Introduction to Real Analysis Infinite Series

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Infinite series Convergent Tests

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Infinite Series

If (x_n) is a sequence, then we can define the infinite series

$$\sum_{n=1}^\infty x_n = x_1+x_2+x_3+\ldots$$

Image: A mathematical states and a mathem

Partial Sums

Definition

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$$S_1 = x_1$$

$$S_2 = x_1 + x_2$$

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$$S_n = x_1 + x_2 + \ldots + x_n = \sum_{k=1}^n x_k$$

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then the sequence $\left(S_{n}\right)$ is called the sequence of partial sums of $\left(x_{n}\right).$

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Convergent nfinite Series

If
$$(S_n)$$
 is convergent then $\sum_{n=1}^\infty x_n$ converges and

$$\sum_{n=1}^{\infty} x_n = \lim S_n$$

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Geometric Series

Examples

Geometric Series

$$\sum_{n=0}^{\infty}a^n, \quad a\in \mathbb{R}$$

$$S_n=1+a+a^2+\ldots+a^r$$

What is its sum?

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Image: A mathematical states and a mathem

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Telescoping series

Examples

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$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

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Theorem

If $\sum x_n$ and $\sum y_n$ converges then

 ${\small \bigcirc } \ \sum (x_n+y_n) \text{ converges and }$

$$\sum (x_n+y_n)=\sum x_n+\sum y_n$$

2 $\sum cx_n$ converges and

$$\sum cx_n = c\sum x_n$$

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What about $\sum x_n y_n$?

Cauchy Criterion

Theorem

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$\sum x_n$ is convergent iff $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}:$ $|S_n - S_m| = |x_{m+1} + \ldots + x_n| < \varepsilon \qquad \forall n > m \geq N$

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Cauchy Criterion

Corollary

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If $\sum x_n$ is convergent then $\lim x_n = 0$

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Infinite series

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• $\sum_{n=1}^{\infty} \frac{n}{n+1}$

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Examples

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$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

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Examples

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• Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

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Examples

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• Harmonic Series

$$\sum_{n=1}^{\infty} (-1)^n$$

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Absolutely Convergent

Definition

 $\sum x_n$ is absolutely convergent if $\sum |x_n|$ is convergent

Theorem

If a series is absolutely convergent then it is convergent

If All terms are of the same sign then there is no difference

Examples

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The series is convergent but not absolutely convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Examples

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The series is absolutely convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n}$$

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Conditionally Convergent

Definition

 $\sum x_n$ is conditionally convergent if $\sum x_n$ is convergent but $\sum |x_n|$ is divergent

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Examples

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The series is conditionally convergent .

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

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Rearrangement

Definition

 $\sum y_n$ is a rearrangement of $\sum x_n$ is there is a bijection $f:\mathbb{N}\to\mathbb{N}$ such that

$$y_n = x_{f(n)}$$

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Conditionally Convergent

Examples

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \dots$$
$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots$$
$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \dots$$

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Conditionally Convergent

If $\sum x_n$ is conditionally convergent (y_n) is the subsequence of positive terms of (x_n) and (z_n) is the subsequence of negative terms of (x_n) then $\sum y_n$ and $\sum z_n$ are divergent

Conditionally Convergent

Theorem

If $\sum x_n$ is conditionally convergent and $c\in \bar{\mathbb{R}}$, then there is a rearrangement of the series that converges to c

Theorem

If $\sum x_n$ is absolutely convergent then all rearrangements of the series converge absolutely to the same limit

Comparison Test

Theorem

If $\left(x_{n}\right)$ and $\left(y_{n}\right)$ are two sequences and

$$0 \le x_n \le y_n \qquad \forall n \ge N \in \mathbb{N}$$

then

• If
$$\sum y_n$$
 is convergent then $\sum x_n$ is convergent
• If $\sum x_n$ is divergent then $\sum y_n$ is divergent.

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Comparison Test

Examples

$$\sum_{n=1}^{\infty} \frac{1}{n+3^n}$$

 $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

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Limit Comparison Test

Theorem

If (x_n) and (y_n) are positive sequences and

$$\lim \frac{x_n}{y_n}$$

exists

O If $\lim \frac{x_n}{y_n} \neq 0$ then $\sum x_n$ and $\sum y_n$ either both converge of both diverge

② If
$$\lim \frac{x_n}{y_n} = 0$$
 and $\sum y_n$ converges then $\sum x_n$ converges

Limit Comparison Test

Examples

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$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

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p -series

Examples

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad p \ge 2$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$	•	$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad 0$
$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad p \ge 2$	0	$\sum_{n=1}^{\infty} \frac{1}{n^2}$
$\sum_{n=1}^{\infty} \frac{1}{n^p}$	8	$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad p \ge 2$
	4	$\sum_{n=1}^{\infty} \frac{1}{n^p}$

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Theorem

If $\sum x_n$ is a series, and

$$r = \lim \sqrt[n]{|x_n|}$$

then

$$\label{eq:starsest} \mathbf{D} \, \sum x_n \text{ is absolutely convergent if } r < 1$$

2
$$\sum x_n$$
 is divergent if $r > 1$

• the test fails if
$$r = 1$$

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Examples

 $\sum_{n=1}^{\infty} \frac{1}{n^n}$

Examples

 $\sum_{n=1}^{\infty} \frac{2^n}{n^4}$

Examples

 $\sum_{n=1}^{\infty} \frac{1}{n^4}$

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Theorem

If $\sum x_n$ is a series, and

$$r = \lim \left| \frac{x_{n+1}}{x_n} \right|$$

then

•
$$\sum x_n$$
 is absolutely convergent if $r < 1$
• $\sum x_n$ is divergent if $r > 1$
• the test fails if $r = 1$

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Examples

 $\sum_{n=1}^{\infty} \frac{2^n}{n^4}$

Examples

 $\sum_{n=1}^{\infty} \frac{1}{n^n}$

Examples

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

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Ratio Test

Examples

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$$\sum_{n=1}^{\infty} \frac{n}{2^n} \qquad \qquad \sum_{n=1}^{\infty} \frac{2^n}{n}$$

Examples

 $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Definition

The sequence (x_n) is alternating if the sign of x_n is different than the sign of x_{n+1} for all n. In this case the series $\sum x_n$ is an alternating series

Theorem

If $\left(x_{n}\right)$ is a positive decreasing sequence whose limit is 0 then the alternating series

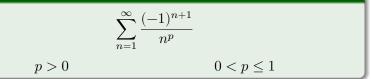
$$\sum (-1)^{n+1} x_n$$

is convergent and

$$\left|\sum_{k=n+1}^\infty (-1)^{k+1} x_k\right| \leq x_{n+1}$$

Examples

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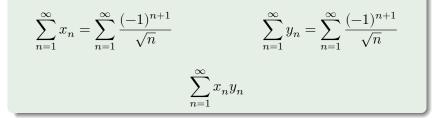


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Examples

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Integral Test

Theorem

If $f:[1,\infty)\to[0,\infty)$ is decreasing and integrable on [1,a] for all a>1 then the series $\sum_{n=1}^\infty f(n)$ is convergent iff

$$\int_1^\infty f(x)dx = \lim_{a\to\infty}\int_1^a f(x)dx$$

exist

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Integral Test

Examples

 $\sum_{n=1}^{\infty} \frac{1}{n^p}$

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Exercises

- Prove that changing a finite number of terms of a series does not affect its convergence.
- ② If $x_n ≥ 0$ for all n ∈ N, prove that the series $\sum x_n$ converges if, and only if, the sequence $S_n = \sum_{k=1}^n x_k$ is bounded above

③ Prove that if a > 0, then

$$\sum_{n=0}^{\infty} \frac{1}{(n+a)(n+1+a)} = \frac{1}{a}$$

Exercises

Test the following series for convergence

 $\sum_{n=1}^\infty \frac{1}{(n+1)(n+1)}$

 $\sum_{n=1}^{\infty}(-1)^n\frac{\log n}{n}$

 $\sum_{n=1}^{\infty} \frac{n^2}{n!}$

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 $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

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- 0 Prove that the convergence of $\sum a_n^2$ and $\sum b_n^2$ implies the convergence of $\sum a_n b_n$
- O Prove that the convergence of $\sum a_n^2$ implies the convergence of $\sum \frac{a_n}{n}$
- § Let the sequence (x_n) be positive and decreasing. If $\sum x_n$ converges, prove that $\lim nx_n=0$