Introduction to Real Analysis

Sequences

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Sequences and Convergence

Definition

A sequence is a function whose domain is $\mathbb{N}$.

$$x : \mathbb{N} \to \mathbb{R}$$

$$x(n) = x_n$$

$$(x_1, x_2, x_3, \dots)$$

$$(x_n)_{n=1}^{\infty}$$

$$(x_n)$$

$\{x_n : n \in \mathbb{N}\}$ is the range of the sequence.

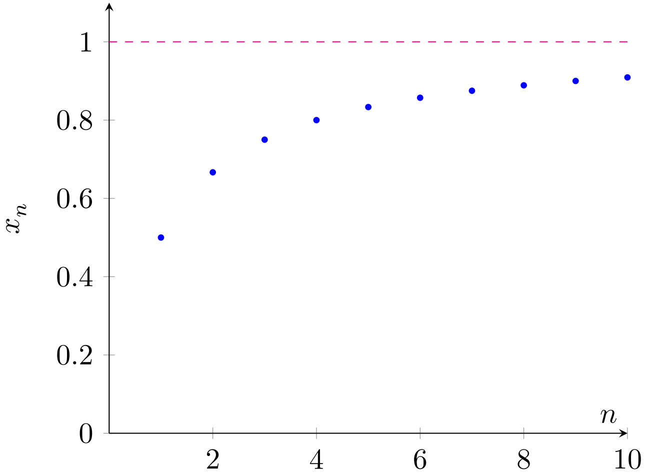
Examples

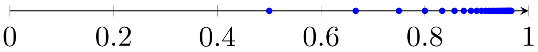
1. $(2) = (2, 2, 2, ...)$
2. $(2n) = (2, 4, 6, ...)$
3. $((-1)^n) = (-1, 1, -1, ...)$ is a sequence whose range is $\{-1, 1\}$
4. $(\frac{1}{n}) = (1, \frac{1}{2}, \frac{1}{3}, ...)$
5. The sequence can be defined by induction

$$a_1 = 1, \quad a_{n+1} = a_n + \frac{1}{n}, \quad n \in \mathbb{N}$$

$$x_n = \frac{n}{n+1}$$

$$\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & \dots & n & \dots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\
\frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{5}{6} & \frac{6}{7} & \dots & \frac{n}{n+1} & \dots
\end{array}$$


1
0.8
0.6
0.4
0.2
0
x_n
n
2
4
6
8
10


0
0.2
0.4
0.6
0.8
1

Convergence

Definition

The sequence (x_n) is said to be convergent if there is $x \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} :$$
$$|x_n - x| < \varepsilon \qquad \forall n \geq N$$

and we write

$$\lim_{n \to \infty} x_n = x$$

$$\lim x_n = x$$

$$x_n \to x$$

Examples

$$\lim \frac{1}{n} = 0$$

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Neighborhood

Definition

If $a \in \mathbb{R}$ then $V \subset \mathbb{R}$ is called a neighboorhood of a if there is $\varepsilon > 0$ such that

$$(a - \varepsilon, a + \varepsilon) \subset V$$

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Remarks

- If we found $N \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon \qquad \forall\ n \geq N$$

then any number greater than N will satisfy the condition.

Remarks

- If we change ε we might need to change N.

Remarks

- If (x_n) satisfies for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ and a constant $C > 0$ such that

$$|x_n - x| < C\varepsilon \qquad \forall\ n \geq N$$

then $x_n \to x$.

Examples

1. $\lim \dfrac{1}{2^n} = 0$
2. $\lim \dfrac{3n}{5n+9} = \dfrac{3}{5}$
3. $((-1)^n)$
4. (n)

Uniquness

Theorem

If the sequence (x_n) is convergent then its limit is unique.

Bounded sequences

Definition

A sequence (x_n) is bounded if there is a $K > 0$ such that

$$|x_n| \leq K \qquad \forall n \in \mathbb{N}$$

Theorem

If a sequence is convergent then it is bounded.

The converse is not true.

Algebraic operations on sequences

Theorem

If $x_n \to x \neq 0$ then there exists $M > 0$ and $N \in \mathbb{N}$ such that

$$|x_n| > M \qquad \forall n \geq N$$

.

Algebraic operations on sequences

Theorem

If (x_n) converges to x and (y_n) converges to y then

1. $(x_n + y_n)$ converges to $x + y$.
2. $(x_n y_n)$ converges to xy.
3. If $y_n \neq 0$ for all $n \in \mathbb{N}$ and $y \neq 0$ then $(\frac{x_n}{y_n})$ converges to $\frac{x}{y}$.

Convergent Sequences

Theorem

If $x_n \to x$ and $y_n \to y$ and If

$$x_n \leq y_n \quad \forall n \in \mathbb{N}$$

then $x \leq y$

Squeeze Theorem

Theorem

If

$$x_n \leq y_n \leq z_n \quad \forall n \geq N_0$$

and $\lim x_n = \lim z_n = l$ then (y_n) converges to l

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Examples

- If $x_n \to x$ then $|x_n| \to |x|$
  Is the converse true?

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Examples

- If $0 < a < 1$ then $\lim a^n = 0$

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Examples

- If $c > 0$ then $\lim c^{\frac{1}{n}} = 1$

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Examples

- If $x_n \geq 0$ for all $n \in \mathbb{N}$ and $x_n \to x$ then $\sqrt{x_n} \to \sqrt{x}$

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Examples

- $\lim n^{\frac{1}{n}} = 1$

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Monotonic sequences

Definition

A sequence (x_n) is

1. increasing if

$$x_{n+1} \geq x_n \quad \forall n \in \mathbb{N}$$

2. strictly increasing if

$$x_{n+1} > x_n \quad \forall n \in \mathbb{N}$$

Monotonic sequences

Definition

A sequence (x_n) is

1. decreasing if
$$x_{n+1} \leq x_n \quad \forall n \in \mathbb{N}$$
2. strictly decreasing if
$$x_{n+1} < x_n \quad \forall n \in \mathbb{N}$$

if a sequence is increasing or decreasing, it is called monotonic sequence.

Note: (x_n) is increasing iff $(-x_n)$ is decreasing

Examples

1. $\left(\frac{1}{n}\right)$
2. (n^2)
3. $((-1)^n)$
4. $\left(\frac{(-1)^n}{n}\right)$

Monotonic Sequences

Theorem

A monotonic sequence is convergent iff it is bounded.

1. if (x_n) is increasing and bounded then
$$\lim x_n = \sup\{x_n : n \in \mathbb{N}\}$$
2. If (x_n) is decreasing and bounded then
$$\lim x_n = \inf\{x_n : n \in \mathbb{N}\}$$

Examples

Prove that $x_1 = 1, \; x_{n+1} = \sqrt{2x_n}$ is convergent then find its limit

.

Extended Real Number System

$$\bar{\mathbb{R}} = \mathbb{R} \bigcup \{-\infty, \infty\} = [-\infty, \infty]$$

Cauchy sequence

Definition

A sequence (x_n) is called a Cauchy sequence if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} :$$
$$|x_n - x_m| < \varepsilon \qquad \forall n, m \geq N$$

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Cauchy criterion

Theorem

A sequence (x_n) is convergent iff it is a Cauchy sequence

Cluster Point

Definition

1. A point $x \in \mathbb{R}$ is called a cluster (an accumulation) point of $A \subset \mathbb{R}$ if every neighborhood V of x contains an element in A different than x.
   The set of all cluster point is $\widehat{A}$.
2. A point in A which is not a cluster point of A is an isolated point of A.

Cluster Point

Examples

1. $\{1, 2, 3\}$
2. $\mathbb{Z}$
3. $[0, 1)$
4. $\{\frac{1}{n} : n \in \mathbb{N}\}$
5. $\mathbb{Q}$

Bolzano - Weierstrass

Theorem

Every infinite and bounded subset of $\mathbb{R}$ has at least one cluster point in $\mathbb{R}$.

Cauchy sequences

Examples

- Show that

$$x_n = \frac{2n}{3n+1}$$

is a Cauchy sequence

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Cauchy sequences

Examples

- If $x_1 = 1, x_2 = 2$,

$$x_n = \frac{1}{2}(x_{n-1} + x_{n-2}), \quad n = 3, 4, ...$$

Prove that (x_n) is convergent.

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Subsequence

Definition

If (x_n) is a sequence, and (n_k) is strictly increasing sequence of natural numbers

$$n_1 < n_2 < n_3 < \dots$$

then the sequence

$$(x_{n_k}) = (x_{n_1}, x_{n_2}, \dots)$$

is a subsequence of (x_n)

Subsequences

Examples

1. $(x_4, x_5, x_6, \dots)$
2. $(x_1, x_3, x_5, \dots)$
3. Is $(\frac{1}{2}, 1, \frac{1}{5}, \dots)$ a subsequence of $(\frac{1}{n})$
4. Is $(4, 8, 9, \dots)$ a subsequence of $(2n)$
5. $(\frac{1}{k^2})$ is a subsequence of $(\frac{1}{n})$

Subsequence

Theorem

If (x_n) converges to x, then every subsequence of (x_n) converges to x

Subsequence

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If (x_n) converges to x, then every subsequence of (x_n) converges to x

Theorem

If (x_n) is convergent and has a subsequence that converges to x, then (x_n) converges to x

Subsequence

Theorem

If (x_n) converges to x, then every subsequence of (x_n) converges to x

Theorem

If (x_n) is convergent and has a subsequence that converges to x, then (x_n) converges to x

Theorem: Bolzano-Weierstrass

Every bounded sequence has a convergent subsequence.

Open Set

Definition

A set $A \subset \mathbb{R}$ is open if for all $x \in A$ there is $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset A$

Open Sets

Examples

1. (a, b)
2. $\mathbb{R}\backslash\{y\}$ where $y \in \mathbb{R}$
3. $[a, b)$
4. $\mathbb{Z}$
5. $\mathbb{Q}$

Open Sets

Theorem

1. $\mathbb{R}$ and ϕ are both open
2. Any union of open sets in $\mathbb{R}$ is open
3. Any finite intersection of open sets in $\mathbb{R}$ is open

Closed Set

Definition

A set $F \subset \mathbb{R}$ is closed if its complement A^c is open.

Closed Sets

Examples

1. $[a, b], [a, \infty), (-\infty, b]$
2. $[a, b)$
3. $\mathbb{Z}$
4. $\mathbb{Q}$

Exercises (Convergent sequences)

1. Use the definition to show that
    1. $\lim \dfrac{2n-1}{3n+2} = \dfrac{2}{3}$
    2. $\lim \dfrac{n^3+1}{2n^3+n} = \dfrac{1}{2}$.
2. Prove that $\lim x_n = 0$ if and only if $\lim |x_n| = 0$
3. If the two sequences (x_n) and (y_n) converge to c, and we define the shuffled sequence (z_n) by

$$(z_1, z_2, z_3, z_4, \dots) = (x_1, y_1, x_2, y_2, \dots)$$

show that the sequence (z_n) also converges to c

Exercises (Convergent sequences)

1. Give an example of two sequences (x_n) and (y_n) such that $(x_n + y_n)$ is convergent and (x_n) is divergent.
2. If the sequences (x_n) and $(x_n + y_n)$ are both convergent, prove that (y_n) is also convergent and determine its limit. Can you state a corresponding result for the sequence $(x_n \cdot y_n)$?
3. Give an example of a divergent sequence (x_n) such that $(|x_n|)$ is convergent.
   When does the convergence of $(|x_n|)$ imply the convergence of (x_n),
   and what is the relation between $\lim |x_n|$ and $\lim x_n$ when both exist?
4. If $\lim \dfrac{x_n - 1}{x_n + 1} = 0$, prove that $\lim x_n = 1$

Exercises (Monotonic sequences)

1. Prove that $x_n = \frac{n^n}{n!}$ is monotonic
2. Prove that each of the following sequences is monotonic and bounded, then find its limit
    1. $x_1 = 1,\ x_{n+1} = \sqrt{3 + x_n}$, for all $n \in \mathbb{N}$
    2. $x_1 = 1,\ x_{n+1} = \dfrac{4x_n + 2}{x_n + 3}$, for all $n \in \mathbb{N}$
3. Given $x_n = \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n}$, prove that (x_n) is decreasing and bounded then conclude it is convergent.

Exercises (Cauchy criterion)

1. Show by definition that (x_n) is a Cauchy sequence
$$x_n = \frac{5n}{n+3}$$
2. If (x_n) satisfies
$$|x_{n+1} - x_n| < \frac{1}{2^n}$$
prove that (x_n) is a Cauchy Sequence.
3. If
$$x_n = \sum_{k=1}^{n} \frac{1}{k^2}$$
prove that (x_n) is a Cauchy sequence.

Exercises (Subsequences)

1. Prove that the sequence (x_n) has a convergent subsequence

$$x_n = \frac{(n^2 + 20n + 30)\sin(n^3)}{n^2 + n + 1}$$

2. Give an example of
   1. a sequence with no convergent subsequence
   2. an unbounded sequence which has a convergent subsequence
3. If every subsequence of (x_n) has a subsequence which converges to 0, prove that $\lim x_n = 0$