Introduction to Real Analysis

Infinite Series

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Infinite Series

If (x_n) is a sequence, then we can define the infinite series

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + x_3 + \dots$$

Partial Sums

Definition

If

$$S_1 = x_1$$

$$S_2 = x_1 + x_2$$

$$\vdots$$

$$S_n = x_1 + x_2 + \ldots + x_n = \sum_{k=1}^{n} x_k$$

then the sequence (S_n) is called the sequence of partial sums of (x_n).

Convergent nfinite Series

If (S_n) is convergent then $\sum_{n=1}^{\infty} x_n$ converges and

$$\sum_{n=1}^{\infty} x_n = \lim S_n$$

Examples

1. Geometric Series
$$\sum_{n=0}^{\infty} a^n, \quad a \in \mathbb{R}$$
$$S_n = 1 + a + a^2 + ... + a^n$$

Series

Theorem

If $\sum x_n$ and $\sum y_n$ converges then

1. $\sum(x_n + y_n)$ converges and
$$\sum(x_n + y_n) = \sum x_n + \sum y_n$$
2. $\sum cx_n$ converges and
$$\sum cx_n = c\sum x_n$$

What about $\sum x_n y_n$?

Cauchy Criterion

Theorem

$\sum x_n$ is convergent iff

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} :$$
$$|S_n - S_m| = |x_{m+1} + \ldots + x_n| < \varepsilon \qquad \forall n > m \geq N$$

Corollary

If $\sum x_n$ is convergent then $\lim x_n = 0$

Examples

1. $$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

2. $$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

3. Harmonic Series
$$\sum_{n=1}^{\infty} \frac{1}{n}$$

4. $$\sum_{n=1}^{\infty} (-1)^n$$

Absolutely Convergent

Definition

$\sum x_n$ is absolutely convergent if $\sum |x_n|$ is convergent

Theorem

If a series is absolutely convergent then it is convergent

If All terms are of the same sign then there is no difference

Examples

1. $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

   is not absolutely convergent.

2. $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n}$$

   is absolutely convergent.

Conditionally Convergent

Definition

$\sum x_n$ is conditionally convergent if $\sum x_n$ is convergent but $\sum |x_n|$ is divergent

Definition

$\sum y_n$ is a rearrangement of $\sum x_n$ is there is a bijection $f : \mathbb{N} \to \mathbb{N}$ such that

$$y_n = x_{f(n)}$$

If All terms are of the same sign then there is no difference

Conditionally Convergent

Examples

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \dots$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots$$

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \dots$$

Conditionally Convergent

If $\sum x_n$ is conditionally convergent
(y_n) is the subsequence of positive terms of (x_n)
and (z_n) is the subsequence of negative terms of (x_n)
then $\sum y_n$ and $\sum z_n$ are divergent

Conditionally Convergent

Theorem

If $\sum x_n$ is conditionally convergent and $c \in \bar{\mathbb{R}}$, then there is a rearrangement of the series that converges to c

Theorem

If $\sum x_n$ is absolutely convergent then all rearrangements of the series converge absolutely to the same limit

Comparison Test

Theorem

If (x_n) and (y_n) are positive sequences and

$$x_n \leq y_n \qquad \forall n \geq N \in \mathbb{N}$$

then

1. If $\sum y_n$ is convergent then $\sum x_n$ is convergent.
2. If $\sum x_n$ is divergent then $\sum y_n$ is divergent.

Comparison Test

Examples

1. $$\sum_{n=1}^{\infty} \frac{1}{n+3^n}$$

2. $$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

Limit Comparison Test

Theorem

If (x_n) and (y_n) are positive sequences and

$$\lim \frac{x_n}{y_n}$$

exists

1. If $\lim \frac{x_n}{y_n} \neq 0$ then $\sum x_n$ and $\sum y_n$ either both converge of both diverge
2. If $\lim \frac{x_n}{y_n} = 0$ and $\sum y_n$ converges then $\sum x_n$ converges

Telescoping series

Examples

1. $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

2. $$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

p -series

Examples

1. $$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad 0 < p \leq 1$$

2. $$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

3. $$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad p \geq 2$$

4. $$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Root Test

Theorem

If $\sum x_n$ is a series, and

$$r = \lim \sqrt[n]{|x_n|}$$

then

1. $\sum x_n$ is absolutely convergent if $r < 1$
2. $\sum x_n$ is divergent if $r > 1$
3. the test fails if $r = 1$

Root Test

Examples

1. $$\sum_{n=1}^{\infty} \frac{1}{n^n}$$

2. $$\sum_{n=1}^{\infty} \frac{2^n}{n^4}$$

Ratio Test

Theorem

If $\sum x_n$ is a series, and

$$r = \lim \left| \frac{x_{n+1}}{x_n} \right|$$

then

1. $\sum x_n$ is absolutely convergent if $r < 1$
2. $\sum x_n$ is divergent if $r > 1$
3. the test fails if $r = 1$

Ratio Test

Examples

1. $$\sum_{n=1}^{\infty} \frac{1}{n!}$$

2. $$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

3. $$\sum_{n=1}^{\infty} \frac{2^n}{n}$$

4. $$\sum_{n=1}^{\infty} \frac{1}{n}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Root test and Ratio Test

Corollary

If $\sum x_n$ is a series, and

$$\lim \left| \frac{x_{n+1}}{x_n} \right| = r$$

exists, then

$$\lim \sqrt[n]{|x_n|} = r$$

1. $\sum x_n$ is absolutely convergent if $r < 1$
2. $\sum x_n$ is divergent if $r > 1$
3. the test fails if $r = 1$

Alternating Series

Definition

The sequence (x_n) is alternating if the sign of x_n is different than the sign of x_{n+1} for all n. In this case the series $\sum x_n$ is an alternating series

Alternating Series

Theorem

If (x_n) is a positive decreasing sequence whose limit s 0 then the alternating series

$$\sum(-1)^{n+1}x_n$$

is convergent and

$$\left|\sum_{k=n+1}^{\infty}(-1)^{k+1}x_k\right| \leq x_{n+1}$$

Alternating Series

Examples

1.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$$

$$p > 0$$

$$0 < p \leq 1$$

Integral Test

Theorem

If $f:[1,\infty)\to[0,\infty)$ is decreasing and integrable on $[1,b]$ for all $b>1$ then
the series $\sum_{n=1}^{\infty} f(n)$ is convergent iff

$$\int_1^{\infty} f(t)dt = \lim_{a\to\infty}\int_1^{a} f(t)dt$$

exist

Integral Test

Examples

1. $$\sum_{n=1}^{\infty} \frac{1}{n^p}$$