Ex 2.6. :

$f(x)=\sum_{n=1}^{\infty}\frac{1}{n^3+n^4x^2}$, $f_n(x)=\frac{1}{n^3+n^4x^2}$

Show that $f'(x)=-2x\sum_{n=1}^{\infty}\frac{1}{(n^3+n^4x^2)^2}$

at $x=0$

$$\sum_{n=1}^{\infty}\frac{1}{n^3+n^4(0)^2}=\sum_{n=1}^{\infty}\frac{1}{n^3}.$$

Converges from P-test ($p=3$)

$$f_n'(x)=\left(\frac{1}{n^3+n^4x^2}\right)'=\left((n^3+n^4x^2)^{-1}\right)'$$

$$=-2n^4x\,(n^3+n^4x^2)^{-2}$$

$$=\frac{-2n^4x}{(n^3+n^4x^2)^2}$$

$\sum_{n=1}^{\infty}\frac{-2xn^4}{(n^3+n^4x^2)^2}$ converge uniformly on $[a,b]$,

$x\in[a,b]$

$$\left|\frac{-2xn^4}{(n^3+n^4x^2)^2}\right|=\frac{2|x|\,n^4}{(n^3+n^4x^2)^2}\leq\frac{2\max(|a|,|b|)\,n^4}{}$$

$n^3+n^4x^2\geq n^3$

$(n^3+n^4x^2)^2\geq n^6>0$

$$\leq\underbrace{\frac{2\max(|a|,|b|)}{n^2}}_{=M_n}$$

$$\frac{1}{(n^3+n^4x^2)^2}\leq\frac{1}{n^6}$$

$$\sum_{n=1}^{\infty}M_n=2\max(|a|,|b|)\sum_{n=1}^{\infty}\frac{1}{n^2}\quad\underline{cv}$$

By M-test $\sum_{n=1}^{\infty}\left(\frac{1}{n^3+n^4x^2}\right)'$ cv uniformally on $[a,b]$

By theorem of diffrentiation of series

we deduce f is differenciable on $[a,b]$

and $f'(x)=\sum_{n=1}^{\infty}\left(\frac{1}{n^3+n^4x^2}\right)'$

$$=-2x\sum_{n=1}^{\infty}\frac{n^4}{(n^3+n^4x^2)^2}$$

Since a,b are arbitrary

then it's differenciable on $\mathbb{R}$.

Ex 2.7. :

$f_n(x)=2^n x e^{-nx^2}$ for $x\in R$

a) for each $A>\delta>\sqrt{\ln 2}$ Prove that

$\sum_{n=1}^{\infty}f_n$ converges uniformally on $[\delta,A]$

$$\left|2^n x e^{-nx^2}\right|\leq A\,2^n e^{-n\delta^2}=A\,(2e^{-\delta^2})^n$$

$$\delta\leq x\leq A$$

$$\delta^2\leq x^2\leq A^2$$

$$n\delta^2\leq nx^2\leq nA^2$$

$$-nA^2\leq -nx^2\leq -n\delta^2$$

$$e^{-nx^2}\leq e^{-n\delta^2}$$

$$2e^{-\delta^2}<1\iff e^{-\delta^2}<\frac{1}{2}$$

$$\iff -\delta^2<-\ln 2$$

$$\iff \delta^2>\ln(2)$$

$$\iff \delta>\sqrt{\ln(2)}$$

- $\sum_{n=1}^{\infty}A(2e^{-\delta^2})^n$ Converges

geometric series, $r=2e^{-\delta^2}$, $|r|<1$

By M-test $\sum_{n=1}^{\infty}f_n$ converges uniformally on $[\delta,A]$

b) $\sum_{n=1}^{\infty}2^n x e^{-nx^2}$, $x>\sqrt{\ln 2}$

Converge pointwise on $(\sqrt{\ln 2},\infty)$

$$\sum_{n=1}^{\infty}2^n x e^{-nx^2}=x\sum_{n=1}^{\infty}(2e^{-x^2})^n$$

$$\sum_{n=0}^{\infty}r^n=\frac{1}{1-r},\ |r|<1$$

$$=x\sum_{n=0}^{\infty}(2e^{-x^2})^{n+1}=x\sum_{n=0}^{\infty}(2e^{-x^2})^n(2e^{-x^2})$$

$$=2xe^{-x^2}\sum_{n=0}^{\infty}(2e^{-x^2})^n$$

$$=2xe^{-x^2}\frac{1}{1-2e^{-x^2}}=\frac{2xe^{-x^2}}{1-2e^{-x^2}}$$

Ex 2.8. :

$f_n(x)=(-1)^n\frac{1}{nx}$, $x\in(0,\infty)$

Show that $\sum_{n=1}^{\infty}f_n$, is continuous

and differenciable on $(0,\infty)$.

Continuity on $(0,\infty)$

Remark $\sum_{n=1}^{\infty}(-1)^n\frac{1}{nx}$ Converges P.W. on $(0,\infty)$

from alternating series test.

Since $f(x)$ has the form $\sum_{n=1}^{\infty}(-1)^n a_n$

$a_n=\frac{1}{nx}$

- $a_n\geq 0\Rightarrow\frac{1}{nx}\geq 0$, $\forall n\in\mathbb{N}$, $\forall x\in(0,\infty)$
- $a_{n+1}\leq a_n$
- $a_n\to 0$ Since x is fixed

So by altenating test for numerical

Series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{nx}$ C.V. Pointwise on $(0,\infty)$

$\sum_{n=1}^{\infty}\frac{(-1)^n}{nx}$ Converge uniformally on $[a,\infty)$, for every $a>0$

Let $f_n(x)=\frac{1}{nx}$

- $f_n(x)\geq 0$, $\forall n\in\mathbb{N}$, $\forall x\geq a$
- $f_{n+1}(x)\leq f_n(x)$, $\forall x\geq a$, $\forall n\geq 1$
- $\sup_{x\geq a}|f_n(x)|\leq\frac{1}{na}\to 0$, $n\to\infty$

(So $f_n\xrightarrow{U}0$ on $[a,\infty)$)

By Continuity theorem for seq

$f(x)=\sum_{n=1}^{\infty}\frac{(-1)^n}{nx}$ is continuous on $[a,\infty)$

for every $a>0$

Therefore f is continuous on $(0,\infty)$