Introduction to Real Analysis Infinite Series

Ibraheem Alolyan

King Saud University

Table of Contents





Ibraheem Alolyan Real Analysis

æ

Infinite Series

If (x_n) is a sequence, then we can define the infinite series

$$\sum_{n=1}^\infty x_n = x_1+x_2+x_3+\ldots$$

Image: A mathematical states and a mathem

Partial Sums

Definition

lf

$$S_1 = x_1$$

$$S_2 = x_1 + x_2$$

:

$$S_n = x_1 + x_2 + \ldots + x_n = \sum_{k=1}^n x_k$$

 α

then the sequence $\left(S_{n}\right)$ is called the sequence of partial sums of $\left(x_{n}\right).$

э

< A > <

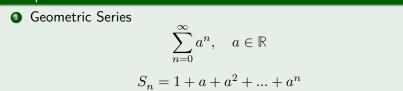
Convergent nfinite Series

If
$$(S_n)$$
 is convergent then $\sum_{n=1}^\infty x_n$ converges and

$$\sum_{n=1}^{\infty} x_n = \lim S_n$$

æ

Examples



æ



Theorem

If $\sum x_n$ and $\sum y_n$ converges then

 ${\small \bigcirc } \ \sum (x_n+y_n) \text{ converges and }$

$$\sum (x_n+y_n)=\sum x_n+\sum y_n$$

2 $\sum cx_n$ converges and

$$\sum cx_n = c\sum x_n$$

▲ 同 ▶ ▲ 三

3)) B

What about $\sum x_n y_n$?

Cauchy Criterion

Theorem

 $\sum x_n$ is convergent iff

$$\begin{aligned} \forall \varepsilon > 0 \ \exists N \in \mathbb{N}: \\ |S_n - S_m| = |x_{m+1} + \ldots + x_n| < \varepsilon \qquad \forall n > m \geq N \end{aligned}$$

Corollary

If $\sum x_n$ is convergent then $\lim x_n=0$

< A > <

Examples 1 $\sum_{n=1}^{\infty} \frac{n}{n+1}$ 2 $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ B Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ 4 $\sum_{n=1}^{\infty} (-1)^n$

æ

(日)

Absolutely Convergent

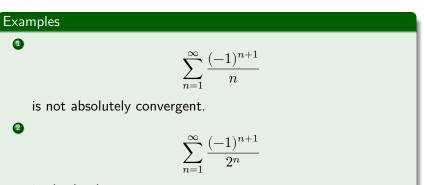
Definition

 $\sum x_n$ is absolutely convergent if $\sum |x_n|$ is convergent

Theorem

If a series is absolutely convergent then it is convergent

If All terms are of the same sign then there is no difference



is absolutely convergent.

æ

Conditionally Convergent

Definition

 $\sum x_n$ is conditionally convergent if $\sum x_n$ is convergent but $\sum |x_n|$ is divergent

Definition

 $\sum y_n$ is a rearrangement of $\sum x_n$ is there is a bijection $f:\mathbb{N}\to\mathbb{N}$ such that

$$y_n = x_{f(n)}$$

If All terms are of the same sign then there is no difference

Conditionally Convergent

Examples

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \dots$$
$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots$$
$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \dots$$

・ロト ・ 同ト ・ ヨト ・

문어 문

Conditionally Convergent

If $\sum x_n$ is conditionally convergent (y_n) is the subsequence of positive terms of (x_n) and (z_n) is the subsequence of negative terms of (x_n) then $\sum y_n$ and $\sum z_n$ are divergent

Conditionally Convergent

Theorem

If $\sum x_n$ is conditionally convergent and $c\in \bar{\mathbb{R}}$, then there is a rearrangement of the series that converges to c

Theorem

If $\sum x_n$ is absolutely convergent then all rearrangements of the series converge absolutely to the same limit

Comparison Test

Theorem

If $\left(x_{n}\right)$ and $\left(y_{n}\right)$ are positive sequences and

$$x_n \leq y_n \qquad \forall n \geq N \in \mathbb{N}$$

then

• If
$$\sum y_n$$
 is convergent then $\sum x_n$ is convergent
• If $\sum x_n$ is divergent then $\sum y_n$ is divergent.

æ

∍⊳

< 4 P < 3

Comparison Test

Examples $\sum_{n=1}^{\infty} \frac{1}{n+3^n}$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

æ

Limit Comparison Test

Theorem

If (x_n) and (y_n) are positive sequences and

$$\lim \frac{x_n}{y_n}$$

exists

O If $\lim \frac{x_n}{y_n} \neq 0$ then $\sum x_n$ and $\sum y_n$ either both converge of both diverge

② If
$$\lim \frac{x_n}{y_n} = 0$$
 and $\sum y_n$ converges then $\sum x_n$ converges

Telescoping series

Examples $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ $\sum_{n=1}^{\infty} \frac{1}{n^2}$

æ

(日)

p -series

Examples

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad p \ge 2$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$	•	$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad 0$
$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad p \ge 2$	0	$\sum_{n=1}^{\infty} \frac{1}{n^2}$
• $\sum_{n=1}^{\infty} \frac{1}{n^p}$	8	$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad p \ge 2$
	4	$\sum_{n=1}^{\infty} \frac{1}{n^p}$

Ibraheem Alolyan

Root Test

Theorem

If $\sum x_n$ is a series, and

$$r = \lim \sqrt[n]{|x_n|}$$

then

$$\label{eq:starsest} \mathbf{D} \, \sum x_n \text{ is absolutely convergent if } r < 1$$

2
$$\sum x_n$$
 is divergent if $r > 1$

• the test fails if
$$r = 1$$

æ

Root Test

Examples $\sum_{n=1}^{\infty} \frac{1}{n^n}$ $\sum_{n=1}^{\infty} \frac{2^n}{n^4}$

æ

Ratio Test

Theorem

If $\sum x_n$ is a series, and

$$r = \lim \left| \frac{x_{n+1}}{x_n} \right|$$

then

•
$$\sum x_n$$
 is absolutely convergent if $r < 1$
• $\sum x_n$ is divergent if $r > 1$
• the test fails if $r = 1$

æ

'≣ ▶

(日)

Ratio Test

Examples

Examples	
$\sum_{n=1}^{\infty} \frac{1}{n!}$	
$\sum_{n=1}^{\infty} \frac{n}{2^n}$	
$\sum_{n=1}^{\infty} \frac{2^n}{n}$	
$\sum_{n=1}^{\infty} \frac{1}{n}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$	50
Ibraheem Alolyan Real Analysis	
ibrancent Aloiyan — Real Analysis	

Root test and Ratio Test

Corollary

If $\sum x_n$ is a series, and

$$\lim \left| \frac{x_{n+1}}{x_n} \right| = r$$

exists, then

$$\lim \sqrt[n]{|x_n|} = r$$

- ${\small \bigcirc } \ \sum x_n \text{ is absolutely convergent if } r < 1$
- 2 $\sum x_n$ is divergent if r > 1
- O the test fails if r=1

Alternating Series

Definition

The sequence (x_n) is alternating if the sign of x_n is different than the sign of x_{n+1} for all n. In this case the series $\sum x_n$ is an alternating series

Alternating Series

Theorem

If $\left(x_{n}\right)$ is a positive decreasing sequence whose limit s 0 then the alternating series

$$\sum (-1)^{n+1} x_n$$

is convergent and

$$\left|\sum_{k=n+1}^\infty (-1)^{k+1} x_k\right| \leq x_{n+1}$$

Alternating Series

Examples $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ p > 0 0

Ibraheem Alolyan Real Analysis

æ

Integral Test

Theorem

If
$$f: [1,\infty) \to [0,\infty)$$
 is decreasing and integrable on $[1,b]$ for all $b > 1$ then
the series $\sum_{n=1}^{\infty} f(n)$ is convergent iff
$$\int_{1}^{\infty} f(t)dt = \lim_{a \to \infty} \int_{1}^{a} f(t)dt$$

exist

æ

Integral Test

Examples $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Ibraheem Alolyan Real Analysis

æ