Chapter 3: Series of Functions

Series of Functions

Uniform Convergence

Let $D \subseteq \mathbb{R}$ and let $(f_k)_{k\geq 1}$ be functions $f_k : D \to \mathbb{R}$. Define the **partial sum functions**

$$s_n(x) := \sum_{k=1}^{n} f_k(x) \qquad (n \in \mathbb{N},\ x \in D).$$

If the limit exists for every $x \in D$, define the **pointwise sum function**

$$f(x) := \lim_{n\to\infty} s_n(x) \quad \Big(\text{equivalently } f(x) = \sum_{k=1}^{\infty} f_k(x)\Big).$$

We say $\sum_{k=1}^{\infty} f_k$ **converges uniformly** on D to f if

$$\sup_{x\in D} |s_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0.$$

Uniform Cauchy Criterion

Theorem

The series $\sum_{k=1}^{\infty} f_k$ converges uniformly on D **if and only if** for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m > n \geq N$,

$$\sup_{x \in D} \left| \sum_{k=n+1}^{m} f_k(x) \right| < \varepsilon.$$

Uniform Cauchy Criterion

Proof

Let $s_n(x) = \sum_{k=1}^{n} f_k(x)$. Then, for $m > n$,

$$s_m(x) - s_n(x) = \sum_{k=n+1}^{m} f_k(x).$$

Uniform convergence of (s_n) is equivalent to (s_n) being **uniformly Cauchy**: for every $\varepsilon > 0$ there exists N such that for all $m > n \geq N$,

$$\sup_{x \in D} |s_m(x) - s_n(x)| < \varepsilon,$$

which is exactly the stated condition.

Weierstrass M-Test

Theorem

Let $f_n : D \to \mathbb{R}$ and assume there exist numbers $M_n \geq 0$ such that

$$\sup_{x \in D} |f_n(x)| \leq M_n \quad (n \in \mathbb{N}),$$

and the numerical series $\sum_{n=1}^{\infty} M_n$ converges. Then $\sum_{n=1}^{\infty} f_n$ converges uniformly on D, and

$$\sup_{x \in D} \left| \sum_{n=1}^{\infty} f_n(x) \right| \leq \sum_{n=1}^{\infty} M_n.$$

Weierstrass M-Test

Proof

For $m > n$ and any $x \in D$,

$$\left| \sum_{k=n+1}^{m} f_k(x) \right| \leq \sum_{k=n+1}^{m} |f_k(x)| \leq \sum_{k=n+1}^{m} M_k.$$

Taking $\sup_{x \in D}$ yields

$$\sup_{x \in D} \left| \sum_{k=n+1}^{m} f_k(x) \right| \leq \sum_{k=n+1}^{m} M_k.$$

Since $\sum M_k$ converges, its tails go to 0, so the partial sums are uniformly Cauchy. Hence $\sum f_n$ converges uniformly on D.

M-Test Example

Example

Consider

$$\sum_{n=1}^{\infty} \frac{n \sin(nx)}{e^n}, \qquad x \in \mathbb{R}.$$

Since $|\sin(nx)| \leq 1$,

$$\sup_{x \in \mathbb{R}} \left| \frac{n \sin(nx)}{e^n} \right| \leq \frac{n}{e^n}.$$

Because $\sum_{n=1}^{\infty} \frac{n}{e^n}$ converges, the series converges uniformly on $\mathbb{R}$.

Dirichlet Test (Uniform Convergence)

Theorem

Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$. Assume:

1. The partial sums (S_n) are uniformly bounded on D: there exists $M > 0$ such that

$$|S_n(x)| \leq M \quad \text{for all } x \in D \text{ and all } n \in \mathbb{N}.$$

2. For each fixed $x \in D$, the sequence $(g_n(x))$ is nonincreasing:

$$g_{n+1}(x) \leq g_n(x) \quad \text{for all } n.$$

3. The sequence (g_n) converges uniformly to 0 on D:

$$\sup_{x \in D} |g_n(x)| \xrightarrow[n\to\infty]{} 0.$$

Then the series

$$\sum_{n=1}^{\infty} f_n(x)\, g_n(x)$$

Dirichlet Example (1/2): Bounded sums + monotone g_n

Proof

Consider

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{3n+1}, \qquad x \in [0,1],$$

and write it as $\sum f_n(x)\, g_n(x)$ with

$$f_n(x) = (-1)^n, \qquad g_n(x) = \frac{x^{3n+1}}{3n+1}.$$

Let $S_N = \sum_{n=0}^{N} (-1)^n$. Then $S_N = \begin{cases} 1, & N \text{ even}, \\ 0, & N \text{ odd}, \end{cases} \quad \Rightarrow \quad |S_N| \leq 1$. Fix $x \in [0,1]$. Since $0 \leq x \leq 1$, we have $x^{3n+4} \leq x^{3n+1}$ and $3n+4 > 3n+1$, hence

$$g_{n+1}(x) = \frac{x^{3n+4}}{3n+4} \leq \frac{x^{3n+1}}{3n+4} < \frac{x^{3n+1}}{3n+1} = g_n(x).$$

Dirichlet Example (2/2): Uniform $g_n \to 0$ + conclusion

Proof

For $x \in [0,1]$, we have $0 \le x^{3n+1} \le 1$, so

$$0 \le g_n(x) = \frac{x^{3n+1}}{3n+1} \le \frac{1}{3n+1}.$$

Therefore

$$\sup_{x\in[0,1]} |g_n(x)| = \frac{1}{3n+1} \xrightarrow[n\to\infty]{} 0,$$

so $g_n \to 0$ uniformly.
All Dirichlet assumptions hold, hence

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{3n+1}$$

converges **uniformly** on $[0,1]$.

Abel's Test (Uniform Convergence)

Theorem

Let $f_n, g_n : D \to \mathbb{R}$. Assume:

1. The series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on D.
2. The sequence (g_n) is uniformly bounded: there exists $M > 0$ such that
$$\sup_{x \in D} |g_n(x)| \leq M \quad \text{for all } n.$$
3. For each fixed $x \in D$, the numerical sequence $(g_n(x))$ is nonincreasing:
$$g_{n+1}(x) \leq g_n(x) \quad \text{for all } n.$$

Then the series
$$\sum_{n=1}^{\infty} f_n(x)\, g_n(x)$$
converges uniformly on D.

Dirichlet vs. Abel (1/2)

Both: study $\sum_{n\geq 1} f_n(x)\, g_n(x)$ on D.

Dirichlet :

1. $S_n(x) = \sum_{k=1}^{n} f_k(x)$ bounded: $|S_n(x)| \leq M$ $(\forall x, n)$.
2. $g_{n+1}(x) \leq g_n(x)$ ($\forall x$, each fixed x).
3. $\sup_{x\in D} |g_n(x)| \to 0$.

Abel :

1. $\sum_{n\geq 1} f_n(x)$ converges uniformly on D.
2. $\sup_{x\in D} |g_n(x)| \leq M$ $(\forall n)$.
3. $g_{n+1}(x) \leq g_n(x)$ (each fixed x).

Dirichlet vs. Abel (2/2)

When to use?

Main difference:

- **Dirichlet:** needs $g_n \to 0$ (uniform), only bounded S_n.
- **Abel:** no need $g_n \to 0$, but needs uniform convergence of $\sum f_n$.

Quick choice:

- Use **Dirichlet** if f_n (e.g. $(-1)^n$, $\sin(nx)$), and g_n decreases to 0 uniformly.
- Use **Abel** if $\sum f_n$ is already uniform (often by M-test or independent of x), and g_n is bounded + decreasing (may not $\to 0$).

$$\text{Dir: } S_n \text{ bnd \& } \sup|g_n| \to 0 \qquad \text{Abel: } \sum f_n \text{ unif \& } \sup|g_n| \leq M.$$

Example

Example

Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n, \qquad x \in [0,1].$$

Write it as $\sum f_n(x)\, g_n(x)$ with

$$f_n(x) = \frac{(-1)^n}{n} \quad \text{(independent of } x\text{)}, \qquad g_n(x) = x^n.$$

Check Abel assumptions:

1. $\sum f_n$ **converges uniformly on** $[0,1]$. Since $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges (alternating series), and f_n does not depend on x, the convergence is uniform on any set, in particular on $[0,1]$.
2. (g_n) **is uniformly bounded.** For all $x \in [0,1]$, $|g_n(x)| = |x^n| \le 1$.
3. $(g_n(x))$ **is nonincreasing in** n **for each fixed** x. If $0 \le x < 1$, then $x^{n+1} \le x^n$; if $x = 1$, then $x^n = 1$ for all n.

Interchanging Limit and Sum

Theorem

Let x be a cluster point of D and suppose $\lim_{t\to x} f_n(t)$ exists for each $n \in \mathbb{N}$. If $\sum f_n$ converges uniformly on $D \setminus \{x\}$, then

$$\lim_{t\to x} \sum_{n=1}^{\infty} f_n(t) = \sum_{n=1}^{\infty} \lim_{t\to x} f_n(t).$$

In particular, if each f_n is continuous at x, then $\sum f_n$ is continuous at x.

Example

Show:

$$\lim_{x \to 0} \sum_{n=0}^{\infty} \frac{e^{-n^3 x^2}}{2^n} = 2.$$

Proof

Let $f_n(x) = \frac{e^{-n^3x^2}}{2^n}$. Then $0 < e^{-n^3x^2} \le 1$, so

$$|f_n(x)| \le \frac{1}{2^n} =: M_n \qquad (\forall x \in \mathbb{R}).$$

Since $\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} \frac{1}{2^n}$ converges, the Weierstrass M-test gives **uniform convergence** of $\sum f_n$ (hence we may pass the limit through the sum). Also, for each n,

$$\lim_{x\to 0} f_n(x) = \frac{1}{2^n}.$$

Therefore

$$\lim_{x\to 0} \sum_{n=0}^{\infty} \frac{e^{-n^3x^2}}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2.$$

Term-by-Term Integration

Theorem

Suppose $f_n \in \mathcal{R}(a, b)$ for all $n \in \mathbb{N}$. If $\sum f_n$ converges uniformly on $[a, b]$, then $\sum f_n \in \mathcal{R}(a, b)$ and

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\, dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\, dx.$$

Example

Show that

$$\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}\, dx = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^3}.$$

Plan:

- (i) Use M-test to justify $\int \sum = \sum \int$ on $[0, \pi]$.
- (ii) Compute $\int_0^{\pi} \frac{\sin(nx)}{n^2}\, dx$.

Swap $\int$ and $\sum$

Proof

Let $f_n(x) = \dfrac{\sin(nx)}{n^2}$ on $[0, \pi]$. Then

$$|f_n(x)| \le \frac{1}{n^2} =: M_n \qquad (\forall x \in [0, \pi]).$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the M-test gives **uniform convergence** of $\sum f_n$ on $[0, \pi]$. Therefore we may integrate term-by-term:

$$\int_0^{\pi} \sum_{n=1}^{\infty} f_n(x)\, dx = \sum_{n=1}^{\infty} \int_0^{\pi} f_n(x)\, dx.$$

Compute the single-term integral

Proof

For $n \geq 1$,

$$\int_0^\pi \frac{\sin(nx)}{n^2}\,dx = \frac{1}{n^2}\int_0^\pi \sin(nx)\,dx = \frac{1}{n^2}\left[\frac{-\cos(nx)}{n}\right]_0^\pi.$$

Hence

$$\int_0^\pi \frac{\sin(nx)}{n^2}\,dx = \frac{1}{n^3}(1-\cos(n\pi)) = \frac{1}{n^3}(1-(-1)^n).$$

So

$$\int_0^\pi \sum_{n=1}^\infty \frac{\sin(nx)}{n^2}\,dx = \sum_{n=1}^\infty \frac{1-(-1)^n}{n^3}.$$

Proof

Since

$$1-(-1)^n = \begin{cases} 0, & n \text{ even}, \\ 2, & n \text{ odd}, \end{cases}$$

we get

$$\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} = \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \frac{2}{n^3}.$$

Write odd $n = 2k-1$. Then

$$\sum_{\substack{n\geq 1 \\ n \text{ odd}}} \frac{2}{n^3} = \sum_{k=1}^{\infty} \frac{2}{(2k-1)^3}.$$

Therefore

$$\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}\, dx = \sum_{k=1}^{\infty} \frac{2}{(2k-1)^3}.$$

Term-by-Term Differentiation

Theorem

Let f_n be differentiable on $[a, b]$ and assume $\sum f_n(x_0)$ converges at some $x_0 \in [a, b]$. If $\sum f_n'(x)$ converges uniformly on $[a, b]$, then $\sum f_n$ converges uniformly on $[a, b]$, its sum is differentiable, and

$$\left(\sum_{n=1}^{\infty} f_n\right)'(x) = \sum_{n=1}^{\infty} f_n'(x), \qquad x \in [a, b].$$

Example

Let

$$f(x) := \sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2} \qquad (x \in \mathbb{R}).$$

Show that

$$f'(x) = -2x \sum_{n=1}^{\infty} \frac{n^4}{(n^3 + n^4 x^2)^2}.$$

Proof

Write $f(x) = \sum_{n=1}^{\infty} f_n(x)$ with

$$f_n(x) := \frac{1}{n^3 + n^4 x^2} = \frac{1}{n^3(1 + nx^2)}.$$

Each f_n is differentiable and

$$f_n'(x) = \frac{d}{dx}\left(n^3 + n^4 x^2\right)^{-1} = -\frac{2n^4 x}{(n^3 + n^4 x^2)^2}.$$

Fix any $a > 0$ and work on the bounded interval $[-a, a]$. For $|x| \le a$,

$$|f_n'(x)| = \frac{2n^4|x|}{(n^3 + n^4 x^2)^2} \le \frac{2a\, n^4}{(n^3)^2} = \frac{2a}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{2a}{n^2}$ converges, the Weierstrass M-test implies $\sum f_n'(x)$ converges uniformly on $[-a, a]$.
Also, $\sum f_n(0) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges. Therefore, by the term-by-term differentiation theorem, $\sum f_n$ converges uniformly on $[-a, a]$, its sum f is differentiable on $[-a, a]$, and

Example

Let

$$f_n(x) := (-1)^n \frac{1}{nx}, \qquad x > 0.$$

Show that the series $\sum_{n=1}^{\infty} f_n(x)$ defines a function that is **continuous** and **differentiable** on $(0, \infty)$.

Proof

Fix $a > 0$ and consider $D = [a, \infty)$.
Write $f_n(x) = u_n\, v_n(x)$ with

$$u_n := \frac{(-1)^n}{n}, \qquad v_n(x) := \frac{1}{x}.$$

Then $\sum_{n=1}^{\infty} u_n$ converges (alternating harmonic series), hence its partial sums are bounded.
Also $v_n(x)$ does not depend on n, so it is trivially nonincreasing in n, and

$$\sup_{x\in[a,\infty)} |v_n(x)| = \sup_{x\in[a,\infty)} \frac{1}{x} = \frac{1}{a} < \infty.$$

Therefore, by **Abel's test (uniform)**, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{nx}$$

converges uniformly on $[a, \infty)$.

Continuity: each f_n is continuous on $(0, \infty)$, so the uniform limit on $[a, \infty)$ is continuous there. Since $a > 0$ is arbitrary, the sum is continuous on $(0, \infty)$.

Proof

For $x > 0$, $f_n'(x) = -\frac{(-1)^n}{nx^2}$. Fix $a > 0$ and work on $[a, \infty)$. Write

$$f_n'(x) = u_n\, v(x), \qquad u_n = \frac{(-1)^{n+1}}{n}, \quad v(x) = \frac{1}{x^2}.$$

Then:

- $\sum u_n$ has bounded partial sums (alternating harmonic).
- $v(x)$ is bounded on $[a, \infty)$: $\sup_{x \geq a} |v(x)| = \frac{1}{a^2}$.
- $v(x)$ does not depend on n (so monotonicity holds trivially).

Hence, by **Abel/Dirichlet uniform test**,

$$\sum f_n'(x)$$

converges uniformly on $[a, \infty)$.
Therefore, the sum $\sum f_n(x)$ is differentiable on $(0, \infty)$.