SERIES OF NUMBERS

If $x_n \geq 0$ for all n, prove that the convergence of $\sum x_n$ implies the convergence of $\sum x_n^2$, but that the converse is false.

ANSWER:

If the series $\sum_{n\geq 1} x_n$ converges, then $\lim_{n\to+\infty} x_n = 0$. There is $N \in \mathbb{N}$ such that for $n \geq N$, $x_n \leq 1$, which implies that $x_n^2 \leq x_n$ for all $n \geq N$. Then the series $\sum_{n\geq 1} x_n^2$ converges.

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The series $\sum_{n\geq 1} \frac{1}{n^2}$ is convergent but $\sum_{n\geq 1} \frac{1}{n}$ is not convergent.

Let $x_n > 0$ for all n and $y_n = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$. Prove that the series $\sum y_n$ is divergent.

ANSWER:

$y_n = \frac{1}{n}\sum_{k=1}^{n} x_k \geq \frac{x_1}{n}$. Then $\sum_{k=1}^{n} y_k \geq x_1 \sum_{k=1}^{n} \frac{1}{k}$, which proves that the series $\sum_{n\geq 1} y_n$ is divergent.

Prove that the convergence of $\sum a_n^2$ and $\sum b_n^2$ implies the convergence of $\sum a_n b_n$.

ANSWER:

Since $|2a_n b_n| \leq a_n^2 + b_n^2$, then if the series $\sum_{n\geq 1} a_n^2$ and $\sum_{n=1}^{+\infty} b_n^2$ converge the series $\sum_{n=1}^{+\infty} a_n b_n$ is absolutely convergent.

Let the sequence (x_n) be positive and decreasing. If $\sum x_n$ converges prove that $\lim nx_n = 0$.

ANSWER:

If the sequence $(x_n)_n$ is positive decreasing and the series $\sum_{n\geq 1} x_n$ is convergent, then by the Cauchy property $\lim_{n\to+\infty} \sum_{k=[\frac{n}{2}]}^{n} x_k = 0$. But $\sum_{k=[\frac{n}{2}]}^{n} x_k \geq \frac{n}{2} x_n$. Then $\lim_{n\to+\infty} nx_n = 0$.

If the series $\sum x_n$ diverges show that $\sum \frac{x_n}{1+x_n}$ also diverges.

ANSWER:

If the sequence $(x_n)_n$ is bounded, say $x_n \leq M$, then $\frac{x_n}{1+x_n} \geq \frac{x_n}{1+M}$, which proves that the series $\sum_{n\geq 1} \frac{x_n}{1+x_n}$ is divergent.

If the sequence $(x_n)_n$ is not bounded, there exists a subsequence $(x_{\varphi(n)})_n$ which tends to $+\infty$, then $\lim_{n\to+\infty} \frac{x_{\varphi(n)}}{1+x_{\varphi(n)}} = 1$ and the series $\sum_{n\geq 1} \frac{x_n}{1+x_n}$ is divergent.
