Chapter 2

Ex. 2.1:

$$y_{jk} \sim N(\mu_j, \sigma^2);\ j = 1,2$$

i.e. $y_{1k} \sim N(\mu_1, \sigma^2);\ k = 1, \dots, n_1$ and $y_{2k} \sim N(\mu_2, \sigma^2); k = 1,2, \dots, n_2$

y_{jk} are independent.

$$H_0: \mu_1 = \mu_2 = \mu$$
$$H_1: \mu_1 \neq \mu_2$$

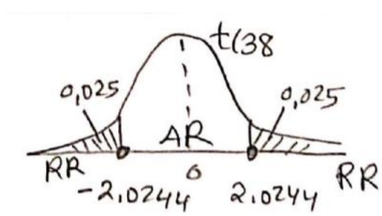
(a) use Minitab.

Group	
Trteatment	control
$\bar{y}_{1.} = 4.86$	$\bar{y}_{2.} = 4.726$
$s_1^2 = 0.626$	$s_2^2 = 0.746$
$n_1 = 20$	$n_2 = 20$

$$N = n_1 + n_2 = 40$$
$$\bar{y}_{..} = 4.793$$
$$s^2 = 0.673$$
$$s_p^2 = 0.686$$
$$s_p = 0.82825$$
$$df = n_1 + n_2 - 2 = 38$$

(b) T-test:

$$t = \frac{\bar{y}_{1.} - \bar{y}_{2.}}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{4.86 - 4.726}{0.82825\sqrt{\frac{1}{20} + \frac{1}{20}}} = \frac{0.134}{0.2619156} = 0.5116$$
$$t_{\frac{\alpha}{2}(n_1+n_2-2)} = t_{0,025}(38) = 2.0244$$
$$|t| = 0.5116 < t_{0.025}(38)$$



$\Rightarrow$ Do not reject to H_0: $\mu_1 = \mu_2$ at significance Lavel $\alpha = 0,05$

(c)

(i) MLE: (Maximum Likelihood Estimatim):

<u>First:</u> For Model under $H_0: \mu_1 = \mu_2 = \mu_0$.

$$f(y_{jk}; \mu) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_{jk}-\mu)^2}$$
$$\Rightarrow \ln\ f(y_{jk}; \mu) = \ln\left[\frac{1}{\sqrt{2\pi\sigma^2}}\right] - \frac{1}{2\sigma^2}(y_{jk} - \mu)^2$$

$$l(\mu)=\sum_{j=1}^{J}\sum_{k=1}^{K}\ln f(y_{jk};\mu)=\sum_{j}\sum_{k}\left\{\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)-\frac{1}{2\sigma^2}\left(y_{jk}-\mu\right)^2\right\}$$

$$\Rightarrow\frac{\partial l(\mu)}{\partial\mu}=-\frac{1}{2\sigma^2}\sum\sum 2\left(y_{jk}-\mu\right)(-1)=\frac{1}{\sigma^2}\sum\sum\left(y_{jk}-\mu\right)$$

$$\frac{\partial l(\mu)}{\partial\mu}\overset{\text{set}}{=}0\Leftrightarrow\frac{1}{\sigma^2}\sum\sum\left(y_{jk}-\mu\right)=0\Leftrightarrow\sum_{j=1}^{J}\sum_{k=1}^{K}\left(y_{jk}-\mu\right)=0\Leftrightarrow\sum_{j=1}^{J}\sum_{k=1}^{K}y_{jk}-JK\mu=0$$

$$\Leftrightarrow\sum\sum y_{jk}=JK\mu\Leftrightarrow\hat{\mu}=\frac{\sum\sum y_{jk}}{Jk}=\frac{\sum\sum y_{jk}}{N}=\bar{y}_{..}.$$

$J=2;k=20\Rightarrow N=JK=(2)(20)=40$

$\therefore$ MLE of μ (under H_0) is:

$$\hat{\mu}=\bar{y}_{..}$$

Second: For Model under $H_1:\mu_1\neq\mu_2$:

$$f\left(y_{jk};\mu_j\right)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2\sigma^2}\left(y_{jk}-\mu_j\right)^2};j=1,2\ ,k=1,\cdots,20$$

$$\Rightarrow\ln f\left(y_{jk};\mu_j\right)=\ln\left[\frac{1}{\sqrt{2\pi\sigma^2}}\right]-\frac{1}{2\sigma^2}\left(y_{jk}-\mu_j\right)^2$$

$$\ell(\mu_1,\mu_2)=\sum_{j=1}^{J}\sum_{k=1}^{K}\ln\left(y_{jk};\mu_j\right)=\sum\sum\left\{\ln\left[\frac{1}{\sqrt{2\pi\sigma^2}}\right]-\frac{1}{2\sigma^2}\left(y_{jk}-\mu_j\right)^2\right\}$$

$$=\sum_{j=1}^{J}\sum_{k=1}^{k}\ln\left[\frac{1}{\sqrt{2\pi\sigma^2}}\right]-\frac{1}{2\sigma^2}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(y_{jk}-\mu_j\right)^2$$

$$=\sum\sum\left[\frac{1}{\sqrt{2\pi\sigma^2}}\right]-\frac{1}{2\sigma^2}\left[\sum_{k=1}^{K}(y_{1k}-\mu_1)^2+\sum_{k=1}^{K}(y_{2k}-\mu_2)^2\right]$$

$$\Rightarrow\frac{\partial l(\mu_1,\mu_2)}{\partial\mu_1}=\frac{1}{\sigma^2}\sum_{k=1}^{K}(y_{1k}-\mu_1)\text{ and }\frac{\partial l(\mu_1,\mu_2)}{\partial\mu_2}=\frac{1}{\sigma^2}\sum_{k=1}^{K}(y_{2k}-\mu_2)$$

$$\frac{\partial\ell(\mu_1,\mu_2)}{\partial\mu_1}\overset{\text{set}}{=}0\text{ and }\frac{\partial\ell(\mu_1,\mu_2)}{\partial\mu_2}\overset{\text{set}}{=}0$$

$$\Rightarrow\tfrac{1}{\sigma^2}\textstyle\sum_{k=1}^{K}(y_{1k}-\mu_1)=0\text{ and }\tfrac{1}{\sigma^2}\sum_{k=1}^{K}(y_{2k}-\mu_2)=0$$

$$\Rightarrow\textstyle\sum_{k=1}^{K}(y_{1k}-\mu_1)=0\text{ and }\sum_{k=1}^{K}(y_{2k}-\mu_2)=0$$

$$\Rightarrow\textstyle\sum_{k=1}^{K}y_{1k}-K\mu_1=0\text{ and }\sum_{k=1}^{K}y_{2k}-K\mu_2=0$$

$$\Rightarrow\textstyle\sum_{k=1}^{K}y_{1k}=K\mu_1\text{ and }\sum_{k=1}^{K}y_{2k}=K\mu_2$$

$$\hat{\mu}_1=\frac{\sum_{k=1}^{K}y_{1k}}{K}=\bar{y}_{1.}\text{ and }\hat{\mu}_2=\frac{\sum_{k=1}^{K}y_{2k}}{K}=\bar{y}_{2.}$$

$\therefore$ The MLE of μ_1 and μ_2 under H_1 are

$$\hat{\mu}_1=\bar{y}_{1.}$$
$$\hat{\mu}_2=\bar{y}_{2.}$$

(ii) LSE (Least Square Estimation):

First: For Model under $H_0:\mu_1=\mu_2=\mu$:

$$S_0 = \sum_{j=1}^{J} \sum_{k=1}^{K} \left(y_{jk} - \mu\right)^2$$

$$\Rightarrow \frac{\partial S_0}{\partial \mu} = -2 \sum \sum \left(y_{jk} - \mu\right)$$

$$\frac{\partial S_0}{\partial \mu} \overset{\text{set}}{=} 0 \Leftrightarrow -2 \sum \sum \left(y_{jk} - \mu\right) = 0 \Leftrightarrow \sum_{j=1}^{J} \sum_{k=1}^{K} \left(y_{jk} - \mu\right) = 0 \Leftrightarrow \sum_{j=1}^{J} \sum_{k=1}^{K} y_{jk} - JK\mu = 0$$

$$\Leftrightarrow \mu = \frac{\sum \sum y_{jk}}{JK} = \frac{\sum \sum y_{jk}}{N} = \bar{y}_{..} \quad \text{(as } N = JK \text{ where } J = 2 \text{ and } K = 20\text{)}$$

$\therefore$ The LSE of μ under H_0 is:

$$\hat{\mu} = \bar{y}_{..} \text{ (same as } MLE\text{)}$$

Second: For Model under $H_1\colon \mu_1 \neq \mu_2$:

$$S_1 = \sum_{j=1}^{J} \sum_{k=1}^{K} \left(y_{jk} - \mu_j\right)^2 = \sum_{k=1}^{K} (y_{1k} - \mu_1)^2 + \sum_{k=1}^{K} (y_{2k} - \mu_2)^2$$

$$\Rightarrow \frac{\partial S_1}{\partial \mu_1} = -2 \textstyle\sum_{k=1}^{K} (y_{1k} - \mu_1) \text{ and } \frac{\partial S_1}{\partial \mu_2} = -2 \sum_{k=1}^{K} (y_{2k} - \mu_2)$$

$$\frac{\partial S_1}{\partial \mu_1} \overset{\text{set}}{=} 0 \text{ and } \frac{\partial S_1}{\partial \mu_2} \overset{\text{set}}{=} 0$$

$$\Rightarrow -2\Sigma(y_{1k} - \mu_1) = 0 \text{ and } -2\Sigma(y_{2k} - \mu_1) = 0$$

$$\Rightarrow \textstyle\sum_{k=1}^{K} (y_{1k} - \mu_1) = 0 \text{ and } \sum_{k=1}^{K} (y_{2k} - \mu_2) = 0$$

$$\Rightarrow \textstyle\sum_{k=1}^{K} y_{1k} - K\mu_1 = 0 \text{ and } \sum_{k=1}^{K} y_{2k} - K\mu_2 = 0$$

$$\Rightarrow \textstyle\sum_{k=1}^{K} y_{1k} = K\mu_1 \text{ and } \sum_{k=1}^{K} y_{2k} = K\mu_2$$

$$\Rightarrow \mu_1 = \frac{\sum_{k=1}^{K} y_{1k}}{K} = \bar{y}_{1.} \text{ and } \mu_2 = \frac{\sum_{k=1}^{K} y_{2k}}{K} = \bar{y}_{2.}$$

$\therefore$ The LSE of μ_1 and μ_2 under H_1 are:

$$\hat{\mu}_1 = \bar{y}_{1.} \text{ and } \hat{\mu}_2 = \bar{y}_{2.}$$

which is the same as MLE

(d) Recall:

- The LSE of μ under ($H_0\colon \mu_1 = \mu_2 = \mu$) is $\hat{\mu} = \bar{y}_{..}$
- The LSE of μ_1 and μ_2 under ($H_1\colon \mu_1 \neq \mu_2$) are:

$$\hat{\mu}_1 = \bar{y}_{1.} \text{ and } \hat{\mu}_2 = \bar{y}_{2.}$$

(i) The minimum value of $S_0 = \sum \sum \left(y_{jk} - \mu\right)^2$ is:

$$\hat{S}_0 = \textstyle\sum_{j=1}^{J} \sum_{k=1}^{K} \left(y_{jk} - \hat{\mu}\right)^2 = \sum_{j=1}^{J} \sum_{k=1}^{K} \left(y_{jk} - \bar{y}_{..}\right)^2 ; \bar{y}_{..} = \frac{\sum \sum Y_{jk}}{JK} \text{ (grand average)}$$

(ii) The minimum value of $S_1 = \sum \sum \left(y_{jk} - \mu_j\right)^2$ is:

$$\hat{S}_1 = \sum_{j=1}^{J} \sum_{k=1}^{K} \left(y_{jk} - \hat{\mu}_j\right)^2 = \sum_{j=1}^{J} \sum_{k=1}^{K} \left(y_{jk} - \bar{y}_{j.}\right)^2 ; \bar{y}_{j.} = \frac{\sum_{k=1}^{K} y_{jk}}{K}$$

$$= \textstyle\sum_{k=1}^{K} (y_{1k} - \bar{y}_{1.})^2 + \sum_{k=1}^{K} (y_{2k} - \bar{y}_{2.})^2$$

(e)

(1)

$$S_1 = \sum_{j=1}^{J} \sum_{k=1}^{K} \left(y_{jk} - \mu_j\right)^2 \text{ ; (under } H_1\text{: } \mu_1 \neq \mu_2)$$

$$= \sum \sum \left(y_{jk} - \bar{y}_{j.} + \bar{y}_{j.} - \mu_j\right)^2 ; \left(\bar{y}_{j.} = \frac{\sum_{k=1}^{K} y_{jk}}{K}; j = 1,2\right)$$

$$= \sum \sum \left[\left(y_{jk} - \bar{y}_{j.}\right) + \left(\bar{y}_{j.} - \mu_j\right)\right]^2$$

$$= \sum_{j=1}^{J} \sum_{k=1}^{K} \left(y_{jk} - \bar{y}_{j.}\right)^2 + \sum_{j=1}^{J} \sum_{k=1}^{K} \left(\bar{y}_{j.} - \mu_j\right)^2 + 2\sum_{j=1}^{J} \sum_{k=1}^{K} \left(y_{jk} - \bar{y}_{j.}\right)\left(\bar{y}_{j.} - \mu_j\right)$$

$$= \sum_{j=1}^{J} \sum_{k=1}^{K} \left(y_{jk} - \bar{y}_{j.}\right)^2 + K\sum_{j=1}^{J} \left(\bar{y}_{j.} - \mu_j\right)^2 + 2\sum_{j=1}^{J} \left(\bar{y}_{j.} - \mu_j\right)\underbrace{\sum_{k=1}^{K} \left(y_{jk} - \bar{y}_{j.}\right)}_{=0}$$

Note that:

$$\sum_{k=1}^{K} \left(y_{jk} - \bar{y}_{j.}\right) = 0 \text{ for } j = 1, \dots, \mathrm{J}$$

$$\Rightarrow S_1 = \sum \sum \left(y_{jk} - \bar{y}_{j.}\right)^2 + K\sum \left(\bar{y}_{j.} - \mu_j\right)^2$$

or

$$\sum_{j=1}^{J} \sum_{k=1}^{K} \left(y_{jk} - \mu_j\right)^2 = \sum \sum \left(y_{jk} - \bar{y}_{j.}\right)^2 + K\sum \left(\bar{y}_{j.} - \mu_j\right)^2$$

$$\Leftrightarrow \underbrace{\sum_{j} \sum_{k} \left(y_{jk} - \bar{y}_{j.}\right)^2} = \sum_{j} \sum_{k} \left(y_{jk} - \mu_j\right)^2 - K\sum_{j} \left(\bar{y}_{j.} - \mu_j\right)^2$$

$$\Leftrightarrow \hat{S}_1 = \sum_{j} \sum_{k} \left(y_{jk} - \mu_j\right)^2 - K\sum_{j} \left(\bar{y}_{j.} - \mu_j\right)^2$$

$$\Leftrightarrow \frac{\hat{S}_1}{\sigma^2} = \frac{1}{\sigma^2}\sum \sum \left(y_{jk} - \mu_j\right)^2 - \frac{K}{\sigma^2}\sum_{j} \left(\bar{y}_{j.} - \mu_j\right)^2$$

Note:
LSE of μ_j is $\hat{\mu}_j = \bar{y}_{j.}$ under H_1:

$$\hat{S}_1 = \sum \sum \left(y_{jk} - \hat{\mu}_j\right)^2 = \sum \sum \left(y_{jk} - \bar{y}_{j.}\right)^2$$

Distribution of $\frac{\hat{S}_1^2}{\sigma^2}$ (under H_1):We found that:

$$\frac{1}{\sigma^2}\hat{S}_1 = \frac{1}{\sigma^2}\sum \sum \left(y_{jk} - \mu_j\right)^2 - \frac{K}{\sigma^2}\sum_{j} \left(\bar{y}_{j.} - \mu_j\right)^2 = \sum_{j=1}^{J} \sum_{k=1}^{K} \left(\frac{y_{jk} - \mu_j}{\sigma}\right)^2 - \sum_{j=1}^{J} \left(\frac{\bar{y}_{j.} - \mu_j}{\sigma/\sqrt{K}}\right)^2$$

Recall:
If $y_1, y_2, \cdots, y_n$ iid $N(\mu, \sigma^2)$ then $\bar{y} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ since y_{jk} are independent and $y_{jk} \sim N\left(\mu_j, \sigma^2\right)$,

(1) We have $\bar{y}_{j.} \sim N\left(\mu_j, \frac{\sigma^2}{K}\right)$; $j = 1,2 \Rightarrow \frac{\bar{y}_{j.} - \mu_j}{\frac{\sigma}{\sqrt{K}}} \sim N(0,1)$ $\left(\frac{\bar{y}_{j.} - \mu_j}{\frac{\sigma}{\sqrt{K}}}\right)^2 = \frac{K}{\sigma^2}\left(\bar{y}_{j.} - \mu_j\right)^2 \sim \chi^2_{(1)}$

$$\Rightarrow \sum_{j=1}^{2}\left(\frac{\bar{y}_{j\cdot}-\mu_j}{\sigma/\sqrt{K}}\right)^2 \sim \chi^2_{(2)}$$

(2) $y_{jk} \sim N(\mu_j, \sigma^2) \Rightarrow \frac{y_{jk}-\mu_j}{\sigma} \sim N(0,1) \Rightarrow \left(\frac{y_{jk}-\mu_j}{\sigma}\right)^2 \sim \chi^2_{(1)} \Rightarrow \sum_{j=1}^{J}\sum_{k=1}^{K}\left(\frac{y_{jk}-\mu_j}{\sigma}\right)^2 \sim \chi^2_{(JK)}$

Therefore:

$$\frac{1}{\sigma^2}\hat{S}_1 = \chi^2_{(JK)} - \chi^2_{(2)} \Rightarrow \chi^2_{(JK)} = \frac{\hat{S}_1}{\sigma^2} + \chi^2_{(2)} \Rightarrow \frac{\hat{S}_1}{\sigma^2} \sim \chi^2_{(JK-2)}$$

(2)

$S_0 = \sum\sum (y_{jk}-\mu)^2$; (under $H_0: \mu_1 = \mu_2 = \mu$)

$$= \sum\sum (y_{jk} - \bar{y}_{..} + \bar{y}_{..} - \mu)^2 = \sum\sum [(y_{jk}-\bar{y}_{..}) + (\bar{y}_{..}-\mu)]^2 ; \left(\bar{y}_{..} = \frac{\sum\sum y_{jk}}{JK}\right)$$

$$= \sum_{j=1}^{J}\sum_{k=1}^{K}(y_{jk}-\bar{y}_{..})^2 + \sum_{j=1}^{J}\sum_{k=1}^{K}(\bar{y}_{..}-\mu)^2 + 2\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{jk}-\bar{y}_{..})(\bar{y}_{..}-\mu)$$

$$= \sum_{j=1}^{J}\sum_{k=1}^{K}(y_{jk}-\bar{y}_{..})^2 + JK(\bar{y}_{..}-\mu)^2 + 2(\bar{y}_{..}-\mu)\underbrace{\sum\sum(y_{jk}-\bar{y}_{..})}_{=0}$$

$$\Rightarrow S_0 = \sum\sum (y_{jk}-\bar{y}_{..})^2 + JK(\bar{y}_{..}-\mu)^2$$

or

$$\sum\sum(y_{jk}-\mu)^2 = \sum\sum(y_{jk}-\bar{y}_{..})^2 + JK(\bar{y}_{..}-\mu)^2$$

$$\Leftrightarrow \underbrace{\sum\sum(y_{jk}-\bar{y}_{..})^2} = \sum\sum(y_{jk}-\mu)^2 - JK(\bar{y}_{..}-\mu)^2$$

$$\hat{\hat{S}}_0 = \sum\sum(y_{jk}-\mu)^2 - JK(\bar{y}_{..}-\mu)^2$$

$$\Leftrightarrow \frac{\hat{S}_0}{\sigma^2} = \frac{1}{\sigma^2}\sum\sum(y_{jk}-\mu)^2 - \frac{JK}{\sigma^2}(\bar{y}_{..}-\mu)^2$$

Note:

The LSE of μ is $\hat{\mu} = \bar{y}_{..}$ under H_0:

$$\hat{S}_0 = \sum\sum(y_{jk}-\hat{\mu})^2 = \sum\sum(y_{jk}-\bar{y}_{..})^2$$

Distribution of $\frac{\hat{S}_0}{\sigma^2}$ under H_0:

We have found that

$$\frac{\hat{S}_0}{\sigma^2} = \frac{1}{\sigma^2}\sum\sum(y_{jk}-\mu)^2 - \frac{Jk}{\sigma^2}(\bar{y}_{..}-\mu)^2$$

$$= \underbrace{\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\frac{y_{jk}-\mu}{\sigma}\right)^2} - \underbrace{\left(\frac{\bar{y}_{..}-\mu}{\sigma/\sqrt{Jk}}\right)^2}$$

$$= \chi^2_{(JK)} - \chi^2_{(1)}$$

$$\Leftrightarrow \chi^2_{(JK)} = \frac{\hat{S}_0}{\sigma^2} + \chi^2_{(1)} \Rightarrow \frac{\hat{S}_0}{\sigma^2} \sim \chi^2_{(JK-1)}$$

(f) We have found that

$$\frac{\hat{S}_0}{\sigma^2} \sim \chi^2_{(JK-1)} \quad (\text{ under } H_0)$$
$$\frac{\hat{S}_1}{\sigma^2} \sim \chi^2_{(JK-2)} \quad (\text{under } H_1) \quad \longrightarrow *$$

Now, if H_0 is true,

$$\frac{\hat{S}_0-\hat{S}_1}{\sigma^2} \sim \chi^2_{(1)} \ \ (\text{as } (JK-1)-(JK-2)=1) \quad \longrightarrow **$$

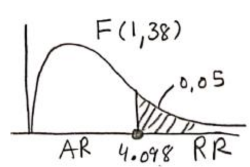
From *and **:

$$F = \frac{\frac{\hat{S}_0-\hat{S}_1}{\sigma^2}/(1)}{\frac{\hat{S}_1}{\sigma^2}/(JK-2)} = \frac{\hat{S}_0-\hat{S}_1}{\hat{S}_1/(JK-2)} \sim F(1, JK-2)$$

(g)

$$H_0: \mu_1 = \mu_2 = \mu$$
$$H_1: \mu_1 \neq \mu_2$$
$$\sum\sum y_{jk} = 191.73, N = JK = (2)(20) = 40$$
$$\hat{S}_0 = \sum\sum \left(y_{jk} - \hat{\mu}\right)^2 = \sum\sum \left(y_{jk} - \bar{y}_{..}\right)^2 = \sum\sum y_{jk}^2 - JK(\bar{y}_{..})^2$$
$$= \sum\sum y_{jk}^2 - (2)(20)(\bar{y}_{..})^2 = 945.241 - (40)(4.793)^2 = 26.32704$$
$$\hat{S}_1 = \sum_{j=1}^{2}\sum_{k=1}^{20}\left(y_{jk} - \bar{y}_{j.}\right)^2 = \sum_{k=1}^{20}(y_{1k} - \bar{y}_{1.})^2 + \sum_{k=1}^{20}(y_{2k} - \bar{y}_{2.})^2$$
$$= \left[\sum_{k=1}^{20} y_{1k}^2 - K(\bar{y}_1.)^2\right] + \left[\sum_{k=1}^{20} y_{2k}^2 - K(\bar{y}_2.)^2\right]$$
$$= [484.278 - (20)(4.86)^2] + [460.964 - (20)(4.726)^2]$$
$$F_0 = \frac{\hat{S}_0 - \hat{S}_1}{\hat{S}_1/(JK\text{-}2)} = \frac{26.32704 - 26.14848}{26.14848/38} = 0.2595$$
$$F_0 = 0.2595 < F_{0,05}(1{,}38) = 4.098$$
$$F_0 \in AR$$

$\therefore$ We do not reject H_0



$F_{0,05}(1,38) = 4.098$

(h)

$$F_0 = 0.2595 \quad (\sim F(1{,}38))$$
$$t_0 = 0.5116 \quad (\sim t(38))$$

$$t_0^2 = (0.5116)^2 = 0.2617$$
$t_0^2 = F_0$ (The difference is because of rounding errors)

(i) Calculating the residuals: $e_{jk} = y_{jk} - \bar{y}_{..}$.
[Minitab For Model H_0].
The residuals are consistent with the assumptions of independence, equal variances and Normality.

Ex. 2.2
(a)

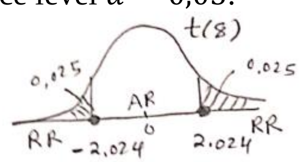
$$H_0: \mu_1 = \mu_2$$
$$H_1: \mu_1 \neq \mu_2$$

Unpaired t-test:

Before	After
$\bar{y}_1 = 103.25$	$\bar{y}_2 = 100.60$
$s_1^2 = 182.72$	$s_2^2 = 155.61$
$n_1 = 20$	$n_2 = 20$

$$N = n_1 + n_2 = 40$$
$$\bar{y}_{..} = 4.793$$
$$s^2 = 0.673$$
$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} = 169.165$$
$$s_p = 13.006$$
$$df = n_1 + n_2 - 2 = 38$$
$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{103.25 - 100.60}{13.006\sqrt{\frac{1}{20} + \frac{1}{20}}} = 0.644$$
$$|t_0| < t_{0.025,(38)} = 2.024$$
$$t_0 \in AR$$

$\therefore$ We do not reject $H_0: \mu_1 = \mu_2$ at significance level $\alpha = 0{,}05$.



(b) $\mu_D = \mu_1 - \mu_2; E(D) = E(y_1 - y_2) = E(y_1) - E(y_2) = \mu_1 - \mu_2$

$$H_0: \mu_D = 0$$
$$H_1: \mu_D \neq 0$$

paired t-test

Man	Before Y_{1k}	After Y_{2k}	Difference $D_k = Y_{1k} - Y_{2k}$
1	1008.8	97.0	38.0
2	102.0	107.5	-5.5
⋮	⋮	⋮	⋮
20	93.0	93.0	0.0

$$t_0 = \frac{\bar{D}}{S_D/\sqrt{n}} = \frac{\bar{D}}{S_{\bar{D}}} = \frac{2.645}{0.9206} = 2.873$$

$$\bar{D} = 2.645$$

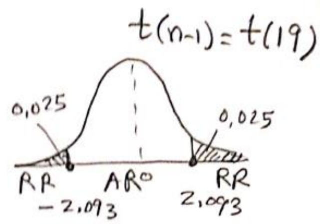
$$S_{\bar{D}} = \frac{S_D}{\sqrt{n}} = \frac{4.117}{\sqrt{20}} = 0.9206$$

$$S_D^2 = 16.947 \Rightarrow S_D = 4.117$$

$$|t_0| > t_{0,025}(19) = 2.093$$

$$t_0 \in RR$$

$\therefore$ we reject $H_0: \mu_1 = \mu_2$ at significance level $\alpha = 0.05$.



(c) The conclusions are:

(1) For t-test: $\begin{cases} H_0: \mu_1 = \mu_2 \text{ is not rejected} \\ \text{at } \alpha = 0.05 \end{cases}$

(2) For paired t-test: $\begin{cases} H_0: \mu_1 = \mu_2 \text{ is rejected} \\ \text{at } \alpha = 0.05 \end{cases}$

(d)

- We reached different conclusions.
- The paired t-test is more approprite than t-test.
- Assumptions:

(1) For t-test: $\begin{cases} 1. \quad y_{jk} \sim N(\mu_j, \sigma_j^2) \\ 2. \quad y_{jk} \text{ are independent} \\ 3. \qquad \sigma_1^2 = \sigma_2^2 \end{cases}$

i.e. For (a) it is assumed that the Y_{jk} 's are independent and $Y_{jk} \sim N(\mu_j, \sigma^2)$ for all j and for all k.

(2) For paired t-test $\begin{cases} 1. y_{jk} \sim N(\mu_j, \sigma_j^2) \\ 2. y_{11}, \ldots, y_{1k} \text{ are independent} \\ 3. y_{21}, \ldots, y_{2k} \text{ ane independent} \end{cases}$

but y_{1k} and y_{2k} are not independent.

i.e. For (b) it is assumed that the D_k 's are independent with $D_k \sim N(\mu_D, \sigma_D^2)$. The analysis in (b) does not involve assuming that y_{1k} and y_{2k} (i.e., 'before' and 'after' weights for the same man) are independent, so it is more appropriate.

Ex 2.4:

$$E(y) = \ln \quad (\beta_0 + \beta_1 x + \beta_2 x^2)$$
$$\Rightarrow e^{E(y)} = \beta_0 + \beta_1 x + \beta_2 x^2$$
$$\Rightarrow g[E(y)] = \beta_0 + \beta_1 x + \beta_2 x^2; (g(x) = e^x)$$
$$\Rightarrow g\left[E\left(\underset{\sim}{y}\right)\right] = \underline{x}\ \underset{\sim}{\beta}$$

$$\underset{\sim}{y} = \begin{bmatrix} 3.15 \\ 4.85 \\ 6.50 \\ 7.20 \\ 8{,}25 \\ 16.50 \end{bmatrix}, \underline{x} = \begin{array}{c} \begin{array}{ccc} \beta_0 & \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} 1 & 1.0 & 1.0 \\ 1 & 1.2 & 1.44 \\ 1 & 1.4 & 1.96 \\ 1 & 1.6 & 2.56 \\ 1 & 1.8 & 3.24 \\ 1 & 2.0 & 4.00 \\ & x & x^2 \end{bmatrix} \end{array}, \underset{\sim}{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$$

Ex 2.5:

$$E\left(y_{jk}\right) = \mu_{jk} = \mu + \alpha_j + \beta_k \begin{cases} y_{jk} \sim N\left(\mu_{jk}, \sigma^2\right); j = 1{,}2 \text{ and } k = 1{,}2{,}3 \\ \text{independent} \end{cases}$$
$$\Rightarrow \mathrm{E}(\underset{\sim}{y}) = \underline{x}\ \underset{\sim}{\beta}$$

constraints:

$$\sum \alpha_j = 0 \Leftrightarrow \alpha_2 = -\alpha_1$$
$$\sum \beta_k = 0 \Leftrightarrow \beta_3 = -\beta_1 - \beta_2$$

$$\underset{\sim}{y} = \begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{22} \\ y_{23} \end{bmatrix}, \underline{x} = \begin{array}{c} \begin{array}{cccc} \mu & \alpha_1 & \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix} \end{array} \text{and } \underset{\sim}{\beta} = \begin{bmatrix} \mu \\ \alpha_1 \\ \beta_1 \\ \beta_2 \end{bmatrix}.$$