Introduction to Real Analysis

Sequences and Series of Fucntions

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Sequence of functions

Definition

For each $n \in \mathbb{N}$, let $f_n : D \to \mathbb{R}$ be a function. Then the sequence (f_n) is a sequence of functions. For a fixed point $x \in D$, $(f_n(x))$ is a sequence of numbers. If the sequence $(f_n(x))$ converges for every $x \in D$, then (f_n) converges pointwise and

$$f(x) = \lim_{n\to\infty} f_n(x)$$

$$\lim f_n = f$$

Pointwise Convergent

Definition

The sequence of functions (f_n) converges pointwise to f on D if , given $\varepsilon > 0$, then for each $x \in D$ there is $N = N(\varepsilon, x) \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad |f_n(x) - f(x)| < \varepsilon$$

Sequence of functions

Examples

Find the poitnwise limit of the functions
$f_n : \mathbb{R} \to \mathbb{R}$

$$f_n(x) = \frac{x}{n}$$

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Sequence of functions

Examples

$f_n : [0, 1] \to \mathbb{R}$

$$f_n(x) = x^n$$

.

Sequence of functions

Examples

$f_n : [0,1] \to \mathbb{R}$

$$f_n(x) = nx(1-x^2)^n$$

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Sequence of functions

Examples

$f_n : \mathbb{R} \to \mathbb{R}$

$$f_n(x) = \begin{cases} -1 & x < -1/n \\ \sin(n\pi x/2) & -1/n \le x \le 1/n \\ 1 & x > 1/n \end{cases}$$

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Uniform Convergent

Definition

The sequence of functions (f_n) converges uniformly to f on D if , given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$n \geq N \implies |f_n(x) - f(x)| < \varepsilon \quad \forall x \in D$$

$$f_n \xrightarrow{u} f$$

Uniform Convergent

Theorem

The sequence of functions (f_n) converges uniformly to f on D iff

$$\sup_{x \in D} |f_n(x) - f(x)| \to 0, \qquad \text{as} \quad n \to \infty$$

Uniform Convergent

Lemma

The sequence of functions (f_n) on D does not converge uniformly to f on D iff there is $\varepsilon > 0$ and a sequence (x_n) in D such that

$$|f_n(x_n) - f(x_n)| \geq \varepsilon \quad \forall n \in \mathbb{N}$$

Sequence of functions

Examples

Determine the pointwise limit of (f_n) then decide whether the convergence is uniform or not

$$f_n(x) = \frac{x}{n}, \quad x \in [0,1]$$

.

Sequence of functions

Examples

$$f_n(x) = \frac{x}{n}, \quad x \in \mathbb{R}$$

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Sequence of functions

Examples

$$f_n(x) = x^n, \quad x \in [0, 1]$$

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Sequence of functions

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Sequence of functions

Examples

$f_n : \mathbb{R} \to \mathbb{R}$

$$f_n(x) = \begin{cases} -1 & x < -1/n \\ \sin(n\pi x/2) & -1/n \le x \le 1/n \\ 1 & x > 1/n \end{cases}$$

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Sequence of functions

Examples

$$f_n(x) = \frac{\sin nx}{n}, \quad x \in \mathbb{R}$$

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Cauchy Criterion Convergent

Theorem

The sequence of functions (f_n) converges uniformly on D iff for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that

$$m, n \geq N \implies \sup_{x \in D} |f_n(x) - f_m(x)| < \varepsilon$$

Properties of Uniform convergence

Theorem

If (f_n) is a sequence of continuous functions on D, and

$$f_n \xrightarrow{u} f$$

on D, then f is also continuous on D.

Properties of Uniform convergence

Theorem

If $f_n \in \mathcal{R}(a,b)$ and

$$f_n \overset{u}{\to} f$$

on $[a,b]$, then $f \in \mathcal{R}(a,b)$, and

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$$

What about differentiation?

Examples

$f_n : [-1, 1] \to \mathbb{R}$

$$f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$$

.

Properties of Uniform convergence

Theorem

Let f_n be differentiable on $[a, b]$ and converges at some point $c \in [a, b]$. If the sequence $(f_n^{'})$ is uniformly convergent on $[a, b]$ then (f_n) is uniformly convergent on $[a, b]$ to a function f and

$$f_n^{'} \xrightarrow{u} f'$$

Sequence of functions

Examples

$$f_n(x) = \frac{nx}{nx+1}, \quad x \geq 0$$

1. Find $f(x) = \lim f_n(x)$

Sequence of functions

Examples

$$f_n(x) = \frac{nx}{nx+1}, \quad x \geq 0$$

1. Find $f(x) = \lim f_n(x)$
2. Show that (f_n) converges uniformly to f on $[a, \infty)$ for $a > 0$

Sequence of functions

Examples

$$f_n(x) = \frac{nx}{nx+1}, \quad x \geq 0$$

1. Find $f(x) = \lim f_n(x)$
2. Show that (f_n) converges uniformly to f on $[a, \infty)$ for $a > 0$
3. Show that (f_n) does no converge uniformly on $[0, \infty]$

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Series of functions

Definition

Let (f_n) be a sequence of function on D. The sum

$$\sum_{n=1}^{\infty} f_n(x)$$

is a series of functions.

Series of functions

Definition

The sequence of partial sums is

$$S_n(x) = \sum_{k=1}^{n} f_k(x)$$

If (S_n) is convergent (pointwise) on D, we say that the series converges pointwise and we have

$$S(x) = \lim_{n\to\infty} S_n(x) = \sum_{k=1}^{\infty} f_k(x)$$

Series of functions

Definition

If (S_n) is divergent, then the series is divergent.
If (S_n) is uniformly convergent, then the series is uniformly convergent
If $\sum_{k=1}^{\infty} |f_k(x)|$ is convergent, then $\sum_{k=1}^{\infty} f_k(x)$ is absolutely convergent.

Series of functions

Theorem

If $x \in \hat{D}$ and the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on $D \backslash \{x\}$, and suppose that $\lim_{t \to x} f_n(t)$ exists for all $n \in \mathbb{N}$, then

$$\lim_{t \to x} \sum_{n=1}^{\infty} f_n(t) = \sum_{n=1}^{\infty} \lim_{t \to x} f_n(t)$$

Therefore, if each f_n is continuous at x then $\sum f_n$ is continuous.

Series of functions

Theorem

If $f_n \in \mathcal{R}(a,b)$ for all $n \in \mathbb{N}$ and the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on $[a,b]$, then $\sum f_n \in \mathcal{R}(a,b)$ and

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)dx$$

Series of functions

Theorem

If f_n is differentiable on $[a, b]$ for all $n \in \mathbb{N}$ and the series $\sum f_n(x_0)$ converges at some point $x_0 \in [a, b]$, then If the series $\sum_{n=1}^{\infty} f_n^{'}$ is uniformly convergent on $[a, b]$, then $\sum_{n=1}^{\infty} f_n$ is also uniformly convergent on $[a, b]$ and

$$\left(\sum_{n=1}^{\infty} f_n\right)^{'}(x) = \sum_{n=1}^{\infty} f_n^{'}(x) \qquad \forall x \in [a, b]$$

Cauchy criterion

Theorem

The series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on D iff for
for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that

$$n > m \geq N \implies |S_n(x) - S_m(x)| = \left| \sum_{k=m+1}^{n} f_k(x) \right| < \varepsilon \quad \forall x \in D$$

Weierstrass M-test

Theorem

Let (f_n) be a sequence of functions on D, and (M_n) be a sequence of positive numbers such that

$$|f_n(x)| \leq M_n \quad \forall x \in D, n \in \mathbb{N}$$

If the series $\sum M_n$ converges, then $\sum f_n$ and $\sum |f_n|$ converge uniformly on D.

Series of functions

Examples

Discuss uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{\sin(3^n x)}{2^n}$$

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Series of functions

Examples

$$\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right), \quad x \in [a, b]$$

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Series of functions

Examples

Discuss uniform convergence of the series

$$\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right), \quad x \in \mathbb{R}$$

.

Series of functions

Examples

$$\sum_{n=1}^{\infty} \frac{1}{n^2 x^2}$$

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Power Series

The series $\sum_{n=0}^{\infty} f_n$ is called a power series if the function f_n has the form

$$f_n(x) = a_n(x-c)^n, \quad n \in \mathbb{N} \cup \{0\}$$

$$f_0(x) = a_0$$

Power Series

We will consider the series

$$\sum_{n=0}^{\infty} a_n x^n$$

If $x = 0$, then

$$\sum_{n=0}^{\infty} a_n x^n \to a_0$$

Power Series

Examples

$$\sum_{n=0}^{\infty} n!x^n$$

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Power Series

Examples

$$\sum_{n=0}^{\infty} x^n$$

.

Power Series

Examples

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

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Radius of convergence

For any power series

$$\sum_{n=0}^{\infty} a_n x^n$$

We define

$$\rho = \lim |a_n|^{\frac{1}{n}}$$

and

$$R = \frac{1}{\rho}. \quad 0 < \rho < \infty$$

$R = \infty$, if $\rho = 0$
$R = 0$ if $\rho = \infty$

Radius of convergence

If

$$\lim \left| \frac{a_{n+1}}{a_n} \right|$$

exists then

$$\rho = \lim \left| \frac{a_{n+1}}{a_n} \right|$$

and

$$R = \lim \left| \frac{a_n}{a_{n+1}} \right|$$

Radius of convergence

Cauchy-Hadamard Theorem

The series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent if $|x| < R$ and divergent if $|x| > R$.

Uniform Convergent

Theorem

Let R be the radius of convergent of the power seres $\sum_{n=0}^{\infty} a_n x^n$. If $0 < r < R$ then the series converges uniformly on $[-r, r]$.

Power Series

Examples

$$\sum_{n=0}^{\infty} \frac{x^n}{n}$$

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Power Series

Examples

$$\sum_{n=0}^{\infty} \frac{x^n}{n^2}$$

.

Power Series

Examples

$$\sum_{n=1}^{\infty} \frac{x^n}{2^n}$$

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Exercises

1. Determine the pointwise limit of (f_n), then decide whether the convergence is uniform or not.
    1. $f_n(x) = \dfrac{\sin(nx)}{nx}, \qquad x \in (0,1)$
    2. $f_n(x) = \dfrac{1}{nx+1}, \qquad x \in (0,1)$
    3. $f_n(x) = \dfrac{x}{nx+1}, \qquad x \in [0,1]$
2. Find the limit of the sequence $f_n(x) = \dfrac{x^n}{x^n+1}$ on $[0,2]$, and determine whether the convergence is uniform.

Exercises

1. The functions f_n on $[-1,1]$ are defined by

$$f_n(x) = \frac{x}{n^2x^2+1}$$

Show that (f_n) converges uniformly and that its limit f is differentiable, but that the equality $f'(x) = \lim f_n'(x)$ does not hold for all $x \in [-1,1]$.

Exercises

1. Determine where the series $\sum f_n$ converges pointwise and where it converges uniformly
   1. $f_n(x) = \dfrac{1}{x^2 + n^2}$
   2. $f_n(x) = \dfrac{1}{x^n + 1}, \qquad x \neq -1$
2. If $\sum a_n$ is absolutely convergent, prove that $\sum a_n \cos nx$ and $\sum a_n \sin nx$ uniformly convergent on $\mathbb{R}$.

Exercises

1. Determine the radius of convergence of the series $\sum \frac{x^n}{n^n}$.
2. For what values of c is the series $\sum \frac{x^n}{n^{\frac{1}{n}}}$ uniformly convergent on $[-c, c]$?