CHAPTER 5 · INTEGRALS

5.3 The Definite Integral

From Riemann sums to a single number: the definition, the notation, the rules, and area under a curve

$$J = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \int_a^b f(x)\,dx$$

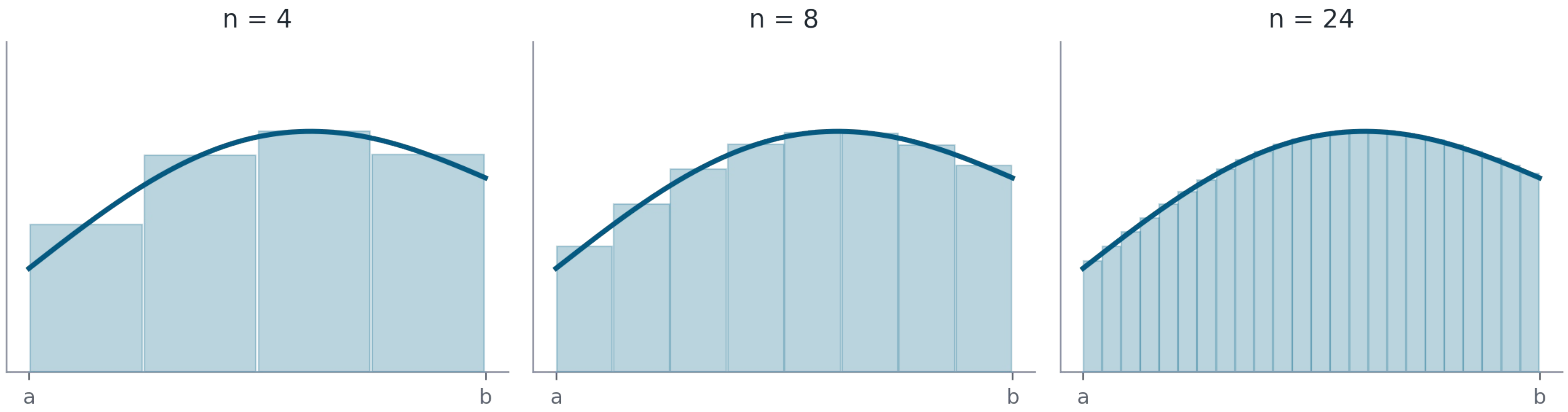
WHERE WE ARE GOING

The road map of this section

1. **The definition** — A number J that every Riemann sum approaches as the partition gets finer.
2. **The notation** — Leibniz's integral sign, the integrand, and the limits of integration.
3. **Which functions work** — Continuous and piecewise-continuous functions are integrable.
4. **The rules** — Seven properties that let us compute integrals without returning to the limit.
5. **Area** — For a nonnegative function, the integral is the area under the graph.

RECALL

Riemann sums with finer and finer partitions



As the norm ‖P‖ of the partition approaches zero, the rectangles approximate the region between the graph and the x-axis more and more closely. Section 5.3 asks what happens in the limit.

Riemann sums with an equal partition

Let f be continuous and nonnegative on the closed interval [a, b], and divide [a, b] into n subintervals of equal width.

1 **Fix the width**

$$\Delta x = \frac{b-a}{n}$$

2 **Locate the partition points**

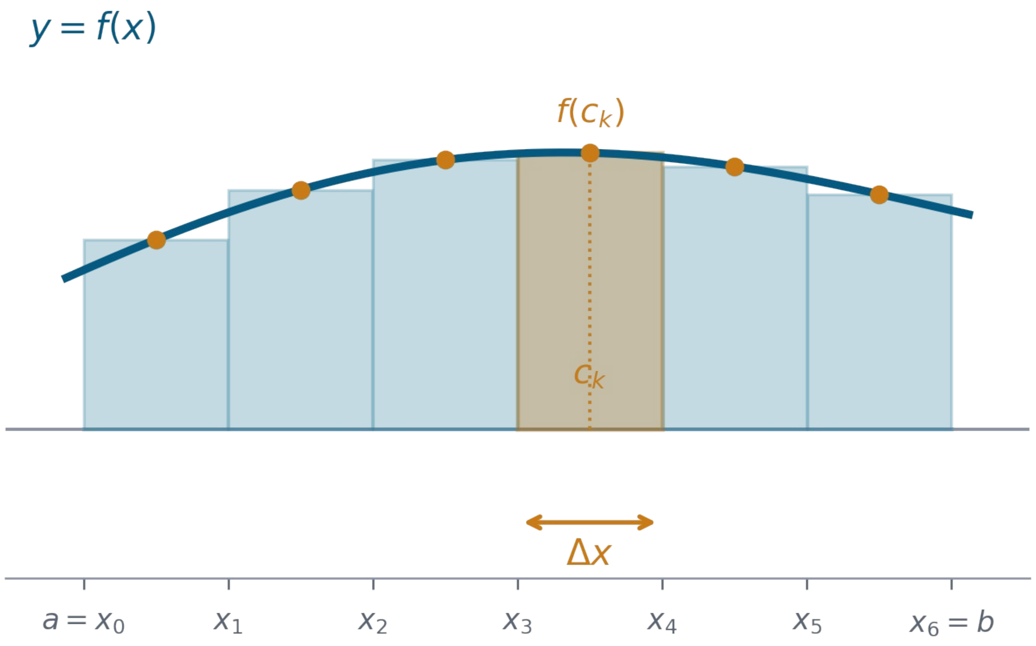
$$x_k = a + k\,\Delta x, \qquad k = 0, 1, 2, \ldots, n$$

3 **Sample a point in each subinterval**

$$x_{k-1} \le c_k \le x_k$$

4 **Add the rectangle areas**

$$S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x = \sum_{k=1}^{n} f(c_k)\,\frac{b-a}{n}$$



Left endpoints, right endpoints, or midpoints all give the same limit.

5 $\lim_{n\to\infty} S_n = \int_a^b f(x)\,dx =$ area under the graph

f continuous ⇒ the limit exists, whatever c_k you pick

DEFINITION

The definite integral

Let f be defined on the closed interval [a, b]. The number J is the definite integral of f over [a, b] if:

for every ε > 0 there is a δ > 0 such that for every partition P of [a, b] with ‖P‖ < δ, and for every choice of c_k in the k-th subinterval,

$$\left|\sum_{k=1}^{n} f(c_k)\,\Delta x_k - J\right| < \varepsilon$$

When this J exists we say f is integrable over [a, b] and we write

$$J = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \int_a^b f(x)\,dx$$

The definite integral exists only when every admissible choice of partition and of the points c_k gives the same limit.

Reading the integral symbol

upper limit b

$$\int_a^b f(x)\,dx$$

lower limit a

integrand f (x) · variable of integration x

Read as

"the integral from a to b of f of x, dee x"

The variable of integration is a dummy variable

$$\int_a^b f(t)\,dt = \int_a^b f(u)\,du = \int_a^b f(x)\,dx$$

The value of the integral depends on f and on the interval — not on the letter used for the variable.

Where the symbol comes from: the sum sign Σ becomes the elongated S of $\int$, the widths Δx_k become dx, and the sampled values $f(c_k)$ become f (x).

Equal-width subintervals

If every subinterval has the same width, the Riemann sum simplifies:

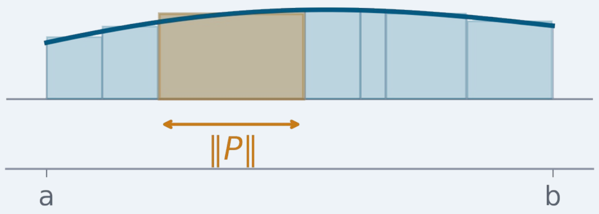
$$S_n = \sum_{k=1}^{n} f(c_k)\,\frac{b-a}{n}, \qquad \Delta x = \frac{b-a}{n}$$

Choosing c_k at the right endpoint of each subinterval gives an explicit formula:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\!\left(a + k\,\frac{b-a}{n}\right)\frac{b-a}{n} \tag{1}$$

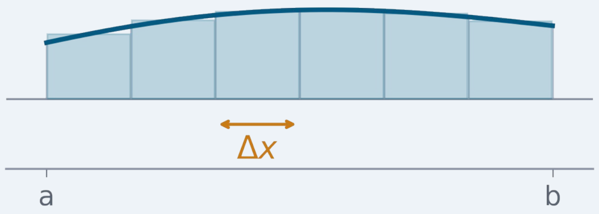
With equal widths, the condition $\|P\| \to 0$ is the same as $n \to \infty$.

A general partition



Widths may differ — the norm ‖P‖ is the largest of them.

An equal partition

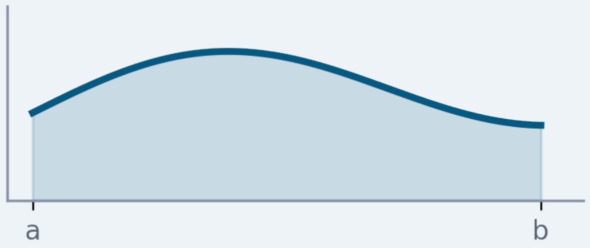


All widths equal, so ‖P‖ = Δx = (b − a)/n.

THEOREM 1

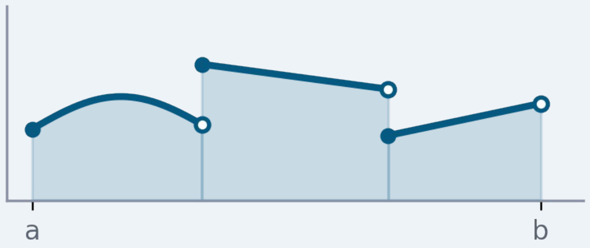
Continuous functions are integrable

If f is continuous on [a, b], then f is integrable on [a, b] and the definite integral exists.



Integrable

Continuous on [a, b]



Integrable

Finitely many jump discontinuities

May fail

Too discontinuous to trap between thin rectangles

Why it works: choosing each c_k to give the largest value of f produces an upper sum, and the smallest value a lower sum. For continuous f these converge to the same limit, and every Riemann sum is trapped between them.

EXAMPLE 1

A function that is not integrable

$$f(x) = 1 \quad \text{if } x \text{ is rational}$$

$$f(x) = 0 \quad \text{if } x \text{ is irrational}$$

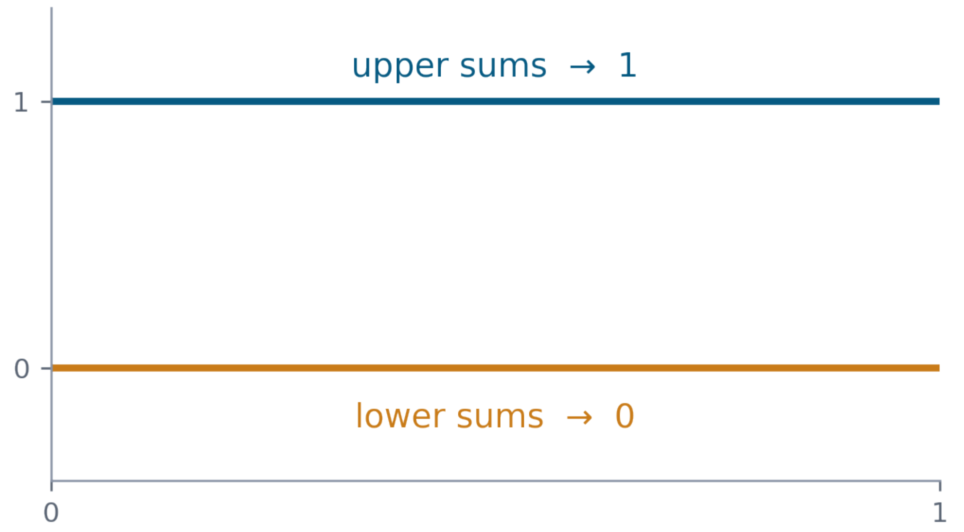
Between any two real numbers there is both a rational and an irrational number, so this function jumps up and down too erratically for thin rectangles to approximate the region beneath it.

Pick every c_k rational → $f(c_k) = 1$

$$U = \sum_{k=1}^{n} (1)\,\Delta x_k = 1$$

Pick every c_k irrational → $f(c_k) = 0$

$$L = \sum_{k=1}^{n} (0)\,\Delta x_k = 0$$



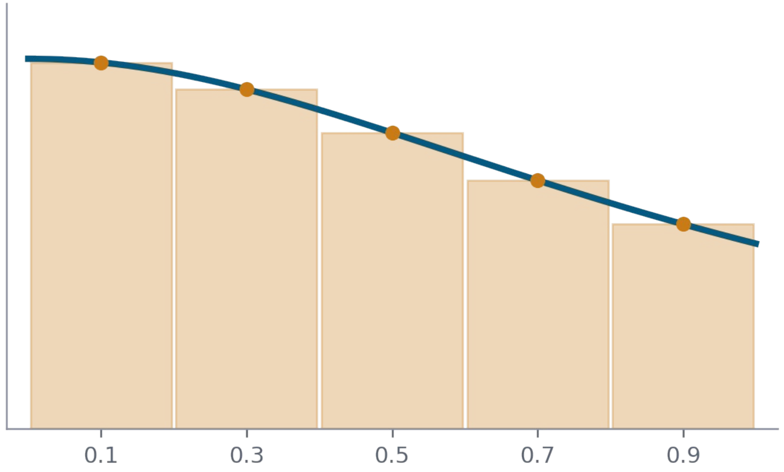
Two choices, two different limits — so no single number J exists and f is not integrable on [0, 1].

The midpoint rule

Taking cₖ at the midpoint of each subinterval usually approximates the integral better than using an endpoint, because on a short interval the value of a linear function at the midpoint is its average value.

$$\int_a^b f(x)\,dx \approx [f(c_1) + f(c_2) + \cdots + f(c_n)]\frac{b-a}{n}$$

$$c_k = \frac{x_{k-1} + x_k}{2}, \qquad x_k = a + k\frac{b-a}{n}$$



y = 1 / (1 + x²) with midpoint rectangles

EXAMPLE 2

Five equal subintervals on [0, 1], midpoints 0.1, 0.3, 0.5, 0.7, 0.9:

$$\int_0^1 \frac{1}{1+x^2}\,dx \approx \left[\frac{1}{1.01} + \frac{1}{1.09} + \frac{1}{1.25} + \frac{1}{1.49} + \frac{1}{1.81}\right](0.2) \approx 0.786$$

TABLE 5.6

Rules satisfied by definite integrals

1 **Order of integration**

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

a definition

2 **Zero width interval**

$$\int_a^a f(x)\,dx = 0$$

a definition

3 **Constant multiple**

$$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$$

any constant k

4 **Sum and difference**

$$\int_a^b (f \pm g)dx = \int_a^b f\,dx \pm \int_a^b g\,dx$$

5 **Additivity**

$$\int_a^c f\,dx + \int_c^b f\,dx = \int_a^b f\,dx$$

[a,c] ∪ [c,b] = [a,b]

6 **Max-min inequality**

$$(\min f)(b-a) \le \int_a^b f\,dx \le (\max f)(b-a)$$

7 **Domination**

$$f \ge g \;\Rightarrow\; \int_a^b f\,dx \ge \int_a^b g\,dx$$

Rules 1 and 2 are definitions; Rules 3–7 are theorems, and all seven hold whenever f and g are integrable on [a, b].

EXAMPLE 3

Using the rules

Suppose

$$\int_{-1}^{1} f(x)dx = 5, \quad \int_{1}^{4} f(x)dx = -2, \quad \int_{-1}^{1} h(x)dx = 7$$

1. $\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = 2$ — **Rule 1**

2. $\int_{-1}^{1} [2f(x) + 3h(x)]dx = 2(5) + 3(7) = 31$ — **Rules 3 and 4**

3. $\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f\,dx + \int_{1}^{4} f\,dx = 5 + (-2) = 3$ — **Rule 5**

EXAMPLE 4

Bounding an integral without evaluating it

Show that the integral below is at most 2.

The max-min inequality (Rule 6) bounds an integral by the extreme values of f:

On [0, 1] the cosine is at most 1, so

$$\cos x \le 1 \;\Rightarrow\; 1 + \cos x \le 2 \;\Rightarrow\; \sqrt{1 + \cos x} \le \sqrt{2}$$

Taking max f = √2 in Rule 6:

$$\int_0^1 \sqrt{1 + \cos x}\, dx \le \sqrt{2}\,(1 - 0) = \sqrt{2} \le 2$$

Rules 6 and 7 bound an integral without evaluating it — the only tools in this section that do.

Rule 6

$$(\min f)(b - a) \le \int_a^b f\, dx \le (\max f)(b - a)$$

The integral is never smaller than the minimum of f times the length of the interval, and never larger than the maximum of f times that length.

Why it matters

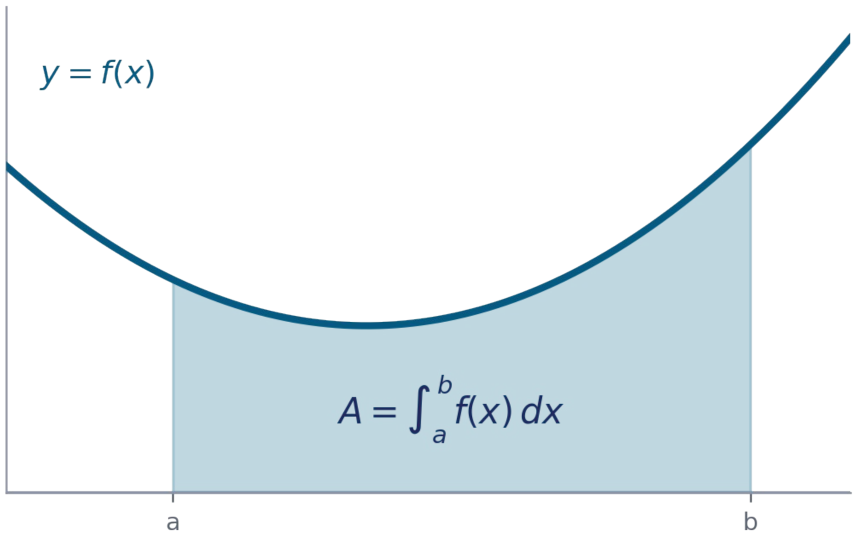
A bound is often all an application needs — and it costs nothing even when the integral has no closed form.

Area under the graph of a nonnegative function

DEFINITION

If y = f (x) is nonnegative and integrable over [a, b], the area under the curve over [a, b] is

$$A = \int_a^b f(x)\,dx$$



This is the first rigorous definition we have for the area of a region whose boundary is a curve. Theorem 1 is what makes it well posed: every Riemann sum — upper, lower, or midpoint — converges to the same integral.

If f takes negative values the integral still exists, but it no longer measures area — it counts area below the x-axis negatively.

EXAMPLE 5

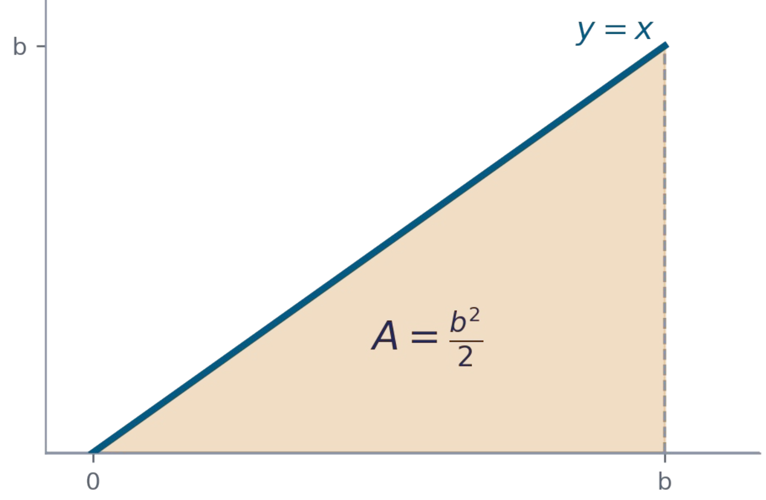
Computing ∫ x dx from the definition

Partition [0, b] into n equal subintervals and take c_k at the right endpoint:

$$\sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} \frac{kb}{n} \cdot \frac{b}{n} = \frac{b^2}{n^2} \cdot \frac{n(n+1)}{2} = \frac{b^2}{2}\left(1 + \frac{1}{n}\right)$$

Letting n → ∞ gives

$$\int_0^b x\,dx = \frac{b^2}{2}$$



The region is a triangle of base b and height b, so the geometry formula gives the same answer — our definition agrees with the area we already knew.

Three integrals you can now quote

$$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2} \tag{2}$$

valid for a < b, and also when a or b is negative

$$\int_a^b c\,dx = c\,(b-a) \tag{3}$$

c is any constant — the region is a rectangle

$$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3} \tag{4}$$

obtained by the same Riemann sum argument

With Table 5.6 these three results evaluate a large share of the exercises in this section.

SUMMARY

What section 5.3 gives us

A number, not a sum — The definite integral is the common limit of all Riemann sums as $\|P\| \to 0$.

A guarantee — Continuous functions — and functions with finitely many jumps — are integrable.

A toolkit — Seven rules in Table 5.6 let us combine and bound integrals algebraically.

A definition of area — For $f \geq 0$, the area under the graph is the integral.

Next: Section 5.4 — the Fundamental Theorem of Calculus, which turns the definition into a method of calculation.