Chapter 2: Sequences of functions

Sequence of functions

Definition

A **sequence of functions** is a list

$$f_1,\ f_2,\ f_3,\ \ldots$$

All functions use the same domain D:

$$f_n : D \to \mathbb{R}.$$

Goal

Find a function f such that f_n becomes close to f as $n \to \infty$.

Error (closeness)

Error at x: $|f_n(x) - f(x)|$.

Example of a sequence of functions

Example

Let $D = \mathbb{R}$ and define

$$f_n(x) = \frac{x}{n}.$$

First terms:

$$f_1(x) = x, \quad f_2(x) = \frac{x}{2}, \quad f_3(x) = \frac{x}{3}, \ \dots$$

For each fixed $x \in \mathbb{R}$,

$$\frac{x}{n} \xrightarrow[n\to\infty]{} 0.$$

So the limit function is

$$f(x) = 0 \quad \text{for all } x \in \mathbb{R}.$$

What does "$f_n \to f$" mean?

For each fixed x, the values

$$f_1(x),\ f_2(x),\ f_3(x),\ \ldots$$

are just a **sequence of numbers**.

We write $f_n \to f$ when, as n gets large, the number $f_n(x)$ gets closer to $f(x)$.

Main question

Do we get close **point by point** (pointwise), or **everywhere at once** (uniform)?

Pointwise convergence

Pointwise means: one point at a time

Pick a point x. Then $f_n(x)$ becomes close to $f(x)$ when n is large.

$$\forall x \in D,\ \forall \varepsilon > 0,\ \exists N :\ |f_n(x) - f(x)| < \varepsilon \text{ for } n \geq N.$$

Important (why it is called *pointwise*)

In **pointwise** convergence, you fix a point x first.

Then the number N you need can depend on that point:

$$N = N(x, \varepsilon).$$

So two different points x_1 and x_2 may need different values of N.

Uniform convergence

Definition

Given $\varepsilon > 0$, there exists N such that for all $n \geq N$ and all $x \in D$,

$$|f_n(x) - f(x)| < \varepsilon.$$

quantifiers

$$\forall \varepsilon > 0 \; \exists N \; \forall n \geq N \; \forall x \in D : \; |f_n(x) - f(x)| < \varepsilon.$$

N depends on ε, not on x:

$$N = N(\varepsilon) \quad \text{(same } N \text{ for all } x \in D\text{)}.$$

Main difference

- **Pointwise:** Fix x. Then choose N. (Different x can have different N.)
- **Uniform:** Choose N once. It works for all x.

Uniform convergence test

Define the largest error on D:

$$\text{Largest error} = \sup_{x \in D} |f_n(x) - f(x)|.$$

sup test

$f_n \to f$ uniformly on D if and only if

$$\sup_{x \in D} |f_n(x) - f(x)| \to 0.$$

Uniform convergence means: **the largest error goes to zero.**

$f_n(x) = x^n$ on $[0,1]$: pointwise

Sequence

$$f_n(x) = x^n, \qquad x \in [0,1].$$

Pointwise limit

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x = 1. \end{cases}$$

So

$$f(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x = 1. \end{cases}$$

$f_n(x) = x^n$ on $[0, 1]$: not uniform

Uniform test

Uniform convergence would require

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| \to 0.$$

Compute the supremum

For $0 \le x < 1$, $f(x) = 0$, so $|f_n(x) - f(x)| = x^n$. Hence

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| = \sup_{0 \le x < 1} x^n = 1.$$

Conclusion

The supremum is always 1, so it does not go to 0. Therefore, the convergence is not uniform.

Example 2

Example

Let $D = \mathbb{R}$ and define $f_n(x) = \frac{x}{n}$.

$$f_1(x) = x, \quad f_2(x) = \frac{x}{2}, \quad f_3(x) = \frac{x}{3}, \ldots$$

Pointwise: for fixed x, $\frac{x}{n} \to 0$. So $f(x) = 0$.

Uniform convergence

Uniform convergence

Not uniform on $\mathbb{R}$:

$$\sup_{x\in\mathbb{R}} \left|\frac{x}{n} - 0\right| = \infty.$$

Uniform on compact $[a, b]$:

$$\sup_{x\in[a,b]} \left|\frac{x}{n}\right| = \frac{\max\{|a|, |b|\}}{n} \to 0.$$

Example 3 : $f_n(x) = x^n$ on three domains

Domains

$$D_1 = \left[0, \tfrac{1}{2}\right], \qquad D_2 = [0, 1), \qquad D_3 = [0, 1], \qquad f_n(x) = x^n.$$

Uniform test

Let $f(x) = \lim_{n\to\infty} f_n(x)$ (pointwise).
Uniform convergence on D means:

$$\sup_{x\in D} |f_n(x) - f(x)| \to 0.$$

Domain $D_1 = \left[0, \frac{1}{2}\right]$: uniform

Pointwise limit

For every $x \in D_1$, we have $x \leq \frac{1}{2} < 1$, so $x^n \to 0$. Thus $f(x) = 0$ on D_1.

$$\sup_{0 \leq x \leq 1/2} |x^n - 0| = \sup_{0 \leq x \leq 1/2} x^n = \left(\frac{1}{2}\right)^n \to 0.$$

Conclusion

Uniform convergence on D_1.

Domain $D_2 = [0, 1)$: not uniform

Pointwise limit

For every fixed $x < 1$, we still have $x^n \to 0$. Thus $f(x) = 0$ on D_2.

Values of x very close to 1 make x^n close to 1. So

$$\sup_{0 \le x < 1} x^n = 1.$$

Conclusion

So no uniform convergence on D_2.

Domain $D_3 = [0,1]$: not uniform

Pointwise limit

$$f(x) = \lim_{n\to\infty} x^n = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x = 1. \end{cases}$$

Take $x_n = 1 - \frac{1}{n}$. Then $f(x_n) = 0$ and

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| \ge |f_n(x_n) - f(x_n)| = \left(1 - \frac{1}{n}\right)^n \to e^{-1} > 0.$$

So $\sup_{x\in[0,1]} |f_n(x) - f(x)| \not\to 0$.

Conclusion

No uniform convergence on D_3.

Example 4

Domain and functions

On $D = [0, 1]$, define

$$f_n(x) = \begin{cases} 1 - nx, & 0 \leq x < \frac{1}{n}, \\ 0, & \frac{1}{n} \leq x \leq 1, \end{cases} \qquad f(x) = \begin{cases} 1, & x = 0, \\ 0, & x > 0. \end{cases}$$

Pointwise convergence

For $x = 0$: $f_n(0) = 1 \to 1 = f(0)$.
For $x > 0$: choose $n > \frac{1}{x}$, then $x \geq \frac{1}{n}$ so $f_n(x) = 0 \to 0 = f(x)$.

Example 4

For $x \in [0,1]$,

$$|f(x) - f_n(x)| = \begin{cases} 0, & x = 0 \text{ or } x \geq \frac{1}{n}, \\ 1 - nx, & 0 < x < \frac{1}{n}. \end{cases}$$

Largest error on $[0,1]$

The largest error happens near $x = 0$:

$$\sup_{x \in [0,1]} |f(x) - f_n(x)| = \sup_{0<x<1/n} (1 - nx) = 1.$$

Conclusion

Since $\sup_{x \in [0,1]} |f(x) - f_n(x)| = 1$ for all n, we do **not** have uniform convergence.

Exercise

Problem

Show that
$$f_n(x) = \frac{x\cos(nx)}{x+n}$$
satisfies:

- $f_n(x) \to 0$ for each $x \in [0, \infty)$ (pointwise),
- but $f_n \not\to 0$ uniformly on $[0, \infty)$.

Solution: pointwise

Pointwise

Fix $x \geq 0$. Since $|\cos(nx)| \leq 1$,

$$\left| \frac{x \cos(nx)}{x+n} \right| \leq \frac{x}{x+n} \xrightarrow[n\to\infty]{} 0.$$

So $f_n(x) \to 0$ for every fixed x.

Solution: not uniform

Choose $x_n = n\pi$

Then $\cos(nx_n) = \cos(n^2\pi) = 1$, so

$$|f_n(x_n)| = \frac{n\pi}{n\pi + n} = \frac{\pi}{\pi + 1}.$$

Supremum does not go to 0

$$\sup_{x\in[0,\infty)} |f_n(x)| \geq |f_n(x_n)| = \frac{\pi}{\pi + 1}.$$

So the convergence is not uniform.

Uniform Cauchy condition

Definition

A sequence of functions (f_n) on D is **uniform Cauchy** if:

For every $\varepsilon > 0$, there exists N such that for all $n, m \geq N$ and all $x \in D$,

$$|f_n(x) - f_m(x)| < \varepsilon.$$

Quantifiers

$$\forall \varepsilon > 0 \; \exists N \; \forall n, m \geq N \; \forall x \in D : \; |f_n(x) - f_m(x)| < \varepsilon.$$

Uniform Cauchy criterion

Supremum form

The uniform Cauchy condition is equivalent to:

$$\forall \varepsilon > 0 \; \exists N \; \forall n, m \geq N : \; \sup_{x \in D} |f_n(x) - f_m(x)| \leq \varepsilon.$$

Theorem (Uniform Cauchy criterion)

$$f_n \text{ converges uniformly on } D \quad \Longleftrightarrow \quad (f_n) \text{ is uniform Cauchy on } D.$$

Pointwise convergence: what may fail?

Main idea

Pointwise convergence means: **for each fixed** x, $f_n(x) \to f(x)$.

But the speed can be different at different points. So the limit f may lose properties that all f_n have.

Possible problems

- The limit may not be continuous.
- We may not be able to change the order of limits.
- We may not be able to move the limit inside an integral.
- The limit may not be differentiable.

(a) Continuity is not preserved

Example: $f_n(x) = x^n$ on $[0, 1]$

Each f_n is continuous.
Pointwise limit:

$$f(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x = 1. \end{cases}$$

Conclusion

The limit f is not continuous at $x = 1$. So pointwise convergence does not preserve continuity.

(b) Limits are not preserved (order matters)

Same example: $f_n(x) = x^n$ on $(0,1)$

$$\lim_{n\to\infty} f_n(x) = 0 \quad \text{for each fixed } x \in (0,1).$$

But for every fixed n,

$$\lim_{x\to 1^-} f_n(x) = 1.$$

Two different answers

$$\lim_{n\to\infty}\left(\lim_{x\to 1^-} f_n(x)\right) = 1, \qquad \lim_{x\to 1^-}\left(\lim_{n\to\infty} f_n(x)\right) = 0.$$

So the order of limits cannot be swapped using only pointwise convergence.

(c) Integrals are not preserved

Example on $[0, 1]$

Define

$$f_n(x) = \begin{cases} n, & 0 \le x \le \frac{1}{n}, \\ 0, & \frac{1}{n} < x \le 1. \end{cases}$$

Then $f_n(x) \to 0$ for every fixed $x \in [0, 1]$.

Integral of f_n

$$\int_0^1 f_n(x)\, dx = \int_0^{1/n} n\, dx = 1.$$

Different results

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 1, \qquad \int_0^1 \lim_{n\to\infty} f_n(x)\, dx = 0.$$

(d) Differentiability is not preserved

Example on $\mathbb{R}$

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}}.$$

Each f_n is differentiable for all x.

Pointwise limit

$$\lim_{n\to\infty} f_n(x) = |x|.$$

Conclusion

The limit $f(x) = |x|$ is not differentiable at $x = 0$. So pointwise convergence does not preserve differentiability.

Hereditary theorems (idea)

A **hereditary theorem** says:

If every f_n has a property (continuity, integrability, differentiability, ...), then the limit f also has the same property.

Pointwise convergence is usually too weak for inheritance. Uniform convergence gives much better inheritance results.

Uniform convergence preserves continuity

Theorem

Let $D \subset \mathbb{R}$. If each $f_n : D \to \mathbb{R}$ is continuous and $f_n \to f$ **uniformly** on D, then f is continuous on D.

Uniform convergence allows us to pass continuity from f_n to the limit f.

Proof idea

Goal

Fix $x \in D$. Show: for every $\varepsilon > 0$, we can choose $\delta > 0$ such that

$$|x - x'| < \delta \ \Rightarrow \ |f(x) - f(x')| < \varepsilon.$$

(uniform convergence)

Choose N so that for all $\xi \in D$,

$$|f(\xi) - f_N(\xi)| < \frac{\varepsilon}{3}.$$

(continuity of one function)

Since f_N is continuous at x, choose $\delta > 0$ such that

$$|x - x'| < \delta \ \Rightarrow \ |f_N(x) - f_N(x')| < \frac{\varepsilon}{3}.$$

Proof

Triangle inequality

For $|x - x'| < \delta$,

$$|f(x) - f(x')| \le |f(x) - f_N(x)| + |f_N(x) - f_N(x')| + |f_N(x') - f(x')|.$$

Each part is $< \varepsilon/3$

$$|f(x) - f_N(x)| < \frac{\varepsilon}{3}, \quad |f_N(x) - f_N(x')| < \frac{\varepsilon}{3}, \quad |f_N(x') - f(x')| < \frac{\varepsilon}{3}.$$

So

$$|f(x) - f(x')| < \varepsilon.$$

f is continuous at x. Since x is arbitrary, f is continuous on D.

Riemann integration

Upper and lower sums

Partition: $a = x_0 < \cdots < x_n = b,\ \Delta x_i = x_i - x_{i-1}$.

On $[x_{i-1}, x_i]$:

$$m_i = \inf f, \qquad M_i = \sup f.$$

Sums:

$$L(f, P) = \sum m_i \Delta x_i, \qquad U(f, P) = \sum M_i \Delta x_i.$$

Riemann criterion

f is integrable iff for every $\varepsilon > 0$ there is a partition P with

$$U(f, P) - L(f, P) < \varepsilon.$$

A useful estimate

If g is Riemann integrable on $[a, b]$, then

$$\left| \int_a^b g(x)\, dx \right| \le \int_a^b |g(x)|\, dx \le (b-a) \sup_{x \in [a,b]} |g(x)|.$$

If $\sup_{[a,b]} |g|$ is small, then $\int_a^b g$ is small.

Uniform convergence and integrals

Theorem (theorem for integrability)

Let $[a, b]$ be a bounded interval. If each f_n is Riemann integrable on $[a, b]$ and $f_n \to f$ **uniformly**, then:

- f is Riemann integrable,
- and the limit can pass inside the integral:

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proof

Use uniform convergence

Given $\varepsilon > 0$, choose N such that for all $n \geq N$,

$$\sup_{x\in[a,b]} |f(x) - f_n(x)| < \varepsilon.$$

Apply the inequality

For $n \geq N$,

$$\left|\int_a^b f - \int_a^b f_n\right| = \left|\int_a^b (f - f_n)\right| \leq \int_a^b |f - f_n| \leq (b-a)\sup_{[a,b]} |f - f_n| < \varepsilon(b-a).$$

$$\int_a^b f_n \to \int_a^b f.$$

Uniform convergence and differentiability

Theorem

Let f_n be differentiable on $[a, b]$. Assume:

- $f_n' \to g$ **uniformly** on $[a, b]$,
- $f_n(x_0)$ converges for one point $x_0 \in [a, b]$.

Then f_n converges **uniformly** to a differentiable function f, and

$$f'(x) = g(x) = \lim_{n\to\infty} f_n'(x) \quad \text{for all } x \in [a, b].$$

Proof Sketch under $f_n \in C^1$

Use the Fundamental Theorem of Calculus

For any $x \in [a, b]$,

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(t)\, dt.$$

Take the limit

Since $f_n(x_0)$ converges and $f_n' \to g$ uniformly,

$$f(x) := \lim_{n \to \infty} f_n(x) \quad \text{exists and} \quad f(x) = f(x_0) + \int_{x_0}^{x} g(t)\, dt.$$

f is differentiable and $f'(x) = g(x)$.

Why we need $f_n(x_0)$ to converge

Important

Uniform convergence of f_n' alone is not enough.

Reason

If c_n is a divergent sequence and $g_n(x) = f_n(x) + c_n$, then

$$g_n'(x) = f_n'(x).$$

So derivatives behave the same, but $g_n(x)$ cannot converge (even pointwise).

Meaning

We need one fixed value $f_n(x_0)$ to "choose the constant".

Example

Define

On $[0, 1]$, let

$$f_n(x) = x - \frac{\sin(nx)}{n^2}.$$

At $x_0 = 0$,

$$f_n(0) = 0,$$

so $f_n(x_0)$ converges.

Derivative

$$f_n'(x) = 1 - \frac{\cos(nx)}{n}.$$

Example (finish)

Uniform convergence of derivatives

$$\sup_{x\in[0,1]} |f_n'(x) - 1| = \sup_{x\in[0,1]} \frac{|\cos(nx)|}{n} \leq \frac{1}{n} \to 0.$$

So $f_n' \to 1$ uniformly, hence $g(x) = 1$.

Limit of f_n

$$|f_n(x) - x| = \left|\frac{\sin(nx)}{n^2}\right| \leq \frac{1}{n^2} \to 0,$$

so $f_n \to f$ uniformly with

$$f(x) = x.$$

Uniform convergence lets us swap limits

Theorem

Let $x \in \widehat{D}$ (a cluster point) and assume $f_n \to f$ **uniformly** on $D \setminus \{x\}$. If the limit

$$\ell_n = \lim_{t \to x} f_n(t)$$

exists for each n, then:

- (ℓ_n) converges,
- and the limit of f at x exists and equals $\lim \ell_n$:

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} \ell_n, \qquad \lim_{n \to \infty} \lim_{t \to x} f_n(t) = \lim_{t \to x} \lim_{n \to \infty} f_n(t).$$

Proof idea (part 1)

Uniform Cauchy

Uniform convergence on $D \setminus \{x\}$ gives: for every $\varepsilon > 0$ there is N such that

$$m, n \geq N \;\Rightarrow\; |f_n(t) - f_m(t)| < \varepsilon \quad \text{for all } t \in D \setminus \{x\}.$$

Let $t \to x$

Taking $t \to x$ and using $\ell_n = \lim_{t \to x} f_n(t)$,

$$m, n \geq N \;\Rightarrow\; |\ell_n - \ell_m| \leq \varepsilon.$$

So (ℓ_n) is Cauchy, hence convergent.

Proof idea (part 2)

Let $\ell = \lim_{n\to\infty} \ell_n$

Choose N such that for all $t \in D \setminus \{x\}$,

$$|f(t) - f_N(t)| \le \varepsilon, \qquad |\ell - \ell_N| \le \varepsilon.$$

Use the limit of f_N

Since $f_N(t) \to \ell_N$ as $t \to x$, choose $\delta > 0$ such that

$$|t - x| < \delta \;\Rightarrow\; |f_N(t) - \ell_N| < \varepsilon.$$

Finish (triangle inequality)

For $|t - x| < \delta$,

$$|f(t) - \ell| \le |f(t) - f_N(t)| + |f_N(t) - \ell_N| + |\ell_N - \ell| < 3\varepsilon.$$

Corollary: continuity is preserved

Corollary

If $f_n \to f$ uniformly on an interval I and each f_n is continuous at $c \in I$, then f is continuous at c.

Reason

$$\lim_{t\to c} f(t) = \lim_{n\to\infty} \lim_{t\to c} f_n(t) = \lim_{n\to\infty} f_n(c) = f(c).$$

Example

Example on $[0, 1]$

$$f_n(x) = \frac{x}{1 + nx}.$$

Then $f_n \to 0$ uniformly on $[0, 1]$, so $f(x) \equiv 0$.

Swap the limits at $x = 0$

$$\lim_{x\to 0}\Big(\lim_{n\to\infty} f_n(x)\Big) = \lim_{x\to 0} 0 = 0, \qquad \lim_{n\to\infty}\Big(\lim_{x\to 0} f_n(x)\Big) = \lim_{n\to\infty} 0 = 0.$$

Both iterated limits agree.

Exercises (1.1–1.6)

Tasks

For each sequence (f_n): find the **pointwise limit** and decide **uniform convergence** (on the stated domain and on $[a, 1]$ when asked).

Exercises

Ex 1.1 $f_n(x) = x^n$ on $\mathbb{R}$.

Ex 1.2 $f_n(x) = \dfrac{\sin(nx)}{nx}$ on $(0, 1)$ and $[a, 1]$.

Ex 1.3 $f_n(x) = \dfrac{1}{nx + 1}$ on $(0, 1)$ and $[a, 1]$.

Ex 1.4 $f_n(x) = \dfrac{x}{nx + 1}$ on $[0, 1]$.

Ex 1.5 $f_n(x) = \dfrac{nx^3}{1 + nx}$ on $[0, 1]$.

Ex 1.6 $f_n(x) = x^n(1 - x)$ on $[0, 1]$.

Exercises (1.7–1.12)

Exercises

Ex 1.7 $f_n(x) = x^n(1 - x^n)$ on $[0, 1]$.

Ex 1.8 $f_n(x) = \begin{cases} nx, & 0 \le x \le 1/n \\ 0, & 1/n < x \le 1 \end{cases}$ on $[0, 1]$.

Ex 1.9 $f_n(x) = \begin{cases} \sqrt{nx}, & 0 \le x \le 1/n \\ 0, & 1/n < x \le 1 \end{cases}$ on $[0, 1]$.

Ex 1.10 If uniform on D and E, prove uniform on $D \cup E$.

Ex 1.11 If $f_n \to f$, $g_n \to g$ uniformly, prove $\alpha f_n + \beta g_n \to \alpha f + \beta g$ uniformly.

Ex 1.12 If f_n, g_n bounded and converge uniformly, prove $f_n g_n \to fg$ uniformly; give counterexample if boundedness fails.

Exercises (1.13–1.18)

Exercises

Ex 1.13 (Dini) On compact D, if $f_n \downarrow f$ and all continuous, prove uniform convergence. Show compactness is needed.

Ex 1.14 $f_n(x) = \dfrac{x^n}{1 + x^n}$ on $[0, 2]$.

Ex 1.15 Build discontinuous f_n on $[0, 1]$ with uniform limit f continuous.

Ex 1.16 $f_n(x) = \varphi(x)x^n$, $\varphi \in C[0, 1]$: uniform iff $\varphi(1) = 0$. Deduce $nx(1 - x)^n \to 0$ pointwise not uniformly.

Ex 1.17 Piecewise "triangle" f_n on $[0, 1]$: pointwise limit, uniform?, compare $\int f_n$ vs $\int \lim f_n$.

Ex 1.18 $f_n(x) = \dfrac{nx}{1 + n^2 x^p}$ on $[0, 1]$: uniform? and integral for $p = 2$.

Answers (1.1–1.4)

1.1 x^n: pointwise 0 if $|x| < 1$, 1 if $x = 1$, diverges if $|x| > 1$ or $x = -1$. Uniform on $[-a, a]$ for $a < 1$; not uniform on $(-1, 1)$.

1.2 $\frac{\sin(nx)}{nx} \to 0$ pointwise on $(0, 1)$. Not uniform on $(0, 1)$. Uniform on $[a, 1]$ $(a > 0)$.

1.3 $\frac{1}{nx+1} \to 0$ pointwise. Not uniform on $(0, 1)$. Uniform on $[a, 1]$ $(a > 0)$.

1.4 $\frac{x}{nx+1} \to 0$ pointwise on $[0, 1]$, and uniform since $\sup_{[0,1]} \frac{x}{nx+1} = \frac{1}{n+1} \to 0$.

Quick checks (1.5–1.9)

Answers

1.5 $\frac{nx^3}{1+nx} \to x^2$ pointwise. Uniform since $\sup |f_n - x^2| \leq \frac{1}{n+1} \to 0$.

1.6 $x^n(1-x) \to 0$ pointwise and uniformly on $[0,1]$.

1.7 $x^n(1-x^n) \to 0$ pointwise, but not uniform ($\sup = 1/4$).

1.8 spike nx on $[0,1/n]$: pointwise $\to 0$, not uniform ($\sup = 1$).

1.9 $\sqrt{nx}$ on $[0,1/n]$: pointwise $\to 0$, not uniform ($\sup = 1$).

Answers (1.10–1.13)

1.10 Uniform on D and $E \Rightarrow$ take $N = \max(N_D, N_E) \Rightarrow$ uniform on $D \cup E$.

1.11 Uniform limits are stable under linear combinations: $\alpha f_n + \beta g_n \to \alpha f + \beta g$ uniformly.

1.12 If f_n, g_n are bounded and converge uniformly, then $f_n g_n \to fg$ uniformly. Boundedness is needed (counterexamples exist on unbounded domains).

1.13 Dini: on compact D, continuous $f_n \downarrow f$ continuous $\Rightarrow$ uniform. Not true on non-compact sets (example x^n on $(0, 1)$).

Answers (1.14–1.18)

1.14 $\frac{x^n}{1+x^n}$: pointwise 0 for $x < 1$, $1/2$ at $x = 1$, 1 for $x > 1$. Not uniform on $[0, 2]$.

1.15 Example: $f_n = \mathbf{1}_{\mathbb{Q}\cap[0,1]}/n$. Each discontinuous everywhere; $f_n \to 0$ uniformly.

1.16 $\varphi(x)x^n$ uniform on $[0, 1]$ iff $\varphi(1) = 0$. Also $nx(1 - x)^n \to 0$ pointwise but not uniform.

1.17 For the given piecewise f_n: pointwise $\to 0$, not uniform; and $\int f_n$ may not equal $\int \lim f_n$.

1.18 $f_n = \frac{nx}{1+n^2x^p}$: pointwise $\to 0$. Uniform if $0 < p < 2$; not uniform if $p = 2$; not uniform if $p > 2$.