Infinite Series

Mongi BLEL

King Saud University

March 25, 2024

Table of contents

Sequences

Definition

An infinite sequence (or a sequence) of real numbers is a real-valued function defined on a set of integers $\{n \in \mathbb{N};\ n \geq k\}$. We call the values of the function the terms of the sequence. We denote a sequence by listing its terms in order, $(u_n)_{n \geq k}$. (u_n is called the general term of the sequence.)

Definition

A sequence $(u_n)_n$ is called convergent to the real number ℓ if

$$\forall\ \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } |u_n - \ell| < \varepsilon \quad \forall n \geq N.$$

ℓ will be called the limit of the sequence $(u_n)_n$ and denoted by $\ell = \lim\limits_{n \to +\infty} u_n$.
A sequence which is not convergent is called divergent.

Remarks

1. The limit of a sequence if it exists is unique.
2. If a sequence $(u_n)_n$ converges to a limit ℓ, then the sequence $(v_n)_n$ defined by $v_n = u_{n+p}$ converges also to ℓ.

Definition

1. A sequence $(u_n)_n$ is called upper bounded, if there exists a $M \in \mathbb{R}$ such that, $u_n \leq M$, $\forall n \in \mathbb{N}$.
2. A sequence $(u_n)_n$ is called lower bounded, if there exists a real number m such that, $u_n \geq m$, $\forall\ n \in \mathbb{N}$.
3. A sequence $(u_n)_n$ is called bounded, if it is upper and lower bounded.

Theorem

Any convergent sequence is bounded.
The sequence $(u_n)_n$ defined by: $u_n = (-1)^{n+1}$ is bounded but not convergent.

Monotone Sequences

Definition

1. A sequence $(u_n)_n$ is called increasing if $u_n \leq u_{n+1}, \ \forall n \in \mathbb{N}$.
2. A sequence $(u_n)_n$ is called decreasing if $u_n \geq u_{n+1}, \ \forall \ n \in \mathbb{N}$.

Theorem

1. Any real upper bounded sequence is convergent.
2. Any decreasing lower bounded sequence is convergent.

Subsequence

Definition

let $(u_n)_n$ be a sequence. A sequence $(v_n)_n$ is called a subsequence of the sequence $(u_n)_n$ if there exists a strictly increasing map $\varphi\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that $v_n = u_{\varphi(n)}$.

Example

$(u_{2n})_n$, $(u_{2n+1})_n$, $(u_{3n})_n$ are subsequences of $(u_n)_n$.

Theorem

Any subsequence of a convergent sequence is convergent and converges to the same limit.

Remark

A sequence $(u_n)_n$ can be divergent and has a convergent subsequence.

Example

$u_n = (-1)^n$, with $n \in \mathbb{N}$. $u_{2n} = 1$ and $u_{2n+1} = -1$, the sequences $(u_{2n})_n$ and $(u_{2n+1})_n$ are convergent, but the sequence $(u_n)_n$ divergent.

Theorem

[Bolzano-Weierstrass Theorem] Any bounded real sequence has a convergent subsequence.

Cauchy Sequences

Definition

A sequence $(u_n)_n$ is called a Cauchy sequence if

$$\forall \varepsilon > 0 \ \exists \ N \in \mathbb{N} \text{ such that } \forall \ n, m \geq N \quad |u_n - u_m| \leq \varepsilon. \qquad (1)$$

Theorem

[Cauchy's Convergence Criterion]
A real sequence is Cauchy sequence if and only if it is convergent.

General Properties of Convergent series

Definition

1. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. We consider the sequence $(S_n)_n$ defined by: $S_n = \sum_{k=1}^{n} u_k$.
   We say that the series $\sum_{n\geq 1} u_n$ is convergent if the sequence $(S_n)_n$ is convergent.
   The limit of the sequence $(S_n)_n$ if it exists is denoted by $\sum_{n=1}^{+\infty} u_n$.
2. The series $\sum_{n\geq 1} u_n$ is called divergent if the sequence $(S_n)_n$ is divergent.

Remarks

1. If the series $\sum_{n\geq 1} u_n$ converges, then $\lim_{n\longrightarrow+\infty} u_n = 0$. $(u_n = S_n - S_{n-1}.)$
2. The condition $\lim_{n\longrightarrow+\infty} u_n = 0$ is not, however, sufficient to ensure the convergence of the series $\sum_{n\geq 1} u_n$. For instance, the series $\sum_{n\geq 1} \sqrt{n+1} - \sqrt{n}$ is divergent because for every $n \in \mathbb{N}$; $S_n = \sqrt{n+1} - 1$ and $\lim_{n\to+\infty} u_n = 0$.

Theorem

[Cauchy Criterion]
Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. The series $\sum_{n\geq 1} u_n$ converges if, and only, if,

$$\forall \ \ \varepsilon > 0, \ \exists \ N_\varepsilon \in \mathbb{N}; \ \ \forall \ q \geq p \geq N_\varepsilon; \ |\sum_{n=p}^{q} u_n| \leq \varepsilon.$$

Definition

A series $\sum_{n\geq 1} u_n$ is called absolutely convergent if the series $\sum_{n\geq 1} |u_n|$ converges.

Remark

Any absolutely convergent series is convergent but the converse is false, it suffices to take the series $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}$.
If $S_n = \sum_{p=1}^{n} \frac{(-1)^{p+1}}{p}$, then $S_{2n+1} - S_{2n} = \frac{-1}{2n+1} \xrightarrow[p\to+\infty]{} 0$. To prove that the series $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}$ converges, it suffices to prove that the sequence $(S_{2n})_n$ is convergent.
We have $S_{2n+2} - S_{2n} = \frac{1}{2n+2} - \frac{1}{2n+1} \leq 0$ and $S_{2n+1} - S_{2n-1} = \frac{1}{2n} - \frac{1}{2n+1} \geq 0$, then the sequences $(S_{2n})_n$ and $(S_{2n+1})_n$ are adjacent, which shows that the sequence $(S_n)_n$ is convergent.
We remark that $\sum_{k=n+1}^{2n} \frac{1}{k} \geq \frac{n}{2n} = \frac{1}{2}$, thus the series $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}$ is not absolutely convergent.

Tests of Convergence

There are several standard tests for convergence of a series of non negative terms:
The following comparison criterions are based primarily on the fact that an increasing sequence is convergent if, and only, if, it is bounded above. It follows that a series $\sum_{n\geq 1} u_n$ with non negative terms is convergent if, and only, if, the sequence $(S_n)_n$ defined by $S_n = \sum_{k=1}^{n} u_k$ is bounded.

Theorem

[Comparison Test]
Let $(u_n)_n$ and $(v_n)_n$ be two sequences with non negative numbers. Assume that there exists an integer $k \in \mathbb{N}$ such that for every $n \geq k$, $u_n \leq v_n$, then if the series $\sum_{n\geq 1} v_n$ is convergent, the series $\sum_{n\geq 1} u_n$ is also convergent.

Corollary

Let $(u_n)_n$ and $(v_n)_n$ be two sequences with non negative numbers. Assume that there exists $a > 0$ and $b > 0$ such that $au_n \leq v_n \leq bu_n$ for every $n \geq k$, then the series $\sum_{n\geq 1} u_n$ and $\sum_{n\geq 1} v_n$ have the same nature.

Corollary

Let $(u_n)_n$ and $(v_n)_n$ be two sequences with non negative numbers. Assume that

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = \ell.$$

1. If $\ell > 0$, the series $\sum_{n\geq 1} u_n$ and $\sum_{n\geq 1} v_n$ have the same nature.
2. If $\ell = 0$, the convergence of the series $\sum_{n\geq 1} v_n$ involves the convergence of the series $\sum_{n\geq 1} u_n$.
3. If $\ell = +\infty$, the convergence of the series $\sum_{n\geq 1} u_n$ involves the convergence of the series $\sum_{n\geq 1} v_n$.

Integral Test

Theorem

[Integral Test]
Let f be a decreasing continuous function on $[1, +\infty[$. For all $n \in \mathbb{N}$, we define $u_n = f(n)$, then

$$\int_1^{+\infty} f(x)dx \text{ converges } \iff \sum_{n\geq 1} u_n \text{ converges.}$$

Corollary

[Convergence of Riemann series]
The series $\sum_{n\geq 1} \frac{1}{n^\alpha}$ converges if, and only, if, $\alpha > 1$.

Theorem

[Application Comparison with Riemann series]
Let $(u_n)_n$ be a sequence with non negative terms. Assume that there exists $0 < a < b$ such that for every n large enough, $0 < a \leq n^\alpha u_n \leq b < +\infty$, then the series $\sum_{n\geq 1} u_n$ converges if, and only, if, $\alpha > 1$.

This Theorem results from Theorem 23

Exercise

Show that the Bertrand series $\sum_{n\geq 2} \frac{1}{n^\alpha \ln^\beta n}$ converges if, and only, if, $\alpha > 1$ or $\alpha = 1$ and $\beta > 1$.

Solution

If $\alpha \leq 0$, $\displaystyle\lim_{n\to+\infty} \frac{n}{n^\alpha (\ln n)^\beta} = +\infty$, then the series diverges.

If $0 < \alpha < 1$, we get $\alpha < \gamma < 1$ and consider the sequence $v_n = \frac{1}{n^\gamma}$.

$\displaystyle\lim_{n\to+\infty} \frac{n^\gamma}{n^\alpha (\ln n)^\beta} = +\infty$, then the series $\displaystyle\sum_{n\geq 2} \frac{1}{n^\alpha (\ln n)^\beta}$ diverges.

If $\alpha > 1$, we get $1 < \gamma < \alpha$ and consider the sequence $v_n = \frac{1}{n^\gamma}$, $\lim\limits_{n\to+\infty} \dfrac{n^\gamma}{n^\alpha(\ln n)^\beta} = 0$, then the series $\displaystyle\sum_{n\geq 2} \frac{1}{n^\alpha(\ln n)^\beta}$ converges.

If $\alpha = 1$, we consider the sequence $u_n = \dfrac{1}{n\ln^\beta n}$ and $f(x) = \dfrac{1}{x\ln^\beta x}$, for $x \geq 2$. f is decreasing for x large. Then the series $\displaystyle\sum_{n\geq 2} \frac{1}{n(\ln n)^\beta}$ converges if and only if $\displaystyle\int_2^\infty \frac{dx}{x\ln^\beta x}$.

The integral

$$\int_2^\infty \frac{dx}{x\ln^\beta x} \overset{t=\ln x}{=} \int_{\ln 2}^\infty \frac{dt}{t^\beta}$$

is convergent if and only if $\beta > 1$. $\square$

Root Test or Cauchy Test

Theorem

[Root Test or the Cauchy Test]
Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and
$\ell = \overline{\lim}_{n\to+\infty} \sqrt[n]{|u_n|}$.

1. If $\ell < 1$, the series $\displaystyle\sum_{n\geq 1} u_n$ is absolutely convergent.
2. If $\ell > 1$, the general term of the series does not tends to 0 and the series $\sum_{n\geq 1} u_n$ diverges.
3. If $\ell = 1$, we can not conclude if the series is convergent or not.

The Ratio Test or D'Alembert's Test

Theorem

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Assume that $\lim\limits_{n\longrightarrow+\infty}|\frac{u_{n+1}}{u_n}| = \ell$. Then

i) If $\ell < 1$, the series $\sum\limits_{n\geq 1} u_n$ is absolutely convergent.

ii) If $\ell > 1$ the general term of the series does not tend to 0 and the series $\sum\limits_{n\geq 1} u_n$ diverges.

iii) If $\ell = 1$, we can not conclude if the series is convergent or not. In this case,

$$\lim_{n\longrightarrow+\infty} \sqrt[n]{|u_n|} = \ell.$$

Examples

1. Let $z \in \mathbb{C}$, the series $\sum_{n\geq 0} \frac{z^n}{n!}$ is absolutely convergent on $\mathbb{C}$, because for every $z \in \mathbb{C}$; $|\dfrac{u_{n+1}}{u_n}| = \dfrac{|z|}{n+1} \xrightarrow[n\longrightarrow+\infty]{} 0$. We denote e^z the sum of this series.
2. For $|z| < 1$, the series $\sum_{n\geq 1} \frac{z^n}{n}$ is absolutely convergent.

The Abel Test

Theorem

[The Abel Test]
Let $(u_n)_n$ be a sequence of real numbers and let $(v_n)_n$ be a sequence of non negative real numbers such that
i) the sequence $(v_n)_n$ is decreasing and tends to 0 when $n \to +\infty$.
ii) the sequence $\left(S_n = \sum_{k=1}^{n} u_k\right)_n$ is bounded.
Then the series $\sum_{n\geq 1} u_n v_n$ is convergent.

Proof

we use the Cauchy criterion (19) for the existence of the limit. Let $q > p \geq 1$,

$$\begin{aligned}\sum_{k=p+1}^{q} u_k v_k &= \sum_{k=p+1}^{q} (S_k - S_{k-1})v_k = \sum_{k=p+1}^{q} S_k v_k - \sum_{k=p}^{q-1} S_k v_{k+1} \\ &= \sum_{k=p+1}^{q-1} (v_k - v_{k+1}) + S_q v_q - S_p v_{p+1}\end{aligned}$$

Since $|S_k| \leq M$, then we have
$|\sum_{k=p+1}^{q} u_k v_k| \leq 2Mv_{k+1} \xrightarrow[k\to+\infty]{} 0.$

Remark

The result still true if the sequence $(S_n)_n$ is bounded and the sequence $(b_n)_n$ tends to 0 and the series $\sum_{n=0}^{+\infty}(b_n - b_{n+1})$ is convergent.

Examples

1. Let $b_n = \dfrac{(-1)^{[\sqrt{n}]}}{n}$, for $n \geq 1$ and $a_n = e^{\mathrm{i}n\theta}$ for $0 < \theta < 2\pi$.
   $|\sum_{n=p}^{q} a_n| \leq \dfrac{1}{\sin\theta/2}$ and we can prove that
   $\sum_{n\geq 2} |b_n - b_{n-1}| \leq \sum_{n\geq 2} \dfrac{2}{(n-1)^2}$. It results that the series
   $\sum_{n\geq 1} \dfrac{(-1)^{[\sqrt{n}]} e^{\mathrm{i}n\theta}}{n}$ converges for all $0 < \theta < 2\pi$.
2. Let $s_n = \sum_{k=1}^{n} \dfrac{1}{k} - \ln n$, $n \geq 1$. We set $u_1 = S_1 = 1$ and for all
   $n \geq 2$; $u_n = S_n - S_{n-1} = \dfrac{1}{n} + \ln\dfrac{n-1}{n} = \dfrac{1}{n} + \ln(1 - \dfrac{1}{n}) =$

The Series Product

Definition

Let $(u_n)_n$ and $(v_n)_n$ be two sequences of real numbers. For $n \in \mathbb{N}$, we set

$$c_n = \sum_{k=1}^{n} u_k v_{n-k}. \tag{2}$$

The series $\sum_{n\geq 1} c_n$ is called the product of the two given series $\sum_{n\geq 1} u_n$ and $\sum_{n\geq 1} v_n$.

In this definition we are not interested in whether the product of the series exists, because it depends of some conditions. Indeed we have the following example
The series $\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n+1}}$ is convergent. The product of this series with itself is a series with general term c_n, such that

$$c_n = \sum_{k=1}^{n} \frac{(-1)^k}{\sqrt{k+1}} \frac{(-1)^{n-k}}{\sqrt{n-k+1}} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\sqrt{n-k+1}}. \tag{3}$$

But $|c_n| \geq 1$, thus the series $\sum_{n\geq 1} c_n$ is divergent.
The following theorem affirms the existence of the series product under certain conditions.

Theorem

Let $(u_n)_n$ and $(v_n)_n$ be two sequences of real numbers.

1. We assume that the series $\sum_{n\geq 1} u_n$ and $\sum_{n\geq 1} v_n$ are absolutely convergent. Then the series $\sum_{n\geq 1} c_n$ is absolutely convergent and one has
$$\sum_{n=1}^{+\infty} c_n = (\sum_{n=1}^{+\infty} u_n)(\sum_{n=1}^{+\infty} v_n). \qquad (4)$$
2. We assume that the series $\sum_{n\geq 1} u_n$ is absolutely convergent and the series $\sum_{n\geq 1} v_n$ is convergent. Then the series $\sum_{n\geq 1} c_n$ is

Proof

It suffices to proves 2).
We set

$$A_n = \sum_{k=1}^{n} u_k, \qquad B_n = \sum_{k=1}^{n} v_k, \qquad C_n = \sum_{k=1}^{n} c_k,$$

$$A = \sum_{n=1}^{+\infty} u_n, \qquad \alpha = \sum_{n=1}^{+\infty} |u_n| \quad \text{and} \quad B = \sum_{n=1}^{+\infty} v_n.$$

Then

$$C_n = \sum_{j=1}^{n} c_j = \sum_{j=1}^{n} u_j B_{n-j} = \sum_{j=1}^{n} u_j(B_{n-j} - B) + BA_n.$$

Since $\lim_{n\to+\infty} B.A_n = A.B$, then to show that $\lim\limits_{n\longrightarrow+\infty} C_n = A.B$, it suffices to show that the sequence $(\Delta_n)_n$ converges to 0, where $\Delta_n = \sum\limits_{j=1}^{n} a_j(B_{n-j} - B)$.

Let $\varepsilon > 0$ $\exists N \in \mathbb{N}$ such that $\forall\ n \geq N$; $|B_n - B| < \frac{\varepsilon}{2\alpha}$ and $\sum\limits_{j=N}^{+\infty} |a_j| \leq \dfrac{\varepsilon}{2M}$, thus for every $n \geq 2N$,

$$|\Delta_n| \leq \sum_{j=1}^{N} |a_j||B_{n-j} - B| + \sum_{j=N+1}^{n} |a_j||B_{n-j} - B| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Power Series

Definition

Let $f_n(x) = a_n(x - x_0)^n$; with $(a_n)_n$ a sequence of real numbers. The series $\sum_{n\geq 1} a_n(x - x_0)^n$ is called a power series centered at x_0.

We denote $S(x) = \sum_{n=1}^{+\infty} a_n(x - x_0)^n$, whenever x where the series converges.

Let $\sum_{n\geq 1} a_n(x-x_0)^n$ be a power series, we look for its domain of convergence. The series converges at least for $x = x_0$. In which follows, we consider the series centered at 0.

Theorem

[Abel's lemma]

If the power series $\sum_{n\geq 1} a_n x^n$ is convergent for $x = x_0$, $x_0 \neq 0$, then

1. the series $\sum_{n\geq 1} a_n x^n$ is absolutely convergent on the interval $]-|x_0|, |x_0|[$,
2. for every $r < |x_0|$, the series $\sum_{n\geq 1} a_n x^n$ converges uniformly on $[-r, r]$.

Proof

1. Let $x \in]-|x_0|,|x_0|[$, $\sum_{n=1}^{+\infty}|a_n x^n| \leq \sum_{n=1}^{+\infty}|a_n x_0^n||\frac{x}{x_0}|^n$. Since the series $\sum_{n\geq 1} a_n x_0^n$ is convergent, the sequence $(a_n x_0^n)_n$ is bounded. Moreover the series $\sum_{n\geq 1}|\frac{x}{x_0}|^n$ is convergent, then the series $\sum_{n\geq 1} a_n x^n$ is absolutely convergent on $]-|x_0|,|x_0|[$.
2. Let $r < |x_0|$ and $x \in [-r, r]$, $|a_n x^n| \leq |a_n| r^n$ and $\sum_{n=1}^{+\infty}|a_n| r^n < +\infty$, thus the series $\sum_{n\geq 1} a_n x^n$ converges uniformly on $[-r, r]$.

$\square$

Corollary

If the power series $\sum_{n\geq 1} a_n x^n$ diverges for $x = x_0$, then it diverges for every x such that $|x| > |x_0|$.

Radius of Convergence of Power Series

Theorem

For every power series $\sum_{n\geq 1} a_n x^n$, there exists a unique $R \in [0, +\infty]$ which fulfills

1. For every $x \in \mathbb{R}$, such that $|x| < R$, the series $\sum_{n\geq 1} a_n x^n$ is absolutely convergent.
2. For every $x \in \mathbb{R}$, such that $|x| > R$, the sequence $(a_n x^n)_n$ is not bounded and then the series $\sum_{n\geq 1} a_n x^n$ diverges.

The number R is called the radius of convergence of the power series and $]-R, R[= \{x \in \mathbb{R};\ |x| < R\}$ is called the

Proof

The uniqueness results from the Abel's lemma. We set $R = \sup\{r \geq 1;\ \sum_{n=1}^{+\infty} |a_n| r^n < +\infty\}$.
If $|x| < R$, the series $\sum\limits_{n\geq 1} a_n x^n$ is absolutely convergent.
If $|x| > R$, the series $\sum\limits_{n\geq 1} a_n x^n$ diverges. If not the series $\sum\limits_{n\geq 1} |a_n| r^n$ converges for every $R < r < |x|$, which is impossible. $\square$

Remark

By the proof of the Theorem 7, we deduce that if R is the radius of convergence of the series $\sum\limits_{n\geq 1} a_n x^n$, then the series converges uniformly on any interval $[-r, r]$ with $0 < r < R$.

Theorem

[Cauchy 1821, used by Hadamard] (Cauchy-Hadamard Rule) Let $\sum\limits_{n\geq 1} a_n x^n$ be a power series with R its radius of convergence. Then

1. $R = \sup\{r > 0;\ \sum\limits_{n=1}^{+\infty} |a_n| r^n < +\infty\} = \sup\{r \geq 0;$ the sequence $(a_n r^n)_n$is bounded $\}$.
2. If $\lim\limits_{n\to+\infty} |\frac{a_n}{a_{n+1}}| = \beta \in [0, +\infty]$, then $R = \beta$.

Theorem

Let f be the function defined by the power series $\sum_{n\geq 0} a_n x^n$ which has $R > 0$ as radius of convergence, then the function g defined by the power series $\sum_{n\geq 1} na_n x^{n-1}$ has R as radius of convergence. The function f is differentiable on $]-R, R[$ and $f'(x) = g(x)$.

For the proof of this theorem, we need the following lemma

Lemma

Let $x \in \mathbb{R}$ and $h \in \mathbb{R}$ such that $0 < |h| \leq r$, then for any $n \in \mathbb{N}$

Proof

From the inequality (6)

$$\begin{aligned}
\left|(x+h)^n - x^n - nhx^{n-1}\right| &= \left|\sum_{k=0}^{n} C_n^k h^k x^{n-k} - x^n - nhx^{n-1}\right| = \left|\sum_{k=2}^{n} C_n^k\right. \\
&\le |h|^2 \sum_{k=2}^{n} C_n^k |x|^{n-k} |h|^{k-2} \le \frac{|h|^2}{r^2} \sum_{k=2}^{n} C_n^k |x|^n \\
&\le \frac{|h|^2}{r^2} (|x|+r)^n.
\end{aligned}$$

We have $|(x+h)^n - x^n - nhx^{n-1}| \ge nr|x|^{n-1} - |x|^n - (|x|+r)^n$.
From the relation (6), we deduce

$$nr|x|^{n-1} \le |x|^n + (|x|+r)^n + |(x+r)^n - x^n - nrx^{n-1}| \le |x|^n + 2(|x|+r)^n.$$

Proof of the theorem 49

We denote R' the radius of convergence of the power series $\sum_{n\geq 1} na_n x^{n-1}$.

It is obvious that $R' \leq R$. Let $r > 0$ such that $|x| + r < R$. From the lemma 49; we have $|na_n x^{n-1}| \leq \frac{1}{r}(2|a_n|(|x| + r)^n + |a_n||x|^n)$ and thus $\sum_{n\geq 1} na_n x^{n-1}$ converges absolutely on $]-R, R[$. Thus the radius of convergence of the series defining g is greater than R. Thus $R = R'$.

From the inequality (6) one has $|\frac{f(x+h) - f(x)}{h} - g(x)| \leq \frac{|h|}{r} \sum_{n=1}^{+\infty} |a_n|(|x| + r)^n$; this proves that when h tends to 0; $f'(x) = g(x)$; for any $x \in]-R, R[$. $\square$

Corollary

If $f(x) = \sum_{n=0}^{+\infty} a_n x^n$, then f is infinitely continuously differentiable on $]-R, R[$; $a_n = \dfrac{f^{(n)}(0)}{n!}$ and $f(x) = \sum_{n=0}^{+\infty} \dfrac{f^{(n)}(0)}{n!} x^n$. (This series is called the Taylor's series of f at 0.)

Examples

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \qquad \forall x \in \mathbb{R}.$$

$$e^{-x} = \sum_{n=0}^{+\infty} \frac{(-1)^n x^n}{n!} \qquad \forall x \in \mathbb{R}.$$

$$\cosh x = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \qquad \forall x \in \mathbb{R}.$$

$$\sinh x = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad \forall x \in \mathbb{R}.$$

For $|x| < 1$,

By integration, we have

$$\ln(1+x)=\sum_{n=0}^{+\infty}(-1)^n\frac{x^{n+1}}{(n+1)} \quad \text{and} \quad \ln(1-x)=-\sum_{n=0}^{+\infty}\frac{x^{n+1}}{(n+1)}.$$

$$\tanh^{-1}x=\frac{1}{2}\ln\frac{1+x}{1-x}=\sum_{n=0}^{+\infty}\frac{x^{2n+1}}{(2n+1)}.$$

$$\frac{1}{1+x^2}=\sum_{n=0}^{+\infty}(-1)^n x^{2n} \text{ and } \tan^{-1}x=\sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)}, \quad |x|<1.$$

$$\cos x = \sum_{n=0}^{+\infty}(-1)^n \frac{x^{2n}}{(2n)!} \text{ and } \sin x = \sum_{n=0}^{+\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Let $f(x) = (1+x)^\alpha$ with α a real number, $\alpha \notin \mathbb{N}$. For $x \in]-1, 1[$; $f'(x) = \alpha(1+x)^{\alpha-1}$, then f satisfies the following differential equation

$$(1+x)y' - \alpha y = 0. \tag{8}$$

We look for a power series $\sum_{n\geq 0} a_n x^n$ as solution of the differential equation (8).

If $S = \sum_{n=0}^{+\infty} a_n x^n$ is a solution, we have

$$(1+x)\sum_{n=0}^{+\infty} n a_n x^{n-1} - \alpha \sum_{n=0}^{+\infty} a_n x^n = 0,$$

then $(n+1)a_{n+1} + na_n - \alpha a_n = 0 \iff a_{n+1} = \frac{\alpha - n}{n+1} a_n \ \forall n \geq 0,$ which yields that

$$a_n = \frac{\alpha(\alpha-1)\dots(\alpha-n)}{2.3\dots(n+1)} a_0.$$

Then

By the uniqueness of the solution of the differential equation

$$(1-x)^{\alpha}=\sum_{n=0}^{+\infty}a_nx^n, \qquad \text{for } |x|<1,$$

where $a_n=\dfrac{\alpha(\alpha-1)\dots(\alpha-n)}{2.3\dots(n+1)}$.

For $\alpha=\frac{-1}{2}$, we have

$$\frac{1}{\sqrt{1-x}}=\sum_{n=0}^{+\infty}\frac{C_{2n}^n}{4^n}x^n$$

$$\frac{1}{\sqrt{1+x}}=\sum_{n=0}^{+\infty}\frac{(-1)^nC_{2n}^n}{4^n}x^n.$$

$$\sqrt{1+x} = 1 + \frac{1}{2}\sum_{n=0}^{+\infty} \frac{(-1)^n C_{2n}^n}{4^n} \frac{x^{n+1}}{n+1}.$$

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{+\infty} \frac{C_{2n}^n}{4^n} x^{2n}.$$

$$\frac{1}{\sqrt{1+x^2}} = \sum_{n=0}^{+\infty} \frac{(-1)^n C_{2n}^n}{4^n} x^{2n}.$$

$$\sin^{-1} x = \sum_{n=0}^{+\infty} \frac{C_{2n}^n}{4^n} \frac{x^{2n+1}}{2n+1}.$$

$$\cos^{-1} x = \frac{\pi}{2} - \sum_{n=0}^{+\infty} \frac{C_{2n}^n}{4^n} \frac{x^{2n+1}}{2n+1}.$$

$$\sinh^{-1} x = \sum_{n=0}^{+\infty} \frac{(-1)^n C_{2n}^n}{4^n} \frac{x^{2n+1}}{2n+1}.$$