# Introduction to Real Analysis

## Infinite Series

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# Infinite Series

If $(x_n)$ is a sequence, then we can define the infinite series

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + x_3 + \ldots$$

# Partial Sums

## Definition

If

$$S_1 = x_1$$

$$S_2 = x_1 + x_2$$

$$\vdots$$

$$S_n = x_1 + x_2 + \ldots + x_n = \sum_{k=1}^{n} x_k$$

then the sequence $(S_n)$ is called the sequence of partial sums of $(x_n)$.

# Convergent nfinite Series

If $(S_n)$ is convergent then $\sum_{n=1}^{\infty} x_n$ converges and

$$\sum_{n=1}^{\infty} x_n = \lim S_n$$

# Geometric Series

## Examples

Geometric Series

$$\sum_{n=0}^{\infty} a^n, \quad a \in \mathbb{R}$$

$$S_n = 1 + a + a^2 + ... + a^n$$

What is its sum?

# Telescoping series

## Examples

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

.

# Series

**Theorem**

If $\sum x_n$ and $\sum y_n$ converges then

1. $\sum (x_n + y_n)$ converges and
$$\sum (x_n + y_n) = \sum x_n + \sum y_n$$
2. $\sum c x_n$ converges and
$$\sum c x_n = c \sum x_n$$

What about $\sum x_n y_n$?

# Cauchy Criterion

**Theorem**

$\sum x_n$ is convergent iff

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} :$$

$$|S_n - S_m| = |x_{m+1} + ... + x_n| < \varepsilon \qquad \forall n > m \geq N$$

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# Cauchy Criterion

## Corollary

If $\sum x_n$ is convergent then $\lim x_n = 0$

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## Examples

- $$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

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## Examples

- $$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

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## Examples

- Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

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## Examples

- Harmonic Series

$$\sum_{n=1}^{\infty}(-1)^n$$

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# Absolutely Convergent

**Definition**

$\sum x_n$ is absolutely convergent if $\sum |x_n|$ is convergent

**Theorem**

If a series is absolutely convergent then it is convergent

If All terms are of the same sign then there is no difference

## Examples

The series is convergent but not absolutely convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

.

## Examples

The series is absolutely convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n}$$

.

# Conditionally Convergent

**Definition**

$\sum x_n$ is conditionally convergent if $\sum x_n$ is convergent but $\sum |x_n|$ is divergent

## Examples

The series is conditionally convergent .

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

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# Rearrangement

## Definition

$\sum y_n$ is a rearrangement of $\sum x_n$ is there is a bijection $f : \mathbb{N} \to \mathbb{N}$ such that

$$y_n = x_{f(n)}$$

# Conditionally Convergent

## Examples

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \dots$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots$$

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \dots$$

# Conditionally Convergent

If $\sum x_n$ is conditionally convergent
$(y_n)$ is the subsequence of positive terms of $(x_n)$
and $(z_n)$ is the subsequence of negative terms of $(x_n)$
then $\sum y_n$ and $\sum z_n$ are divergent

# Conditionally Convergent

**Theorem**

If $\sum x_n$ is conditionally convergent and $c \in \bar{\mathbb{R}}$, then there is a rearrangement of the series that converges to $c$

**Theorem**

If $\sum x_n$ is absolutely convergent then all rearrangements of the series converge absolutely to the same limit

# Comparison Test

**Theorem**

If $(x_n)$ and $(y_n)$ are two sequences and

$$0 \leq x_n \leq y_n \qquad \forall n \geq N \in \mathbb{N}$$

then

1. If $\sum y_n$ is convergent then $\sum x_n$ is convergent.
2. If $\sum x_n$ is divergent then $\sum y_n$ is divergent.

# Comparison Test

## Examples

$$\sum_{n=1}^{\infty} \frac{1}{n+3^n} \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

.

# Limit Comparison Test

**Theorem**

If $(x_n)$ and $(y_n)$ are positive sequences and

$$\lim \frac{x_n}{y_n}$$

exists

1. If $\lim \frac{x_n}{y_n} \neq 0$ then $\sum x_n$ and $\sum y_n$ either both converge of both diverge
2. If $\lim \frac{x_n}{y_n} = 0$ and $\sum y_n$ converges then $\sum x_n$ converges

# Limit Comparison Test

## Examples

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

.

# $p$ -series

## Examples

1. $$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad 0 < p \leq 1$$

2. $$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

3. $$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad p \geq 2$$

4. $$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

# Root Test

## Theorem

If $\sum x_n$ is a series, and

$$r = \lim \sqrt[n]{|x_n|}$$

then

1. $\sum x_n$ is absolutely convergent if $r < 1$
2. $\sum x_n$ is divergent if $r > 1$
3. the test fails if $r = 1$

# Root Test

## Examples

$$\sum_{n=1}^{\infty} \frac{1}{n^n}$$

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# Root Test

## Examples

$$\sum_{n=1}^{\infty} \frac{2^n}{n^4}$$

.

# Root Test

## Examples

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

.

# Ratio Test

## Theorem

If $\sum x_n$ is a series, and

$$r = \lim \left| \frac{x_{n+1}}{x_n} \right|$$

then

1. $\sum x_n$ is absolutely convergent if $r < 1$
2. $\sum x_n$ is divergent if $r > 1$
3. the test fails if $r = 1$

# Ratio Test

## Examples

$$\sum_{n=1}^{\infty} \frac{2^n}{n^4}$$

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# Ratio Test

## Examples

$$\sum_{n=1}^{\infty} \frac{1}{n^n}$$

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# Ratio Test

## Examples

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

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# Ratio Test

## Examples

$$\sum_{n=1}^{\infty} \frac{n}{2^n} \qquad\qquad \sum_{n=1}^{\infty} \frac{2^n}{n}$$

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# Ratio Test

## Examples

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

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# Alternating Series

## Definition

The sequence $(x_n)$ is alternating if the sign of $x_n$ is different than the sign of $x_{n+1}$ for all $n$. In this case the series $\sum x_n$ is an alternating series

# Alternating Series

## Theorem

If $(x_n)$ is a positive decreasing sequence whose limit is $0$ then the alternating series

$$\sum (-1)^{n+1} x_n$$

is convergent and

$$\left| \sum_{k=n+1}^{\infty} (-1)^{k+1} x_k \right| \leq x_{n+1}$$

# Alternating Series

## Examples

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$$

$p > 0$ $0 < p \leq 1$

.

# Alternating Series

## Examples

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \qquad \sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} x_n y_n$$

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# Integral Test

## Theorem

If $f : [1, \infty) \to [0, \infty)$ is decreasing and integrable on $[1, a]$ for all $a > 1$ then the series $\sum_{n=1}^{\infty} f(n)$ is convergent iff

$$\int_1^{\infty} f(x)dx = \lim_{a \to \infty} \int_1^{a} f(x)dx$$

exist

# Integral Test

## Examples

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

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# Exercises

1. Prove that changing a finite number of terms of a series does not affect its convergence.
2. If $x_n \geq 0$ for all $n \in N$, prove that the series $\sum x_n$ converges if, and only if, the sequence $S_n = \sum_{k=1}^{n} x_k$ is bounded above
3. Prove that if $a > 0$, then

$$\sum_{n=0}^{\infty} \frac{1}{(n+a)(n+1+a)} = \frac{1}{a}$$

# Exercises

1. Test the following series for convergence
   1. $$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+1)}$$
   2. $$\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$$
   3. $$\sum_{n=1}^{\infty} \frac{n^2}{n!}$$
   4. $$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

# Exercises

1. Prove that the convergence of $\sum a_n^2$ and $\sum b_n^2$ implies the convergence of $\sum a_n b_n$
2. Prove that the convergence of $\sum a_n^2$ implies the convergence of $\sum \frac{a_n}{n}$
3. Let the sequence $(x_n)$ be positive and decreasing. If $\sum x_n$ converges, prove that $\lim n x_n = 0$