Sequences

Lecture 1: Convergent Sequences

Definition

A **sequence** is a function

$$a : \mathbb{N} \longrightarrow \mathbb{R},$$

which assigns to each natural number n a real number a_n.

A sequence is simply a **list of numbers** written in a specific order. The index n tells us the position of a number in the list.

A sequence is usually written as

$$\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \ldots$$

Each number a_n is called the n**th term** of the sequence.

Examples of Sequences

Example

- $a_n = \dfrac{1}{n} \Rightarrow \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ The terms become smaller and smaller.
- $a_n = (-1)^n \Rightarrow \{-1, 1, -1, 1, \ldots\}$ The sequence alternates between two values.
- $a_n = n^2 \Rightarrow \{1, 4, 9, 16, \ldots\}$ The terms grow rapidly.

Convergence of a Sequence

Definition

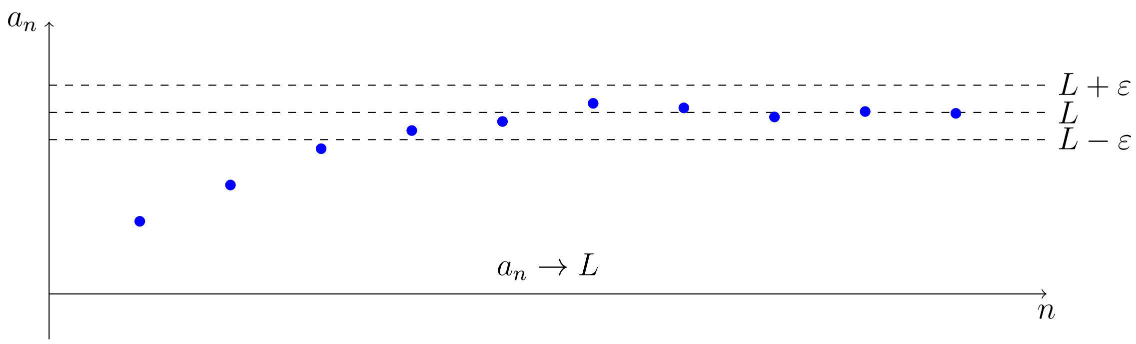
A sequence $\{a_n\}$ **converges** to a real number L if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow |a_n - L| < \varepsilon.$$

We write

$$\lim_{n\to\infty} a_n = L \quad \text{or simply} \quad a_n \to L.$$

Convergence means that the terms of the sequence get closer and closer to a fixed number L, and after some point they stay close to it forever.



After some index N, all the terms of the sequence lie inside a narrow horizontal band around the limit L.

Divergent Sequences

Important

If a sequence does not converge to any real number, it is said to **diverge**.

Example

- $a_n = n$ diverges to infinity because the terms grow without bound.
- $a_n = (-1)^n = -1, 1, -1, 1, \ldots$ does not converge.

The second sequence oscillates between two values and never settles near a single number. Therefore, it has no limit.

Properties of Convergent Sequences

Rule

If $a_n \to a$ and $b_n \to b$, then:

- $a_n + b_n \to a + b$
- $a_n b_n \to ab$
- If $b \neq 0$ and $b_n \neq 0$, then $\dfrac{a_n}{b_n} \to \dfrac{a}{b}$
- $|a_n| \to |a|$

These properties allow us to compute limits using familiar algebraic rules. Instead of using the definition of convergence every time, we can work directly with limits.

Lecture 2: Rational Sequences

Rational Sequences

A **rational sequence** is a sequence of the form

$$a_n = \frac{P(n)}{Q(n)},$$

where $P(n)$ and $Q(n)$ are polynomials in n and $Q(n) \neq 0$.

A First Example

Example

Consider the sequence

$$a_n = \frac{2n^2 - 3n}{3n^2 + 5n + 3}.$$

To make the dominant terms visible, divide the numerator and denominator by n^2:

$$a_n = \frac{2 - \frac{3}{n}}{3 + \frac{5}{n} + \frac{3}{n^2}}.$$

Since

$$\frac{1}{n} \to 0 \quad \text{and} \quad \frac{1}{n^2} \to 0,$$

all lower-order terms disappear in the limit. Therefore,

$$\lim_{n\to\infty} a_n = \frac{2}{3}.$$

Interpretation

For large n, the sequence behaves like

$$\frac{2n^2}{3n^2}.$$

General Rule for Rational Sequences

Rule

Let

$$a_n = \frac{P(n)}{Q(n)}.$$

Then:

- If $\deg P < \deg Q$, then $\lim_{n\to\infty} a_n = 0$
- If $\deg P = \deg Q$, the limit equals the ratio of leading coefficients
- If $\deg P > \deg Q$, the sequence diverges to $+\infty$ or $-\infty$

Illustrative Examples

$\deg P < \deg Q$

Consider

$$a_n = \frac{3n+5}{n^2+1}.$$

Dividing by n^2,

$$a_n = \frac{\frac{3}{n} + \frac{5}{n^2}}{1 + \frac{1}{n^2}}.$$

All terms in the numerator tend to zero, hence

$$\lim_{n\to\infty} a_n = 0.$$

$\deg P = \deg Q$

Consider

$$a_n = \frac{4n^2 - n + 1}{2n^2 + 3}.$$

Dividing by n^2,

$$a_n = \frac{4 - \frac{1}{n} + \frac{1}{n^2}}{2 + \frac{3}{n^2}}.$$

Only the leading coefficients remain, so

$$\lim_{n\to\infty} a_n = 2.$$

$\deg P > \deg Q$

Consider

$$a_n = \frac{n^3+1}{n^2-1}.$$

The numerator grows faster than the denominator, so

$$\lim_{n\to\infty} a_n = +\infty.$$

Now consider

$$a_n = \frac{-3n^3 + n}{2n^2 + 1}.$$

The dominant term in the numerator is negative, therefore

$$\lim_{n\to\infty} a_n = -\infty.$$

The sign of the leading term determines whether the sequence diverges to $+\infty$ or $-\infty$.

Lecture 3: Geometric Sequences

Definition

A **geometric sequence** is a sequence of the form

$$a_n = r^n,$$

where r is a real number called the **ratio**.

Case 1: $0 \leq r < 1$ (Decay to Zero)

Example: $r = \frac{1}{2}$

$$\left(\tfrac{1}{2}\right)^1 = \tfrac{1}{2}, \quad \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}, \quad \left(\tfrac{1}{2}\right)^3 = \tfrac{1}{8}, \quad \ldots$$

The sequence is decreasing and converges to zero.

Conclusion

When $0 \leq r < 1$, $r^n \to 0$.

Case 2: $r = 1$ (Constant Sequence)

$$1^n = 1 \quad \text{for all } n.$$

Conclusion

The sequence is constant, so

$$\lim_{n \to \infty} r^n = 1.$$

Case 3: $r > 1$ (Growth to Infinity)

Example: $r = 2$

$$2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8, \quad 2^4 = 16, \quad \ldots$$

Each multiplication increases the size of the terms rapidly.

Conclusion

When $r > 1$, the sequence grows without bound:

$$r^n \to +\infty.$$

Case 4: $-1 < r < 0$

Example: $r = -\frac{1}{2}$

$$-\tfrac{1}{2}, \quad +\tfrac{1}{4}, \quad -\tfrac{1}{8}, \quad +\tfrac{1}{16}, \quad \ldots$$

The sign alternates, but the absolute value decreases to zero.

Conclusion

Although the signs alternate, the terms still approach zero:

$$r^n \to 0.$$

Case 5: $r = -1$ (Oscillation)

$$(-1)^n = -1,\ 1,\ -1,\ 1, \ldots$$

Conclusion

The sequence oscillates between two values and never approaches a single number. The limit does not exist.

Case 6: $r < -1$ (Alternating and Unbounded)

Example: $r = -2$

$$-2, \quad 4, \quad -8, \quad 16, \quad \ldots$$

The magnitude grows without bound, while the sign alternates.

Conclusion

The sequence is unbounded and diverges.

Summary

Let $a_n = r^n$. Then:

$$\begin{aligned}
|r| < 1 &\Rightarrow r^n \to 0 && \text{(terms converge to zero)},\\
r = 1 &\Rightarrow r^n = 1 && \text{(constant sequence)},\\
r > 1 &\Rightarrow r^n \to +\infty && \text{(grows without bound)},\\
-1 < r < 0 &\Rightarrow r^n \to 0 && \text{(alternating but convergent)},\\
r = -1 &\Rightarrow r^n \text{ oscillates between } -1 \text{ and } 1 && \text{(no limit)},\\
r < -1 & && \text{(alternating and unbounded, no limit)}.
\end{aligned}$$

The nth Root of a Positive Number

If $\alpha > 0$, then

$$\lim_{n\to\infty} \sqrt[n]{\alpha} = 1.$$

Let

$$a_n = \sqrt[n]{\alpha}.$$

Taking logarithms,

$$\ln a_n = \frac{\ln \alpha}{n}.$$

Since $\ln \alpha$ is constant and $\frac{1}{n} \to 0$, we get

$$\ln a_n \to 0.$$

Exponentiating,

$$a_n = e^{\ln a_n} \to e^0 = 1.$$

The Sequence $\sqrt[n]{n}$

The sequence $\sqrt[n]{n}$ converges to 1:

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Let

$$a_n = \sqrt[n]{n}.$$

Then

$$\ln a_n = \frac{\ln n}{n}.$$

Since the logarithm grows much more slowly than n,

$$\frac{\ln n}{n} \to 0.$$

Hence,

$$a_n = e^{\ln a_n} \to e^0 = 1.$$

Lecture 4: Monotone Sequences

Definition

A sequence $\{a_n\}$ is said to be:

- **Increasing** if $a_{n+1} \geq a_n$ for all n,
- **Decreasing** if $a_{n+1} \leq a_n$ for all n,
- **Monotone** if it is either increasing or decreasing.

Monotone Convergence Theorem

If a sequence is **monotone** and **bounded**, then it converges.

Why this theorem is important

This theorem allows us to prove convergence *without* explicitly computing the limit. It is enough to check two simple properties: monotonicity and boundedness.

Example: An Increasing and Bounded Sequence

Example

Consider the sequence

$$a_n = 1 - \frac{1}{n}.$$

- Since $\frac{1}{n+1} \leq \frac{1}{n}$, we have

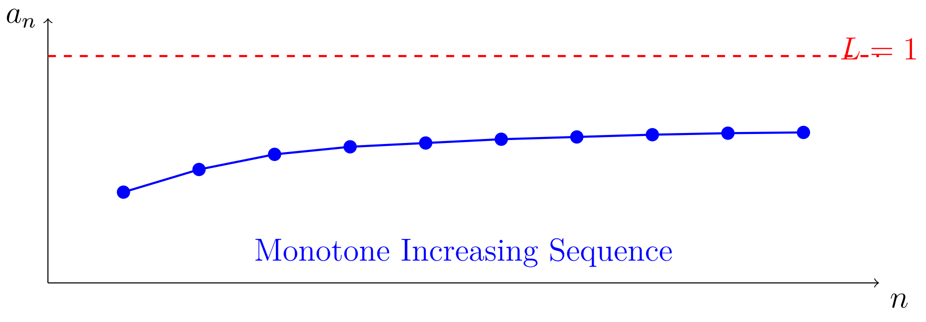
$$a_{n+1} = 1 - \frac{1}{n+1} \geq 1 - \frac{1}{n} = a_n,$$

  so the sequence is **increasing**.

- For all n, we have $a_n < 1$, so the sequence is **bounded above by** 1.

Conclusion

By the Monotone Convergence Theorem, the sequence (a_n) is convergent.



Example: Sequence of Partial Sums

Given a sequence (a_n), the associated sequence of partial sums is defined by

$$S_N = \sum_{k=1}^{N} a_k.$$

This definition means that we add the terms one by one:

$$\begin{aligned} S_1 &= a_1, \\ S_2 &= a_1 + a_2, \\ S_3 &= a_1 + a_2 + a_3, \\ &\vdots \end{aligned}$$

Key Identity

$$S_{N+1} = S_N + a_{N+1}.$$

Case: Non-Negative Terms

Assume that

$$a_n \geq 0 \quad \text{for all } n.$$

Since $a_{N+1} \geq 0$, we have

$$S_{N+1} = S_N + a_{N+1} \geq S_N.$$

Conclusion

When all terms are non-negative, the sequence of partial sums (S_N) is increasing.

Bounded Partial Sums

If there exists a number M such that

$$S_N \leq M \quad \text{for all } N,$$

then the partial sums increase but never exceed M.
By the Monotone Convergence Theorem, the sequence (S_N) converges.

Unbounded Partial Sums

If no such upper bound exists, the partial sums keep increasing without limit.

$$\lim_{N \to \infty} S_N = +\infty.$$

Case : Non-Positive Terms

If instead

$$a_n \leq 0 \quad \text{for all } n,$$

then each new term subtracts a non-positive quantity from the sum.

In this case,

$$S_{N+1} = S_N + a_{N+1} \leq S_N,$$

so the sequence of partial sums (S_N) is decreasing.

Lecture 5: The Squeeze Theorem

Squeeze Theorem

Let $(a_n), (b_n), (c_n)$ be sequences such that

$$a_n \leq b_n \leq c_n \quad \text{for all } n \geq N,$$

for some integer N. If

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then

$$\lim_{n\to\infty} b_n = L.$$

If a sequence is always trapped between two other sequences that both approach the same limit, then it is forced to approach that limit as well.

Example:

Show that

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

Finding Bounds

For every real number x, we know that

$$-1 \leq \sin x \leq 1.$$

This inequality is also true when $x = n$.

Dividing by the positive number n, we obtain

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}.$$

Behavior of the Bounds

The two bounding sequences

$$-\frac{1}{n} \quad \text{and} \quad \frac{1}{n}$$

are simple and easy to analyze.

Since

$$\lim_{n\to\infty} \left(-\frac{1}{n}\right) = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} = 0,$$

both bounds converge to the same limit.

Conclusion

By the Squeeze Theorem,

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

Exercises

Exercise 1

Decide whether the following sequences converge or diverge. If they converge, find the limit:

$$a_n = \frac{5n}{e^n}, \qquad b_n = \frac{n^2}{2^n + 1}, \qquad c_n = \frac{\ln n}{n}.$$

Solution

(1) $a_n = \frac{5n}{e^n}$

Consider the function $f(x) = \frac{5x}{e^x}$. As $x \to \infty$, both numerator and denominator tend to infinity.

By L'Hopital's rule,

$$\lim_{x\to\infty} \frac{5x}{e^x} = \lim_{x\to\infty} \frac{5}{e^x} = 0.$$

Therefore,

$$\lim_{n\to\infty} a_n = 0.$$

(2) $b_n = \frac{n^2}{2^n+1}$

Applying L'Hopital's rule twice,

$$\lim_{x\to\infty} \frac{x^2}{2^x + 1} = \lim_{x\to\infty} \frac{2x}{2^x \ln 2} = \lim_{x\to\infty} \frac{2}{(\ln 2)^2\, 2^x} = 0.$$

Hence,

$$\lim_{n\to\infty} b_n = 0.$$

(3) $c_n = \frac{\ln n}{n}$

Using L'Hopital's rule,

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Thus,

$$\lim_{n\to\infty} c_n = 0.$$

Exercise 2

Discuss the convergence of the following sequences:

(i) $(-1.2)^n$ (ii) $\dfrac{1}{2^n}$ (iii) $\dfrac{\cos^2(n)}{3^n}$

(iv) $\dfrac{4n^3+5n+1}{2n^3+n^2+5}$ (v) $\left(1+\dfrac{1}{n}\right)^n$ (vi) $n^{1/n}$

Solution

(i) $(-1.2)^n$

This is a geometric sequence with common ratio $r = -1.2$.

Since $|r| = 1.2 > 1$, the absolute value of the terms grows without bound. Therefore, the sequence diverges.

(ii) $\frac{1}{2^n}$

This can be written as a geometric sequence

$$\frac{1}{2^n} = \left(\frac{1}{2}\right)^n.$$

Here $|r| = \frac{1}{2} < 1$, so the sequence converges and

$$\lim_{n\to\infty} \frac{1}{2^n} = 0.$$

(iii) $\frac{\cos^2(n)}{3^n}$

Since $0 \le \cos^2(n) \le 1$, we have

$$0 \le \frac{\cos^2(n)}{3^n} \le \frac{1}{3^n}.$$

Because $\frac{1}{3^n} \to 0$, the squeeze theorem gives

$$\lim_{n\to\infty} \frac{\cos^2(n)}{3^n} = 0.$$

(iv) $\frac{4n^3+5n+1}{2n^3+n^2+5}$

Dividing numerator and denominator by n^3, we obtain

$$\frac{4+\frac{5}{n^2}+\frac{1}{n^3}}{2+\frac{1}{n}+\frac{5}{n^3}} \longrightarrow \frac{4}{2} = 2.$$

Hence the sequence converges to 2.

(v) $\left(1+\frac{1}{n}\right)^n$

This is a classical sequence that defines the number e.

Let

$$e_n = \left(1+\frac{1}{n}\right)^n.$$

Taking logarithms,

$$\ln(e_n) = n\ln\left(1+\frac{1}{n}\right) = \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}.$$

Let $x = \frac{1}{n}$. As $n \to \infty$, we have $x \to 0^+$, and

$$\ln(e_n) = \frac{\ln(1+x)}{x}.$$

Using the standard limit

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1,$$

we obtain

$$\lim_{n\to\infty} \ln(e_n) = 1.$$

Since $\ln(e_n) \to 1$, it follows that

$$e_n = e^{\ln(e_n)} \to e.$$

(vi) $n^{1/n}$

Taking logarithms,

$$\ln\left(n^{1/n}\right) = \frac{\ln n}{n}.$$

As $n \to \infty$, this quotient tends to 0, so

$$n^{1/n} \to e^0 = 1.$$

Exercise 3

Decide whether the following sequences converge or diverge:

$$a_n = \sqrt{n+1} - \sqrt{n}, \qquad b_n = \frac{n^2}{2n-1} - \frac{n^2}{2n+1}.$$

Solution

(1) $a_n = \sqrt{n+1} - \sqrt{n}$

$$a_n = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{2\sqrt{n}} \to 0.$$

By the squeeze theorem,

$$\lim_{n\to\infty} a_n = 0.$$

(2) $b_n = \frac{n^2}{2n-1} - \frac{n^2}{2n+1}$

$$b_n = \frac{2n^2}{4n^2-1} \to \frac{1}{2}.$$

Infinite Series

Lecture 1: What Is an Infinite Series?

An **infinite series** is the sum of the terms of a sequence:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

Partial Sums

The **sequence of partial sums** is defined by

$$S_N = \sum_{k=1}^{N} a_k.$$

$$S_1 = a_1, \qquad S_2 = a_1 + a_2, \qquad S_3 = a_1 + a_2 + a_3, \quad \ldots$$

Definition of Convergence

The series $\sum_{n=1}^{\infty} a_n$ **converges** if the sequence (S_N) converges to a finite limit. In that case,

$$\sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} S_N.$$

Divergence

If the sequence of partial sums has no finite limit, the series **diverges**.

Telescoping Series

Example

Evaluate the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

We first rewrite the general term using partial fractions:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Consider the partial sum

$$S_N = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right).$$

$$S_N = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right).$$

Telescoping Effect

All intermediate terms cancel. Only the first and the last terms remain.

$$S_N = 1 - \frac{1}{N+1}.$$

Limit

Taking the limit as $N \to \infty$,

$$\lim_{N\to\infty} S_N = 1.$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1}$$

General Rule: Telescoping Series

Telescoping Rule

If a sequence satisfies

$$a_n = b_n - b_{n+1},$$

then

$$S_N = \sum_{n=1}^{N} a_n = b_1 - b_{N+1}.$$

If $\lim_{N\to\infty} b_{N+1}$ exists, then

$$\sum_{n=1}^{\infty} a_n = b_1 - \lim_{N\to\infty} b_{N+1}.$$

The Geometric Series $\sum_{n=0}^{\infty} r^n$

For which real numbers r does the series

$$\sum_{n=0}^{\infty} r^n$$

converge, and what is its value?

Partial Sums

To answer this question, we study the sequence of partial sums

$$S_N = \sum_{n=0}^{N} r^n.$$

$$\begin{aligned}(1-r)\sum_{n=0}^{N} r^n &= \sum_{n=0}^{N} \left(r^n - r^{n+1}\right) \\ &= (r^0 - r^1) + (r^1 - r^2) + \cdots + (r^N - r^{N+1}) \\ &= r^0 - r^{N+1} = 1 - r^{N+1}.\end{aligned}$$

Dividing both sides by $1-r$, we obtain the formula for the partial sum:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}, \quad r \neq 1.$$

The behavior of the infinite series depends entirely on the limit of r^{N+1} as $N \to \infty$.

$$\begin{cases} |r| < 1 & \Rightarrow \quad r^{N+1} \to 0, \\ |r| \geq 1 & \Rightarrow \quad r^{N+1} \text{ diverges }. \end{cases}$$

Geometric Series Test

$$\sum_{n=0}^{\infty} r^n \text{ converges if and only if } |r| < 1.$$

Sum of the Series

If $|r| < 1$, then

$$\sum_{n=0}^{\infty} r^n = \lim_{N\to\infty} S_N = \frac{1}{1-r}.$$

If $|r| \geq 1$, the series diverges.

Example

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2.$$

Basic Properties of Series

Linearity

If both series converge, then:

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n,$$

$$\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n.$$

Exercises

Exercise

Evaluate the series

$$\sum_{n=3}^{\infty} \frac{1}{n(n+1)}.$$

Use the known value of the full series starting at $n = 1$.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

$$\sum_{n=3}^{\infty} \frac{1}{n(n+1)} = 1 - \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3}\right).$$

$$\frac{1}{2} + \frac{1}{6} = \frac{3}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}.$$

Answer

$$\sum_{n=3}^{\infty} \frac{1}{n(n+1)} = \boxed{\frac{1}{3}}.$$

Exercises

Evaluate:

$$\text{(a)} \quad \sum_{n=0}^{\infty} \frac{2^{n+1}}{3^{n-2}}, \qquad \text{(b)} \quad \sum_{n=0}^{\infty} \frac{2^n + 3^n}{5^n}.$$

Solution (a)

Rewrite the general term:

$$\frac{2^{n+1}}{3^{n-2}} = 18\left(\frac{2}{3}\right)^n.$$

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{3^{n-2}} = 18 \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n.$$

Since $|2/3| < 1$,

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1 - \frac{2}{3}} = 3.$$

Answer

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{3^{n-2}} = \boxed{54}.$$

Solution (b)

Use linearity to split the series:

$$\sum_{n=0}^{\infty} \frac{2^n + 3^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n.$$

$$\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = \frac{5}{3}, \qquad \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n = \frac{5}{2}.$$

Answer

$$\sum_{n=0}^{\infty} \frac{2^n + 3^n}{5^n} = \boxed{\frac{25}{6}}.$$

Necessary Condition for Convergence

Statement of the Necessary Condition

If the series

$$\sum_{n=1}^{\infty} a_n$$

converges, then

$$\lim_{n\to\infty} a_n = 0.$$

Fundamental Identity of Partial Sums

Let

$$S_n = \sum_{k=1}^{n} a_k$$

be the sequence of partial sums. Then, for every $n \geq 2$,

$$\boxed{a_n = S_n - S_{n-1}}.$$

Assume now that the series converges. Then the sequence of partial sums (S_n) converges to a finite limit S, where

$$S = \sum_{n=1}^{\infty} a_n.$$

Using the fundamental identity,

$$a_n = S_n - S_{n-1}.$$

Taking limits on both sides,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

Important

The converse is **false**.
Even if $a_n \to 0$, the series

$$\sum_{n=1}^{\infty} a_n$$

may still diverge.

Lecture 2 : Convergence Tests for Positive Series

Positive Series

We study series of the form

$$\sum_{n=1}^{\infty} a_n,$$

where

$$a_n \geq 0 \quad \text{for all } n.$$

Such series are called **positive series**.

(a) The n^{th}-Term Test

The n^{th}-Term Test

If

$$\lim_{n\to\infty} a_n \neq 0 \quad \text{or the limit does not exist,}$$

then the series

$$\sum_{n=1}^{\infty} a_n$$

diverges.

Example 1

$$a_n = 1.$$

Here

$$\lim_{n\to\infty} a_n = 1 \neq 0.$$

Partial sums:

$$S_1 = 1, \quad S_2 = 2, \quad S_3 = 3, \ \ldots, \quad S_N = N \to \infty.$$

Hence, the series

$$\sum_{n=1}^{\infty} 1$$

diverges.

Example 2

$$a_n = (-1)^n.$$

The limit $\lim_{n\to\infty}(-1)^n$ does not exist (the terms oscillate). Partial sums:

$$S_1 = -1,\ S_2 = 0,\ S_3 = -1,\ S_4 = 0,\ \ldots$$

The sequence (S_n) does not converge, so the series diverges.

Example 3

Consider

$$\sum_{n=1}^{\infty} \frac{n}{n+1}.$$

We compute

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \neq 0.$$

By the n^{th}-term test, the series diverges.

Why This Test Is One-Way

The condition $a_n \to 0$ is **necessary** but **not sufficient**.
For example,

$$a_n = \frac{1}{n} \to 0,$$

but the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

(b) Integral Test

Integral Test

Let $f : [1, \infty) \to \mathbb{R}$ satisfy:

- f is continuous,
- $f(x) \geq 0$ for all $x \geq 1$,
- f is decreasing on $[1, \infty)$,
- $f(n) = a_n$ for all integers $n \geq 1$.

Then

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \quad \Longleftrightarrow \quad \int_1^{\infty} f(x)\, dx \text{ converges.}$$

Application: The p-Series

The p-Series

Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \qquad p > 0.$$

Let

$$f(x) = \frac{1}{x^p},$$

which is continuous, positive, and decreasing on $[1, \infty)$.

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{t \to \infty} \int_1^t \frac{1}{x^p}\, dx.$$

Case $p \neq 1$.

$$\int_1^t \frac{1}{x^p}\, dx = \left[\frac{x^{1-p}}{1-p} \right]_1^t = \frac{t^{1-p} - 1}{1-p}.$$

- If $p > 1$, then $t^{1-p} \to 0$ and the integral converges.
- If $p < 1$, then $t^{1-p} \to \infty$ and the integral diverges.

Case $p = 1$.

$$\int_1^t \frac{1}{x}\, dx = \ln t \to \infty.$$

Final Conclusion

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{Converges}, & p > 1, \\ \text{Diverges}, & p \leq 1. \end{cases}$$

Examples of the p-Series Test

- $\sum_{n=1}^{\infty} \frac{1}{n^2}$ **Converges**, since $p = 2 > 1$.
- $\sum_{n=1}^{\infty} \frac{1}{n}$ **Diverges** (harmonic series), since $p = 1$.
- $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ **Diverges**, since $p = \frac{1}{2} < 1$.

(c) Comparison Test

Comparison Test

Let (a_n) and (b_n) be sequences with

$$0 \leq a_n \leq b_n \quad \text{for all } n.$$

- If $\sum b_n$ **converges**, then $\sum a_n$ also **converges**.
- If $\sum a_n$ **diverges** and $0 \leq a_n \leq b_n$, then $\sum b_n$ also **diverges**.

Idea: A series smaller than a convergent one must converge. A series larger than a divergent one must diverge.

Example (Direct Comparison)

Let

$$a_n = \frac{1}{n^2+1}, \qquad b_n = \frac{1}{n^2}.$$

Since $n^2 + 1 \geq n^2$, we have

$$0 \leq a_n \leq b_n \quad \text{for all } n.$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a p-series with $p = 2 > 1$, hence it **converges**. Therefore, by the Comparison Test,

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \quad \textbf{converges}.$$

(d) Limit Comparison Test

Limit Comparison Test

Let $a_n > 0$ and $b_n > 0$. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \quad \text{with } 0 < c < \infty,$$

then the series $\sum a_n$ and $\sum b_n$ **either both converge or both diverge**.

How to Use the Limit Comparison Test

1. Choose a comparison series $\sum b_n$ that you already know.
2. Compute $L = \lim_{n\to\infty} \dfrac{a_n}{b_n}$.
3. If $0 < L < \infty$, both series behave the same.
4. If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
5. If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Examples: Rational Terms (Comparison with a p-Series)

Example 1 (Convergent)

Consider

$$\sum_{n=1}^{\infty} \frac{2n^3+5}{9n^5+1}.$$

Compare with the p-series $\sum \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{\dfrac{2n^3+5}{9n^5+1}}{1/n^2} = \lim_{n\to\infty} \frac{2n^5+5n^2}{9n^5+1} = \frac{2}{9}.$$

Since $\sum \frac{1}{n^2}$ converges, the given series **also converges**.

Example 2 (Divergent)

Consider

$$\sum_{n=1}^{\infty} \frac{7n+4}{3n^2+1}.$$

Compare with the harmonic series $\sum \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{\dfrac{7n+4}{3n^2+1}}{1/n} = \lim_{n\to\infty} \frac{7n^2+4n}{3n^2+1} = \frac{7}{3}.$$

Since $\sum \frac{1}{n}$ diverges, the given series **diverges**.

Examples: Comparison with a Geometric Series

Example 3 (Convergent)

Let

$$a_n = \frac{3^n+2}{6^n+1}.$$

Compare with the geometric series $b_n = \left(\frac{1}{2}\right)^n$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(3^n+2)2^n}{6^n+1} = 1.$$

Since $\sum \left(\frac{1}{2}\right)^n$ converges, the series $\sum a_n$ also **converges**.

Example 4 (Divergent)

Let
$$c_n = \frac{8^n - 1}{5^n}.$$
Compare with $d_n = \left(\frac{8}{5}\right)^n$:
$$\lim_{n\to\infty} \frac{c_n}{d_n} = \lim_{n\to\infty} \left(1 - \frac{1}{8^n}\right) = 1.$$
Since $\sum \left(\frac{8}{5}\right)^n$ diverges (ratio > 1), the series $\sum c_n$ **diverges**.

- Polynomial terms $\Rightarrow$ compare with a p-series.
- Exponential terms $\Rightarrow$ compare with a geometric series.
- The limit comparison test is often faster than direct comparison.

(e) Ratio Test

Ratio Test

Let (a_n) be a series with $a_n > 0$. Define
$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

- If $L < 1$, the series **converges**.
- If $L > 1$ or $L = \infty$, the series **diverges**.
- If $L = 1$, the test is **inconclusive**.

When to Use the Ratio Test

The Ratio Test is especially effective when the terms involve:

- factorials ($n!$),
- exponential terms (c^n),
- products of powers and exponentials.

Convergent Cases ($L < 1$)

(a) $a_n = \dfrac{5^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{5}{n+1} \longrightarrow 0 < 1.$$

Hence, $\sum a_n$ converges.

(b) $a_n = \dfrac{n}{3^n}$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} \cdot \frac{1}{3} \longrightarrow \frac{1}{3} < 1.$$

Thus, $\sum a_n$ converges.

Divergent Cases ($L > 1$)

(c) $a_n = \dfrac{2^n}{n}$

$$\frac{a_{n+1}}{a_n} = 2 \cdot \frac{n}{n+1} \longrightarrow 2 > 1.$$

Hence, $\sum a_n$ diverges.

(d) $a_n = n!$

$$\frac{a_{n+1}}{a_n} = n + 1 \longrightarrow \infty.$$

Therefore, $\sum a_n$ diverges.

(f) Root Test

Root Test (Rule)

Let $a_n > 0$. Define

$$L = \lim_{n \to \infty} \sqrt[n]{a_n}.$$

- If $L < 1$, the series **converges**.
- If $L > 1$ or $L = \infty$, the series **diverges**.
- If $L = 1$, the test gives **no conclusion**.

- See $n!$: use the **Ratio Test**.
- See $(\text{something})^n$: use the **Root Test**.
- If the limit equals 1, the test fails – try another method.

Lecture 3: Absolute and Conditional Convergence

Definitions

- A series $\sum a_n$ is said to be **absolutely convergent** if

$$\sum |a_n| \quad \text{converges.}$$

- A series $\sum a_n$ is said to be **conditionally convergent** if

$$\sum a_n \text{ converges, but } \sum |a_n| \text{ diverges.}$$

Theorem (Absolute Convergence)

If the series $\sum |a_n|$ converges, then the original series $\sum a_n$ also converges.

Remark. The converse is not true.

Classical Example

Consider the alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

This series converges, but

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

Hence, the alternating harmonic series is **conditionally convergent**.

(g) Alternating Series Test (Leibniz Test)

Alternating Series Test

Let $a_n > 0$. If:

- a_n is decreasing, and
- $\lim_{n\to\infty} a_n = 0$,

then the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

converges.

Examples

- $\sum \frac{(-1)^n}{n^2}$ **Absolutely convergent** (use the p-series test).
- $\sum \frac{(-1)^n}{\ln n}$ **Conditionally convergent** (use the Leibniz test).
- $\sum \frac{n!}{n^n}$ **Convergent** (use the Ratio or Root Test).

Exercises

Instructions

Decide whether each series converges or diverges. If it converges, state whether the convergence is **absolute** or **conditional**, and find the sum when possible.

Problems

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} \quad (p > 0)$

2. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \quad (p > 0)$

3. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3}{n^3-1}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Solutions

1. $\sum_{n=1}^{\infty} \frac{1}{n^p}$

   Integral test. For $p > 0$, set $f(x) = x^{-p}$, positive and decreasing on $[1, \infty)$. Then

   $$\int_1^{\infty} x^{-p}\, dx = \begin{cases} \frac{1}{p-1}, & p > 1, \\ \infty, & p \leq 1. \end{cases}$$

   Hence the series converges iff $p > 1$.

2. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

   Using the integral test with $u = \ln x$,

   $$\int_2^{\infty} \frac{dx}{x(\ln x)^p} = \begin{cases} \frac{(\ln 2)^{1-p}}{p-1}, & p > 1, \\ \infty, & 0 < p \leq 1. \end{cases}$$

   Thus the series converges iff $p > 1$.

3. $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

   $$\int_2^{\infty} \frac{\ln x}{x^2}\, dx = \frac{\ln 2 + 1}{2} < \infty,$$

   so the series converges absolutely.

4. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

   Limit comparison with $\sum \frac{1}{n}$ gives divergence.

5. $\sum_{n=1}^{\infty} \frac{n^2+3}{n^3-1}$

   Limit comparison with $\sum \frac{1}{n}$ gives divergence.

6. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

   Telescoping:

   $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Problems

1. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n$

2. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{3^n}$

3. $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{n!}$

4. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$

5. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

6. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3n+1}{2n-1}\right)^n$

7. $\displaystyle\sum_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^n}{n^2}$

8. $\displaystyle\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$

9. $\displaystyle\sum_{n=2}^{\infty}(-1)^{n-1}\frac{\ln n}{n}$

10. $\displaystyle\sum_{n=1}^{\infty}\frac{n^p}{2^n} \quad (p \in \mathbb{R})$

11. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p} \quad (p > 0)$

Solutions

1. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n$

   Geometric series; sum $= \dfrac{5}{2}$.

2. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{3^n}$

   Ratio test gives convergence; sum $= \dfrac{3}{4}$.

3. $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{n!}$

   Ratio test; sum $= e^2$.

4. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$

   Alternating harmonic series: converges conditionally.

5. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

   Leibniz test applies; convergence is conditional.

6. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3n+1}{2n-1}\right)^n$

   Root test gives limit $= \frac{3}{2} > 1$; diverges.

7. $\displaystyle\sum_{n=1}^{\infty}\frac{(1+\frac{1}{n})^n}{n^2}$

   Comparison with $\sum \frac{e}{n^2}$; converges absolutely.

8. $\displaystyle\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$

   Limit comparison with $\sum \frac{1}{n}$ shows divergence.

9. $\displaystyle\sum_{n=2}^{\infty}(-1)^{n-1}\frac{\ln n}{n}$

   Leibniz test gives convergence; absolute series diverges.

10. $\displaystyle\sum_{n=1}^{\infty}\frac{n^p}{2^n}$

    Ratio test gives absolute convergence for all $p \in \mathbb{R}$.

11. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p}$

    Alternating series converges for $p > 0$; absolute iff $p > 1$.