Exercises chapter 3

3.6 Exercises p. 55 Dobson 2008

Ex. 3.1 :

(a) Y = weight (Responce)

x_1 = age
x_2 = sex
x_3 = height
x_4 = mean daily food intake
x_5 = " " energy expenditure

explanatory variables

$$Y_i \sim N(\mu_i, \sigma^2)$$

$$E(Y_i) = \mu_i$$

Model: $\mu_i = \beta_0 + \beta_1 x_{i1} + \dots + \beta_5 x_{i5}$

$g(\mu_i) = \mu_i$ (link function = identity function)

$\eta(\underline{x}_i) = \sum_{j=0}^{5} \beta_j x_{ij}$ $(x_0 \equiv 1)$ (linear component)

(b) Y = number of mice infected (in each level)

x = exposure level of bacteria (with 5 levels: $x_1, x_2, \dots, x_5$)

$n = 20$ mice for each level.

$Y_i \sim Bino(n, \pi_i)$ $i = 1, 2, \dots, 5$, $n = 20$

$E(Y_i) = \mu_i = n\pi_i$

Let $P_i = \frac{Y_i}{n}$ = proportion of infected mice in the i-th level

$$E(P_i) = \pi_i$$

Model: $g(\pi_i) = \beta_0 + \beta_1 x_i$ for some link function such as $g(\pi) = \ln\left(\frac{\pi}{1-\pi}\right)$

$\eta(\underline{x}_i) = \beta_0 + \beta_1 x_i \leftarrow$ Linear Predictor

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Cont. Ex 3.1

(c) Y = number of trips per week (Responce)

X_1 = number of people in the household

X_2 = household income

X_3 = distance to supermarket

] explanatory variable

$$Y_i \sim Po(\mu_i)$$

$$E(Y_i) = \mu_i$$

model;

$$g(\mu_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3}$$

$$= \eta(\underset{\sim}{x}_i)$$

fore some link function such as

$$g(\mu_i) = \ln(\mu_i)$$

$$\eta(\underset{\sim}{x}_i) = \sum_{j=0}^{3} \beta_j x_{ij} \quad (x_{i0} = 1)$$ ← linear predictor

Ex 3.2 :

$Y \sim \text{Gamma}(\alpha, \beta) \qquad (y>0, \alpha>0, \beta>0)$

- α is assumed to be known

$$f(y;\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} y^{\alpha-1} e^{-y\beta}$$

$$= \beta^{\alpha} [\Gamma(\alpha)]^{-1} y^{\alpha-1} e^{-y\beta}$$

$$= e^{\ln(\beta^{\alpha})} e^{\ln[\Gamma(\alpha)]^{-1}} e^{\ln(y^{\alpha-1})} e^{-y\beta}$$

$$= e^{\alpha \ln(\beta)} e^{-\ln(\Gamma(\alpha))} e^{(\alpha-1)\ln(y)} e^{-y\beta}$$

$$= e^{-y\beta + \alpha \ln(\beta) - \ln(\Gamma(\alpha)) + (\alpha-1)\ln(y)}$$

$$= e^{y(-\beta) + \alpha \ln(\beta) - \ln[\Gamma(\alpha)] + (\alpha-1)\ln(y)}$$

This is the canonical form of EF.

$a(y) = y$

$b(\beta) = -\beta$ (natural parameter)

$c(\beta) = \alpha \ln(\beta)$

$d(y) = -\ln[\Gamma(\alpha)] + (\alpha-1)\ln(y)$

- Natural parameter is $b(\beta) = -\beta$
- $E(Y) = -\dfrac{c'(\beta)}{b'(\beta)} = -\dfrac{(\alpha/\beta)}{(-1)}$

$$= \frac{\alpha}{\beta}$$

- $\text{Var}(Y) = \dfrac{b''(\beta)c'(\beta) - c''(\beta)b'(\beta)}{[b'(\beta)]^3}$

$$= \frac{0 - (-\alpha/\beta^2)(-1)}{(-1)^3}$$

$$= \frac{\alpha}{\beta^2}$$

$$b'(\beta) = \frac{d}{d\beta} b(\beta) = \frac{d}{d\beta}(-\beta) = -1$$

$$c'(\beta) = \frac{d}{d\beta} c(\beta) = \frac{d}{d\beta}(\alpha \ln(\beta)) = \frac{\alpha}{\beta}$$

$$b''(\beta) = 0$$

$$c''(\beta) = -\alpha/\beta^2$$

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Ex 3.3 :

(a) $f(y;\theta) = \theta y^{-\theta-1}$

$= e^{\ln(\theta)} e^{\ln[y^{-\theta-1}]} = e^{\ln(\theta)} e^{(-\theta-1)\ln(y)}$

$= e^{\ln(y)(-\theta-1)+\ln(\theta)}$

$a(y) = \ln(y)$

$b(\theta) = -\theta-1$

$c(\theta) = \ln(\theta)$

$d(y) = 0$

(b) $f(y;\theta) = \theta e^{-y\theta}$

$= e^{\ln(\theta)} e^{-y\theta} = e^{y(-\theta)+\ln(\theta)}$

$a(y) = y$

$b(\theta) = -\theta$

$c(\theta) = \ln(\theta)$

$d(y) = 0$

(c) $f(y;\theta) = \binom{y+r-1}{r-1} \theta^{r} (1-\theta)^{y}$ (r is known)

$= e^{\ln\binom{y+r-1}{r-1}} e^{\ln(\theta^{r})} e^{\ln[(1-\theta)^{y}]}$

$= e^{\ln\binom{y+r-1}{r-1}} e^{r\ln(\theta)} e^{y\ln(1-\theta)}$

$= e^{y\ln(1-\theta)+r\ln(\theta)+\ln\binom{y+r-1}{r-1}}$

$a(y) = y$

$b(\theta) = \ln(1-\theta)$

$c(\theta) = r\ln(\theta)$

$d(y) = \ln\binom{y+r-1}{r-1}$

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Ex 3.4 :

For the distribution belonging to EF, we have:

$$E[a(y)] = -\frac{c'(\theta)}{b'(\theta)}$$

$$Var[a(y)] = \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{[b'(\theta)]^3}$$

(a) For $Y \sim Po(\theta)$:

$$\begin{cases} a(y) = y \\ b(\theta) = \ln(\theta) \end{cases} \qquad c(\theta) = -\theta \qquad d(y) = -\ln(y!)$$

$$\begin{cases} b'(\theta) = \frac{1}{\theta} \\ b''(\theta) = -\frac{1}{\theta^2} \\ c'(\theta) = -1 \\ c''(\theta) = 0 \end{cases}$$

$$E(Y) = \frac{-(-1)}{\left(\frac{1}{\theta}\right)} = \theta$$

$$Var(Y) = \frac{\left(-\frac{1}{\theta^2}\right)(-1) - 0}{\left(\frac{1}{\theta}\right)^3} = \frac{1/\theta^2}{1/\theta^3} = \frac{1}{\left(\frac{1}{\theta}\right)} = \theta$$

(b) For $Y \sim N(\mu, \sigma^2)$: (it is assumed that σ^2 is fixed)

$a(y) = y$

$$b(\mu) = \frac{\mu}{\sigma^2}$$

$$c(\mu) = -\frac{\mu^2}{2\sigma^2}$$

$$d(y) = -\frac{y^2}{2\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2)$$

$$\begin{cases} b'(\mu) = \frac{1}{\sigma^2} \\ b''(\mu) = 0 \\ c'(\theta) = -\frac{\mu}{\sigma^2} \\ c''(\theta) = -\frac{1}{\sigma^2} \end{cases}$$

$$E(Y) = \frac{-(-\mu/\sigma^2)}{(1/\sigma^2)} = \mu$$

$$Var(Y) = \frac{0 - (-1/\sigma^2)(1/\sigma^2)}{(1/\sigma^2)^3} = \frac{1/\sigma^4}{1/\sigma^6} = \sigma^2$$

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(c) For $Y \sim Bino(n, \pi)$:

$$a(y) = y$$

$$b(\pi) = \ln\left(\frac{\pi}{1-\pi}\right)$$

$$c(\pi) = n \ln(1-\pi)$$

$$d(y) = \ln\binom{n}{y}$$

$$b'(\pi) = \frac{1-\pi}{\pi}\left[\frac{(1-\pi) - \pi(-1)}{(1-\pi)^2}\right] = \frac{1-\pi}{\pi} \cdot \frac{1}{(1-\pi)^2} = \frac{1}{\pi(1-\pi)}$$

$$b''(\pi) = \frac{0 - [(1-\pi) + \pi(-1)]}{[\pi(1-\pi)]^2} = \frac{2\pi - 1}{\pi^2(1-\pi)^2}$$

$$c'(\pi) = \frac{n}{1-\pi}(-1) = \frac{-n}{1-\pi}$$

$$c''(\pi) = -n\left[\frac{-(-1)}{(1-\pi)^2}\right] = -\frac{n}{(1-\pi)^2}$$

$$E(Y) = -\frac{\left(-\frac{n}{1-\pi}\right)}{\frac{1}{\pi(1-\pi)}} = n\pi$$

$$\operatorname{Var}(Y) = \frac{\frac{2\pi-1}{\pi^2(1-\pi)^2}\left(\frac{-n}{1-\pi}\right) - \left(-\frac{n}{(1-\pi)^2}\right)\left(\frac{1}{\pi(1-\pi)}\right)}{\left[\frac{1}{\pi(1-\pi)}\right]^3}$$

$$= \frac{\frac{n(1-2\pi)}{\pi^2(1-\pi)^3} + \frac{n}{\pi(1-\pi)^3}}{\frac{1}{\pi^3(1-\pi)^3}} = \frac{\frac{n(1-2\pi) + n\pi}{\pi^2(1-\pi)^3}}{\frac{1}{\pi^3(1-\pi)^3}}$$

$$= [n(1-2\pi) + n\pi]\pi = (n - 2n\pi + n\pi)\pi$$

$$= (n - n\pi)\pi$$

$$= n(1-\pi)\pi$$

$$= n\pi(1-\pi)$$

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Ex 3.5 :

(a) For $Y \sim NBin(r,\theta)$: (See Ex 3.3 part (c))

$a(y) = y$

$b(\theta) = \ln(1-\theta)$

$c(\theta) = r \ln(\theta)$

$d(y) = \ln\binom{y+r-1}{r-1}$

$$b'(\theta) = \frac{-1}{1-\theta}$$

$$b''(\theta) = -\frac{(1-\theta)(0)-(-1)}{(1-\theta)^2} = -\frac{1}{(1-\theta)^2}$$

$$c'(\theta) = \frac{r}{\theta}$$

$$c''(\theta) = -\frac{r}{\theta^2}$$

$$E(Y) = -\frac{\left(\frac{r}{\theta}\right)}{\left(-\frac{1}{1-\theta}\right)} = \frac{r(1-\theta)}{\theta}$$

$$Var(Y) = \frac{-\frac{1}{(1-\theta)^2}\left(\frac{r}{\theta}\right) - \left(-\frac{r}{\theta^2}\right)\left(-\frac{1}{1-\theta}\right)}{\left(-\frac{1}{1-\theta}\right)^3}$$

$$= \frac{-\frac{r}{\theta(1-\theta)^2} - \frac{r}{\theta^2(1-\theta)}}{-\frac{1}{(1-\theta)^3}}$$

$$= \frac{-\left[\frac{\theta r + (1-\theta) r}{\theta^2(1-\theta)^2}\right]}{-\frac{1}{(1-\theta)^3}} = \frac{\frac{r}{\theta^2(1-\theta)^2}}{\frac{1}{(1-\theta)^3}}$$

$$= \frac{\frac{r}{\theta^2}}{\frac{1}{(1-\theta)}} = \frac{r(1-\theta)}{\theta^2}$$

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Ex. 3.6 :

Refere to Example 3.5.3 (P.54 : Mortality Rate)

Model : $E(Y_i) = \mu_i = n_i e^{\theta i}$

$\Leftrightarrow \quad g(\mu_i) = \ln(\mu_i) = \ln(n_i) + \theta i$

$\Leftrightarrow \quad \ln(\mu_i) - \ln(n_i) = \theta i$

$\Leftrightarrow \quad \ln\left(\frac{\mu_i}{n_i}\right) = \theta i$

$\Leftrightarrow \quad \mu_i^* = \theta i$ where $\mu_i^* = \ln\left(\frac{\mu_i}{n_i}\right)$

age x_i	y_i	n_i	$\frac{y_i}{n_i}$	$\ln\left(\frac{y_i}{n_i}\right)$
30	1	17742	5.64×10^{-5}	-9.7837
35	5	16554	3.02×10^{-4}	-8.1049
⋮	⋮	⋮	⋮	⋮
65	65	9933	6.54×10^{-3}	-5.0292

Let $Y_i^* = \ln\left(\frac{y_i}{n_i}\right)$ and $x_i^* = \ln(x_i)$

Regress Y_i^* on i using the model

$Y_i^* = \beta_0 + \beta_1 x_i^*$ (Simple linear Regression)

[Use computer]

$\hat{\beta}_0 = -13.182$

$\hat{\beta}_1 = 0.1331$

Fitted values for Y_i^* : $\hat{Y}_i^* = \hat{\beta}_0 + \hat{\beta}_1 x_i^* = -13.182 + 0.1331 x_i^*$

Fitted values for Y_i : $\hat{Y}_i = n_i e^{\hat{Y}_i^*}$

$R^2 = 0.9685 \leftarrow$ (good fit)

[See computer output]

Ex. 3.7 :

- $Y_i \sim \text{Bernoulli}(\pi_i)$
- $Y_1, Y_2, \ldots, Y_N$ are independent.

$$f(y_i; \pi_i) = \pi_i^{y_i} (1-\pi_i)^{1-y_i} \quad ; \quad y_i = 0, 1$$

(a)
$$f(y, \pi) = \pi^{y} (1-\pi)^{1-y}$$
$$= e^{\ln(\pi^{y})} \, e^{\ln[(1-\pi)^{1-y}]}$$
$$= e^{y \ln(\pi)} \, e^{(1-y)\ln(1-\pi)}$$
$$= e^{y \ln(\pi) + (1-y)\ln(1-\pi)}$$
$$= e^{y \ln(\pi) + \ln(1-\pi) - y\ln(1-\pi)}$$
$$= e^{y[\ln(\pi) - \ln(1-\pi)] + \ln(1-\pi)}$$
$$= e^{y \ln\left(\frac{\pi}{1-\pi}\right) + \ln(1-\pi)} \in EF$$

$a(y) = y$, $b(\pi) = \ln\left(\frac{\pi}{1-\pi}\right)$, $c(\pi) = \ln(1-\pi)$,
$d(y) = 0$

(b) Natural parameter is
$$b(\pi) = \ln\left(\frac{\pi}{1-\pi}\right)$$

(c)
$$E(Y) = -\frac{c'(\pi)}{b'(\pi)}$$
$$= -\frac{\left(-\frac{1}{1-\pi}\right)}{\left(\frac{1}{\pi(1-\pi)}\right)}$$
$$= \pi$$

$c'(\pi) = -\frac{1}{1-\pi}$

$b'(\pi) = \frac{1}{\pi(1-\pi)}$

[See Ex. 3.4 Part (c)]

{Note: $\frac{\pi}{1-\pi}$ is called odds (معامل الترجيح)}

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Cont. Ex. 37 :

(d) Link function $g(\pi) = \ln\left(\frac{\pi}{1-\pi}\right) = \underset{\sim}{x}'\underset{\sim}{\beta}$.

$$\ln\left(\frac{\pi}{1-\pi}\right) = \underset{\sim}{x}'\underset{\sim}{\beta}$$

$$e^{\ln\left(\frac{\pi}{1-\pi}\right)} = e^{\underset{\sim}{x}'\underset{\sim}{\beta}} \qquad (e^{\ln x} = x)$$

$$\frac{\pi}{1-\pi} = e^{\underset{\sim}{x}'\underset{\sim}{\beta}}$$

$$\pi = (1-\pi)\, e^{\underset{\sim}{x}'\underset{\sim}{\beta}} = e^{\underset{\sim}{x}'\underset{\sim}{\beta}} - \pi e^{\underset{\sim}{x}'\underset{\sim}{\beta}}$$

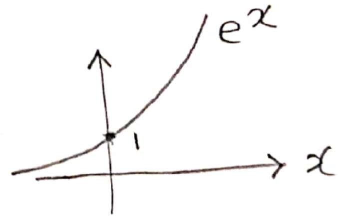
$$\pi + \pi e^{\underset{\sim}{x}'\underset{\sim}{\beta}} = e^{\underset{\sim}{x}'\underset{\sim}{\beta}}$$

$$\pi\left(1 + e^{\underset{\sim}{x}'\underset{\sim}{\beta}}\right) = e^{\underset{\sim}{x}'\underset{\sim}{\beta}}$$

$$\pi = \frac{e^{\underset{\sim}{x}'\underset{\sim}{\beta}}}{1 + e^{\underset{\sim}{x}'\underset{\sim}{\beta}}} \qquad \text{(logistic Regression model)}$$

(e) For $\underset{\sim}{x}'\underset{\sim}{\beta} = \beta_1 + \beta_2 x$, we have

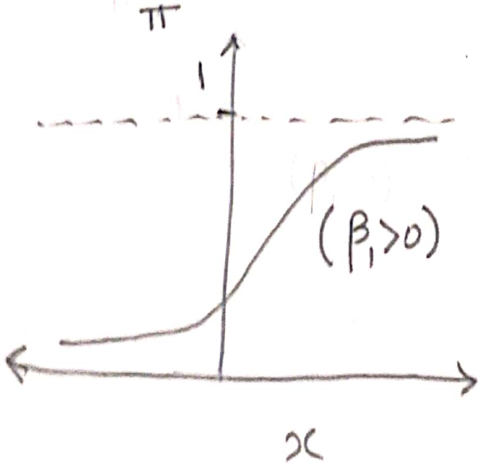
$$\pi = \frac{e^{\beta_1+\beta_2 x}}{1 + e^{\beta_1+\beta_2 x}} \qquad \text{(logistic function)}$$

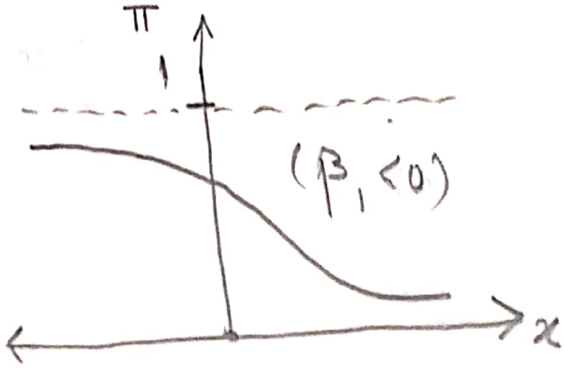


note: $e^{\beta_1+\beta_2 x} > 0$

$$0 < \frac{e^{\beta_1+\beta_2 x}}{1 + e^{\beta_1+\beta_2 x}} < 1$$

$$0 < \pi < 1$$





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Ex. 3.8 :

WLOG, Let $\phi = 1$. (ϕ is nuisance parameter) it is considered as fixed

$$f(y;\theta) = e^{(y-\theta) - e^{(y-\theta)}}$$

$$= e^{y-\theta - e^{(y-\theta)}}$$

$$= e^{y-\theta - e^{y} e^{-\theta}}$$

$$= e^{e^{y}(-e^{-\theta}) - \theta + y}$$

$$= e^{a(y)b(\theta) + c(\theta) + d(y)}$$

$$a(y) = e^{y};\quad b(\theta) = -e^{-\theta},\quad c(\theta) = -\theta,\quad d(y) = y$$

Ex. 3.9

$Y_1, Y_2, \dots, Y_n$ are independent.

$Y_i \sim \text{Pareto}(\theta)$: $f(y;\theta) = \theta y^{-\theta-1} \in EF$

$$a(y) = \ln(y),\quad b(\theta) = -\theta - 1,\quad c(\theta) = \ln(\theta),\quad d(y) = 0$$

(See Ex. 3.3 (a)).

$$E(Y_i) = \mu_i = (\beta_0 + \beta_1 x_i)^2$$

$$\Leftrightarrow \quad \sqrt{\mu_i} = \beta_0 + \beta_1 x_i$$

$$g(\mu_i) = \sqrt{\mu_i} \quad \text{link function}$$

For Pareto, $a(y) = \ln(y) \neq y$

One of the conditions of Generalized linear model is that

$$a(y)$$

(Canonical form)

Therefore, the model is not a generalized linear model.

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Ex. 3,10 :

$Y_1, Y_2, \ldots, Y_N$ are independent

$Y_i \sim N(\mu_i, \sigma^2) \in EF$ in canonical form.

$E(Y_i) = \mu_i = \beta_0 + \ln(\beta_1 + \beta_2 x_i)$

$$\mu_i = \beta_0 + \ln(\beta_1 + \beta_2 x_i)$$

$$e^{\mu_i} = e^{\beta_0 + \ln(\beta_1 + \beta_2 x_i)}$$

$$= e^{\beta_0} e^{\ln(\beta_1 + \beta_2 x_i)}$$

$$= e^{\beta_0}(\beta_1 + \beta_2 x_i)$$

$$= e^{\beta_0}\beta_1 + e^{\beta_0}\beta_2 x_i$$

$$= \underbrace{\beta_1^* + \beta_2^* x_i}_{\text{linear combonent}} \qquad \begin{cases} \beta_1^* = e^{\beta_0}\beta_1 \\ \beta_2^* = e^{\beta_0}\beta_2 \end{cases}$$

The model is a generalized linear model.

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Ex. 3.11 :

$Y \sim \text{Pareto}(\theta)$.

$f(y;\theta) = \theta\, y^{-\theta-1}$ (for one observation)

Let $Y_1, Y_2, \dots, Y_n$ are independent, and

$Y_i \sim \text{Pareto}(\theta) \quad i = 1, 2, \dots, n$

Likelihood function is

$$L(\theta;y) = f(\underset{\sim}{y};\theta) = \prod_{i=1}^{n} f(y_i,\theta)$$

- log-likelihood function is

$$\ell(\theta;\underset{\sim}{y}) = \ln[L(\theta;\underset{\sim}{y})] = \ln\left[\prod_{i=1}^{n} f(y_i;\theta)\right]$$

$$= \sum_{i=1}^{n} \ln[f(y_i,\theta)]$$

$$= \sum_{i=1}^{n} \ln[\theta\, y_i^{-\theta-1}]$$

$$= \sum_{i=1}^{n} \{\ln(\theta) + \ln[y_i^{-\theta-1}]\}$$

$$= \sum_{i=1}^{n} \{\ln(\theta) + (-\theta-1)\ln(y_i)\}$$

$$= n\ln(\theta) - (\theta+1)\sum_{i=1}^{n}\ln(y_i)$$

- The Score Statistic is

$$U(\theta;\underset{\sim}{y}) = \frac{\partial \ell(\theta;\underset{\sim}{y})}{\partial\theta}$$

$$= \frac{n}{\theta} - \sum_{i=1}^{n}\ln(y_i)$$

Recall: $E[U(\theta,\underset{\sim}{y})] = 0$

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Cont. Ex. 3.11 :

- Information:

$$J = Var(U) = E(U^2) = -E(U') = -E\left[\frac{\partial}{\partial\theta} u(\theta; \underset{\sim}{y})\right]$$

$$= -E\left\{\frac{\partial}{\partial\theta}\left(\frac{n}{\theta} - \sum_{i=1}^{n} \ln(y_i)\right)\right\}$$

$$= -E\left[-\frac{n}{\theta^2}\right]$$

$$= \frac{n}{\theta^2}$$

2nd solution :

For Pareto distr, we have found that [See Ex. 3.3.(a)]

$$\begin{cases} a(y) = \ln(y) \\ b(\theta) = -\theta - 1 \\ c(\theta) = \ln(\theta) \end{cases} \qquad \begin{cases} b'(\theta) = -1 \ ; \ b''(\theta) = 0 \\ c'(\theta) = \frac{1}{\theta} \ ; \ c''(\theta) = -\frac{1}{\theta^2} \end{cases}$$

We know that

- $E[a(y)] = -\dfrac{c'(\theta)}{b'(\theta)}$

$$\therefore E[\ln(y)] = -\frac{1/\theta}{(-1)} = \frac{1}{\theta}$$

Note:
$$E(U) = E\left[\frac{n}{\theta} - \sum \ln(y_i)\right] = \frac{n}{\theta} - \sum E[\ln y_i] = \frac{n}{\theta} - \sum \frac{1}{\theta} = \frac{n}{\theta} - \frac{n}{\theta} = 0$$

- $Var[a(y)] = \dfrac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{[b'(\theta)]^3}$

$$\therefore Var[\ln(y)] = \frac{0 - (-1/\theta^2)(-1)}{(-1)^3} = \frac{-1/\theta^2}{-1} = \frac{1}{\theta^2}$$

$$E\{[\ln(y)]^2\} = Var\{\ln(y)\} + \{E[\ln(y)]\}^2 = \frac{1}{\theta^2} + \left(\frac{1}{\theta}\right)^2 = \frac{2}{\theta^2}$$

Recall:
$$\begin{cases} Var(X) = E(X^2) - (E(X))^2 \\ E(X^2) = Var(X) + (E(X))^2 \end{cases}$$

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cont. Ex 3.11

Now,

$$J = Var(U) = E(U^2)$$

$$= E\left\{\left[\frac{n}{\theta} - \sum \ln(y_i)\right]^2\right\}$$

$$= E\left[\left(\frac{n}{\theta}\right)^2 + \left[\sum \ln(y_i)\right]^2 - 2\frac{n}{\theta}\sum \ln(y_i)\right]$$

$$= \left(\frac{n}{\theta}\right)^2 + E\left\{\left[\sum \ln(y_i)\right]^2\right] - 2\frac{n}{\theta} E\left[\sum \ln(y_i)\right]$$

$$= \left(\frac{n}{\theta}\right)^2 + \left[Var\left[\sum \ln(y_i)\right] + \left\{E\left[\sum \ln(y_i)\right]\right\}^2\right]$$

$$- 2\frac{n}{\theta} E\left[\sum \ln(y_i)\right]$$

$$= \left(\frac{n}{\theta}\right)^2 + \left[\sum_{i=1}^{n} Var(\ln(y_i)) + \left[\sum_{i=1}^{n} E(\ln(y_i))\right]^2\right.$$

$$- 2\frac{n}{\theta}\sum E\left[\ln(y_i)\right]$$

$$= \left(\frac{n}{\theta}\right)^2 + \left[\sum_{i=1}^{n}\frac{1}{\theta^2} + \left(\sum_{i=1}^{n}\frac{1}{\theta}\right)^2\right]$$

$$- 2\frac{n}{\theta}\sum_{i=1}^{n}\frac{1}{\theta}$$

$$= \left(\frac{n}{\theta}\right)^2 + \frac{n}{\theta^2} + \left(\frac{n}{\theta}\right)^2 - 2\frac{n}{\theta}\left(\frac{n}{\theta}\right)$$

$$= \frac{n^2}{\theta^2} + \frac{n}{\theta^2} + \frac{n^2}{\theta^2} - \frac{2n^2}{\theta^2}$$

$$= \frac{n}{\theta^2}$$

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3rd Solution:

We know that

$$Var(U) = \frac{b''(\theta)\, c'(\theta)}{b'(\theta)} - c''(\theta) \quad \text{(for one observation)}$$

$$Var(U) = \frac{(0)\left(\frac{1}{\theta}\right)}{(-1)} - \left(-\frac{1}{\theta^2}\right)$$

$$= \frac{1}{\theta^2}$$

- $\because \mathcal{J} = Var(U)$ (for all observations)

$$= \sum_{i=1}^{n} \frac{1}{\theta^2}$$

$$= \frac{n}{\theta^2}$$

- $E(U) = 0$:

$$\left(E(\ln(y_i)) = \frac{1}{\theta}\right)$$

$$U = \frac{n}{\theta} - \sum_{i=1}^{n} \ln(y_i)$$

$$E(U) = E\left[\frac{n}{\theta} - \sum_{i=1}^{n} \ln(y_i)\right]$$

$$= \frac{n}{\theta} - E\left[\sum_{i=1}^{n} \ln(y_i)\right]$$

$$= \frac{n}{\theta} - \sum_{i=1}^{n} E[\ln(y_i)]$$

$$= \frac{n}{\theta} - \sum_{i=1}^{n} \frac{1}{\theta}$$

$$= \frac{n}{\theta} - \frac{n}{\theta} = 0$$

الممسوحة ضوئيا بـ CamScanner