Chapter 1

Ex. 1.1:

$$y_1 \sim N(1{,}3) \text{ and } y_2 \sim N(2{,}5) \text{ independent}$$
$$w_1 = y_1 + 2y_2$$
$$w_2 = 4y_1 - y_2$$

$$\Rightarrow E(w_1) = E(y_1 + 2y_2) = E(y_1) + 2E(y_2) = 1 + (2)(2) = 5$$
$$E(w_2) = E(4y_1 - y_2) = 4E(y_1) - E(y_2) = (4)(1) - 2 = 2$$
$$\text{Var}(w_1) = \text{Var}(y_1 + 2y_2) = \text{Var}(y_1) + 4\text{Var}(y_2) = 3 + (4)(5) = 23$$
$$\text{Var}(w_2) = \text{Var}(4y_1 - y_2) = 16\text{Var}(y_1) + \text{Var}(y_2) = (16)(3) + 5 = 53$$
$$\text{Cov}(w_1, w_2) = \text{Cov}(y_1 + 2y_2, 4y_1 - y_2) = \text{Cov}(y_1, 4y_1) + \text{Cov}(y_1, -y_2) + \text{Cov}(2y_2, 4y_1) + \text{Cov}(2y_2, -y_2)$$
$$= 4Cov(y_1, y_1) - Cov(y_1, y_2) + (2)(4)Cov(y_2, y_1) - (2)Cov(y_2, y_2)$$
$$= 4\text{Var}(y_1) - 0 + 0 - 2\text{Var}(y_2) = (4)(3) - (2)(5) = 12 - 10 = 2$$

$$\underset{\sim}{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$
$$E(\underset{\sim}{w}) = \begin{bmatrix} E(w_1) \\ E(w_2) \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \underset{\sim}{\mu}^*$$
$$\text{Cov}(\underset{\sim}{w}) = \begin{bmatrix} \text{Var}(w_1) & \text{Cov}(w_1, w_2) \\ \text{Cov}(w_1, w_2) & \text{Var}(w_2) \end{bmatrix} = \begin{bmatrix} 23 & 2 \\ 2 & 53 \end{bmatrix} = \underline{v}^*$$
$$\underset{\sim}{w} \sim \text{MVN}\left(\underset{\sim}{\mu}^*, \underline{v}^*\right)$$

<u>Another Solution:</u>

$$\underset{\sim}{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} y_1 + 2y_2 \\ 4y_1 - y_2 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underline{A}\, \underset{\sim}{y}$$
$$E(\underset{\sim}{w}) = E(\underline{A}\, \underset{\sim}{y}) = \underline{A} E(\underset{\sim}{y}) = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \underset{\sim}{\mu}^*$$

$$\text{Cov}(\underset{\sim}{w}) = \text{Cov}\left(\underline{A}\, \underset{\sim}{y}\right) = \underline{A}\, \text{Cov}\left(\underset{\sim}{y}\right) \underline{A}'$$
$$\text{Cov}\left(\underset{\sim}{y}\right) = \begin{bmatrix} \text{Var}(y_1) & \text{Cov}(y_1, y_2) \\ \text{Cov}(y_1, y_2) & \text{Var}(y_1) \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$

Note: y_1 and y_2 are independent $\Rightarrow \text{Cov}(Y_1, Y_2) = 0$

$$\therefore \text{Cov}(\underset{\sim}{w}) = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ 12 & -5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 23 & 2 \\ 2 & 53 \end{bmatrix} = \underline{v}^*$$

$$\underset{\sim}{y} \sim \text{MVN}\left(\underset{\sim}{\mu} \quad , \underline{v} \quad\right), E\left(\underset{\sim}{y}\right) = \underset{\sim}{\mu} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{Var}\left(\underset{\sim}{y}\right) = \underline{v} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$
$$\therefore \underset{\sim}{w} \sim \text{MVN}\left(\underset{\sim}{\mu}^*, \underline{v}^*\right)$$

<u>Another Solution:</u>

$$w_1 = y_1 + 2y_2 = (1 \quad 2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underset{\sim}{a}_1' \underset{\sim}{y}$$
$$w_2 = 4y_1 - y_2 = (4 - 1) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underset{\sim}{a}_2' \underset{\sim}{y}$$

$$\Rightarrow E\left(\underset{\sim}{y}\right) = \underset{\sim}{\mu} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{Var}\left(\underset{\sim}{y}\right) = \underline{v} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$
$$E(w_1) = E\left(\underset{\sim}{a}_1' \underset{\sim}{y}\right) = \underset{\sim}{a}_1' E\left(\underset{\sim}{y}\right) = \underset{\sim}{a}_1' \underset{\sim}{\mu} = (1 \quad 2) \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 + 4 = 5$$
$$E(w_2) = E\left(\underset{\sim}{a}_2' \underset{\sim}{y}\right) = \underset{\sim}{a}_2' E\left(\underset{\sim}{y}\right) = \underset{\sim}{a}_1' \underset{\sim}{\mu} = (4 \quad -1) \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 4 - 2 = 2$$

$$\therefore \underset{\sim}{\mu}^* = E(\underset{\sim}{w}) = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \text{ where } \underset{\sim}{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\mathrm{Var}(w_1) = \mathrm{Var}\left(\underset{\sim}{a}_1' \, \underset{\sim}{y}\right) = \underset{\sim}{a}_1' \, \mathrm{Cov}\left(\underset{\sim}{y}\right) \underset{\sim}{a}_1 = (1 \quad 2)\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 23$$

$$\mathrm{Var}(w_2) = \mathrm{Var}\left(\underset{\sim}{a}_2' \, \underset{\sim}{y}\right) = \underset{\sim}{a}_2' \, \mathrm{Cov}\left(\underset{\sim}{y}\right) \underset{\sim}{a}_2 = (4 \quad -1)\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix} = 53$$

$$Cov(w_1, w_2) = Cov\left(\underset{\sim}{a}_1' \, \underset{\sim}{y}, \underset{\sim}{a}_2' \, \underset{\sim}{y}\right) = \underset{\sim}{a}_1' \, \mathrm{Cov}\left(\underset{\sim}{y}\right) \underset{\sim}{a}_2 = (1 \quad 2)\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix} = 2$$

$$\therefore \underline{v}^* = \mathrm{Cov}(\underset{\sim}{w}) = \begin{pmatrix} 23 & 2 \\ 2 & 53 \end{pmatrix}$$

$$\therefore \underset{\sim}{w} \sim \mathrm{MVN}\left(\underset{\sim}{\mu}^*, \underline{v}^*\right)$$

Ex 1.2:

$$\left.\begin{matrix} Y_1 \sim N(0,1) \\ y_2 \sim N(3,4) \end{matrix}\right\} \text{ independent } \Rightarrow \mathrm{Cov}(y_1, y_2) = 0$$

$$\underset{\sim}{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \underset{\sim}{\mu} = E\left(\underset{\sim}{y}\right) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \underline{v} = \mathrm{Cov}\left(\underset{\sim}{y}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\underset{\sim}{y} \sim \mathrm{MVN}\left(\underset{\sim}{\mu}, \underline{v}\right)$$

(a) $y_1 \sim N(0,1) \Rightarrow y_1^2 \sim \chi_1^2$

(b) Let $\underset{\sim}{w} = \begin{bmatrix} y_1 \\ \frac{y_2 - 3}{2} \end{bmatrix}$

Note: $y_2 \sim N(3,4) \Leftrightarrow \frac{y_2-3}{2} \sim N(0,1) \Leftrightarrow \left(\frac{y_2-3}{2}\right)^2 \sim \chi_1^2$

$$\underset{\sim}{w}' \, \underset{\sim}{w} = \left(y_1, \frac{y_2 - 3}{2}\right)\begin{pmatrix} y_1 \\ \frac{y_2 - 3}{2} \end{pmatrix} = y_1^2 + \left(\frac{y_2 - 3}{2}\right)^2 = \chi_1^2 + \chi_1^2$$

and as y_1^2 and $\left(\frac{y_2-3}{2}\right)^2$ are independent, so,

$$\therefore \underset{\sim}{w}' \, \underset{\sim}{w} \sim \chi_2^2$$

(c)

$$\underset{\sim}{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} ; \ \underset{\sim}{\mu} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}; \ \underline{v} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}; \underline{v}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$x = \underset{\sim}{y}' \, \underline{v}^{-1} \, \underset{\sim}{y} = (y_1, y_2)\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left(y_1, \frac{y_2}{4}\right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = y_1^2 + \frac{1}{4}y_2^2$$

$$x = \underset{\sim}{y}' \, \underline{v}^{-1} \, \underset{\sim}{y} \sim X^2(2, \lambda) ; \ \lambda = \underset{\sim}{\mu}' \, \underline{v}^{-1} \, \underset{\sim}{\mu} = (0,3)\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \left(0, \frac{3}{4}\right)\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \frac{9}{4} = 2.25$$

Ex 1.3:

$$\underset{\sim}{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} ; \ \underset{\sim}{y} \sim \mathrm{MVN}\left(\underset{\sim}{\mu}, \underline{v}\right); \underset{\sim}{\mu} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}; \underline{v} = \begin{bmatrix} 4 & 1 \\ 1 & 9 \end{bmatrix}; \underline{v}^{-1} = \frac{1}{4(9) - 1^2}\begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix}$$

(a)

$$\left(\underset{\sim}{y} - \underset{\sim}{\mu}\right)' \underline{v}^{-1}\left(\underset{\sim}{y} - \underset{\sim}{\mu}\right) \sim ?$$

$$\left(\underset{\sim}{y} - \underset{\sim}{\mu}\right)' = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{pmatrix} y_1 - 2 \\ y_2 - 3 \end{pmatrix}$$

$$\Rightarrow (\underset{\sim}{y}-\underset{\sim}{\mu})'\underline{v}^{-1}(\underset{\sim}{y}-\underset{\sim}{\mu}) = \begin{pmatrix} y_1-2 \\ y_2-3 \end{pmatrix}' \frac{1}{35}\begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix}\begin{pmatrix} y_1-2 \\ y_2-3 \end{pmatrix} = \frac{1}{35}[9(y_1-2)^2 - 2(y_1-2)(y_2-3) + 4(y_2-3)^2]$$

Note that:

$$\underset{\sim}{x}'\,\underline{A}\quad \underset{\sim}{x} = (x_1, x_2)\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}x_ix_j$$

and as $\underline{A}' \; = \underline{A}$ symmetric ($a_{ij} = a_{ji}$),

$$\Rightarrow \underset{\sim}{x}'\,\underline{A}\quad \underset{\sim}{x} = \sum_{i=1}^{2} a_{ii}x_i^2 + a_{12}x_1x_2 + a_{21}x_2x_1 = \sum_{i=1}^{2} a_{i1}x_i^2 + 2a_{12}x_1x_2 = a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2$$

$$\therefore (\underset{\sim}{y}-\underset{\sim}{\mu})'\underline{v}^{-1}(\underset{\sim}{y}-\underset{\sim}{\mu}) \sim \chi_2^2$$

(b)

$$\underset{\sim}{y}'\,\underline{v}^{-1}\,\underset{\sim}{y} = (y_1, y_2)\frac{1}{35}\begin{pmatrix} 9 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{35}[9y_1^2 + 4y_2^2 - 2y_1y_2]$$

$$\Rightarrow \underset{\sim}{y}'\,\underline{v}^{-1}\,\underset{\sim}{y} \sim X^2(2, \lambda)$$

where

$$\lambda = \underset{\sim}{\mu}'\,\underline{v}^{-1}\,\underset{\sim}{\mu} = (2,3)\frac{1}{35}\begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \frac{1}{35}[(9)(2)^2 + (4)(3)^2 - (2)(2)(3)] = \frac{37}{35}$$

Ex 1.4:

$y_1, y_2, \ldots, y_n$ are imdependent and $y_i \sim N(\mu_1\sigma^2)$.

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i\ ;\ S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2;\ \underset{\sim}{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix};\ \underset{\sim}{\mu} = E(\underset{\sim}{y}) = \begin{bmatrix} \mu \\ \vdots \\ \mu \end{bmatrix};\ \mathrm{Cov}(\underset{\sim}{y}) = \begin{bmatrix} \sigma^2 & 0 \\ \vdots & 0 \\ 0^2 & \sigma^2 \end{bmatrix} = \sigma^2 I$$

(a)

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$\Rightarrow E(\bar{y}) = \frac{1}{n}\sum_{i=1}^{n} E(y_i) = \frac{1}{n}\sum_{i=1}^{n} \mu = \frac{1}{n}(n\mu) = \mu$$

$$\Rightarrow \mathrm{Var}(\bar{y}) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) = \frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^{n} y_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(y_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{n}{n^2}\sigma^2 = \frac{\sigma^2}{n}$$

$$\therefore \bar{y} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

<u>Another Solution:</u>

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}(1, \ldots, 1)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \frac{1}{n}\underset{\sim}{1}'\,\underset{\sim}{y}\ ;\ \underset{\sim}{1} \quad = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\Rightarrow E(\bar{y}) = E\left(\frac{1}{n}\underset{\sim}{1}'\,\underset{\sim}{y}\right) = \frac{1}{n}\underset{\sim}{1}'\,E(\underset{\sim}{y}) = \frac{1}{n}\underset{\sim}{1}'\,\underset{\sim}{\mu} = \frac{1}{n}(1, \ldots, 1)\begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix} = \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{n}{n}\mu = \mu$$

$$\Rightarrow \mathrm{Var}(\bar{y}) = \mathrm{Var}\left(\frac{1}{n}\underset{\sim}{1}'\,\underset{\sim}{y}\right) = \frac{1}{n^2}\mathrm{Var}(\underset{\sim}{1}'\,\underset{\sim}{y}) = \frac{1}{n^2}\underset{\sim}{1}'\,\mathrm{Cov}(\underset{\sim}{y})\,\underset{\sim}{1} \quad = \frac{1}{n^2}\underset{\sim}{1}'\,(\sigma^2 I)\,\underset{\sim}{1} \quad = \frac{\sigma^2}{n^2}\underset{\sim}{1}'\,\underset{\sim}{1} \quad = \frac{\sigma^2}{n^2}(1, \ldots, 1)\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$= \frac{\sigma^2}{n^2}(n) = \frac{\sigma^2}{n}$$

(b)

$s^2 = \frac{1}{n-1}\sum_{i=1}^n (y_i - \bar{y})^2 = \frac{1}{n-1}\sum_{i=1}^n (y_i - \mu + \mu - \bar{y})^2 = \frac{1}{n-1}\sum_{i=1}^n [(y_i - \mu) - (\bar{y} - \mu)]^2$
$= \frac{1}{n-1}\sum_{i=1}^n [(y_i - \mu)^2 + (\bar{y} - \mu)^2 - 2(y_i - \mu)(\bar{y} - \mu)]$
$= \frac{1}{n-1}[\sum_{i=1}^n (y_i - \mu)^2 + \sum_{i=1}^n (\bar{y} - \mu)^2 - 2\sum_{i=1}^n (y_i - \mu)(\bar{y} - \mu)]$
$= \frac{1}{n-1}[\sum_{i=1}^n (y_i - \mu)^2 + n(\bar{y} - \mu)^2 - 2(\bar{y} - \mu)\sum_{i=1}^n (y_i - \mu)]$
$= \frac{1}{n-1}\{\sum_{i=1}^n (y_i - \mu)^2 + n(\bar{y} - \mu)^2 - 2(\bar{y} - \mu)[\sum_{i=1}^n y_i - \sum_{i=1}^n \mu]\}$
$= \frac{1}{n-1}\{\sum_{i=1}^n (y_i - \mu)^2 + n(\bar{y} - \mu)^2 - 2(\bar{y} - \mu)[n\bar{y} - n\mu]\}$
$= \frac{1}{n-1}[\sum_{i=1}^n (y_i - \mu)^2 + n(\bar{y} - \mu) - 2(\bar{y} - \mu)n(\bar{y} - \mu)]$
$= \frac{1}{n-1}[\sum_{i=1}^n (y_i - \mu)^2 - n(\bar{y} - \mu)^2]$

(c)

$$S^2 = \frac{1}{n-1}\left[\sum_{i=1}^n (y_i - \mu)^2 - n(\bar{y} - \mu)^2\right]$$

$$\Rightarrow (n-1)S^2 = \sum_{i=1}^n (y_i - \mu)^2 - n(\bar{y} - \mu)^2$$

$$\Rightarrow \sum_{i=1}^n (y_i - \mu)^2 = (n-1)S^2 + n(\bar{y} - \mu)^2$$

$$\Rightarrow \sum_{i=1}^n \frac{(y_i - \mu)^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2} + \frac{n(\bar{y} - \mu)^2}{\sigma^2}$$

$$\Rightarrow \sum_{i=1}^n \left(\frac{y_i - \mu}{\sigma}\right)^2 = \frac{(n-1)S^2}{\sigma^2} + \left(\frac{\bar{y} - \mu}{\sigma/\sqrt{n}}\right)^2$$

Note:

1. $Y_i \sim N(\mu, \sigma^2) \Leftrightarrow \frac{y_i - \mu}{\sigma} \sim N(0,1) \Leftrightarrow \left(\frac{Y_i - \mu}{\sigma}\right)^2 \sim \chi^2_{(1)}, (i = 1,2,\cdots,n)$
   $y_1, y_2, \dots, y_n$ are independent $\Rightarrow \left(\frac{Y_1 - \mu}{\sigma}\right)^2, \dots, \left(\frac{y_n - \mu}{\sigma}\right)^2$ are independent
   $$\therefore \sum_{i=1}^n \left(\frac{y_i - \mu}{\sigma}\right)^2 \sim \chi^2(n)$$
2. $\bar{y} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \Leftrightarrow \frac{\bar{y} - \mu}{\sigma/\sqrt{n}} \sim N(0,1) \Leftrightarrow \left(\frac{\bar{y} - \mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2_{(1)}$

Now;

$$\sum_{i=1}^n \left(\frac{y_i - \mu}{\sigma}\right)^2 = \frac{(n-1)S^2}{\sigma^2} + \left(\frac{\bar{y} - \mu}{\sigma\sqrt{n_n}}\right)^2$$

$$X^2_{(n)} = X^2_{(n-1)} + \chi^2_{(1)} \text{ (Cochran's Theorem)}$$

Therefore, $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$ and $\frac{(n-1)s^2}{\sigma^2}$ and $\left(\frac{\bar{y} - \mu}{\sigma/\sqrt{n}}\right)^2$ are independent $\Rightarrow S^2$ and $\bar{y}$ are independent

Note: This can also show by Cochran's Theorem.

(d) From (c), we have shown that

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$$

This can also be shown by Cocharn's Theorem.

(e) $\frac{\bar{y}-\mu}{s/\sqrt{n}} \sim ?$

Note:

1. $\frac{\bar{y}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$
2. $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$
3. $\bar{Y}$ and S^2 are indepondat
4. If $Z \sim N(0,1), X \sim \chi^2(m)$, and Z and X are independent, then

$$t = \frac{Z}{\sqrt{x/m}} \sim t(m)$$

Now,

$$t = \frac{\dfrac{\bar{y}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\dfrac{(n-1)s^2}{\sigma^2}/(n-1)}} = \frac{\dfrac{\sqrt{n}(\bar{y}-\mu)}{\sigma}}{\sqrt{\dfrac{s^2}{\sigma^2}}} = \frac{\dfrac{\sqrt{n}(\bar{y}-\mu)}{\sigma}}{\dfrac{s}{\sigma}} = \frac{\sqrt{n}(\bar{y}-\mu)}{s} = \frac{\bar{y}-\mu}{s/\sqrt{n}} \sim t(n-1)$$

Ex 1.5: (See Example in Section 1.6.2)

$y_1, y_2, \dots, y_n$ are independent, and $y_i \sim$ Poisson$(\theta) \Leftrightarrow y_1, y_2, \dots, y_n$ iid Poisson(θ) (random sample)

(a) Suppose that $y \sim$ Poissin (θ) The pmf is

$$f_y(y;\theta) = \frac{e^{-\theta}\theta^y}{y!}; y = 0,1,2,\cdots$$

$$E(y) = \sum_{y=0}^{\infty} y f_y(y;\theta) = \sum_{y=0}^{\infty} y \frac{e^{-\theta}\theta^y}{y!} = e^{-\theta}\sum_{y=0}^{\infty} y\frac{\theta^y}{y!} = e^{-\theta}\sum_{y=1}^{\infty}\frac{y\theta^y}{y!} = e^{-\theta}\sum_{y=1}^{\infty}\frac{y\theta^y}{y(y-1)!} = e^{-\theta}\sum_{y=1}^{\infty}\frac{\theta^y}{(y-1)!}$$

$$= e^{-\theta}\theta\sum_{y=1}^{\infty}\frac{\theta^{y-1}}{(y-1)!} = \theta e^{-\theta}\sum_{k=0}^{\infty}\frac{\theta^k}{k}\ ; (k = y-1)$$

Result:

$$\sum_{k=0}^{\infty}\frac{x^k}{k!} = e^x \text{ [Mclaurin Expansion of } f(x) = e^x]$$

$$\therefore E(y) = \theta e^{-\theta}\sum_{k=0}^{\infty}\frac{\theta^k}{k} = \theta e^{-\theta}e^{\theta} = \theta e^{\theta-\theta} = \theta e^0 = \theta.$$

(b) $\theta = e^\beta \Leftrightarrow \beta = \ln(\theta)$

We need to find the maximum likelihood estimator (MLE) of β.

<u>First</u>: MLE of θ : (See Example in Section 1.62)

$$f(y_i,\theta) = \frac{e^{-\theta}\theta^{y_i}}{y_i!}\ y_i = 0,1,2,\dots$$

$$\ln f(y_i,\theta) = \ln\left(e^{-\theta}\right) + \ln(\theta^{y_i}) - \ln(y_i!) = -\theta + y_i\ln(\theta) - \ln(y_i!)$$

The log-likelihood function is:

$$l(\theta; \underset{\sim}{y}) = \sum_{i=1}^{n} \ln\ f(y_i; \theta) = \sum_{i=1}^{n} [-\theta + y_i \ln(\theta) - \ln(y_i!)] = -\theta n + \ln(\theta) \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \ln(y_i!)$$

$$\Rightarrow \frac{\partial l(\theta; \underset{\sim}{y})}{\partial \theta} = -n + \frac{1}{\theta} \sum_{i=1}^{n} y_i$$

$$\frac{\partial l(\theta; \underset{\sim}{y})}{\partial \theta} \overset{\text{set}}{=} 0 \Leftrightarrow 0 = -n + \frac{1}{\theta} \sum_{i=1}^{n} y_i \Leftrightarrow \frac{1}{\theta} \sum_{i=1}^{n} y_i = n \Leftrightarrow \hat{\theta} = \frac{\sum_{i=1}^{n} y_i}{n} = \bar{y}$$

We need to make sure that $\bar{y}$ maximize $l(\theta; \underset{\sim}{y})$ not minimizing,

$$\left.\frac{\partial^2 l(\theta; \underset{\sim}{y})}{\partial \theta^2}\right|_{\theta=\hat{\theta}} = \frac{\partial}{\partial \theta}\left[-n + \frac{1}{\theta} \sum_{i=1}^{n} y_i\right] = -\left.\frac{\sum_{i=1}^{n} y_i}{\theta^2}\right|_{\theta=\bar{y}} = -\frac{\sum y_i}{\bar{y}^2} = -\frac{n\bar{y}}{\bar{y}^2} = -\frac{n}{\bar{y}} < 0$$

$as\ n > 0$ and $\bar{y} > 0$.

$\therefore$ The MLE of θ is $\hat{\theta} = \bar{y}$.

Second: Using invariance property of MLE,
The MLE of $\beta = \ln(\theta)$ is:

$$\hat{\beta} = \ln(\hat{\theta}) = \ln(\bar{y}).$$

(c) $s = \sum_{i=1}^{n} \left(y_i - e^{\beta}\right)^2$
we need to find the value of β that minimizes s : [Least Square Estimator of β]:

$$\frac{\partial s}{\partial \beta} = \frac{\partial}{\partial \beta} \sum_{i=1}^{n} \left(y_i - e^{\beta}\right)^2 = \sum_{i=1}^{n} \frac{\partial}{\partial \beta}\left(y_i - e^{\beta}\right)^2 = \sum_{i=1}^{n} 2\left(y_i - e^{\beta}\right)\left(-e^{\beta}\right) = -2e^{\beta} \sum_{i=1}^{n} \left(y_i - e^{\beta}\right)$$

$$= -2e^{\beta}\left[\sum_{i=1}^{n} y_i - ne^{\beta}\right]$$

$$\frac{\partial s}{\gamma \beta} \overset{\text{set}}{=} 0 \Leftrightarrow -2e^{\beta}\left[\sum_{i=1}^{n} y_i - ne^{\beta}\right] = 0 \Leftrightarrow \sum y_i - ne^{\beta} = 0 \Leftrightarrow \sum_{i=1}^{n} y_i = ne^{\beta} \Leftrightarrow e^{\beta} = \frac{\sum_{i=1}^{n} y_i}{n} = \bar{y}$$

$$\Leftrightarrow \hat{\beta} = \ln(\bar{y}) \rightarrow \text{ Same result as in } (b)$$

We need to make sure that $\hat{\beta} = \ln(\bar{y})$ minimizes s not maximizing:

$$\left.\frac{\partial^2 s}{\partial \beta^2}\right|_{\beta=\beta} = \frac{\partial}{\partial \beta}\left[-2e^{\beta}\left(\Sigma y_i - ne^{\beta}\right)\right] = -2\frac{\partial}{\partial \beta}\left[e^{\beta}\left(\Sigma y_i - ne^{\beta}\right)\right] = -2\left\{e^{\beta}\left(\Sigma y_i - ne^{\beta}\right) + e^{\beta}\left(-ne^{\beta}\right)\right\}$$

$$= -2\left\{e^{\beta}\Sigma y_i - ne^{2\beta} - ne^{2\beta}\right\} = -2\left[e^{\beta}\Sigma y_i - 2ne^{2\beta}\right] = -2e^{\beta}\left[\Sigma y_i - 2ne^{\beta}\right]$$

$$\Rightarrow \left.\frac{\partial^2 s}{\partial \beta^2}\right|_{\hat{\beta}=\ln(\bar{y})} = -2e^{\ln(\bar{y})}\left[\Sigma y_i - 2ne^{\ln(\bar{y})}\right] = -2\bar{y}(\Sigma y_i - 2n\bar{y}) = -2\bar{y}(n\bar{y} - 2n\bar{y}) = -2\bar{y}(-n\bar{y}) = +2\bar{y}n\bar{y}$$

$$= 2n(\bar{y})^2 > 0$$

as $\bar{y} \geqslant 0$.

$\therefore \hat{\beta} = \ln(\bar{y})$ minimizes $s = \sum_{i=1}^{n} \left(y_i - e^{\beta}\right)^2$, and $\hat{\beta}$ is the least Squares Estimator of β.

Ex 1.6:
Table 1. 4
n_i = number of eggs of the i-th insect.

y_i = number of female eggs of the i-th insect.
$n_i - y_i$ = number of male eggs of the i-th insect.
$i = 1,2, \dots ,16$

Progeny Group	Female (y_i)	Male ($n_i - y_i$)	Total (n_i)	p_i
1	18	11	29	0.62
2	31	22	53	0.58
⋮	⋮	⋮	⋮	⋮
16	7	12	19	0.37
Total	$\Sigma y_i = 363$	$\Sigma (n_i - y_i) = 371$	$\Sigma n_i = 734$	0.49

(a) The proportion of female eggs of the i-th insect is:

$$p_i = \frac{y_i}{n_i}$$

$$\begin{cases} p_1 = \dfrac{y_1}{n_1} = \dfrac{18}{29} = 0.62 \\ p_2 = \dfrac{y_2}{n_2} = \dfrac{31}{53} = 0.58 \\ \vdots \\ p_{16} = \dfrac{y_{16}}{n_{16}} = \dfrac{7}{19} = 0,37 \end{cases}$$

(b) y_i = number of female eggs (successes) of the
n_i = Total number of eggs of the i-th insect.

$$\begin{cases} y_i \sim \text{Binomial}(\theta) \\ y_1, y_2, \dots y_n \text{ are independent} \end{cases}$$

We need to find the MLE of θ :

$$f(y_i, \theta) = \binom{n_i}{y_i} \theta^{y_i} (1-\theta)^{n_i - y_i}; y_i = 0, \dots , n_i$$

$$\ln f(y_i, \theta) = \ln \binom{n_i}{y_i} + \ln (\theta^{y_i}) + \ln [(1-\theta)^{n_i - y_i}] = \ln \binom{n_i}{y_i} + y_i \ln (\theta) + (n_i - y_i) \ln (1-\theta).$$

The log-likelikood function is:

$$l(\theta; \underset{\sim}{y}) = \sum_{i=1}^{n} \ln f(y_i; \theta) = \sum_{i=1}^{n} \left\{ \ln \left[\binom{n_i}{y_i} \right] + y_i \ln(\theta) + (n_i - y_i) \ln(1-\theta) \right\}$$

$$= \sum_{i=1}^{n} \ln \left[\binom{n_i}{y_i} \right] + \ln (\theta) \sum_{i=1}^{n} y_i + \ln (1-\theta) \sum_{i=1}^{n} (n_i - y_i)$$

$$\Rightarrow \frac{\partial \ell(\theta; \underset{\sim}{y})}{\partial \theta} = \frac{\sum y_i}{\theta} - \frac{\sum (n_i - y_i)}{1-\theta} = \frac{(1-\theta) \sum y_i - \theta \sum (n_i - y_i)}{\theta(1-\theta)}$$

$$= \frac{\sum_{i=1}^{n} y_i - \theta \sum_{i=1}^{n} y_i - \theta \sum_{i=1}^{n} n_i + \theta \sum_{i=1}^{n} y_i}{\theta(1-\theta)} = \frac{\sum y_i - \theta \sum n_i}{\theta(1-\theta)}; \ (0 < \theta < 1)$$

$$\Rightarrow \frac{\partial l(\theta; \underset{\sim}{y})}{\partial \theta} \overset{\text{Set}}{=} 0 \Leftrightarrow \Sigma y_i = \theta \Sigma n_i \Leftrightarrow \hat{\theta} = \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} n_i} = \frac{363}{734} = 0.49455 \approx 0.495$$

We need to make sure that $\hat{\theta}$ maximizes the likelihood function not minimizing;

$$\frac{\partial^2 l(\theta,\underset{\sim}{y})}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left[\frac{\sum y_i - \theta\Sigma n_i}{\theta(1-\theta)}\right] = \frac{1}{\theta^2(1-\theta)^2}\{\theta(1-\theta)[-\Sigma n_i] - [\Sigma y_i - \theta\Sigma n_i][1-2\theta]\}$$

$$= \frac{1}{\theta^2(1-\theta)^2}\left[-\theta\Sigma n_i + \theta^2\Sigma n_i - \Sigma y_i + 2\theta\Sigma y_i + \theta\Sigma n_i - 2\theta\Sigma n_i^2\right] = \frac{1}{\theta^2(1-\theta)^2}\left[-2\theta^2\sum n_i + 2\theta\sum y_i - \Sigma y_i\right]$$

$$\Rightarrow \left.\frac{\partial^2 l(\theta,\underset{\sim}{y})}{\partial\theta^2}\right|_{\hat{\theta}=\frac{\sum y_i}{\sum n_i}} = \frac{1}{\hat{\theta}^2(1-\hat{\theta})^2}\left[-2\left(\frac{\sum y_i}{\sum n_i}\right)^2\sum n_i + 2\left(\frac{\sum y_i}{\sum n_i}\right)\sum y_i - \sum y_i\right] = \frac{1}{\hat{\theta}^2(1-\hat{\theta})^2}\left[-2\frac{(\sum y_i)^2}{\sum n_i} + 2\frac{(\sum y_i)^2}{\sum n_i} - \sum y_i\right]$$

$$= -\frac{\sum y_i}{\hat{\theta}^2(1-\hat{\theta})^2} < 0 \text{ ; as } \Sigma y_i > 0$$

$\therefore \hat{\theta} = \frac{\sum y_i}{\sum n_i}$ is the MLE of θ.

(c)

<u>Numerical Method (Newton-Raphson Method):</u>

we need to solve

$$\frac{\partial l(\theta;\underset{\sim}{y})}{\partial\theta} = 0 \Leftrightarrow \frac{\sum y_i - \theta\sum n_i}{\theta(1-\theta)} = 0 \Leftrightarrow \frac{363 - 734\theta}{\theta(1-\theta)} = 0$$

Let:

$$g(\theta) = \frac{363 - 734\theta}{\theta(1-\theta)}$$

$$\Rightarrow g'(\theta) = \frac{1}{\theta^2(1-\theta)^2}[-(2)(734)\theta^2 + (2)(363)\theta - 363] = \frac{1}{\theta^2(1-\theta)^2}[726\theta - 1468\theta^2 - 363]$$

The iterative procedure for $r = 1,2,3,\ldots$:

$$\hat{\theta}^{(r+1)} = \hat{\theta}^{(r)} - \frac{g(\hat{\theta}^{(r)})}{g'(\hat{\theta}^{(r)})} = \hat{\theta}(r) - \frac{\frac{363 - 734\hat{\theta}^{(r)}}{\hat{\theta}^{(r)}(1-\hat{\theta}^{(r)})}}{\frac{726\hat{\theta}^{(r)} - 1468(\hat{\theta}^{(r)})^2 - 363}{(\hat{\theta}^{(r)})^2\left(1-(\hat{\theta}^{(r)})\right)^2}} = \hat{\theta}^{(r)} - \frac{363 - 734\hat{\theta}^{(r)}}{\frac{726\hat{\theta}^{(r)} - 1468(\hat{\theta}^{(r)})^2 - 363}{\hat{\theta}^{(r)}(1-\hat{\theta}^{(r)})}}$$

We choose the initial value of θ: $\hat{\theta}^{(0)} = 0.4$

$$\hat{\theta}^{(1)} = 0.4 - \frac{363 - 734(0,4)}{\frac{726(0,4) - 1468(0,4^2) - 363}{(0,4)(0,6)}} = 0.4542$$

$$\hat{\theta}^{(2)} = 0.4542 - \frac{363 - 734(0.4542)}{\frac{726(0.4542) - 1468(0.4542)^2 - 363}{(0.4542)(0.5458)}} = 0.4760$$

$$\hat{\theta}^{(3)} = \cdots = 0.4857$$
$$\hat{\theta}^{(4)} = \cdots = 0.4902$$
$$\theta^{(5)} = \cdots = 0.4924$$
$$\hat{\theta}^{(6)} = \cdots = 0.4935$$
$$\hat{\theta}^{(7)} = \cdots = 0.4940$$
$$\hat{\theta}^{(8)} = \cdots = 0.4943$$
$$\hat{\theta}^{(9)} = \cdots = 0.4944$$
$$\hat{\theta}^{(10)} = \cdots = 0.4945$$

Bisection Method:

The likelihood function is

$$l(\theta; \underset{\sim}{y}) = \underbrace{\sum \ln \binom{n_i}{y_i}}_{not\ dep.\ on\ \theta} + \underbrace{\ln(\theta) \sum\ y_i + \ln(1-\theta) \sum\ (n_i - y_i)}_{dep.\ on\ \theta}$$

Define:

$$l^*(\theta) = \ln(\theta) \sum\ y_i + \ln(1-\theta) \sum\ (n_i - y_i)\ = 363\ln(\theta) + 371\ln(1-\theta) = 363\ln(\theta) + 371\ln(1-\theta)$$

as $\sum\ y_i = 363$ and $\sum\ n_i = 734$.

Let's strat with $\theta^{(1)} = 0.1$ and $\theta^{(2)} = 0.9$:

k	$\theta^{(k)}$	$l^*(\theta^{(k)})$	$\theta^{(k+1)}$
1	0.1	−874.9271	
2	0.9	−892.5049	$\theta^{(3)} = \frac{\theta^{(1)} + \theta^{(2)}}{2} = 0.5$
3	0.5	−508.7700	$\theta^{(4)} = \frac{\theta^{(1)} + \theta^{(3)}}{2} = 0.3$
4	0.3	−569.3685	$\theta^{(5)} = \frac{\theta^{(3)} + \theta^{(4)}}{2} = 0.4$
5	0.4	−522.1298	$\theta^{(6)} = \frac{\theta^{(3)} + \theta^{(5)}}{2} = 0.45$
6	0.45	−511.6558	$\theta^{(7)} = \frac{\theta^{(3)} + \theta^{(6)}}{2} = 0.475$
7	0.475	−508.9352	$\theta^{(8)} = \frac{\theta^{(3)} + \theta^{(7)}}{2} = 0.4875$
8	0.4875	−508.7994	$\theta^{(9)} = \frac{\theta^{(3)} + \theta^{(8)}}{2} = 0.4938$
9	0.4938	−508.7276	$\theta^{(10)} = \frac{\theta^{(3)} + \theta^{(9)}}{2} = 0.4969$
10	0.4969	−508.7345	$\theta^{(11)} = \frac{\theta^{(9)} + \theta^{(10)}}{2} = 0.4954$
11	0.4954	−508.7275	$\theta^{(12)} = \frac{\theta^{(9)} + \theta^{(11)}}{2} = 0.4946$
12	0.4946	−508.7264	$\theta^{(13)} = \frac{\theta^{(11)} + \theta^{(12)}}{2} = 0.4950$
13	$\boxed{0.4950}$	−508.7267	