Ex 2.1:

$$\lim_{x\to 0}\sum_{n=0}^{\infty}\frac{e^{-n^3x^2}}{2^n}=2$$

uniform convergence of the series:

* $\sup_{x\in\mathbb{R}}\left|\frac{e^{-n^3x^2}}{2^n}\right|\leqslant\left(\frac{1}{2}\right)^n : \forall n$

* $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$ conv

$\left(\sqrt[n]{\left(\frac{1}{2}\right)^n}=\frac{1}{2}<1\right)$

By M-test

$\sum_{n=0}^{\infty}\frac{e^{-n^3x^2}}{2^n}$ Converges uniformly on $\mathbb{R}$

$$\lim_{x\to 0}\sum_{n=0}^{\infty}\frac{e^{-n^3x^2}}{2^n}=\sum_{n=0}^{\infty}\lim_{x\to 0}\frac{e^{-n^3x^2}}{2^n}$$

$$=\sum_{n=0}^{\infty}\frac{1}{2^n}$$

$$=\frac{1}{1-\frac{1}{2}}=2$$

Recall that:

- $f_n\xrightarrow{u}f$ on $D\setminus\{x_0\}$
- $\lim_{x\to x_0}f_n(x)$ Exists

then $\lim_{x\to x_0}\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\lim_{x\to x_0}f_n(x)$

$\sum_{n=0}^{\infty}\frac{e^{-n^3x^2}}{2^n}$ cu on $\mathbb{R}$

$$S_n(x)=\sum_{k=0}^{n}\frac{e^{-k^3x^2}}{2^k}\xrightarrow{u}\sum_{n=0}^{\infty}\frac{e^{-n^3x^2}}{2^n}, \text{ on } \mathbb{R}$$

$$\lim_{x\to x_0}\sum_{n=0}^{\infty}\frac{e^{-n^3x^2}}{2^n}=\lim_{x\to x_0}\lim_{n\to\infty}\sum_{k=0}^{n}\frac{e^{-k^3x^2}}{2^k}$$

$$=\lim_{n\to\infty}\lim_{x\to x_0}\sum_{k=0}^{n}\frac{e^{-k^3x^2}}{2^k}$$

$$=\lim_{n\to\infty}\sum_{k=0}^{n}\lim_{x\to x_0}\frac{e^{-k^3x^2}}{2^k}$$

$$=\sum_{k=0}^{\infty}\lim_{x\to 0}\frac{e^{-k^3x^2}}{2^k}$$

$$=\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k=2.$$

Ex 2.2:

$$\int_0^{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^2}\,dx=\sum_{n=1}^{\infty}\frac{2}{(2n-1)^3}$$

Solution:

- $f_n(x)=\frac{\sin(nx)}{n^2}\in R(0,\pi), \forall n$

- $\sup_{x\in[0,\pi]}\left|\frac{\sin(nx)}{n^2}\right|\leqslant\frac{1}{n^2}, \forall n$

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ Converges, $P=2>1$, P-Series.

By M-test $\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^2}$ C.V. uniformly on $[0,\pi]$

therefore

$$\int_0^{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^2}\,dx=\sum_{n=1}^{\infty}\int_0^{\pi}\frac{\sin(nx)}{n^2}\,dx$$

$$=\sum_{n=1}^{\infty}\left[\frac{-\cos(nx)}{n^3}\right]_0^{\pi}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\left[-\cos(n\pi)+\cos(0)\right]$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^3}\left((-1)^{n+1}+1\right)$$

$$=\sum_{n=1}^{\infty}\frac{2}{(2n-1)^3}$$

$1+(-1)^{n+1}=2$ if $n+1$ even
if n is odd

$=0$ if $n+1$ is odd
if n is even

Ex 2.3:

Compute $\int_1^2\sum_{n=1}^{\infty}\frac{1}{(n+x)^2}\,dx$

Solution:

- $f_n(x)=\frac{1}{(n+x)^2}$,
for all n, f_n is Rieman integrable on $[1,2]$
- $\sum_{n=1}^{\infty}\frac{1}{(n+x)^2}$ Converges uniformally on $[1,2]$

Since $\sup_{x\in[1,2]}\left|\frac{1}{(n+x)^2}\right|\leqslant\frac{1}{n^2}$, M-test

$\left(n+x\geqslant n \Rightarrow (n+x)^2\geqslant n^2\Rightarrow\left|\frac{1}{(n+x)^2}\right|\leqslant\frac{1}{n^2}\right)$

By M-test $\sum_{n=1}^{\infty}\frac{1}{(n+x)^2}$ C.V. uniformally

therefore

$$\int_1^2\sum_{n=1}^{\infty}\frac{1}{(n+x)^2}\,dx=\sum_{n=1}^{\infty}\int_1^2\frac{1}{(n+x)^2}\,dx$$

$$=\sum_{n=1}^{\infty}\left[\frac{-1}{n+x}\right]_1^2$$

$$=\sum_{n=1}^{\infty}\left[\frac{-1}{n+2}+\frac{1}{n+1}\right]$$

$$=\lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$

$$=\lim_{N\to\infty}\left(\frac{1}{2}-\frac{1}{N+2}\right) \quad \text{telescoping sum}$$

$$=\frac{1}{2}$$