots$$

$$= (\underline{X}^t \underline{X})^{-1} \underline{X}^t \underset{\sim}{Z}^{(m)}$$

$$\underline{X}^t \underline{W} \underline{X} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_N \end{bmatrix} \underline{I} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_N \end{bmatrix} = \begin{bmatrix} N & \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 \end{bmatrix}$$

$$\therefore \underline{X}^t \underline{W}^{(m)} \underline{X} = \begin{bmatrix} N & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix}$$ for all m (i.e., $\underline{W}$ is fixed matrix)

$$\underline{X}^t \underline{W} = \underline{X}^t \underline{I} = \underline{X}^t$$

$$\therefore \underline{X}^t \underline{W}^{(m)} = \underline{X}^t = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_N \end{bmatrix}$$ for all m (i.e. fixed)

$$\underline{X}^t \underline{W}^{(m)} \underset{\sim}{Z}^{(m)} = \underline{X}^t \underset{\sim}{Z}^{(m)}$$

$$= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_N \end{bmatrix} \begin{bmatrix} Z_1^{(m)} \\ Z_2^{(m)} \\ \vdots \\ Z_N^{(m)} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{N} Z_i^{(m)} \\ \sum_{i=1}^{N} x_i Z_i^{(m)} \end{bmatrix}$$

i	1	2	...	17
x_i	3.36	2.88	...	5.00
y_i	65	156	...	65

$$N = 17 \;,\; \Sigma x_i = 69.63 \;,\; \Sigma x_i^2 = 291.457$$

$$\underline{X}^t \underline{W} \underline{X} = \begin{bmatrix} 17 & 69.63 \\ 69.63 & 291.457 \end{bmatrix} \;,\; (\underline{X}^t \underline{W} \underline{X})^{-1} = \begin{bmatrix} 2.73843 & -0.65422 \\ -0.65422 & 0.159726 \end{bmatrix}$$

Iterative ML procedure :

- Step $(m=0)$

Let $\underset{\sim}{b}^{(0)} = \begin{pmatrix} b_1^{(0)} \\ b_2^{(0)} \end{pmatrix} = \begin{pmatrix} 11 \\ -2 \end{pmatrix}$ $\left\{ \begin{array}{l} b_1^{(0)} = 11 \\ b_2^{(0)} = -2 \end{array} \right\}$ initial values

$$z_i^{(0)} = b_1^{(0)} + b_2^{(0)} x_i + y_i e^{-b_1^{(0)} - b_2^{(0)} x_i} - 1$$

$$= 11 - 2x_i + y_i e^{-11 + 2x_i} - 1$$

$$= 10 - 2x_i + y_i e^{-11 + 2x_i} \qquad i = 1, 2, \dots, N$$

$$\underset{\sim}{Z}^{(0)} = [4.1798 \quad 5.0668 \quad \dots \quad 23.9122]$$

$$\underline{X}^t \underline{W}^{(0)} \underset{\sim}{Z}^{(0)} = \underline{X}^t \underset{\sim}{Z}^{(0)} = \begin{bmatrix} \Sigma Z_i^{(0)} \\ \Sigma x_i Z_i^{(0)} \end{bmatrix} = \begin{bmatrix} 90.8865 \\ 378.255 \end{bmatrix}$$

$$\underset{\sim}{b}^{(1)} = (\underline{X}^t \underline{W}^{(0)} \underline{X})^{-1} \underline{X}^t \underline{W}^{(0)} \underset{\sim}{Z}^{(0)} = \begin{bmatrix} 2.73843 & -0.65422 \\ -0.65422 & 0.159726 \end{bmatrix} \begin{bmatrix} 90.8865 \\ 378.255 \end{bmatrix}$$

$$= \begin{bmatrix} 1.42449 \\ 0.95749 \end{bmatrix}$$

Step $m=1$:

We repeat Step $m=0$, using $\begin{cases} b_1^{(1)} = 1.42449 \\ b_2^{(1)} = 0.95749 \end{cases}$

The following table summarizes the results of the iterative procedure:

m	0	1		8	9
$b_1^{(m)}$	11	1.4245	....	8.4775	8.4775
$b_2^{(m)}$	-2	0.9575		-1.1093	-1.1093

same

We stop the procedure at step $m=9$

[% D:\ExponentialEstimation.txt]

(e) The fitted values are:

$$\hat{y}_i = \hat{\mu}_i = e^{b_1+b_2x_i} = e^{8.4775-1.1093x}$$

y_i	65	156	100	....	65
$\hat{y}_i$	115.61	196.90	85.69	....	18.75

(See Minitab output)

Standardized Residuals:

$$r_i = \frac{y_i - \hat{y}_i}{\hat{y}_i}$$

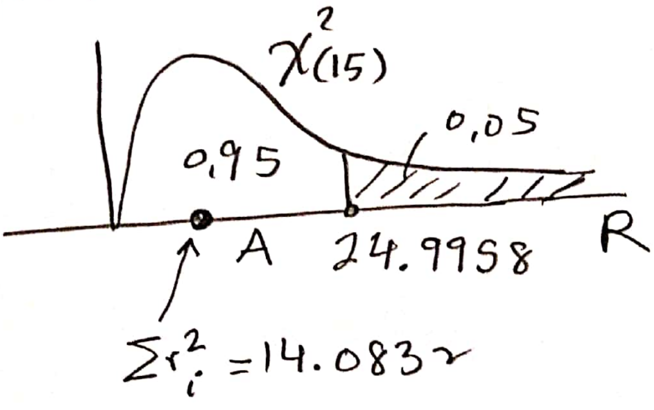
i	1	2	3	....	17
r_i	-0.4378	-0.2077	0.1670	....	2.4673

$$\sum r_i^2 = (-0.4378)^2 + \dots + (2.4673)^2 = 14.0832$$

If the model is correct, we have

$$\sum r_i^2 \sim \chi^2_{(N-p)} = \chi^2_{(15)} \qquad \begin{cases} N = 17 \\ p = 2 \end{cases}$$

$$\chi^2_{0.05,(15)} = 24.9958 \qquad (\alpha = 0,05)$$



Since $\sum r_i^2 < \chi^2_{0.05(15)}$ $(\Leftrightarrow \sum r_i^2 \in A)$, we conclude that the model is adequate for describing the data.

(See Minitab output)

Exercise 4.3 :

$Y_1, Y_2, \dots, Y_N$ are independent

$Y_i \sim N(\ln(\beta), \sigma^2)$; σ^2 is known

$$f(y_i;\beta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}[y_i - \ln(\beta)]}$$

$$l_i = l_i(\beta) = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}[y_i - \ln(\beta)]^2$$

$$l = l(\beta) = \sum_{i=1}^{N} l_i = -\frac{N}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{N}[y_i - \ln(\beta)]^2$$

$$\frac{\partial l}{\partial \beta} = -\frac{1}{2\sigma^2}\left[2\sum_{i=1}^{N}(y_i - \ln(\beta))\left(-\frac{1}{\beta}\right)\right]$$

$$= \frac{1}{\sigma^2\beta}\sum_{i=1}^{N}[y_i - \ln(\beta)]$$

$$\frac{\partial l}{\partial \beta} = 0 \iff \sum_{i=1}^{N}[y_i - \ln(\beta)] = 0$$

$$\iff \sum y_i - N\ln(\beta) = 0$$

$$\iff \sum y_i = N\ln(\beta)$$

$$\iff \ln(\beta) = \frac{\sum y_i}{N}$$

$$\iff \ln(\beta) = \bar{y}$$

$\iff \hat{\beta} = e^{\bar{y}}$ is the MLE of β because:

$$\left.\frac{\partial^2 l}{\partial \beta^2}\right|_{\beta=\hat{\beta}} = \frac{1}{\sigma^2}\left[\frac{\beta(-1/\beta) - \sum[y_i - \ln(\beta)]}{\beta^2}\right]\Bigg|_{\beta=\hat{\beta}}$$

$$= \frac{1}{\sigma^2\hat{\beta}^2}\left[-1 - \sum[y_i - \ln(\hat{\beta})]\right]$$

$$= \frac{1}{\sigma^2\hat{\beta}^2}\left[-1 - \sum_{i=1}^{N}(y_i - \bar{y})\right] = -\frac{1}{\sigma^2\hat{\beta}^2} < 0$$