chapter 3

Ex. 3.1:

(a)

$y =$ weight (Response)
$x_1 =$ age
$x_2 =$ sex
$x_3 =$ height
$x_4 =$ mean daily food intake
$x_5 =$ mean daily energy expenditure
$\Bigg]$ *explanatory variables*
$y_i \sim N(\mu_i, \sigma^2)$
$E(y_i) = \mu_i$

Model:

$$\mu_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_5 x_{i_5}$$

$$g(\mu_i) = \mu_i \text{ (link function = identity function)}$$

$$\eta(\underline{x}_i) = \sum_{j=0}^{5} \beta_j x_{ij} \ (x_0 \equiv 1) \text{ (linear component)}$$

(b)

$$y = \text{number of mice infected (in each level)}$$

$$x = \text{ exposure level of bacteria, with 5 levels:} x_1, x_2, \ldots, x_5$$

$$n = 20 \text{ mice for each level.}$$

$y_i \sim \text{Bino}(n, \pi_i), n = 20, i = 1,2, \ldots ,5$ because 'infection' is a binary outcome (but the plausibility of the assumption of independence of infection for mice depends on the experimental conditions.

$$\Rightarrow E(y_i) = \mu_i = n\pi_i$$

Let $P_i = \frac{y_i}{n}$: proportion of infected mice in the $i - th$ level
$E(P_i) = \pi_i$

<u>Model:</u>

$$g(\pi_i) = \beta_0 + \beta_1 x_i \text{ for some link function such as } g(\pi) = \ln\left(\frac{\pi}{1-\pi}\right)$$

$$\eta(\underline{x}_i) = \beta_0 + \beta_1 x_i \leftarrow \text{ Linear Predictor}$$

or $g(\pi_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \beta_5 x_{i5}$ with the β_k's subject to a corner point or sum-to-zero constraint.

(c)

$y =$ number of trips per week (Response)
$x_1 = x_{i1} =$ number of people in the household.
$x_2 = x_{i2} =$ household income
$x_3 = x_{i3} =$ distance to supermarket
$\Bigg]$ *explanatory variables*

$$y_i \sim P_0(\mu_i) \cong \text{Poisson}(\lambda_i)$$

$$\Rightarrow E(y_i) = \mu_i$$

<u>model;</u>

$$g(\mu_i) = \log \lambda_i = \beta_0 + \beta_1 x_{i1} + \beta_x x_{i2} + \beta_3 x_{i3}$$
$$= \eta(\underline{x}_i)$$

for some link function such as $g(\mu_i) = \ln(\mu_i)$.

$$\eta(\underline{x}_i) = \sum_{j=0}^{3} \beta_j ; x_{ij} \ (x_{i0} = 1) \leftarrow \text{ Linear Predictor}$$

Ex 3. 2:

$y \sim \text{Gamma}(\alpha, \beta)\ (y > 0, \alpha > 0, \beta > 0)$

- α is assumed to be known:

$$\begin{aligned} f(y;\beta) &= \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-y\beta} \\ &= \beta^\alpha [\Gamma(\alpha)]^{-1} y^{\alpha-1} e^{-y\beta} \\ &= e^{\ln(\beta^\alpha)} e^{\ln[\Gamma(\alpha)]^{-1}} e^{\ln(y^{\alpha-1})} e^{-y\beta} \\ &= e^{\alpha\ln(\beta)} e^{-\ln(\Gamma(\alpha))} e^{(\alpha-1)\ln(y)} e^{-y\beta} \\ &= e^{-y\beta+\alpha\ln(\beta)-\ln(\Gamma(\alpha))+(\alpha-1)\ln(y)} \\ &= e^{y(-\beta)+\alpha\ln(\beta)-\ln[\Gamma(\alpha)]+(\alpha-1)\ln(y)} \end{aligned}$$

This is the canonical form of EF:

$$\begin{aligned} a(y) &= y \\ b(\beta) &= -\beta \text{ (natinal panametor)} \\ c(\beta) &= \alpha\ln(\beta) \\ d(y) &= -\ln[\Gamma(\alpha)] + (\alpha-1)\ln(y) \end{aligned}$$

- Natural parameter is $b(\beta) = -\beta$

- $E(y) = -\frac{c'(\beta)}{b'(\beta)} = -\frac{(\alpha/\beta)}{(-1)} = \frac{\alpha}{\beta}$
- $\text{Var}(y) = \frac{b''(\beta)c'(\beta) - c''(\beta)b'(\beta)}{[b'(\beta)]^3} = \frac{0-(-\alpha/\beta^2)(-1)}{(-1)^3} = \frac{\alpha}{\beta^2}$

where:

$$b'(\beta) = \frac{d}{d\beta} b(\beta) = \frac{d}{d\beta}(-\beta) = -1$$
$$c'(\beta) = \frac{d}{d\beta} c(\beta) = \frac{d}{d\beta}(\alpha\ln(\beta)) = \frac{\alpha}{\beta}$$
$$b''(\beta) = 0$$
$$c''(\beta) = -\alpha/\beta^2$$

Ex 3.3:

(a)

$$f(y;\theta) = \theta y^{-\theta-1} = e^{\ln(\theta)} e^{\ln[y^{-\theta-1}]} = e^{\ln(\theta)} e^{(-\theta-1)\ln(y)} = e^{\ln(y)(-\theta-1)+\ln(\theta)}$$

$$\begin{aligned} \Rightarrow a(y) &= \ln(y) \\ b(\theta) &= -\theta - 1 \\ c(\theta) &= \ln(\theta) \\ d(y) &= 0 \end{aligned}$$

(b)

$$f(y\ \ ;\theta) = \theta e^{-y\theta} = e^{\ln(\theta)} e^{-y\theta} = e^{y(-\theta)+\ln(\theta)}$$

$$\begin{aligned} \Rightarrow a(y) &= y \\ b(\theta) &= -\theta \\ c(\theta) &= \ln(\theta) \\ d(y) &= 0 \end{aligned}$$

(c)

$$\begin{aligned} f(y;\theta) &= \binom{y+r-1}{r-1} \theta^r (1-\theta);\ r \text{ is } known \\ &= e^{\ln\binom{y+r-1}{r-1}} e^{\ln(\theta^r)} e^{\ln[(1-\theta)^y]} = e^{\ln\binom{y+r-1}{r-1}} e^{r\ln(\theta)} e^{y\ln(1-\theta)} = e^{y\ln(1-\theta)+r\ln(\theta)+\ln\binom{y+r-1}{r-1}} \end{aligned}$$

$$\Rightarrow a(y) = y$$
$$b(\theta) = \ln(1-\theta)$$
$$c(\theta) = r\ln(\theta)$$
$$d(y) = \ln\binom{y+r-1}{r-1}$$

Ex 3.4:

For the distribution belonging to EF, we have:

$$E[a(y)] = -\frac{c'(\theta)}{b'(\theta)}$$
$$\mathrm{Var}[a(y)] = \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{[b'(\theta)]^3}$$

(a) For y ~ Po(θ) :

$$\Rightarrow a(y) = \mathrm{y}$$
$$b(\theta) = \ln(\theta)$$
$$c(\theta) = -\theta$$
$$b(y) = -\ln(y!)$$
$$\Rightarrow b'(\theta) = \frac{1}{\theta}$$
$$b''(\theta) = -\frac{1}{\theta^2}$$
$$c'(\theta) = -1$$
$$c''(\theta) = 0$$
$$\Rightarrow E(y) = \frac{-(-1)}{\left(\frac{1}{\theta}\right)} = \theta$$
$$\mathrm{Var}(y) = \frac{\left(-\frac{1}{\theta^2}\right)(-1) - 0}{\left(\frac{1}{\theta}\right)^3} = \frac{1/\theta^2}{1/\theta^3} = \frac{1}{\left(\frac{1}{\theta}\right)} = \theta$$

(b) For y ~ $N(\mu, \sigma^2)$: it is assumed that $\sigma^2$is fixed,

$$\Rightarrow a(y) = y$$
$$b(\mu) = \frac{\mu}{\sigma^2}$$
$$c(\mu) = -\frac{\mu^2}{2\sigma^2}$$
$$d(y) = -\frac{y^2}{2\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2)$$
$$\Rightarrow \begin{cases} b'(\mu) = \frac{1}{\sigma^2} \\ b''(\mu) = 0 \\ c'(\theta) = -\frac{\mu}{\sigma^2} \\ c''(\theta) = -\frac{1}{\sigma^2} \end{cases}$$
$$\Rightarrow E(y) = \frac{-(-\mu/\sigma^2)}{(1/\sigma^2)} = \mu$$
$$\mathrm{Var}(y) = \frac{0 - (-1/\sigma^2)(1/\sigma^2)}{(1/\sigma^2)^3} = \frac{1/\sigma^4}{1/\sigma^6} = \sigma^2$$

(c) For y ~ Bino(n, π) :

$$\Rightarrow a(y) = y$$
$$b(\pi) = \ln\left(\frac{\pi}{1-\pi}\right)$$
$$c(\pi) = n\ln(1-\pi)$$
$$d(y) = \ln\binom{n}{y}$$
$$\Rightarrow E(y) = -\frac{\left(-\frac{n}{1-\pi}\right)}{\frac{1}{\pi(1-\pi)}} = n\pi$$
$$\mathrm{Var}(y) = \frac{\frac{2\pi-1}{\pi^2(1-\pi)^2}\left(\frac{-n}{1-\pi}\right) - \left(-\frac{n}{(1-\pi)^2}\right)\left(\frac{1}{\pi(1-\pi)}\right)}{\left[\frac{1}{\pi(1-\pi)}\right]^3} = \frac{\frac{n(1-2\pi)}{\pi^2(1-\pi)^3} + \frac{n}{\pi(1-\pi)^3}}{\frac{1}{\pi^3(1-\pi)^3}}$$
$$= \frac{\frac{n(1-2\pi)+n\pi}{\pi^2(1-\pi)^3}}{\frac{1}{\pi^3(1-\pi)^3}} = [n(1-2\pi)+n\pi]\pi = (n-2n\pi+n\pi)\pi = (n-n\pi)\pi = n(1-\pi)\pi$$
$$= n\pi(1-\pi)$$

where

$$b'(\pi) = \frac{1-\pi}{\pi}\left[\frac{(1-\pi)-\pi(-1)}{(1-\pi)^2}\right] = \frac{1-\pi}{\pi}\frac{1}{(1-\pi)^2} = \frac{1}{\pi(1-\pi)}$$
$$b''(\pi) = \frac{0-[(1-\pi)+\pi(-1)]}{[\pi(1-\pi)]^2} = \frac{2\pi-1}{\pi^2(1-\pi)^2}$$
$$c'(\pi) = \frac{n}{1-\pi}(-1) = \frac{-n}{1-\pi}$$
$$c''(\pi) = -n\left[\frac{-(-1)}{(1-\pi)^2}\right] = -\frac{n}{(1-\pi)^2}$$

Ex 3.5:

(a) For y $\sim NBin(r,\theta)$ (see Ex 3.3 part(c)),

$$\Rightarrow a(y) = y$$
$$b(\theta) = \ln(1-\theta)$$
$$c(\theta) = r\ln(\theta)$$
$$d(y) = \ln\binom{y+r-1}{r-1}$$
$$\Rightarrow\ b'(\theta) = \frac{-1}{1-\theta}$$
$$b''(\theta) = -\frac{(1-\theta)(0)-(-1)}{(1-\theta)^2} = -\frac{1}{(1-\theta)^2}$$
$$c'(\theta) = \frac{r}{\theta}$$
$$c''(\theta) = -\frac{r}{\theta^2}$$
$$\Rightarrow E(y) = -\frac{\left(\frac{r}{\theta}\right)}{\left(-\frac{1}{1-\theta}\right)} = \frac{r(1-\theta)}{\theta}$$

$$\text{Van}(y)=\frac{-\frac{1}{(1-\theta)^2}\left(\frac{r}{\theta}\right)-\left(-\frac{r}{\theta^2}\right)\left(-\frac{1}{1-\theta}\right)}{\left(-\frac{1}{1-\theta}\right)^3}=\frac{-\frac{r}{\theta(1-\theta)^2}-\frac{r}{\theta^2(1-\theta)}}{-\frac{1}{(1-\theta)^3}}=\frac{-\left[\frac{\theta r+(1-\theta)r}{\theta^2(1-\theta)2}\right]}{-\frac{1}{(1-\theta)^3}}$$

$$=\frac{\frac{r}{\theta^2(1-\theta)^2}}{\frac{1}{(1-\theta)^3}}=\frac{\frac{r}{\theta^2}}{\frac{1}{(1-\theta)}}=\frac{r(1-\theta)}{\theta^2}$$

Ex. 3.6:

Refare to Example 3.5.3 (Mortality Rote) Model:

$$E(\mu_i)=\mu_i=n_ie^{\theta i}$$
$$\Leftrightarrow g(\mu_i)=\ln(\mu_i)=\ln(n_i)+\theta i$$
$$\Leftrightarrow \ln(\mu_i)-\ln(n_i)=\theta i$$
$$\Leftrightarrow \ln\left(\frac{\mu_i}{n_i}\right)=\theta i$$
$$\Leftrightarrow \mu_i^*=\theta i \text{ whene } \mu_i^*=\ln\left(\frac{\mu_i}{n_i}\right)$$

age x_i	y_i	n_i	$\frac{y_i}{n_i}$	$\ln\left(\frac{y_i}{n_i}\right)$
30	1	17742	$5.64*10^{-5}$	-9.7837
35	5	16554	$3.02*10^{-4}$	-8.1049
⋮	⋮	⋮	⋮	⋮
65	65	9933	6.54×10^{-3}	-5.0292

Let $y_i^*=\ln\left(\frac{y_i}{n_i}\right)$ and $x_i^*=\ln(x_i)$

Regnen y_i^* on i using the model

$$y_i^*=\beta_0+\beta_1x_i^* \text{ (Simple linear Regression)}$$
$$\hat\beta_0=-13.182 \text{ and } \hat\beta_1=0.1331 \text{ [by using computer]}$$

Fitted value for y_i^*: $\hat y_i^*=\hat\beta_0+\hat\beta_1x_i^*=-13.182+0.1331x_i^*$

Fitted values for y_i: $\hat y_i=n_ie^{\hat y_i^*}$

$$R^2=0.9685\leftarrow(\text{ good fit })\text{ [see computer output]}$$

Ex. 3.7:

- $y_i\sim\text{Bernoulli}(\pi_i)$
- $y_1,y_2,\dots,y_N$ are midependent.

$$f(y_i,\pi_i)=\pi_i^{y_i}(1-\pi_i)^{1-y_i};y_i=0,1$$

(a)

$$f(y,\pi)=\pi^y(1-\pi)^{1-y}=e^{\ln(\pi^y)}e^{\ln[(1-\pi)^{1-y}]}=e^{y\ln(\pi)}e^{(1-y)\ln(1-\pi)}=e^{y\ln(\pi)+(1-y)\ln(1-\pi)}$$
$$=e^{y\ln(\pi)+\ln(1-\pi)-y\ln(1-\pi)}=e^{y[\ln(\pi)-\ln(1-\pi)]+\ln(1-\pi)}=e^{y\ln\left(\frac{\pi}{1-\pi}\right)+\ln(1-\pi)}\in EF$$
$$\Rightarrow a(y)=y$$
$$b(\pi)=\ln\left(\frac{\pi}{1-\pi}\right),c(\pi)=\ln(1-\pi)$$
$$d(y)=0$$

(b) Natural parameter is

$$b(\pi)=\ln\left(\frac{\pi}{1-\pi}\right)$$

(c)

$$E(y) = -\frac{c'(\pi)}{b'(\pi)} = -\frac{\left(-\frac{1}{1-\pi}\right)}{\left(\frac{1}{\pi(1-\pi)}\right)} = \pi$$

where (see Ex. 3.4 Part (c))

$$c'(\pi) = -\frac{1}{1-\pi}$$
$$b'(\pi) = \frac{1}{\pi(1-\pi)}$$

Note: $\frac{\pi}{1-\pi}$ is called odds (معامل الترجيح)

(d) Link function $g(\pi) = \ln\left(\frac{\pi}{1-\pi}\right) = \underset{\sim}{x}'\underset{\sim}{\beta}$.

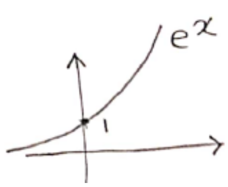
$$\ln\left(\frac{\pi}{1-\pi}\right) = \underset{\sim}{x}'\underset{\sim}{\beta}$$
$$\Leftrightarrow e^{\ln\left(\frac{\pi}{1-\pi}\right)} = e^{\underset{\sim}{x}'\underset{\sim}{\beta}}, \left(e^{\ln x} = x\right)$$
$$\Leftrightarrow \frac{\pi}{1-\pi} = e^{\underset{\sim}{x}'\underset{\sim}{\beta}}$$
$$\Leftrightarrow \pi = (1-\pi)e^{\underset{\sim}{x}'\underset{\sim}{\beta}} = e^{\underset{\sim}{x}'\underset{\sim}{\beta}} - \pi e^{\underset{\sim}{x}'\underset{\sim}{\beta}}$$
$$\Leftrightarrow \pi + \pi e^{\underset{\sim}{x}'\underset{\sim}{\beta}} = e^{\underset{\sim}{x}'\underset{\sim}{\beta}}$$
$$\Leftrightarrow \pi\left(1 + e^{\underset{\sim}{x}'\underset{\sim}{\beta}}\right) = e^{\underset{\sim}{x}'\underset{\sim}{\beta}}$$
$$\Leftrightarrow \pi = \frac{e^{\underset{\sim}{x}'\underset{\sim}{\beta}}}{1+e^{\underset{\sim}{x}'\underset{\sim}{\beta}}} \quad \text{(logistic Regression model)}$$

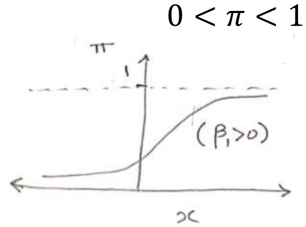
(e) For $\underset{\sim}{x}'\underset{\sim}{\beta} = \beta_1 + \beta_2 x$, we have

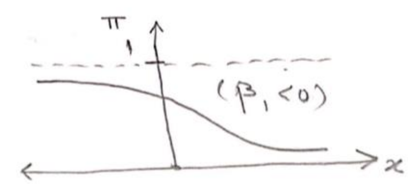
$$\pi = \frac{e^{\beta_1+\beta_2 x}}{1+e^{\beta_1+\beta_2 x}} \quad \text{(logisti cfunction)}$$

note: $e^{\beta_1+\beta_2 x} > 0$

$$0 < \frac{e^{\beta_1+\beta_2 x}}{1+e^{\beta_1+\beta_2 x}} < 1$$
$$0 < \pi < 1$$







Ex. 3.8:

WLOG, let $\phi = 1$ (ϕ is nuisance parameter) and it is considered as fixed

$$f(y;\theta) = e^{(y-\theta)-e^{(y-\theta)}} = e^{y-\theta-e^{(y-\theta)}} = e^{y-\theta-e^{y}e^{-\theta}} = e^{e^{y}(-e^{-\theta})-\theta+y} = e^{a(y)b(\theta)+c(\theta)+d(y)}$$
$$a(y) = e^{y}; b(\theta) = -e^{-\theta}, c(\theta) = -\theta, d(y) = y$$

Ex. 3.9:

$y_1, y_2, \dots, y_n$ are independent.

$$y_i \sim \text{Pareto}(\theta) \therefore\ f(y;\theta) = \theta y^{-\theta-1} \in EF$$
$$\Rightarrow a(y) = \ln(y), b(\theta) = -\theta-1, c(\theta) = \ln(\theta), d(y) = 0 \ \text{ (See Ex 3.3}(a)\text{).}$$

$$\Rightarrow E(y_i) = \mu_i = (\beta_0 + \beta_1 x_i)^2$$
$$\Leftrightarrow \sqrt{\mu_i} = \beta_0 + \beta_1 x_i$$
$$g(\mu_i) = \sqrt{\mu_i} \text{ link function}$$

For Pareto, $a(y) = \ln(y) \neq y$
one of the conditions of Generalized linear model is that $a(y)$ (Canonical form)
Therefore, the model is not a generalized linear Model.

Ex. 3.10:
$y_1, y_2, \dots, y_N$ are independent
$y_i \sim N(\mu_i, \sigma^2) \in EF$ in canonical form.

$$E(y_i) = \mu_i = \beta_0 + \ln(\beta_1 + \beta_2 x_i)$$
$$\mu_i = \beta_0 + \ln(\beta_1 + \beta_2 x_i)$$
$$\Leftrightarrow e^{\mu_i} = e^{\beta_0 + \ln(\beta_1 + \beta_2 x_i)} = e^{\beta_0} e^{\ln(\beta_1 + \beta_2 x_i)} = e^{\beta_0}(\beta_1 + \beta_2 x_i) = \beta_1 e^{\beta_0} + \beta_2 x_i e^{\beta_0} = \underbrace{\beta_1^* + \beta_2^* x_i}_{\text{linear combonent}}$$

where $\beta_1^* = \beta_1 e^{\beta_0}$ and $\beta_2^* = \beta_2 e^{\beta_0}$.The model is a generalized linear model.

Ex. 3.11:
y $\sim$ Pareto(θ).

$$f(y; \theta) = \theta y^{-\theta-1} \text{ (for one observation)}$$

Let $y_1, y_2, \dots, y_n$ are independent, and $y_i \sim$ Pareto$(\theta)$for $i = 1,2,\dots,n$
Likelihood function is:

$$L(\theta; \underset{\sim}{y}) = f(\underset{\sim}{y}; \theta) = \prod_{i=1}^{n} f(y_i; \theta)$$

log-likelihood function is:

$$l(\theta; \underset{\sim}{y}) = \ln[L(\theta; \underset{\sim}{y})] = \ln\left[\prod_{i=1}^{n} f(y_i; \theta)\right] = \sum_{i=1}^{n} \ln[f(y_i, \theta)] = \sum_{i=1}^{n} \ln\left[\theta y_i^{-\theta-1}\right]$$
$$= \sum_{i=1}^{n} \left\{\ln(\theta) + \ln\left[y_i^{-\theta-1}\right]\right\} = \sum_{i=1}^{n} \{\ln(\theta) + (-\theta - 1)\ln(y_i)\}$$
$$= n\ln(\theta) - (\theta + 1)\sum_{i=1}^{n} \ln(y_i)$$

The Score Statistic is:

$$U(\theta, \underset{\sim}{y}) = \frac{\partial l(\theta; \underset{\sim}{y})}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \ln(y_i)$$

Recall: $E[U(\theta, \underset{\sim}{y})] = 0$
Information:

$$J = \text{Var}(u) = E(u^2) = -E(u') = -E\left[\frac{\partial}{\partial \theta} u(\theta; \underset{\sim}{y})\right] = -E\left\{\frac{\partial}{\partial \theta}\left(\frac{n}{\theta} - \sum_{i=1}^{n} \ln(y_i)\right)\right\} = -E\left[-\frac{n}{\theta^2}\right] = \frac{n}{\theta^2}$$

<u>2nd Solution:</u>

For Pareto distr. , we have [sec Ex. $3.3(a)$]

$$a(y) = \ln(y)$$
$$b(\theta) = -\theta - 1$$
$$c(\theta) = \ln(\theta)$$

$$\Rightarrow b'(\theta) = -1; b''(\theta) = 0$$
$$c'(\theta) = \tfrac{1}{\theta}; c''(\theta) = -\tfrac{1}{\theta^2}$$

We know that

$$E[a(y)] = -\frac{c'(\theta)}{b'(\theta)}$$
$$\therefore E[\ln(y)] = -\frac{1/\theta}{(-1)} = \frac{1}{\theta}$$

Note: $E(u) = E\left[\frac{n}{\theta} - \sum \ln(y_i)\right] = \frac{n}{\theta} - \sum E[\ln(y_i)] = \frac{n}{\theta} - \sum \frac{1}{\theta} = \frac{n}{\theta} - \frac{n}{\theta} = 0$

$$\mathrm{Var}[a(y)] = \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{[b'(\theta)]^3}$$
$$\therefore Var[\ln(y)] = \frac{0 - (-1/\theta^2)(-1)}{(-1)^3} = \frac{-1/\theta^2}{-1} = \frac{1}{\theta^2}$$
$$\Rightarrow E\{[\ln(y)]^2\} = Var[\ln(y)] + \{E[\ln(y)]\}^2 = \tfrac{1}{\theta^2} + \left(\tfrac{1}{\theta}\right)^2 = \tfrac{2}{\theta^2}$$

Now,

$$J = \mathrm{Var}(u) = E(u^2) = E\left\{\left[\frac{n}{\theta} - \sum \ln(y_i)\right]^2\right\} = E\left[\left(\frac{n}{\theta}\right)^2 + \left[\sum \ln(y_i)\right]^2 - 2\frac{n}{\theta}\sum \ln(y_i)\right\}$$
$$= \left(\frac{n}{\theta}\right)^2 + E\left\{\left[\sum \ln(y_i)\right]^2\right\} - 2\frac{n}{\theta}E\left[\sum \ln(y_i)\right]$$
$$= \left(\frac{n}{\theta}\right)^2 + \left[\mathrm{Var}\left[\sum \ln(y_i)\right] + \left\{E\left[\sum \ln(y_i)\right]\right\}^2\right] - 2\frac{n}{\theta}E\left[\sum \ln(y_i)\right]$$
$$= \left(\frac{n}{\theta}\right)^2 + \sum_{i=1}^{n} \mathrm{Van}\big(\ln(y_i)\big) + \left[\sum_{i=1}^{n} E\big(\ln(y_i)\big)\right]^2 - 2\frac{n}{\theta}\sum E[\ln(y_i)]$$
$$= \left(\frac{n}{\theta}\right)^2 + \left[\sum_{i=1}^{n} \frac{1}{\theta^2} + \left(\sum_{i=1}^{n} \frac{1}{\theta}\right)^2\right] - 2\frac{n}{\theta}\sum_{i=1}^{n} \frac{1}{\theta} = \left(\frac{n}{\theta}\right)^2 + \frac{n}{\theta^2} + \left(\frac{n}{\theta}\right)^2 - 2\frac{n}{\theta}\left(\frac{n}{\theta}\right)$$
$$= \frac{n^2}{\theta^2} + \frac{n}{\theta^2} + \frac{n^2}{\theta^2} - 2\frac{n^2}{\theta^2} = \frac{n}{\theta^2}$$

<u>3rd Solution:</u>

We know that

$$\mathrm{Var}(u) = \frac{b''(\theta)c'(\theta)}{b'(\theta)} - c''(\theta) \quad \text{(for one observation)}$$
$$\Rightarrow \mathrm{Var}(u) = \frac{(0)\left(\frac{1}{\theta}\right)}{(-1)} - \left(-\frac{1}{\theta^2}\right) = \frac{1}{\theta^2}$$

$$\therefore\ J = \mathrm{Var}(u) \quad \text{(for all observations)}$$
$$\Rightarrow J = \mathrm{Var}(u) = \sum_{i=1}^{n} \frac{1}{\theta^2} = \frac{n}{\theta^2}$$

$E(u) = 0$:

$$u = \frac{n}{\theta} - \sum_{i=1}^{n} \ln(y_i)$$

$$\Rightarrow E(u) = E\left[\frac{n}{\theta} - \sum_{i=1}^{n} \ln(y_i)\right] = \frac{n}{\theta} - E\left[\sum_{i=1}^{n} \ln(y_i)\right] = \frac{n}{\theta} - \sum_{i=1}^{n} E[\ln(y_i)] = \frac{n}{\theta} - \sum_{i=1}^{n} \frac{1}{\theta} = \frac{n}{\theta} - \frac{n}{\theta} = 0$$

where $E\big(\ln(y_i)\big) = \frac{1}{\theta}$.