Chapter 1 : Exercises

1.7 Exercise p. 15 Dobson 2008

Ex. 1.1 :

$Y_1 \sim N(1,3)$
$Y_2 \sim N(2,5)$ } independent

$W_1 = Y_1 + 2Y_2$
$W_2 = 4Y_1 - Y_2$

$$\begin{cases} Var(W_1) = Var(Y_1 + 2Y_2) \\ \quad = Var(Y_1) + 4Var(Y_2) \\ \quad = 3 + (4)(5) = 23 \\ Var(W_2) = Var(4Y_1 - Y_2) \\ \quad = 16Var(Y_1) + Var(Y_2) \\ \quad = (16)(3) + 5 = 53 \end{cases}$$

$$\begin{cases} E(W_1) = E(Y_1 + 2Y_2) = E(Y_1) + 2E(Y_2) = 1 + (2)(2) = 5 \\ E(W_2) = E(4Y_1 - Y_2) = 4E(Y_1) - E(Y_2) = (4)(1) - 2 = 2 \end{cases}$$

$$Cov(W_1, W_2) = Cov(Y_1 + 2Y_2, 4Y_1 - Y_2)$$

$$= Cov(Y_1, 4Y_1) + Cov(Y_1, -Y_2) + Cov(2Y_2, 4Y_1) + Cov(2Y_2, -Y_2)$$

$$= 4\,Cov(Y_1, Y_1) - Cov(Y_1, Y_2) + (2)(4)Cov(Y_2, Y_1) - (2)Cov(Y_2, Y_2)$$

$$= 4\,Var(Y_1) - 0 + 0 - 2\,Var(Y_2)$$

$$= (4)(3) - (2)(5) = 12 - 10$$

$$= 2$$

$$\underset{\sim}{W} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \qquad E(\underset{\sim}{W}) = \begin{bmatrix} E(W_1) \\ E(W_2) \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \underset{\sim}{\mu}^*$$

$$Cov(\underline{W}) = \begin{bmatrix} Var(W_1) & Cov(W_1, W_2) \\ Cov(W_1, W_2) & Var(W_2) \end{bmatrix} = \begin{bmatrix} 23 & 2 \\ 2 & 53 \end{bmatrix} = \underline{V}^*$$

$$\underset{\sim}{W} \sim MVN(\underset{\sim}{\mu}^*, \underline{V}^*)$$

Another Solution:

$$\underset{\sim}{W} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} Y_1 + 2Y_2 \\ 4Y_1 - Y_2 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

$$\underset{\sim}{W} = \underline{A}\,\underset{\sim}{y} \qquad \underset{\sim}{W} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, \; \underset{\sim}{y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \; \underline{A} = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$$

$$E(\underset{\sim}{W}) = E(\underline{A}\underset{\sim}{y}) = \underline{A}\,E(\underset{\sim}{y}) = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \underset{\sim}{\mu}^*$$

$$\text{Cov}(\underset{\sim}{W}) = \text{Cov}(\underline{A}\underset{\sim}{Y}) = A\,\text{Cov}(\underset{\sim}{Y})A'$$

$$\text{Cov}(\underset{\sim}{Y}) = \begin{bmatrix} \text{Var}(Y_1) & \text{Cov}(Y_1,Y_2) \\ \text{Cov}(Y_1,Y_2) & \text{Var}(Y_1) \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$

Note: Y_1 and Y_2 are independent $\Rightarrow \text{Cov}(Y_1,Y_2)=0$

$$\therefore \text{Cov}(\underset{\sim}{W}) = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 10 \\ 12 & -5 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 23 & 2 \\ 2 & 53 \end{bmatrix} = \underline{V}^*$$

$$\underset{\sim}{Y} \sim MVN(\underset{\sim}{\mu}, \underline{V}) \quad \begin{cases} E(\underset{\sim}{Y}) = \underset{\sim}{\mu} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \text{Var}(\underset{\sim}{Y}) = \underline{V} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \end{cases}$$

$$\therefore \underset{\sim}{W} \sim MVN(\underset{\sim}{\mu}^*, \underline{V}^*)$$

<u>Another Solution</u>

$W_1 = Y_1 + 2Y_2$

$= (1 \;\; 2)\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$

$= \underset{\sim}{a}_1' \underset{\sim}{Y}$

$W_2 = 4Y_1 - Y_2$

$= (4 \;\; -1)\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$

$= \underset{\sim}{a}_2' \underset{\sim}{Y}$

where $\underset{\sim}{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$

$\underset{\sim}{a}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \underset{\sim}{a}_2 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

$E(\underset{\sim}{Y}) = \underset{\sim}{\mu} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\text{Cov}(\underset{\sim}{Y}) = \underline{V} = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$

$E(W_1) = E(\underset{\sim}{a}_1' \underset{\sim}{Y}) = \underset{\sim}{a}_1' E(\underset{\sim}{Y}) = \underset{\sim}{a}_1' \underset{\sim}{\mu}$

$= (1 \;\; 2)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1+4 = 5$

$E(W_2) = E(\underset{\sim}{a}_2' \underset{\sim}{Y}) = \underset{\sim}{a}_2' E(\underset{\sim}{Y}) = \underset{\sim}{a}_2' \underset{\sim}{\mu}$

$= (4 \;\; -1)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 4-2 = 2$

$\therefore \underset{\sim}{\mu}^* = E(\underline{W}) = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ where $\underset{\sim}{W} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$

$\text{Var}(W_1) = \text{Var}(\underset{\sim}{a}_1' \underset{\sim}{Y}) = \underset{\sim}{a}_1' \text{Cov}(\underset{\sim}{Y}) \underset{\sim}{a}_1$

$= (1 \;\; 2)\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 23$

$\text{Var}(W_2) = \text{Var}(\underset{\sim}{a}_2' \underset{\sim}{Y}) = \underset{\sim}{a}_2' \text{Cov}(\underset{\sim}{Y}) \underset{\sim}{a}_1$

$= (4 \;\; -1)\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix} = 53$

$\text{Cov}(W_1, W_2) = \text{Cov}(\underset{\sim}{a}_1' \underline{Y}, \underset{\sim}{a}_2' \underline{Y})$

$= \underset{\sim}{a}_1' \text{Cov}(\underset{\sim}{Y}) \underset{\sim}{a}_2 = (1 \;\; 2)\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix} = 2$

$\therefore V^* = \text{Cov}(\underline{W}) = \begin{pmatrix} 23 & 2 \\ 2 & 53 \end{pmatrix}$

$\underline{W} \sim MVN(\underline{\mu}^*, \underline{V}^*)$

Ex 1.2 :

$$\left.\begin{array}{l} Y_1 \sim N(0,1) \\ Y_2 \sim N(3,4) \end{array}\right\} \text{independent} \Rightarrow Cov(Y_1, Y_2) = 0$$

$$\underset{\sim}{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad \underset{\sim}{\mu} = E(\underset{\sim}{Y}) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad \underline{V} = Cov(\underset{\sim}{Y}) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\underset{\sim}{Y} \sim MVN(\underset{\sim}{\mu}, \underline{V})$$

(a) $Y_1 \sim N(0,1) \Rightarrow Y_1^2 \sim \chi^2_{(1)}$

(b) Let $\underset{\sim}{W} = \begin{bmatrix} Y_1 \\ \frac{Y_2 - 3}{2} \end{bmatrix}$

Note: $Y_2 \sim N(3,4) \Leftrightarrow \frac{Y_2 - 3}{2} \sim N(0,1)$

$$\underset{\sim}{W}' \underset{\sim}{W} = \left(Y_1, \frac{Y_2-3}{2}\right) \begin{pmatrix} Y_1 \\ \frac{Y_2-3}{2} \end{pmatrix}$$

$$= Y_1^2 + \left(\frac{Y_2-3}{2}\right)^2$$

$$\left.\begin{array}{l} Y_1^2 \sim \chi^2_{(1)} \\ \left(\frac{Y_2-3}{2}\right)^2 \sim \chi^2_{(1)} \end{array}\right\} \text{independent}$$

$$= \underbrace{\chi^2_{(1)} + \chi^2_{(1)}}_{\text{independent}}$$

$$\therefore \underset{\sim}{W}' \underset{\sim}{W} \sim \chi^2_{(2)}$$

(c) $\underset{\sim}{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$; $\underset{\sim}{\mu} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $\underline{V} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$; $\underline{V}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$

$$X = \underset{\sim}{Y}' \underline{V}^{-1} \underset{\sim}{Y} = (Y_1, Y_2) \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

$$= \left(Y_1, \frac{Y_2}{4}\right) \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

$$= Y_1^2 + \frac{1}{4} Y_2^2$$

$X = \underset{\sim}{Y}' \underline{V}^{-1} \underset{\sim}{Y} \sim \chi^2_{(2,\lambda)}$; $\lambda = \underset{\sim}{\mu}' \underline{V}^{-1} \underset{\sim}{\mu} = (0,3) \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$$= \left(0, \frac{3}{4}\right) \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \frac{9}{4} = 2.25$$

Ex 1.3 :

$\underset{\sim}{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$; $\underset{\sim}{y} \sim MVN(\underset{\sim}{\mu}, \underline{V})$; $\underset{\sim}{\mu} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\underline{V} = \begin{bmatrix} 4 & 1 \\ 1 & 9 \end{bmatrix}$

(a) $(\underset{\sim}{y} - \underset{\sim}{\mu})' \underline{V}^{-1} (\underset{\sim}{y} - \underset{\sim}{\mu}) = ?$

$$\underline{V}^{-1} = \frac{1}{(4)(9) - 1^2} \begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix}$$

$$\underset{\sim}{y} - \underset{\sim}{\mu} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} y_1 - 2 \\ y_2 - 3 \end{pmatrix}$$

$$(\underset{\sim}{y} - \underset{\sim}{\mu})' \underline{V}^{-1} (\underset{\sim}{y} - \underset{\sim}{\mu}) = (y_1 - 2, y_2 - 3) \frac{1}{35} \begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix} \begin{pmatrix} y_1 - 2 \\ y_2 - 3 \end{pmatrix}$$

$$= \frac{1}{35} \left[9(y_1 - 2)^2 + 4(y_2 - 3)^2 - 2(y_1 - 2)(y_2 - 3) \right]$$

Note: $\underset{\sim}{x}' \underline{A} \underset{\sim}{x} = (x_1, x_2) \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ ($A = A'$ symmetric, $a_{ij} = a_{ji}$)

$$= \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} x_i x_j$$

$$= \sum_{i=1}^{2} a_{ii} x_i^2 + a_{12} x_1 x_2 + a_{21} x_2 x_1$$

$$= \sum_{i=1}^{2} a_{ii} x_i^2 + 2 a_{12} x_1 x_2 \quad (a_{12} = a_{21})$$

$$= a_{11} x_1^2 + a_{22} x_2^2 + 2 a_{12} x_1 x_2$$

$$(\underset{\sim}{y} - \underset{\sim}{\mu})' \underline{V}^{-1} (\underset{\sim}{y} - \underset{\sim}{\mu}) \sim \chi^2_{(2)}$$

(b) $\underset{\sim}{y}' \underline{V}^{-1} \underset{\sim}{y} = (y_1, y_2) \frac{1}{35} \begin{pmatrix} 9 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$= \frac{1}{35} \left[9y_1^2 + 4y_2^2 - 2y_1y_2 \right]$$

$$\underset{\sim}{y}' \underline{V}^{-1} \underset{\sim}{y} \sim \chi^2(2, \lambda)$$

$$\lambda = \underset{\sim}{\mu}' \underline{V}^{-1} \underset{\sim}{\mu} = (2, 3) \frac{1}{35} \begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$= \frac{1}{35} \left[(9)(2)^2 + (4)(3)^2 - (2)(2)(3) \right]$$

$$= \frac{37}{35}$$

Ex 1.4 :

$y_1, y_2, \dots, y_n$ are independent and $y_i \sim N(\mu, \sigma^2)$.

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i \quad , \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

$$\underset{\sim}{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \; ; \; \underset{\sim}{\mu} = E(\underset{\sim}{y}) = \begin{bmatrix} \mu \\ \vdots \\ \mu \end{bmatrix} \; , \; Cov(\underset{\sim}{y}) = \begin{bmatrix} \sigma^2 & & 0 \\ & \ddots & \\ 0 & & \sigma^2 \end{bmatrix} = \sigma^2 \underline{I}$$

(a) $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

$$E(\bar{y}) = \frac{1}{n}\sum_{i=1}^{n} E(y_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{1}{n}(n\mu) = \mu$$

$$Var(\bar{y}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} y_i\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(y_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{n}{n^2}\sigma^2 = \frac{\sigma^2}{n}$$

$$\bar{y} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

another Solution

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}(1, \dots, 1)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \frac{1}{n}\underset{\sim}{1}'\underset{\sim}{y}$$

$$\underset{\sim}{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$E(\bar{y}) = E\left(\frac{1}{n}\underset{\sim}{1}'\underset{\sim}{y}\right)$$

$$= \frac{1}{n}\underset{\sim}{1}' E(\underset{\sim}{y}) = \frac{1}{n}\underset{\sim}{1}'\underset{\sim}{\mu}$$

$$= \frac{1}{n}(1, \dots, 1)\begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix} = \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{n}{n}\mu = \mu$$

$$Var(\bar{y}) = Var\left(\frac{1}{n}\underset{\sim}{1}'\underset{\sim}{y}\right) = \frac{1}{n^2} Var(\underset{\sim}{1}'\underset{\sim}{y})$$

$$= \frac{1}{n^2}\underset{\sim}{1}' Cov(\underset{\sim}{y})\underset{\sim}{1} = \frac{1}{n^2}\underset{\sim}{1}'(\sigma^2\underline{I})\underset{\sim}{1} = \frac{\sigma^2}{n^2}\underset{\sim}{1}'\underset{\sim}{1}$$

$$= \frac{\sigma^2}{n^2}(1, \dots, 1)\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \frac{\sigma^2}{n^2}(n) = \frac{\sigma^2}{n}$$

(b) $$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\bar{Y})^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\mu+\mu-\bar{Y})^2$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}\left[(Y_i-\mu)-(\bar{Y}-\mu)\right]^2$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}\left[(Y_i-\mu)^2+(\bar{Y}-\mu)^2-2(Y_i-\mu)(\bar{Y}-\mu)\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}(Y_i-\mu)^2+\sum_{i=1}^{n}(\bar{Y}-\mu)^2-2\sum_{i=1}^{n}(Y_i-\mu)(\bar{Y}-\mu)\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}(Y_i-\mu)^2+n(\bar{Y}-\mu)^2-2(\bar{Y}-\mu)\sum_{i=1}^{n}(Y_i-\mu)\right]$$

$$= \frac{1}{n-1}\left\{\sum(Y_i-\mu)^2+n(\bar{Y}-\mu)^2-2(\bar{Y}-\mu)\left[\sum_{i=1}^{n}Y_i-\sum_{i=1}^{n}\mu\right]\right\}$$

$$= \frac{1}{n-1}\left\{\sum(Y_i-\mu)^2+n(\bar{Y}-\mu)^2-2(\bar{Y}-\mu)\left[n\bar{Y}-n\mu\right]\right\}$$

$$= \frac{1}{n-1}\left[\sum(Y_i-\mu)^2+n(\bar{Y}-\mu)^2-2(\bar{Y}-\mu)n(\bar{Y}-\mu)\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}(Y_i-\mu)^2-n(\bar{Y}-\mu)^2\right]$$

(c) $$S^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n}(Y_i-\mu)^2-n(\bar{Y}-\mu)^2\right]$$

$$(n-1)S^2 = \sum_{i=1}^{n}(Y_i-\mu)^2-n(\bar{Y}-\mu)^2$$

$$\sum_{i=1}^{n}(Y_i-\mu)^2 = (n-1)S^2+n(\bar{Y}-\mu)^2$$

$$\frac{\sum_{i=1}^{n}(Y_i-\mu)^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2}+\frac{n(\bar{Y}-\mu)^2}{\sigma^2}$$

$$\sum_{i=1}^{n}\left(\frac{Y_i-\mu}{\sigma}\right)^2 = \frac{(n-1)S^2}{\sigma^2}+\left(\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}}\right)^2$$

Note:

1. $Y_i \sim N(\mu, \sigma^2) \Leftrightarrow \frac{Y_i - \mu}{\sigma} \sim N(0,1)$

$\Leftrightarrow \left(\frac{Y_i - \mu}{\sigma}\right)^2 \sim \chi^2_{(1)} \quad (i = 1, 2, \dots, n)$

$Y_1, Y_2, \dots, Y_n$ are independent

$\Rightarrow \left(\frac{Y_1 - \mu}{\sigma}\right)^2, \dots, \left(\frac{Y_n - \mu}{\sigma}\right)^2$ are independent

$\therefore \sum_{i=1}^{n} \left(\frac{Y_i - \mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$

2. $\bar{Y} \sim N(\mu, \frac{\sigma^2}{n}) \Leftrightarrow \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$

$\Leftrightarrow \left(\frac{\bar{Y} - \mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2_{(1)}$

Now;

$$\sum_{i=1}^{n} \left(\frac{Y_i - \mu}{\sigma}\right)^2 = \frac{(n-1)S^2}{\sigma^2} + \left(\frac{\bar{Y} - \mu}{\sigma/\sqrt{n}}\right)^2$$

$$\chi^2_{(n)} = \chi^2_{(n-1)} + \chi^2_{(1)}$$

(Cochran's Theorem)

independent

Therefore, $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$

and $\frac{(n-1)S^2}{\sigma^2}$ and $\left(\frac{\bar{Y} - \mu}{\sigma/\sqrt{n}}\right)^2$ are independent

$\Rightarrow S^2$ and $\bar{Y}$ are independent

Note: This can also shown by Cochran's Theorem.

(d) From (c), we have shown that

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

This can also be shown by Cocharn's Theorem.

(e)

$$\frac{\bar{Y}-\mu}{S/\sqrt{n}} \sim ?$$

Note:

1. $$\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$
2. $$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$
3. $\bar{Y}$ and S^2 are independent
4. If $Z \sim N(0,1)$, $X \sim \chi^2(m)$, and Z and X are independent, then
$$t = \frac{Z}{\sqrt{X/m}} \sim t(m)$$

Now,

$$t = \frac{\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2}/(n-1)}} = \frac{\frac{\sqrt{n}(\bar{Y}-\mu)}{\sigma}}{\sqrt{\frac{S^2}{\sigma^2}}} = \frac{\frac{\sqrt{n}(\bar{Y}-\mu)}{\sigma}}{\frac{S}{\sigma}}$$

$$= \frac{\sqrt{n}(\bar{Y}-\mu)}{S} = \frac{\bar{Y}-\mu}{S/\sqrt{n}} \sim t(n-1)$$

Ex 1.5 : (See Example in Section 1.6.2)

$Y_1, Y_2, \ldots, Y_n$ are independent, and $Y_i \sim \text{Poisson}(\theta)$

$\Leftrightarrow$ $Y_1, Y_2, \ldots, Y_n$ iid Poisson(θ) (random sample)

(a) Suppose that $Y \sim \text{Poisson}(\theta)$. The pmf is

$$f_Y(y;\theta) = \frac{e^{-\theta}\theta^{y}}{y!} \quad ; \; y = 0, 1, 2, \ldots$$

$$E(Y) = \sum_{y=0}^{\infty} y f_Y(y;\theta) = \sum_{y=0}^{\infty} y \frac{e^{-\theta}\theta^{y}}{y!}$$

$$= e^{-\theta} \sum_{y=0}^{\infty} \frac{y\theta^{y}}{y!} = e^{-\theta} \sum_{y=1}^{\infty} \frac{y\theta^{y}}{y!} = e^{-\theta} \sum_{y=1}^{\infty} \frac{y\theta^{y}}{y(y-1)!}$$

$$= e^{-\theta} \sum_{y=1}^{\infty} \frac{\theta^{y}}{(y-1)!} = e^{-\theta}\,\theta \sum_{y=1}^{\infty} \frac{\theta^{y-1}}{(y-1)!}$$

$$= \theta e^{-\theta} \sum_{k=0}^{\infty} \frac{\theta^{k}}{k} \qquad (k = y-1)$$

Result:

$$\sum_{k=0}^{\infty} \frac{x^{k}}{k!} = e^{x} \qquad \left[\text{Mclaurin Expansion of } f(x) = e^{x}\right]$$

$$\therefore \; E(Y) = \theta e^{-\theta} \sum_{k=0}^{\infty} \frac{\theta^{k}}{k} = \theta e^{-\theta} e^{\theta} = \theta e^{\theta-\theta} = \theta e^{0}$$

$$= \theta.$$

(b) $\theta = e^{\beta} \Leftrightarrow \beta = \ln(\theta)$

We need to find the maximum likelihood estimator (MLE) of β.

First: MLE of θ: (See Example in Section 1.62)

$$f(y_i;\theta) = \frac{e^{-\theta}\,\theta^{y_i}}{y_i!} \qquad y_i = 0,1,2,\ldots$$

$$\ln f(y_i;\theta) = \ln(e^{-\theta}) + \ln(\theta^{y_i}) - \ln(y_i!)$$

$$= -\theta + y_i \ln(\theta) - \ln(y_i!)$$

The log-likelihood function is:

$$\ell(\theta;\underset{\sim}{y}) = \sum_{i=1}^{n} \ln f(y_i;\theta)$$

$$= \sum_{i=1}^{n} \left[-\theta + y_i \ln(\theta) - \ln(y_i!)\right]$$

$$= -\theta n + \ln(\theta)\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \ln(y_i!)$$

$$\frac{\partial \ell(\theta;\underset{\sim}{y})}{\partial\theta} = -n + \frac{1}{\theta}\sum_{i=1}^{n} y_i$$

$$\frac{\partial \ell(\theta;\underset{\sim}{y})}{\partial\theta} \overset{set}{=} 0$$

$$\Leftrightarrow \quad 0 = -n + \frac{1}{\theta}\sum_{i=1}^{n} y_i$$

$$\Leftrightarrow \quad \frac{1}{\theta}\sum_{i=1}^{n} y_i = n$$

$$\Leftrightarrow \quad \hat{\theta} = \frac{\sum_{i=1}^{n} y_i}{n} = \bar{Y}$$

We need to make sure that $\bar{Y}$ maximize $\ell(\theta;\underset{\sim}{y})$ not minimizing:

$$\left.\frac{\partial^2 \ell(\theta;\underset{\sim}{y})}{\partial\theta^2}\right|_{\theta=\hat{\theta}} = \frac{\partial}{\partial\theta}\left[-n+\frac{1}{\theta}\sum_{i=1}^{n} y_i\right]$$

$$= -\left.\frac{\sum_{i=1}^{n} y_i}{\theta^2}\right|_{\theta=\bar{Y}} = -\frac{\sum Y_i}{\bar{Y}^2} = -\frac{n\bar{Y}}{\bar{Y}^2} \overset{(\hat{\theta}=\bar{Y})}{=} -\frac{n}{\bar{Y}}$$

< 0 $\quad \left(\begin{matrix} n>0 \\ \bar{Y}>0 \end{matrix}\right)$

$\therefore$ The MLE of θ is $\hat{\theta}=\bar{y}$.

Second: Using invariance property of MLE,

The MLE of $\beta=\ln(\theta)$ is:

$$\hat{\beta}=\ln(\hat{\theta})$$
$$=\ln(\bar{y}).$$

(c) $$S=\sum_{i=1}^{n}(y_i-e^{\beta})^2$$

we need to find the value of β that minimizes S: [Least Squares Estimator of β]:

$$\frac{\partial S}{\partial\beta} = \frac{\partial}{\partial\beta}\sum_{i=1}^{n}(y_i-e^{\beta})^2 = \sum_{i=1}^{n}\frac{\partial}{\partial\beta}(y_i-e^{\beta})^2$$

$$= \sum_{i=1}^{n} 2(y_i-e^{\beta})(-e^{\beta})$$

$$= -2e^{\beta}\sum_{i=1}^{n}(y_i-e^{\beta}) = -2e^{\beta}\left[\sum_{i=1}^{n} y_i - ne^{\beta}\right]$$

$$\frac{\partial S}{\partial\beta} \overset{\text{set}}{=} 0$$

$$\Leftrightarrow \quad -2e^{\beta}\left[\sum_{i=1}^{n} y_i - ne^{\beta}\right]=0$$

$$\Leftrightarrow \quad \sum y_i - ne^{\beta} = 0$$

$$\Leftrightarrow \quad \sum_{i=1}^{n} y_i = n e^{\beta}$$

$$\Leftrightarrow \quad e^{\beta} = \frac{\sum_{i=1}^{n} y_i}{n} = \bar{y}$$

$$\Leftrightarrow \quad \hat{\beta} = \ln(\bar{y}) \longrightarrow \text{[same result as in (b)]}$$

We need to make sure that $\hat{\beta} = \ln(\bar{y})$ minimizes S not maximizing:

$$\left.\frac{\partial^2 S}{\partial \beta^2}\right|_{\beta=\hat{\beta}} = \frac{\partial}{\partial \beta}\left[-2e^{\beta}\left(\sum y_i - n e^{\beta}\right)\right]$$

$$= -2 \frac{\partial}{\partial \beta}\left[e^{\beta}\left(\sum y_i - n e^{\beta}\right)\right]$$

$$= -2\left\{e^{\beta}\left(\sum y_i - n e^{\beta}\right) + e^{\beta}\left(-n e^{\beta}\right)\right\}$$

$$= -2\left\{e^{\beta}\sum y_i - n e^{2\beta} - n e^{2\beta}\right\}$$

$$= -2\left[e^{\beta}\sum y_i - 2n e^{2\beta}\right]$$

$$= -2e^{\beta}\left[\sum y_i - 2n e^{\beta}\right]$$

$$\left.\frac{\partial^2 S}{\partial \beta^2}\right|_{\hat{\beta}=\ln(\bar{y})} = -2e^{\ln(\bar{y})}\left[\sum y_i - 2n e^{\ln(\bar{y})}\right]$$

$$= -2\bar{y}\left(\sum y_i - 2n\bar{y}\right)$$

$$= -2\bar{y}\left(n\bar{y} - 2n\bar{y}\right)$$

$$= -2\bar{y}\left(-n\bar{y}\right)$$

$$= +2\bar{y}n\bar{y} = 2n(\bar{y})^2 > 0 \quad (\bar{y} > 0)$$

$\therefore$ $\hat{\beta} = \ln(\bar{y})$ minimizes $S = \sum_{i=1}^{n}(y_i - e^{\beta})^2$, and $\hat{\beta}$ is the least Squares Estimator of β.

Ex 1.6 :

Table 1.4 (p.18)

n_i = number of eggs of the i-th insect.

y_i = number of female eggs of the i-th insect.

$n_i - y_i$ = number of male eggs of the i-th insect.

$i = 1, 2, \ldots, 16$

Progeny Group	Female (y_i)	Male ($n_i - y_i$)	Total (n_i)	p_i
1	18	11	29	0.62
2	31	22	53	0.58
⋮	⋮	⋮	⋮	⋮
16	7	12	19	0.37
Total	$\sum y_i = 363$	$\sum (n_i - y_i) = 371$	$\sum n_i = 734$	0.49

(a) The proportion of female eggs of the i-th insect is:

$$p_i = \frac{y_i}{n_i}$$

$$\begin{cases} p_1 = \frac{y_1}{n_1} = \frac{18}{29} = 0.62 \\ p_2 = \frac{y_2}{n_2} = \frac{31}{53} = 0.58 \\ \vdots \\ p_{16} = \frac{y_{16}}{n_{16}} = \frac{7}{19} = 0.37 \end{cases}$$

(b) Y_i = number of female eggs (Successes) of the i-th insect.

n_i = Total number of eggs of the i-th insect.

$$\begin{cases} Y_i \sim \text{Binomial}(\theta) \\ Y_1, Y_2, \ldots, Y_n \text{ are independent} \end{cases}$$

We need to find the MLE of θ:

$$f(y_i, \theta) = \binom{n_i}{y_i} \theta^{y_i} (1-\theta)^{n_i - y_i} \quad ; \; y_i = 0, 1, \ldots, n_i$$

$$\ln f(y_i, \theta) = \ln\left[\binom{n_i}{y_i}\right] + \ln(\theta^{y_i}) + \ln\left[(1-\theta)^{n_i - y_i}\right]$$

$$= \ln\left[\binom{n_i}{y_i}\right] + y_i \ln(\theta) + (n_i - y_i)\ln(1-\theta)$$

The log-likelihood function is:

$$\ell(\theta;\underset{\sim}{y}) = \sum_{i=1}^{n} \ln f(y_i,\theta)$$

$$= \sum_{i=1}^{n} \left\{ \ln\left[\binom{n_i}{y_i}\right] + y_i \ln(\theta) + (n_i - y_i)\ln(1-\theta) \right\}$$

$$= \sum_{i=1}^{n} \ln\left[\binom{n_i}{y_i}\right] + \ln(\theta)\sum_{i=1}^{n} y_i + \ln(1-\theta)\sum_{i=1}^{n}(n_i - y_i)$$

$$\frac{\partial \ell(\theta;\underset{\sim}{y})}{\partial\theta} = \frac{\sum y_i}{\theta} - \frac{\sum (n_i - y_i)}{1-\theta}$$

$$= \frac{(1-\theta)\sum y_i - \theta\sum(n_i - y_i)}{\theta(1-\theta)}$$

$$= \frac{\sum_{i=1}^{n} y_i - \theta\sum_{i=1}^{n} y_i - \theta\sum_{i=1}^{n} n_i + \theta\sum_{i=1}^{n} y_i}{\theta(1-\theta)}$$

$$= \frac{\sum y_i - \theta\sum n_i}{\theta(1-\theta)} \qquad (0<\theta<1)$$

$$\frac{\partial \ell(\theta;y)}{\partial\theta} \overset{\text{set}}{=} 0$$

$$\Leftrightarrow \quad \sum y_i = \theta\sum n_i$$

$$\Leftrightarrow \quad \hat{\theta} = \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} n_i} = \frac{363}{734} = \boxed{0.49455} \simeq \boxed{0.495}$$

We need to make sure that $\hat{\theta}$ maximizes the likelihood function <u>not</u> minimizing:

$$\frac{\partial^2 \ell(\theta,\underset{\sim}{y})}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left[\frac{\sum y_i - \theta\sum n_i}{\theta(1-\theta)}\right]$$

$$= \frac{1}{\theta^2(1-\theta)^2}\left\{\theta(1-\theta)\left[-\sum n_i\right] - \left[\sum y_i - \theta\sum n_i\right]\left[1-2\theta\right]\right\}$$

$$= \frac{1}{\theta^2(1-\theta)^2}\left[-\theta\sum n_i + \theta^2\sum n_i - \sum y_i + 2\theta\sum y_i + \theta\sum n_i - 2\theta^2\sum n_i\right]$$

$$= \frac{1}{\theta^2(1-\theta)^2}\left[-2\theta^2\sum n_i + 2\theta\sum y_i - \sum y_i\right]$$

$$\left.\frac{\partial^2 l(\theta;\underset{\sim}{y})}{\partial\theta^2}\right|_{\hat\theta=\frac{\sum y_i}{\sum n_i}} = \frac{1}{\hat\theta^2(1-\hat\theta)^2}\left[-2\left(\frac{\sum y_i}{\sum n_i}\right)^2\sum n_i + 2\left(\frac{\sum y_i}{\sum n_i}\right)\sum y_i - \sum y_i\right]$$

$$= \frac{1}{\hat\theta^2(1-\hat\theta)^2}\left[-2\frac{(\sum y_i)^2}{\sum n_i} + 2\frac{(\sum y_i)^2}{\sum n_i} - \sum y_i\right]$$

$$= -\frac{\sum y_i}{\hat\theta^2(1-\hat\theta)^2} < 0 \qquad (\sum y_i > 0)$$

$\therefore\ \hat\theta = \dfrac{\sum y_i}{\sum n_i}$ is the MLE of θ.

(c) Numerical Method (Newton-Raphson Method):

we need to solve

$\boxed{\begin{array}{l}\sum y_i = 363 \\ \sum n_i = 734\end{array}}$

$$\frac{\partial l(\theta;\underset{\sim}{y})}{\partial\theta} = 0 \iff \frac{\sum y_i - \theta\sum n_i}{\theta(1-\theta)} = 0$$

$$\iff \frac{363 - 734\theta}{\theta(1-\theta)} = 0$$

Let:

$$g(\theta) = \frac{363 - 734\theta}{\theta(1-\theta)}$$

$$g'(\theta) = \frac{1}{\theta^2(1-\theta)^2}\left[-(2)(734)\theta^2 + (2)(363)\theta - 363\right]$$

$$= \frac{1}{\theta^2(1-\theta)^2}\left[726\theta - 1468\theta^2 - 363\right]$$

The iterative procedure:

$$\hat{\theta}^{(r+1)} = \hat{\theta}^{(r)} - \frac{g(\hat{\theta}^{(r)})}{g'(\hat{\theta}^{(r)})} \qquad r = 1, 2, 3, \ldots$$

$$= \hat{\theta}^{(r)} - \frac{\dfrac{363 - 734\hat{\theta}^{(r)}}{\hat{\theta}^{(r)}(1-\hat{\theta}^{(r)})}}{\dfrac{726\hat{\theta}^{(r)} - 1468(\hat{\theta}^{(r)})^2 - 363}{(\hat{\theta}^{(r)})^2(1-(\hat{\theta}^{(r)}))^2}}$$

$$= \hat{\theta}^{(r)} - \frac{363 - 734\hat{\theta}^{(r)}}{\dfrac{726\hat{\theta}^{(r)} - 1468\hat{\theta}^2 - 363}{\hat{\theta}^{(r)}(1-\hat{\theta}^{(r)})}}$$

We choose the initial value of θ: $\hat{\theta}^{(0)} = 0.4$

$$\hat{\theta}^{(1)} = 0.4 - \frac{363 - 734(0,4)}{\dfrac{726(0,4) - 1468(0,4^2) - 363}{(0,4)(0,6)}}$$

$$= 0.4542$$

$$\hat{\theta}^{(2)} = 0.4542 - \frac{363 - 734(0.4542)}{\dfrac{726(0.4542) - 1468(0.4542)^2 - 363}{(0.4542)(0.5458)}}$$

$$= 0.4760$$

$\hat{\theta}^{(3)} = \ldots = 0.4857$

$\hat{\theta}^{(4)} = \ldots = 0.4902$

$\hat{\theta}^{(5)} = \ldots = 0.4924$

$\hat{\theta}^{(6)} = \ldots = 0.4935$

$\hat{\theta}^{(7)} = \ldots = 0.4940$

$\hat{\theta}^{(8)} = \ldots = 0.4943$

$\hat{\theta}^{(9)} = \ldots = 0.4944$

$\hat{\theta}^{(10)} = \ldots = \boxed{0.4945}$

Bisection Method:

The likelihood function is

$$l(\theta; \underline{y}) = \underbrace{\sum \ln\binom{n_i}{y_i}}_{\text{لا تعتمد على } \theta} + \underbrace{\ln(\theta)\sum y_i + \ln(1-\theta)\sum(n_i - y_i)}_{\text{تعتمد على } \theta}$$

Define

$$l^*(\theta) = \ln(\theta)\sum y_i + \ln(1-\theta)\sum(n_i - y_i) \qquad \left(\begin{array}{l}\sum y_i = 363 \\ \sum n_i = 734\end{array}\right)$$

$$= 363\ln(\theta) + 371\ln(1-\theta)$$

Let's strat with $\theta^{(1)} = 0.1$ and $\theta^{(2)} = 0.9$

k	$\theta^{(k)}$	$l^*(\theta^{(k)})$	$\theta^{(k+1)}$
1	0.1	-874.9271	
2	0.9	-892.5049	$\rightarrow \theta^{(3)} = \frac{\theta^{(1)} + \theta^{(2)}}{2} = 0.5$
3	0.5	-508.7700	$\rightarrow \theta^{(4)} = \frac{\theta^{(1)} + \theta^{(3)}}{2} = 0.3$
4	0.3	-569.3685	$\rightarrow \theta^{(5)} = \frac{\theta^{(3)} + \theta^{(4)}}{2} = 0.4$
5	0.4	-522.1298	$\rightarrow \theta^{(6)} = \frac{\theta^{(3)} + \theta^{(5)}}{2} = 0.45$
6	0.45	-511.6558	$\rightarrow \theta^{(7)} = \frac{\theta^{(3)} + \theta^{(6)}}{2} = 0.475$
7	0.475	-508.9352	$\rightarrow \theta^{(8)} = \frac{\theta^{(3)} + \theta^{(7)}}{2} = 0.4875$
8	0.4875	-508.7994	$\rightarrow \theta^{(9)} = \frac{\theta^{(3)} + \theta^{(8)}}{2} = 0.4938$
9	0.4938	-508.7276	$\rightarrow \theta^{(10)} = \frac{\theta^{(3)} + \theta^{(9)}}{2} = 0.4969$
10	0.4969	-508.7345	$\rightarrow \theta^{(11)} = \frac{\theta^{(9)} + \theta^{(10)}}{2} = 0.4954$
11	0.4954	-508.7275	$\rightarrow \theta^{(12)} = \frac{\theta^{(9)} + \theta^{(11)}}{2} = 0.4946$
12	0.4946	-508.7264	$\rightarrow \theta^{(13)} = \frac{\theta^{(11)} + \theta^{(12)}}{2} = 0.4950$
13	$\boxed{0.4950}$	-508.7267	