Chapter 2: Exercise

2.5 Exercises : P40 (Dobson 2008):

Ex. 2.1 p.40 (Dobson 2008)

$Y_{jk} \sim N(\mu_j, \sigma^2)$ $\quad j=1,2$ $\quad \begin{cases} Y_{1k} \sim N(\mu_1, \sigma^2) & k=1,\dots,n_1 \\ Y_{2k} \sim N(\mu_2, \sigma^2) & k=1,2,\dots,n_2 \end{cases}$

Y_{jk} are independent

$H_0: \mu_1 = \mu_2 = \mu$

$H_1: \mu_1 \neq \mu_2$

(a) Use Minitab.

(b) T-test:

Group: Treatment	Group: Control
$\bar{Y}_{1\cdot} = 4.86$	$\bar{Y}_{2\cdot} = 4.726$
$S_1^2 = 0.626$	$S_2^2 = 0.746$
$n_1 = 20$	$n_2 = 0$

$N = n_1 + n_2 = 40$

$\bar{Y}_{\cdot\cdot} = 4.793$

$S^2 = 0.673$

$S_p^2 = 0.686$

$S_p = 0.82825$

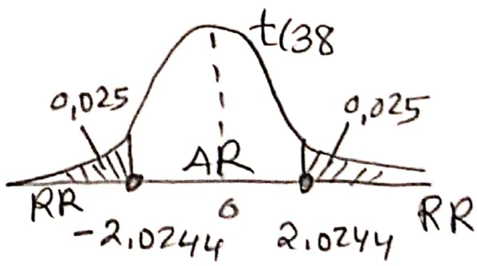
$df = n_1 + n_2 - 2 = 38$

$$t = \frac{\bar{Y}_{1\cdot} - \bar{Y}_{2\cdot}}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$$

$$= \frac{4.86 - 4.726}{0.82825\sqrt{\frac{1}{20}+\frac{1}{20}}}$$

$$= \frac{0.134}{0.2619156}$$

$$= 0.5116$$



$t_{\frac{\alpha}{2}}(n_1+n_2-2) = t_{0.025}(38) = 2.0244$

$|t| = 0.5116 < t_{0.025}(38)$

$\Rightarrow$ Do not reject $H_0: \mu_1 = \mu_2$ at significance Level $\alpha = 0.05$

c) (i) MLE : (Maximum Likelihood Estimation)

First: For Model under $H_0: \mu_1 = \mu_2 = \mu$:-

$$f(y_{jk};\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_{jk}-\mu)^2}$$

$$\ln f(y_{jk};\mu) = \ln\left[\frac{1}{\sqrt{2\pi\sigma^2}}\right] - \frac{1}{2\sigma^2}(y_{jk}-\mu)^2$$

$$l(\mu) = \sum_{j=1}^{J}\sum_{k=1}^{K} \ln f(y_{jk};\mu)$$

$$= \sum_{j}\sum_{k} \left\{ \ln\left[\frac{1}{\sqrt{2\pi\sigma^2}}\right] - \frac{1}{2\sigma^2}(y_{jk}-\mu)^2 \right\}$$

$$\frac{\partial l(\mu)}{\partial \mu} = -\frac{1}{2\sigma^2}\sum\sum 2(y_{jk}-\mu)(-2)$$

$$= \frac{1}{\sigma^2}\sum\sum(y_{jk}-\mu)$$

$$\frac{\partial l(\mu)}{\partial \mu} \overset{set}{=} 0$$

$$\Leftrightarrow \quad \frac{1}{\sigma^2}\sum\sum(y_{jk}-\mu) = 0$$

$$\Leftrightarrow \quad \sum_{j=1}^{J}\sum_{k=1}^{K}(y_{jk}-\mu) = 0$$

$\left\{ \begin{matrix} J=2 \\ K=20 \end{matrix} \right\}$

$N = JK = (2)(20) = 40$

$$\Leftrightarrow \quad \sum_{j=1}^{J}\sum_{k=1}^{K} y_{jk} - JK\mu = 0$$

$$\Leftrightarrow \quad \sum\sum y_{jk} = JK\mu$$

$$\Leftrightarrow \quad \hat{\mu} = \frac{\sum\sum y_{jk}}{JK} = \bar{Y}_{..}$$

$$\bar{Y}_{..} = \frac{\sum\sum y_{jk}}{N}$$

$\therefore$ MLE of μ (under H_0) is:

$$\hat{\mu} = \bar{Y}_{..}$$

Second: For Model under $H_1: \mu_1 = \mu_2$:-

$$f(y_{jk}; \mu_j) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_{jk}-\mu_j)^2} \quad \begin{cases} j = 1,2 \\ k = 1,\dots,20 \end{cases}$$

$$\ln(y_{jk}; \mu_j) = \ln\left[\frac{1}{\sqrt{2\pi\sigma^2}}\right] - \frac{1}{2\sigma^2}(y_{jk}-\mu_j)^2$$

$$\ell(\mu_1,\mu_2) = \sum_{j=1}^{J}\sum_{k=1}^{K} \ln(y_{jk}; \mu_j)$$

$$= \sum\sum \left\{ \ln\left[\frac{1}{\sqrt{2\pi\sigma^2}}\right] - \frac{1}{2\sigma^2}(y_{jk}-\mu_j)^2 \right\}$$

$$= \sum_{j=1}^{J}\sum_{k=1}^{K} \ln\left[\frac{1}{\sqrt{2\pi\sigma^2}}\right] - \frac{1}{2\sigma^2} \sum_{j=1}^{J}\sum_{k=1}^{K} (y_{jk}-\mu_j)^2$$

$$= \sum\sum\left[\frac{1}{\sqrt{2\pi\sigma^2}}\right] - \frac{1}{2\sigma^2}\left[\sum_{k=1}^{K}(y_{1k}-\mu_1)^2 + \sum_{k=1}^{K}(y_{2k}-\mu_2)^2\right]$$

$$\begin{cases} \dfrac{\partial \ell(\mu_1,\mu_2)}{\partial \mu_1} = \dfrac{1}{\sigma^2}\displaystyle\sum_{k=1}^{K}(y_{1k}-\mu_1) \\ \dfrac{\partial \ell(\mu_1,\mu_2)}{\partial \mu_2} = \dfrac{1}{\sigma^2}\displaystyle\sum_{k=1}^{K}(y_{2k}-\mu_2) \end{cases}$$

$$\begin{cases} \dfrac{\partial \ell(\mu_1,\mu_2)}{\partial \mu_1} \overset{\text{set}}{=} 0 \\ \dfrac{\partial \ell(\mu_1,\mu_2)}{\partial \mu_2} \overset{\text{set}}{=} 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \dfrac{1}{\sigma^2}\displaystyle\sum_{k=1}^{K}(y_{1k}-\mu_1) = 0 \\ \dfrac{1}{\sigma^2}\displaystyle\sum_{k=1}^{K}(y_{2k}-\mu_2) = 0 \end{cases} \Leftrightarrow \begin{cases} \displaystyle\sum_{k=1}^{K}(y_{1k}-\mu_1) = 0 \\ \displaystyle\sum_{k=1}^{K}(y_{2k}-\mu_2) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \sum_{k=1}^{K} y_{1k} - K\mu_1 = 0 \\ \sum_{k=1}^{K} y_{2k} - K\mu_2 = 0 \end{cases} \Leftrightarrow \begin{cases} \sum_{k=1}^{K} y_{1k} = K\mu_1 \\ \sum_{k=1}^{K} y_{2k} = K\mu_2 \end{cases}$$

$$\Leftrightarrow \begin{cases} \hat{\mu}_1 = \frac{\sum_{k=1}^{K} y_{1k}}{K} = \bar{Y}_{1.} \\ \hat{\mu}_2 = \frac{\sum_{k=1}^{K} y_{2k}}{K} = \bar{Y}_{2.} \end{cases}$$

$\therefore$ The MLE of μ_1 and μ_2 under H_1 are

$$\hat{\mu}_1 = \bar{Y}_{1.}$$

$$\hat{\mu}_2 = \bar{Y}_{2.}$$

(ii) LSE : (Least Squares Estimation)

For model under $H_0: \mu_1 = \mu_2 = \mu$:-

First: For model under H_0:

$$S_0 = \sum_{j=1}^{J}\sum_{k=1}^{K} (y_{jk} - \mu)^2$$

$$\frac{\partial S_0}{\partial \mu} = -2 \sum\sum (y_{jk} - \mu)$$

$$\frac{\partial S_0}{\partial \mu} \overset{set}{=} 0$$

$$\Leftrightarrow -2 \sum\sum (y_{jk} - \mu) = 0 \Leftrightarrow \sum_{j=1}^{J}\sum_{k=1}^{K} (y_{jk} - \mu) = 0$$

$$\Leftrightarrow \sum_{j=1}^{J}\sum_{k=1}^{K} y_{jk} - JK\mu = 0$$

$$\Leftrightarrow \mu = \frac{\sum\sum y_{jk}}{JK} = \frac{\sum\sum y_{jk}}{N} = \bar{Y}_{..} \quad (N = JK)$$

$\therefore$ The LSE of μ under H_0 is:

$$\hat{\mu} = \bar{Y}_{..} \quad \text{(same as MLE)}$$

Second: For Model under $H_1: \mu_1 \neq \mu_2$:-

$$S_1 = \sum_{j=1}^{J}\sum_{k=1}^{K}(y_{jk}-\mu_j)^2 \qquad \left(\begin{matrix} J=2 \\ K=20 \end{matrix}\right)$$

$$= \sum_{k=1}^{K}(y_{1k}-\mu_1)^2 + \sum_{k=1}^{K}(y_{2k}-\mu_2)$$

$$\begin{cases} \dfrac{\partial S_1}{\partial \mu_1} = -2\sum_{k=1}^{K}(y_{1k}-\mu_1) \\ \dfrac{\partial S_1}{\partial \mu_2} = -2\sum_{k=1}^{K}(y_{2k}-\mu_2) \end{cases}$$

$$\begin{cases} \dfrac{\partial S_1}{\partial \mu_1} \overset{set}{=} 0 \\ \dfrac{\partial S_1}{\partial \mu_2} \overset{set}{=} 0 \end{cases} \Leftrightarrow \begin{cases} -2\sum(y_{1k}-\mu_1) = 0 \\ -2\sum(y_{2k}-\mu_1) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \sum_{k=1}^{K}(y_{1k}-\mu_1) = 0 \\ \sum_{k=1}^{K}(y_{2k}-\mu_2) = 0 \end{cases} \Leftrightarrow \begin{cases} \sum_{k=1}^{K} y_{1k} - K\mu_1 = 0 \\ \sum_{k=1}^{K} y_{2k} - K\mu_2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \sum_{k=1}^{K} y_{1k} = K\mu_1 \\ \sum_{k=1}^{K} y_{2k} = K\mu_2 \end{cases} \Leftrightarrow \begin{cases} \mu_1 = \dfrac{\sum_{k=1}^{K} y_{1k}}{K} = \overline{Y}_{1.} \\ \mu_2 = \dfrac{\sum_{k=1}^{K} y_{2k}}{K} = \overline{Y}_{2.} \end{cases}$$

$\therefore$ The LSE of μ_1 and μ_2 under H_1 are:

$$\begin{cases} \hat{\mu}_1 = \overline{Y}_{1.} \\ \hat{\mu}_2 = \overline{Y}_{2.} \end{cases} \qquad \text{(same as MLE)}$$

(d) Recall:

- The LSE of μ under ($H_0: \mu_1 = \mu_2 = \mu$) is $\hat{\mu} = \bar{Y}_{..}$.
- The LSE of μ_1 and μ_2 under ($H_1: \mu_1 \neq \mu_2$) are

$$\hat{\mu}_1 = \bar{Y}_{1.} \quad \text{and} \quad \hat{\mu}_2 = \bar{Y}_{2.}$$

(i) The minimum value of $S_0 = \sum\sum (Y_{jk} - \mu)^2$ is:

$$\hat{S}_0 = \sum_{j=1}^{J}\sum_{k=1}^{K} (y_{jk} - \hat{\mu})^2$$

$$= \sum\sum (y_{jk} - \bar{Y}_{..})^2 \quad ; \quad \bar{Y}_{..} = \frac{\sum\sum y_{jk}}{JK}$$

(grand average)

(ii) The minimum value of $S_1 = \sum\sum (y_{jk} - \mu_j)^2$ is:

$$\hat{S}_1 = \sum\sum (y_{jk} - \hat{\mu}_j)^2$$

$$= \sum_{j=1}^{J}\sum_{k=1}^{K} (y_{jk} - \bar{Y}_{j.})^2 \quad ; \quad \bar{Y}_{j.} = \frac{\sum_{k=1}^{K} y_{1k}}{K}$$

$$= \sum_{k=1}^{K} (y_{1k} - \bar{Y}_{1.})^2 + \sum_{k=1}^{K} (y_{2k} - \bar{Y}_{2.})^2$$

(The sample mean of the j-th sample)

(e) (1) $S_1 = \sum_{j=1}^{J}\sum_{k=1}^{K}(y_{jk}-\mu_j)^2$ (under $H_1: \mu_1 \neq \mu_2$)

$$= \sum\sum(y_{jk}-\bar{y}_{j.}+\bar{y}_{j.}-\mu_j)^2 \qquad \left(\bar{y}_{j.} = \frac{\sum_{k=1}^{K} y_{jk}}{K}\right) \quad (j=1,2)$$

$$= \sum\sum\left[(y_{jk}-\bar{y}_{j.})+(\bar{y}_{j.}-\mu_j)\right]^2$$

$$= \sum_{j=1}^{J}\sum_{k=1}^{K}(y_{jk}-\bar{y}_{j.})^2 + \sum_{j=1}^{J}\sum_{k=1}^{K}(\bar{y}_{j.}-\mu_j)^2 + 2\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{jk}-\bar{y}_{j.})(\bar{y}_{j.}-\mu_j)$$

$$= \sum_{j=1}^{J}\sum_{k=1}^{K}(y_{jk}-\bar{y}_{j.}) + K\sum_{j=1}^{J}(\bar{y}_{j.}-\mu_j)^2 + 2\sum_{j=1}^{J}(\bar{y}_{j.}-\mu_j)\underbrace{\sum_{k=1}^{K}(y_{jk}-\bar{y}_{j.})}_{=0}$$

Note that:

$$\sum_{k=1}^{K}(y_{jk}-\bar{y}_{j.}) = 0 \quad \text{for } j=1,\dots,J$$

$$S_1 = \sum\sum(y_{jk}-\bar{y}_{j.})^2 + K\sum(\bar{y}_{j.}-\mu_j)^2$$

or

$$\sum\sum(y_{ij}-\mu_j)^2 = \sum\sum(y_{jk}-\bar{y}_{j.})^2 + K\sum(\bar{y}_{j.}-\mu_j)^2$$

$$\Leftrightarrow \underbrace{\sum_j\sum_k(y_{jk}-\bar{y}_{j.})^2}_{} = \sum_j\sum_k(y_{jk}-\mu_j)^2 - K\sum_j(\bar{y}_{j.}-\mu_j)^2$$

$$\Leftrightarrow \hat{S}_1 = \sum_j\sum_k(y_{jk}-\mu_j)^2 - K\sum_j(\bar{y}_{j.}-\mu_j)^2$$

$$\Leftrightarrow \frac{\hat{S}_1}{\sigma^2} = \frac{1}{\sigma^2}\sum\sum(y_{jk}-\mu_j)^2 - \frac{K}{\sigma^2}\sum_j(\bar{y}_{j.}-\mu_j)^2$$

(Note: LSE of μ_j is $\hat{\mu}_j = \bar{y}_{j.}$ under H_1)

$$\hat{S}_1 = \sum\sum(y_{jk}-\hat{\mu}_j)^2 = \sum\sum(y_{jk}-\bar{y}_{j.})$$

(2) $S_0 = \sum\sum (y_{jk} - \mu)^2$ (under $H_0: \mu_1 = \mu_2 = \mu$)

$$= \sum\sum (y_{jk} - \bar{y}_{..} + \bar{y}_{..} - \mu)^2$$

$$= \sum\sum \left[(y_{jk} - \bar{y}_{..}) + (\bar{y}_{..} - \mu) \right]^2 \qquad \left(\bar{y}_{..} = \frac{\sum\sum y_{jk}}{JK} \right)$$

$$= \sum_{j=1}^{J}\sum_{k=1}^{K} (y_{jk} - \bar{y}_{..})^2 + \sum_{j=1}^{J}\sum_{k=1}^{K} (\bar{y}_{..} - \mu)^2 + 2 \sum_{j=1}^{J}\sum_{k=1}^{K} (y_{jk} - \bar{y}_{..})(\bar{y}_{..} - \mu)$$

$$= \sum\sum (y_{jk} - \bar{y}_{..})^2 + JK(\bar{y}_{..} - \mu)^2 + 2(\bar{y}_{..} - \mu) \underbrace{\sum\sum (y_{jk} - \bar{y}_{..})}_{=0}$$

$$S_0 = \sum\sum (y_{jk} - \bar{y}_{..})^2 + JK(\bar{y}_{..} - \mu)^2$$

or $\sum\sum (y_{jk} - \mu)^2 = \sum\sum (y_{jk} - \bar{y}_{..})^2 + JK(\bar{y}_{..} - \mu)^2$

$$\Leftrightarrow \underbrace{\sum\sum (y_{jk} - \bar{y}_{j.})^2}_{\hat{S}_0} = \sum\sum (y_{jk} - \mu)^2 - JK(\bar{y}_{..} - \mu)^2$$

$$\hat{S}_0 = \sum\sum (y_{jk} - \mu)^2 - JK(\bar{y}_{..} - \mu)^2$$

$$\Leftrightarrow \frac{\hat{S}_0}{\sigma^2} = \frac{1}{\sigma^2} \sum\sum (y_{jk} - \mu)^2 - \frac{JK}{\sigma^2} (\bar{y}_{..} - \mu)^2$$

(Note: The LSE of μ is $\hat{\mu} = \bar{y}_{..}$ under H_0)

$$\hat{S}_0 = \sum\sum (y_{jk} - \hat{\mu})^2 = \sum\sum (y_{jk} - \bar{y}_{..})^2$$

- Distribution of $\frac{\hat{S}_1^2}{\sigma^2}$ (under H_1):—

We found that:

$$\frac{1}{\sigma^2}\hat{S}_1 = \frac{1}{\sigma^2}\sum\sum(Y_{jk}-\mu_j)^2 - \frac{K}{\sigma^2}\sum(\bar{Y}_{j\cdot}-\mu_j)^2$$
$$= \sum_{j=1}^{J}\sum_{k=1}^{K}\left(\frac{Y_{jk}-\mu_j}{\sigma}\right)^2 - \sum_{j=1}^{J}\left(\frac{\bar{Y}_{j\cdot}-\mu_j}{\sigma/\sqrt{K}}\right)^2$$

Recall:
If $Y_1, Y_2, \dots, Y_n$ iid $N(\mu,\sigma^2)$ then $\bar{Y} \sim N(\mu, \frac{\sigma^2}{n})$

Since Y_{jk} are independent and $Y_{jk} \sim N(\mu_j, \sigma^2)$,

(1) We have $\bar{Y}_{j\cdot} \sim N(\mu_j, \frac{\sigma^2}{K})$ $(j=1,2)$

Therefore: $\frac{\bar{Y}_{j\cdot}-\mu_j}{\sigma/\sqrt{K}} \sim N(0,1)$

and $\left(\frac{\bar{Y}_{j\cdot}-\mu_j}{\sigma/\sqrt{K}}\right)^2 = \frac{K}{\sigma^2}(\bar{Y}_{j\cdot}-\mu_j)^2 \sim \chi^2_{(1)}$

(2) $\frac{Y_{jk}-\mu_j}{\sigma} \sim N(0,1)$

Therefore $\left(\frac{Y_{jk}-\mu_j}{\sigma}\right)^2 \sim \chi^2_{(1)}$

From (1): $\sum_{j=1}^{2}\left(\frac{\bar{Y}_{j\cdot}-\mu_j}{\sigma/\sqrt{K}}\right)^2 \sim \chi^2_{(2)}$

From (2) $\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\frac{Y_{jk}-\mu_j}{\sigma}\right)^2 \sim \chi^2_{(JK)}$

Therefore:

$$\frac{1}{\sigma^2}\hat{S}_1 = \chi^2_{(JK)} - \chi^2_{(2)}$$

$$\Leftrightarrow \chi^2_{(JK)} = \frac{\hat{S}_1}{\sigma^2} + \chi^2_{(2)}$$

$$\Rightarrow \hat{S}_1/\sigma^2 \sim \chi^2(JK-2)$$

- Distribution of $\frac{\hat{S}_0}{\sigma^2}$ under H_0 :-

We have found that

$$\frac{\hat{S}_0}{\sigma^2} = \frac{1}{\sigma^2}\sum\sum(y_{jk}-\mu)^2 - \frac{JK}{\sigma^2}(\bar{y}_{..}-\mu)^2$$

$$= \underbrace{\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\frac{y_{jk}-\mu}{\sigma}\right)^2}_{} - \underbrace{\left(\frac{\bar{y}-\mu}{\sigma/\sqrt{JK}}\right)^2}_{}$$

$$= \chi^2(JK) - \chi^2(1)$$

$$\Leftrightarrow \quad \chi^2(JK) = \frac{\hat{S}_0}{\sigma^2} + \chi^2(1)$$

$$\Rightarrow \quad \frac{\hat{S}_0}{\sigma^2} \sim \chi^2(JK-1)$$

(f) We have found that

$$\frac{\hat{S}_0}{\sigma^2} \sim \chi^2(JK-1) \qquad (\text{under } H_0)$$

$$\frac{\hat{S}_1}{\sigma^2} \sim \chi^2(JK-2) \qquad (\text{under } H_1) \text{ --- } (*)$$

Now, If H_0 is True,

$$\frac{\hat{S}_0-\hat{S}_1}{\sigma^2} \sim \chi^2(1) \text{ --- } (**) \qquad (JK-1)-(JK-2) = 1$$

From (*) and (**) :

$$F = \frac{\frac{\hat{S}_0-\hat{S}_1}{\sigma^2}/(1)}{\frac{\hat{S}_1}{\sigma^2}/(JK-2)} = \frac{\hat{S}_0-\hat{S}_1}{\hat{S}_1/(JK-2)} \sim F(1, JK-2)$$

(g) $H_0: \mu_1 = \mu_2 = \mu$

$H_1: \mu_1 \neq \mu_2$

$$\left\{ \begin{array}{l} \sum\sum y_{jk} = 191.73 \\ \sum\sum y_{jk}^2 = 945.241 \\ \bar{y}_{..} = 4.793 \\ N = JK = (2)(20) = 40 \end{array} \right.$$

$$\hat{S}_0 = \sum\sum(y_{jk} - \hat{\mu})^2 = \sum\sum(y_{jk} - \bar{y}_{..})$$
$$= \sum\sum y_{jk}^2 - Jk(\bar{y}_{..})^2$$
$$= 945.241 - (40)(4.793)^2$$
$$= 26.32704$$

$$\hat{S}_1 = \sum_{j=1}^{2}\sum_{k=1}^{20}(y_{jk} - \bar{y}_{j.})^2 = \sum_{k=1}^{20}(y_{1k} - \bar{y}_{1.})^2 + \sum_{k=1}^{20}(y_{2k} - \bar{y}_{2.})^2$$
$$= \left[\sum_{k=1}^{20} y_{1k}^2 - K(\bar{y}_{1.})^2\right] + \left[\sum_{k=1}^{20} y_{2k}^2 - K(\bar{y}_{2.})^2\right]$$
$$= \left[484.278 - (20)(4.86)^2\right] + \left[460.964 - (20)(4.726)^2\right]$$
$$= 26.14848$$

$$\left\{ \begin{array}{l} \sum_{k=1}^{20} y_{1k}^2 = 484.278 \\ \bar{y}_{1.} = 4.86 \end{array} \right.$$

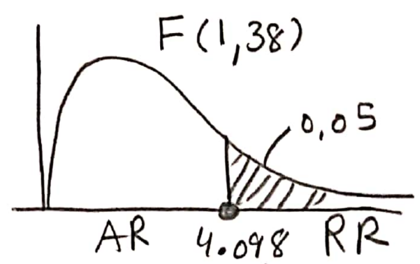
$$\left\{ \begin{array}{l} \sum_{k=1}^{20} y_{2k}^2 = 460.964 \\ \bar{y}_{2.} = 4.726 \end{array} \right.$$

$$F_0 = \frac{\hat{S}_0 - \hat{S}_1}{\hat{S}_1 / (JK-2)} = \frac{26.32704 - 26.14848}{26.14848 / 38}$$
$$= 0.2595$$

$F_0 = 0.2595 < F_{0.05}(1,38)$

$F_0 \in AR$

$\therefore$ We do not reject H_0.



$F_{0.05}(1,38) = 4.098$

(h) $F_0 = 0.2595 \quad (\sim F(1,38))$

$t_0 = 0.5116 \quad (\sim t(38))$

$t_0^2 = (0.5116)^2 = 0.2617$

$t_0^2 = F_0$ (The difference is because of rounding errors)

(i) Calculating the residuals : $e_{jk} = Y_{jk} - \bar{Y}_{..}$
[Minitab]. (For Model H_0)

Ex 2.2 p.42 (Dobson 2008)

Before	After
$n_1 = 20$	$n_2 = 20$
$\bar{Y}_1 = 103.25$	$\bar{Y}_2 = 100.60$
$S_1^2 = 182.72$	$S_2^2 = 155.61$

(a) $H_0: \mu_1 = \mu_2$

$H_1: \mu_1 \neq \mu_2$

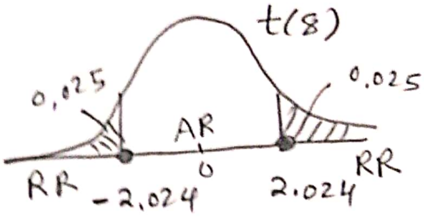
Unpaired t-test:

$$t_0 = \frac{\bar{Y}_1 - \bar{Y}_2}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{103.25 - 100.60}{13.006\sqrt{\frac{1}{20}+\frac{1}{20}}} = 0.644$$

$$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2} = 169.165$$

$S_p = 13.006$

$df = n_1 + n_2 - 2 = 18$



$|t_0| < t_{0.025}(18)$

$t_0 \in AR$

$t_{0.025}(38) = 2.024$

$\therefore$ We do not reject $H_0: \mu_1 = \mu_2$ at significance level $\alpha = 0.05$.

(b) $\mu_D = \mu_1 - \mu_2$; $E(D) = E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = \mu_1 - \mu_2$

$H_0: \mu_D = 0$

$H_1: \mu_D \neq 0$

Paired t-test

Man	Before Y_{1k}	After Y_{2k}	Difference $D_k = Y_{1k} - Y_{2k}$
1	1008.8	97.0	38.0
2	102.0	107.5	-5.5
⋮	⋮	⋮	⋮
20	93.0	93.0	0.0

$$t_0 = \frac{\bar{D}}{S_D/\sqrt{n}} = \frac{\bar{D}}{S_{\bar{D}}} = \frac{2.645}{0.9206} = 2.873$$

$\bar{D} = 2.645$

$S_D^2 = 16.947$

$S_D = 4.117$

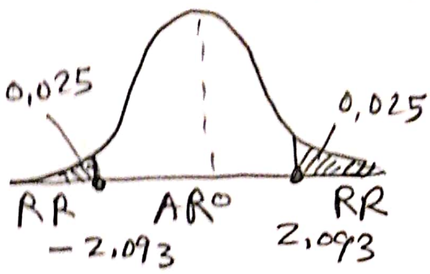
$$S_{\bar{D}} = \frac{S_D}{\sqrt{n}} = \frac{4.117}{\sqrt{20}} = 0.9206$$

$t(n-1) = t(19)$

$|t_0| > t_{0.025}(19)$

$t_0 \in RR$

$\therefore$ we reject $H_0: \mu_1 = \mu_2$
at significance level $\alpha = 0.05$



$t_{0.025}(19) = 2.093$

(c) • The conclusions are:

(1) For t-test : $\begin{cases} H_0: \mu_1 = \mu_2 \text{ is not rejected} \\ \text{at } \alpha = 0.05 \end{cases}$

(2) For Paired t-test : $\begin{cases} H_0: \mu_1 = \mu_2 \text{ is rejected} \\ \text{at } \alpha = 0.05 \end{cases}$

(d) • We reached different conclusions.

• The Paired t-test is more appropriate than t-test.

• Assumptions:

① For t-test : $\begin{cases} 1.\ Y_{jk} \sim N(\mu_j, \sigma_j^2) \\ 2.\ Y_{jk} \text{ are independent} \\ 3.\ \sigma_1^2 = \sigma_2^2 \end{cases}$

② For paired t-test $\begin{cases} 1.\ Y_{jk} \sim N(\mu_j, \sigma_j^2) \\ 2.\ Y_{11}, \dots, Y_{1k} \text{ are independent} \\ 3.\ Y_{21}, \dots, Y_{2k} \text{ are independent} \end{cases}$

Y_{1k} and Y_{2k} are not independent

Ex 2.4 p.43 (Dobson 2008)

$$E(Y) = \ln(\beta_0 + \beta_1 x + \beta_2 x^2)$$

$$e^{E(Y)} = \beta_0 + \beta_1 x + \beta_2 x^2$$

$$g[E(Y)] = \beta_0 + \beta_1 x + \beta_2 x^2 \qquad (g(x) = e^x)$$

$$\underset{\sim}{Y} = \begin{bmatrix} 3.15 \\ 4.85 \\ 6.50 \\ 7.20 \\ 8.25 \\ 16.50 \end{bmatrix} \qquad \underline{X} = \begin{array}{c} \begin{array}{ccc} \beta_0 & \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} 1 & 1.0 & 1.0 \\ 1 & 1.2 & 1.44 \\ 1 & 1.4 & 1.96 \\ 1 & 1.6 & 2.56 \\ 1 & 1.8 & 3.24 \\ 1 & 2.0 & 4.00 \end{bmatrix} \\ \begin{array}{ccc} & \uparrow & \uparrow \\ & x & x^2 \end{array} \end{array} , \quad \underset{\sim}{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$$

Ex 2.5:

$$E(Y_{jk}) = \mu_{jk} = \mu + \alpha_j + \beta_k \quad ; \begin{cases} Y_{jk} \sim N(\mu_{jk}, \sigma^2) \\ \text{independent} \end{cases} \begin{cases} j = 1, 2 \\ k = 1, 2, 3 \end{cases}$$

constraints:

$$\sum \alpha_j = 0 \iff \alpha_2 = -\alpha_1$$

$$\sum \beta_k = 0 \iff \beta_3 = -\beta_1 - \beta_2$$

$$\underset{\sim}{Y} = \begin{bmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{21} \\ Y_{22} \\ Y_{33} \end{bmatrix} ; \quad \underline{X} = \begin{array}{c} \begin{array}{cccc} \mu & \alpha_1 & \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix} \end{array} , \quad \underset{\sim}{\beta} = \begin{bmatrix} \mu \\ \alpha_1 \\ \beta_1 \\ \beta_2 \end{bmatrix}$$